Existence, Uniqueness, and Symmetry of Free-Entry Cournot Equilibrium: The Importance of Market Size and Technology Choice

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Abstract

This article adds technology choice to a free-entry Cournot model with linear demand and constant marginal costs. Firms can choose from a discrete set of technologies. This simple framework yields non-existence of (pure strategy) equilibrium, existence of multiple equilibria and equilibria in which ex-ante identical firms choose different technologies as possible outcomes. The (non-)existence problem disappears if vertical market size is large. Non-existence is largely a 'small number' phenomenon. Asymmetric equilibria emerge either because of indivisibilities or due to similarity of different technologies in terms of the average costs realized.

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1 Introduction

Existence and uniqueness of Cournot equilibrium are topics of a long and ongoing debate. The problem is that even for well-behaved preferences, ‘examples of duopoly models in which no Cournot equilibrium (in pure strategies) exist are easily produced’ (Vives, 1999, p. 94). Typically, the approaches dealing with existence start from a given number of firms. They consider the quantity setting game, taking as given technologies, i.e. the cost functions. While firms may differ, these differences are usually given exogenously (see, for instance, Novshek, 1985). From an Industrial Organization perspective it is interesting to know whether and under what conditions both market structure and technology can be endogenized without running into existence problems. Is it easily possible to add an additional stage which deals with entry and technology choice?

From the above quote as well as from the well-known problem of non-existence of a pure strategy equilibrium in research tournaments without uncertainty (see Dasgupta and Stiglitz, 1980a), one might conclude that the non-existence problem becomes even more severe. I assess how important the problem is by analyzing the simple and standard case of linear demand and constant marginal costs. As regards technology choice, I assume that firms can choose from a set of two technologies, a large-scale and a small-scale technology.

This simple framework is quite rich in terms of the patterns of existence and uniqueness of (pure strategy) equilibrium it yields. Non-existence of equilibrium in pure strategies, existence of multiple equilibria and equilibria in which ex-ante identical firms choose different technologies are possible outcomes depending on the parameters. I characterize parameter sets for which these outcomes arise.

There are two main findings with respect to non-existence of equilibrium: First, the

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1See the recent article by Long and Soubeyran (2000) for a list of contributions.
2Long and Soubeyran (2000) claim that an advantage of their approach is that the equilibrium is characterized in terms of marginal costs. According to these authors, this facilitates the study of a class of two-stage Cournot games (see p. 345).
existence problem disappears if vertical market size is larger than a certain threshold. Vertical market size is measured here by the vertical intercept of the demand curve. Second, non-existence is largely a 'small number’ phenomenon. One may well construct examples for which an equilibrium fails to exist for arbitrarily large numbers of firms. However, the range of parameter values where non-existence may occur is large only if the market supports only a few large-scale firms. This result is quite important in the light of the wide-spread use of duopoly models. An interesting example is a recent paper by Mills and Smith (1996). In a Cournot duopoly model with technology choice, Mills and Smith characterize conditions under which ex-ante identical firms choose different technologies. My results show that the characterization is incomplete as long as entry is not explicitly accounted for. The article also derives a condition, which provides an easy check for the existence of equilibrium.

Concerning uniqueness of equilibrium I characterize the market size range in which multiple equilibria are likely to exist. It turns out that multiplicity of equilibrium requires that the different types of firms do not differ much in terms of the average costs realized in an equilibrium in which only one technology is available. Consequently the performance, i.e., prices and market output, of very different industry structures may be almost identical, a point also made by Davis (1999).

A final topic addressed in this article concerns the question under what conditions ex-ante identical potential entrants end up employing different technologies. I derive a sufficient condition for an equilibrium to exist in which all firms choose the same technology. I call such an equilibrium symmetric. From an empirical point of view, the most interesting parameter values seem to be those for which this sufficient condition does not hold. In the respective range a heterogeneous industry structure arises endogenously. The article shows that the results of Mills and Smith (1996) generalize even to the case of free entry. My model provides an endogenous explanation of the differences in firm size frequently found in many industries (see, e.g., Sutton, 1998) within a framework of technology choice. By explicitly allowing for heterogeneity, it differs from the
main body of the literature on technology choice and R&D activities, respectively, under Cournot competition. In that literature, most authors either directly assume that a unique and symmetric equilibrium exists (see, e.g., Okuno-Fujiwara and Suzumura, 1993) or they ensure existence of such an equilibrium by making strong assumptions on the - typically continuous - set of available technologies (see, e.g., Dasgupta and Stiglitz, 1980b). Consequently, these models cannot account for heterogeneity among firms by assumption.

The remainder of the paper is organized as follows. Sections 2 and 3 present the basic model and introduce entry. Section 4 derives a sufficient condition for a unique and symmetric equilibrium to exist. Section 5 discusses non-existence of equilibrium, co-existence of different types of firms in equilibrium and non-uniqueness of equilibrium. It proves that non-existence vanishes for large values of vertical market size and presents an example on the importance of non-existence. The example also demonstrates for which parameter values both asymmetric and multiple equilibria arise. Section 6 concludes.

2 The model

Consider an industry which produces a homogeneous product. The inverse demand function is

\[ p(y) = a - \frac{y}{s}, \]

where \( p \) and \( y \) respectively denote the price and the aggregate demand of the product. The demand function exhibits two market size parameters, \( a \) and \( s \). The parameter \( a \) accounts for what I call vertical market size. It measures the maximum willingness to pay for that product. The parameter \( s \) is a measure of horizontal market size. One can think of it as the number of (identical) consumers. Firms may choose from a set of two technologies, a 'small-scale' and a 'large-scale' technology. The constant marginal costs associated with the small-scale technology are \( c > 0 \). Firms entering with the small-
scale technology incur fixed costs $f_S$. The marginal costs for large-scale producers are zero. Their fixed costs are denoted as $f_L$. Of course, $f_L > f_S$. The overall number of (active) firms is denoted by $n$, $m$ describes the number of small-scale (or $S$-) firms and $n-m$ the number of large-scale (or $L$-) firms.

Firms’ profits depend on the technology they have chosen. The profit functions are:

$$\Pi_j(y_1, \ldots, y_n) = (a - \frac{\sum_{i=1}^{n} y_i}{s})y_j - cy_j - f_S, \quad j \in M$$ (2)

$$\Pi_i(y_1, \ldots, y_n) = (a - \frac{\sum_{j=1}^{n} y_j}{s})y_i - f_L, \quad i \in N \setminus M,$$ (3)

where $y_k$ is the output of firm $k = 1, \ldots, n$. $M$ is the set of $S$-firms and $N := \{1, \ldots, n\}$.

The equilibrium quantity of an $S$-firm is

$$y_S(m, n) = \frac{s(a - c - (n - m)c)}{n + 1} \quad \forall m = 1, \ldots, n.$$ (4)

The equilibrium quantity of an $L$-firm is

$$y_L(m, n) = \frac{s(a + mc)}{n + 1} \quad \forall m = 0, \ldots, n - 1.$$ (5)

Substituting $y_l$ and $y_h$ into (2) and (3) leads to equilibrium profits of $S$ and $L$ firms as a function of the respective firm numbers:

$$\Pi_S(m, n) = \frac{s}{(n + 1)^2} (a - c - (n - m)c)^2 - f_S \quad \forall m = 1, \ldots, n,$$ (6)

and

$$\Pi_L(m, n) = \frac{s}{(n + 1)^2} (a + mc)^2 - f_L \quad \forall m = 0, \ldots, n - 1.$$ (7)

3 Entry

Suppose entry into the market is free, and a large number of identical potential entrants exists. A potential firm can either stay out or enter as an $S$-firm or an $L$-firm. In a free-entry-equilibrium $(m, n)$ the zero-profit conditions

$$\Pi_S(m + 1, n + 1) < 0 \leq \Pi_S(m, n)$$ (8)
and
\[ \Pi_L(m, n + 1) < 0 \leq \Pi_L(m, n) \quad (9) \]
must hold. Given the equilibrium configuration \((m, n)\) a potential entrant must not have an incentive to enter as either an \(S\) or an \(L\)-type firm.

**Definition 1** A candidate equilibrium is a configuration \((m, n)\) that satisfies the zero-profit conditions (8) and (9).

A candidate equilibrium is of some interest on its own. It constitutes the equilibrium of a game with a large population of potential entrants of two different types. The types, i.e. the technology of the respective firms, are exogenous. Later on I clarify the difference between models with exogeneous and endogenous technology, respectively.

In my model with technology choice, an equilibrium configuration \((m, n)\) must additionally satisfy the no-switching conditions
\[ \Pi_S(m, n) \geq \Pi_L(m - 1, n) \quad (10) \]
and
\[ \Pi_L(m, n) \geq \Pi_S(m + 1, n). \quad (11) \]

\(S\)-firms must not have an incentive to employ the \(L\)-technology, and \(L\)-firms must not have an incentive to employ the \(S\)-technology.

As the above description makes clear, I consider a two-stage game. In stage 1, firms decide on entry and technology. In the second stage, firms choose their output levels. Each equilibrium of the game with endogenous technology is, of course, an equilibrium of a game with exogenous technologies and given types. Therefore, the equilibria of the endogenous technology case also indicate possible equilibrium configurations for an environment with exogenous heterogeneity, i.e. ex-ante heterogeneity.
4 Existence of a unique and symmetric equilibrium

In this section I provide a sufficient condition for a unique equilibrium to exist in which all active firms choose the same technology. I call such an equilibrium symmetric. The derivation of the condition first proceeds in a graphical way, before it is stated and proved in more formal terms. The graphical analysis provides some intuition for the requirements of a symmetric equilibrium.

Let \( T_S = c + \sqrt{f_S/s} \) and \( T_L = \sqrt{f_L/s} \). \( T_S \) and \( T_L \) denote the average costs realized by an \( L \)-firm and an \( S \)-firm, respectively, in a free entry equilibrium in which only the respective technology is used. \( T_L = T_S \) is clearly a knife-edge case. The relation between \( T_S \) and \( T_L \) determines which of the two technologies is, roughly speaking, the efficient one. Furthermore, define \( D \equiv (a/T_L) - (a/T_S) \). As will become clear below, \( D \) gives a measure of the cost difference between the two technologies.

4.1 Graphical analysis

4.1.1 \( T_S > T_L, D > 1 \): Only the \( L \)-technology is employed in equilibrium

For the graphical analysis I display the equations \( \Pi_S(m, n) = 0 \) and \( \Pi_L(m, n) = 0 \) in \((n, m)\)-space. Using equation (6), \( \Pi_S(m, n) = 0 \) yields \( m = n(T_S/c) + (T_S - a)/c \). Using equation (7), \( \Pi_L(m, n) = 0 \) yields \( m = n(T_L/c) + (T_L - a)/c \). The two lines intersect at \((n, m) = (-1, -a/c)\). Figure 1 depicts the two lines for \( T_S > T_L \). They intersect the horizontal axis at \(-1 + a/T_S \) and \(-1 + a/T_L \). Therefore, the horizontal distance between the two lines at \( m = 0 \) is \( D \). Note that the lines have a positive slope greater than 1. This follows from the fact that profits can be kept constant with increasing \( n \) if \( m \), the number of small firms, increases faster than \( n \). Profits of the respective firm types are negative in the area below the respective zero-profit

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3 The equilibrium is unique up to a change of the 'identity' of the active firms.

4 Average cost are calculated here neglecting the integer constraint.

5 The following derivation ignores the integer constraint. It is taken into account in the formal derivation below.

6 The slope of locus \( \Pi_L(m, n) = 0 \) is smaller than one if \( c > T_L \). In this case the reduced profit function derived in equation (7) does not apply, as it would require negative output of small firms.
line. Since less small firms are active in that region, while the total number of firms is unchanged profits must be smaller than on and above the locus.

Figure 1 about here

Given the configuration in Figure 1 S type firms cannot be active. To see this, note that the use of technology $S$ by some firms would require a configuration $(m, n)$ on or above the locus $\Pi_S(m, n) = 0$. Otherwise $S$-type firms would make losses. However, such a configuration would trigger entry of large-scale firms. Take, for instance, the horizontal arrow in Figure 1. Entry of an $L$-firm implies that $(m, n)$ changes in the direction of this arrow. It immediately follows that at least one firm could enter using the $L$-technology provided that the horizontal distance between the two lines at the respective value of $m$ is greater than 1. The resulting configuration $(m, n + 1)$ implies positive profits for the entrant. A sufficient condition for the horizontal distance between the two lines to always be greater than one is that $D$ is greater than one.

Next, I show that only $L$ firms are active is indeed an equilibrium. For this configuration to be an equilibrium an $L$-type firm must not have an incentive to switch to the $S$-technology. Otherwise an equilibrium would fail to exist since a switch would trigger further entry. The vertical arrow in Figure 1 indicates the direction of the change if a large firm were to switch. The arrow starts at $(n, m) = (-1 + a/T_L, 0)$, i.e., at the free entry number of $L$-type firms if only this technology is used. If the vertical distance between the two lines for $n = -1 + a/T_L$ is greater than 1 such a move cannot be profitable. To see this, note that the resulting configuration $(1, n)$ would lie below the zero-profit line for $S$-technology firms. Again, the assumption $D > 1$ together with the fact that the slope of the locus $\Pi_S(m, n) = 0$ is greater than the slope of locus $\Pi_L(m, n) = 0$ guarantees that the (vertical) distance is greater than 1. This establishes the existence of a symmetric equilibrium with only $L$-type firms. The condition on $D$ shows what it takes in terms of a cost disadvantage in order to keep 'inefficient' firms out of the market. This is equivalent to guaranteeing the existence of a unique
and symmetric equilibrium.

4.1.2 $T_S \leq T_L$

For the respective parameter values only $S$-type firms can be active in equilibrium. To see this, consider the knife-edge case $T_S = T_L$. In this case the zero-profit curves coincide. Consequently, $L$ type firms cannot be active in equilibrium. This follows from the fact that profits of all firms inclusive of the switching firm increase as soon as an $L$-type firm switches to the $S$-technology. Starting from a situation of zero profits it is always profitable for a large firm to switch to the small-scale technology. Irrespective of what the number of $L$-firms is in the candidate equilibrium given $n$, deviation is a dominant strategy. The equilibrium is reached when only $S$-firms are active. Neither switching nor entry would be profitable. Switching to a large-scale technology given the total number of firms depresses profits of all firms inclusive of the switching firm. Further entry is not possible by construction of the zero-profit curves.

The graphical analysis reveals an important difference between the cases with endogenous and exogenous technology, respectively. With exogenous technology the condition $T_S \leq T_L$ is insufficient to guarantee that the 'inefficient' technology, the $L$-technology, is not employed in equilibrium. 7 Consider again the knife-edge case $T_S = T_L$. With exogenous technology only large firms are active is an equilibrium. The entry of neither large nor small firms is possible. Starting from a situation when only large firms are active, entry of a small firm would lead to a movement along an arrow with slope 1. As the slope of locus $\Pi_S(m, n) = 0$ is greater than 1, a small entrant cannot break even. With exogenous heterogeneity, uniqueness of equilibrium requires a sufficient distance between the two loci in the case $T_S \leq T_L$ as well. Only if the cost advantage of the 'efficient' technology, the $S$-technology, is sufficiently large, only large firms are active cannot be an equilibrium.

The analysis also reveals an asymmetry between the case where the $S$-technology

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7Note that only small firms are active is an equilibrium anyway as it is the equilibrium in the endogenous technology case.
and that where the *L*-technology is the efficient one. If the small-scale technology is efficient, the result obtained is similar to that of models with a continuum of firms (see Elberfeld and Götz, 2002). Only the efficient technology is employed in equilibrium.\(^8\) If the *L*-technology is the efficient one the results differ from, for instance, a model of perfect competition with a continuum of firms. In our oligopoly model with large agents even 'inefficient' firms may be viable in the long run, if the cost difference is not too large.

4.2 Formal analysis

Proposition 1 states and proves the results for the endogenous technology case in more formal terms. It also takes the integer constraint into account. Therefore the value of \(D\) which is sufficient for existence and symmetry is greater than in Figure 1.

**Proposition 1** If \(D \geq 2\), a unique equilibrium exists and in equilibrium all firms use the *L*-technology. If \(T_s \leq T_L\), a unique equilibrium exists and in equilibrium all firms use the *S*-technology.

**Proof** see Appendix.

Having derived the conditions under which a unique and symmetric equilibrium exists, I examine how different variables affect these conditions. This is closely related to the question how technology choice depends on the parameters in general.

The two market size parameters \(a\) and \(s\) have quite different effects as far as uniqueness and symmetry are concerned. Vertical market size \(a\) does not affect technology choice directly as it does not enter \(T_s\) or \(T_L\). However, increasing \(a\) makes it ever more likely that a symmetric equilibrium exists. This follows immediately from the fact that \(D\) is strictly increasing in \(a\). Horizontal market size \(s\) has a clear impact on technology choice. Increasing \(s\) eventually makes the large-scale technology superior. It is with respect to horizontal market size that the statement holds that large markets give rise

\(^8\)Note that one obtains a unique equilibrium even for the knife-edge case \(T_L = T_S\). This is the only case in which equilibrium is not unique in a framework with a continuum of firms!
to the use of large-scale technologies. Symmetric equilibria exist for small and for large values of $s$ but not for intermediate values. In the limit, as either $a$ or $s$ approach infinity, a unique and symmetric equilibrium exists. Large vertical market size does not determine which technology is used. This property of the linear demand model has also been documented by Neumann et al. (2001). They show that changes in the parameter $a$ leave firm size and firm R&D expenditures constant. Only the number of firms changes with $a$.

The cost parameters affect technology choice in the way one would expect. If either marginal or fixed costs of a technology decrease it is more likely that the respective technology is used. A change in a cost parameter that increases the differences in average costs makes existence of a unique and symmetric equilibrium more likely.

### 4.3 More than two technologies

Now I extend Proposition 1 to the case of $k$ different technologies. For that purpose I modify the above notation in a straightforward way. Suppose technology type $t$, where $t = 1, \ldots, k$, has fixed costs $f_t$ and (constant) marginal costs $c_t$. Let $T_t = c_t + \sqrt{f_t}/s$.

Again, $T_t$ denotes average costs in the free-entry equilibrium with technology $t$. Define $T_i = \min\{T_1, \ldots, T_k\}$, $T_j = \min\{T_i, \ldots, T_i - 1, T_i + 1, \ldots, T_k\}$ and $D_t \equiv a/T_t - (a/T_i)$ for all $t = 1, \ldots, k; t \neq i$.

**Proposition 2** If $D_j \geq 2$, then an equilibrium exists and in equilibrium all firms use technology $i$.

**Proof** Note that $D_j < D_t$ for all $t = 1, \ldots, k; t \neq i, j$ by definition. The Proposition then follows immediately from the proof of Proposition 1. By the assumption on $D_j$ a deviation to technologies with greater marginal costs and smaller fixed costs cannot be profitable. Taking into account that $D_j \geq 2$ implies $T_i < T_t$ for all $t = 1, \ldots, k; t \neq i$, technologies with lower marginal but higher fixed costs cannot be profitable either. $\square$

The condition employed in Proposition 2 is more restrictive than the respective condition of Proposition 1. The main purpose of Proposition 2 is to show that the
above arguments easily extend to more general cases. Two consequences of Proposition 2 are worth mentioning. First, technology choice and therefore industry structure may change quite often as a function of horizontal market size $s$. Of course, this requires that the various technologies are important in the sense that they provide the minimum average costs for some values of $s$. Second, the range for which the sufficient conditions for existence of a unique and symmetric equilibrium does not hold increases if more technologies exist. Thus, it is even more important in the case with $k$ technologies to examine what happens if the sufficient condition is not satisfied. I turn to this in the next section. The analysis will be constrained to the case of two technologies, which as above are labeled $L$- and $S$-technology.

5 Non-existence, non-uniqueness, and asymmetry of equilibrium

In this Section I examine the outcomes in the range where the sufficient condition is not satisfied. The result that is probably the most interesting one from a theoretical and methodological view concerns the non-existence of a pure strategy equilibrium for certain parameter ranges. From an empirical point of view, the asymmetric equilibria arising in large part of the range where the sufficient condition does not hold, are well worth mentioning. They reproduce the heterogeneous structures found in many industries (see, e.g., Sutton 1998). In the relevant range ex-ante identical firms end up with different amounts of output. Thus, one obtains an endogenous explanation of firm size differences leading to an asymmetric industry structure. The final outcome I shall discuss concerns uniqueness of equilibrium. Examples show that one often obtains multiple equilibria in the range considered in this section. This is particularly true if the candidate equilibria involve several large firms. The interesting thing about these equilibria is that industry performance measured by the equilibrium price is approximately the same in the different equilibria while the implied industry configurations may be quite different.
In the following subsections, I examine the three possible outcomes in some detail. The analysis proceeds mainly by means of examples. After excluding the range for which general results are easy to derive, general results are hardly possible. There is one important exception, however, as far as non-existence is concerned. In the next subsection I show that an equilibrium always exists if vertical market size \( a \) is greater than some threshold value.

### 5.1 Non-existence of equilibrium

I start with two examples of non-existence. In order to evaluate how important non-existence is, I depict the ranges of horizontal and vertical market size for which non-existence arises, given the cost parameters of the examples. Next, I show that non-existence does not arise for 'large' values of vertical market size \( a \). Finally, I discuss the parameters which determine how large \( a \) must be in order to guarantee existence.

The examples assume the following cost parameters: \( f_S = 5, c = 2, f_L = 2050 \).

**Example 1:** A candidate equilibrium with one small and one large firm.

The vector of vertical and horizontal market size reads \((a, s) = (4.3443, 517.954)\). With these values the free entry number of \( S \) and \( L \)-type firms if only the respective technology were available is 22 and 1 firm, respectively. To see that \((m, n) = (1, 2)\) is a candidate equilibrium note that the profit of both the small and the large firm is positive \((\Pi_S(1, 2) = 1.822, \Pi_L(1, 2) = 266.414\), resp.), and that neither a small \((\Pi_S(2, 3) = -1.163)\) nor a large firm \((\Pi_L(1, 3) = -963.854)\) could profitably enter. To see that an equilibrium with endogenous technology does not exist, consider the entry decision and technology choice of the pivotal firm. Given a situation in which (only) one \( S \)-type firm will enter the market, the optimal decision of the pivotal firm is to enter as an \( S \) firm since \((\Pi_S(2, 2) = 311.282 > \Pi_L(1, 2) = 266.414)\). In loose terms one could say that the large firm of the candidate equilibrium has an incentive to switch to

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\( ^9 \) In the case of exogenous technology this experiment leads to existence. Just pick an \( S \)-type and an \( L \)-type entrant and assume that all other potential entrants stay out of the market. Given the described behavior of all other agents, it is optimal for all agents to choose the (candidate) equilibrium strategies.
the small scale technology. Non-existence of an equilibrium follows from the fact that further entry would occur if only two small firms were active.\footnote{Note that a single large firm could always break even as long as only small firms are active. This prevents 'only small firms are active' from being an equilibrium.}

Figure 2 presents the region in the \((a, s)\)-space in which non-existence applies since a candidate equilibrium with one large and either no, one or two small firm fails to be an equilibrium of the game with endogenous technology. As in the example above the reason for non-existence is the incentive to switch. Locus \(A\) depicts the condition that one large firm is viable. In the region to the right of \(V_1\) and \(V_2\) one and two small firms, resp. are viable given that one \(L\)-type firm is active. Non-existence arises in the area between \(A\) and \(s_1\) due to the fact that a large firm (which is viable to the right of \(A\)) has an incentive to switch to the \(S\) technology. Locus \(s_1\) captures the respective no-switching condition for the case in which no small firm is active; switching applies below the locus. If one small firm is active, the respective switching condition is denoted as \(s_{1,1}\), if two \(S\) firms are active, the condition is \(s_{1,2}\). The non-existence area is bounded from below by locus \(C\), which captures the constraint that 'only small firms are active' is an equilibrium of the game. For values of (horizontal) market size above locus \(C\) the pivotal firm has always an incentive to switch from the small scale to the large scale technology. Economically, an increase in \(s\) makes it ever more attractive to switch to the large scale technology. The arrows indicate the non-existence areas. The above example is taken from the region where the center arrow applies, the non-existence area indicated by the right arrow is so small that it is hardly visible. For parameter values where three or more small firms were viable non-existence cannot arise.

**Example 2:** Candidate equilibria with 18 and 19 large firms, resp.

Now I consider the market size vector \((a, s) = (40.079, 510.587)\). The free entry number of \(S\) and \(L\)-type firms if only the respective technology were available is 383 and 19 firm, respectively. This parameters give rise to two candidate equilibria, namely
In order to show non-existence I start with the second configuration in which 18 large firms are active. In this case, it is a small firm that has an incentive to switch to the large scale technology. To see this, note that \( \Pi_S(2,20) = 0.004 < \Pi_L(1,20) = 0.038 \). Therefore, neither the candidate equilibrium \((m,n) = (2,20)\) nor the configuration \((m,n) = (1,20)\) is an equilibrium. The latter result follows from the fact that the profit of the small firm would be negative (-4.993). But the candidate equilibrium in which only large firms are active is not an equilibrium either. With this configuration a large firm has an incentive to switch to the small scale technology. This follows from \( \Pi_S(1,19) = 0.517 > \Pi_L(0,19) = 0.423 \). This switch would trigger entry; an entrant choosing the \( L \) technology would earn profits of 0.038. An equilibrium does not exist.

The example also yields an intuition as to why non-existence arises: As long as many rivals are active (in the example: 18 \( L \) type and 1 \( S \) type firms), the pivotal firm chooses the large-scale technology. Given this level of investment, rivals (in the example: one rival) do not have an incentive to enter in the first place. However, if there are less rivals (18 \( L \) type firms), it is optimal for the pivotal firm to switch to the small-scale technology.

**Figure 3 about here**

Figure 3 depicts the regions in the \((a,s)\)-space where non-existence applies for our set of cost parameters. As will be shown below, the maximum number of large firms, for which non-existence can arise, equals 20 for the values of fixed cost assumed above. The triangular-shaped areas are the areas where non-existence occurs. The largest 'triangle' is a reproduction of the region depicted in Figure 2; one large firm is active in the respective candidate equilibrium. The non-existence regions end with an area which looks more like a dot in the diagram. The values of Example 2 are taken from this area, the respective candidate equilibrium exhibits (at the most) 19 large firms. Figure 3 shows that non-existence arises for all candidate equilibria in which between
one and 19 large firms (and no small firms) are active. The constraint provided by locus $V_1$ (a small firm is viable given the respective number of large firms) is binding only for the case of one to four large firms.\footnote{Note that in the part of the triangular regions of Figure 3 that lies to the right of the (dashed) locus $V_1$ non-existence does not always arise. As in Figure 2 one would have to consider additional constraints (analogous to $s_{1,1}$). The relevant areas would hardly be visible.}

As shown above, non-existence requires that (either $S$ or $L$-type) entry occurs once a large incumbent switches to the small scale technology. The importance of this condition becomes apparent in the case of a candidate equilibrium with 20 large (and no small) firms. Non-existence does not arise because neither a small nor a large entrant would be viable if the pivotal firm were to switch to the $S$-technology. Example 2 shows that 'destroying' the candidate equilibrium with only large firms active (call the respective number of large firms $\tilde{n}$) is not sufficient to generate non-existence. An equilibrium could exist in which $\tilde{n} - 1$ large and some small firms are active. Locus $B$ is the lower envelope of curves which ensure in such a situation that a small firm had an incentive to switch to the $L$-technology. Candidate equilibria in which $L$-type and $S$-type firms coexist are destroyed by the incentive to switch for a small firm for (horizontal) market size values above $B$. An array of curves results when the number of small firms in such a candidate equilibrium is varied (from one to the maximum number of small firms viable if $\tilde{n} - 1$ large firms are active).

The above example indicates two things. First, non-existence does not seem to arise for large values of vertical market size $a$. Indeed, Proposition 3 below shows that, given cost parameters, a threshold value $a^*$ exists such that non-existence cannot occur for $a > a^*$. Second, it demonstrates that the area where non-existence occurs is 'small', in general, with the exception of candidate equilibria with a small number of large firms. Non-existence is mainly a small number problem.

In the following I demonstrate that an equilibrium of the whole game always exists in markets of a sufficiently large vertical size. What is 'sufficient' is shown below.

**Proposition 3** Given the cost parameters of the model, an $a^*$ exists such that for all
$a > a^*$ an equilibrium of the two-stage game exists.

Proof see Appendix.

It is instructive to look at how the proof proceeds. The proof starts with the necessary conditions for a candidate equilibrium $(m, n)$ not to be an equilibrium of the game (equations (29) and (30)). The conditions yield a maximum value for $a$ denoted as $\bar{a}$ such that non-existence can arise. I then show that for sufficiently large $n$, $n$ firms are not viable, given the market size vector ensuring non-existence. This is a contradiction to the assumption that the configuration is a candidate equilibrium. This result yields an intuitive explanation as well. Remember that changes in $a$ leave the relative profitability of the two technologies unaffected. Therefore, an increase in the number of large firms requires a roughly proportional change in market size for the 'switching'-condition still to hold. Increases in the number of firms, however, imply that the market becomes more competitive in the sense that price-cost margins decrease. Thus, a given change in the number of firms requires a more than proportional increase in market size in order for the larger number of firms to be viable.\(^{12}\) As a result, the conditions for non-existence to arise eventually cease to hold as the number of (large) firms increases.

It is possible to explicitly determine the maximum number $n^*$ of $L$-type firms for which non-existence may arise. It can be shown that $n^*$ is the (positive) root of the equation\(^{13}\)

\[
(1 + 4n + 12n^2 + 20n^3 + 24n^4 + 16n^5 + 4n^6) f_S \\
- (1 + 8n + 20n^2 + 16n^3 + 4n^4) f_L = 0. \quad (12)
\]

Two points deriving from equation (12) are worth being mentioned. First, $n^*$ depends only on the ratio $f_L/f_S$. Second, $n^*$ is of order $\sqrt{f_L/f_S}$. $n^*$ may well grow without

\(^{12}\)Actually, $\bar{a}$, capturing the non-existence condition, is strictly concave in the number of large firms (see equation (33)), whereas the zero-profit condition is linear in $n$.

\(^{13}\)The equation derives from setting $\Pi_L(0, n)$ as defined in equation (34) in the Appendix equal to 0.
bound. However, this requires quite a large difference between the two technologies in terms of their respective fixed costs. Empirically, a ratio of $f_L/f_S \approx 400$ as in the above example seems to be quite a large number. And even in this case, $n^*$ is only 20. The reason why $n^*$ increases with $f_L/f_S$ is straightforward: switching technologies is more likely to trigger entry (or exit) if the differences between technologies, and therefore also firm output, are large.

Given $n^*$, we can immediately derive $a^*$. Substituting $n^*$ in equation (33) and setting $m = 1$ yields

$$a^* = c \left( n^* + 1 + \frac{3 + 2 n^*}{2 (-1 + n^* + n^*^2)} \right).$$

For $n^* \geq 2$, $a^*$ is smaller than $c(\sqrt{f_L/f_S} + 2)$. Similar to the condition for $D$ in Proposition 1, we obtain an easy check for the question whether an equilibrium exists. The way in which $a^*$ depends on $c$, the difference in marginal costs, is straightforward. If the difference is large, $a$ must be large in order for small firms to be viable at all. The same argument applies for large values of $n^*$. In order for a single small firm to produce a positive amount of output in the case of $n^*$-type rivals, $a$ must be greater than $cn^*$ (see equation (4)).

### 5.2 Asymmetric and multiple equilibria

In this subsection I discuss both under what conditions firms employing different technologies may co-exist in equilibrium and the circumstances which give rise to the existence of multiple equilibria. As noted above, the empirical importance of asymmetric equilibria, i.e. of equilibria in which firms of different size are active, stems from the well established empirical finding that firm sizes differ greatly within industries (see, e.g., Sutton, 1998 and Cabral and Mata, 2003). My model stresses the importance of both lumpy technology and market size in the explanation of these facts. In what follows I describe the market size vectors for which asymmetric and multiple equilibria arise, given cost parameters.

In general, asymmetric equilibria arise due to two rather different configurations.
First, a situation in which indivisibilities of technology determine the type of equilibrium. Due to the indivisibilities (and the related integer constraint) entry of the more efficient $L$ type firm might not be possible, as it could take place only on a large scale. On the other hand, a small firm may be able to enter the market even though its average costs are higher. Small-scale entry is possible even in the case of a small residual demand.

Second, a situation in which firms are not 'too' different in terms of average costs. Here we can further distinguish two scenarios. First, a configuration in which a candidate equilibrium with only large firms active is destroyed by an incentive for the pivotal firm to switch to the small-scale technology. Due to the similarity in average costs, the switching firm incurs only a small cost disadvantage, but can at the same time reduce competition as the optimal output with the $S$-technology will be much smaller. The resulting equilibrium is an asymmetric one.\footnote{Multiple asymmetric equilibria might exist in this case.} Second, multiple equilibria due to the possibility of 'producing' different equilibria by replacing, for instance, one large firm by the number of small firms, which is sufficient to produce approximately the same output. Therefore, one obtains multiple equilibria, consisting in general both of symmetric and asymmetric outcomes.

In the following I illustrate the two possibilities by examples.

5.2.1 **Asymmetric equilibrium due to indivisibilities**

Starting from the cost parameters in the preceding Section, I consider the market size vector $(a, s) = (6.44, 788)$.\footnote{The parameters are derived from Figure 3 by extending locus $V_1$ up to the locus where a further large firm becomes viable. The vector is close to the (not drawn) intersection of the two curves.} The free entry number of $S$ and $L$-type firms if only the respective technology were available is 54 and 2 firm, respectively. There exists a unique equilibrium in which two small and two large firms are active. Closer examination of the example shows two things. First, the cost disadvantage of the $S$-type firms is quite large. Average costs are 2.072 for the small scale and 1.245 for the large scale firms.
Nevertheless, entry is profitable with the $S$ technology as the indivisibilities in terms of the large fixed costs prevent entry with the ‘more efficient’ technology. Second, the possibility of small scale entry restricts the excess profits which accrue to large scale incumbents due to the integer constraint. The entry of the two small scale firms reduces profits of the two large incumbents from 1581 to 1385.

Co-existence of different technologies in equilibrium is not restricted to cases with a small number of (large) firms. To see this, consider the market size vector $(a, s) = (1941.91, 464.129)$. For these parameters, an equilibrium with one small firm and 922 large firms exists. This shows that $a$ must be quite high to guarantee existence of a unique symmetric equilibrium.

### 5.2.2 Asymmetric and multiple equilibria when average costs are similar

As mentioned above, asymmetric equilibria may also exist due to other reasons than indivisibilities. To see this, consider the vector $(a, s) = (18.283, 505.956)$. In this case, two equilibria exist: $(m, n) = (0, 8)$ and $(m, n) = (14, 21)$. The first equilibrium implies that a small firm is not viable, given the maximum number of large firms is active. Together, the two equilibria show that multiple equilibria exist and that one of the equilibria will be an asymmetric one, in general.

The above example is chosen such that the value of horizontal market size is about the highest for which multiplicity can arise. The small scale technology exhibits average costs which are about 9% higher than that related to the $L$ technology. This difference is the highest which is compatible with multiple equilibria. A unique asymmetric equilibrium can be found for even higher values of horizontal market size, the maximum value is slightly above the $s$ value from the market size vector $(a, s) = (18.02, 511.98)$. The equilibrium exhibits 12 small and seven large firms. Average costs are 10% higher for the small scale technology. The interesting point here is that the cost difference

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16It is straightforward to derive an upper bound for $s$ for the example of Figures 2 and 3. It is sufficient to derive the constraint that a small scale firm does not have an incentive to switch to the large scale technology in a candidate equilibrium in which the active number of large firms is one short of the maximum feasible number (of large firms).
is much smaller when looking at the candidate equilibrium in which 8 large firms are active. If one of the eight large firms switches to the $S$ technology, its average costs increase by less than 2.5%. The respective switch is profitable, destroying the equilibrium in which only large firms are active.

The above equilibrium configurations exhibit the maximum difference between the two technologies which is compatible with multiple equilibria. It remains to examine equilibrium configurations which arise when technologies are very similar. From Proposition 1 we know that the lower bound $s^*$ of horizontal market size for which multiple equilibria can arise is the value of $s$ for which $T_S = T_L$ holds. In the example we obtain $s^* = 463.129$. It is easy to find examples for values slightly greater than $s^*$ for which multiple equilibria arise. Take, for example, $(a, s) = (20, 464.129)$. The equilibrium configurations are:

$$(m, n) = \{(30, 37), (50, 56), (71, 76), (91, 95), (111, 114), (131, 133), (152, 153), (172, 172)\}.$$ 

From the value of $s$ it is clear that the two technologies exhibit rather similar average costs. As asserted above, this similarity makes it possible to produce different equilibria by replacing, for instance, one large firm by the number of small firms, which is sufficient to produce approximately the same output. In the example one large firm can be replaced by about 20 small firms. Note that the equilibrium output of a large firm is about 977, small firms produce about one twentieth, namely about 49 units of output. This result is very much in vein of Davis (1999) who provides conditions under which market output is uniquely determined within the set of (multiple) equilibria.

A final point related to multiple equilibria, but also to my model of technology choice in general, concerns the relation between market concentration and market power, respectively, and performance. The multiple equilibria exhibit nearly identical performance in terms of equilibrium prices. Prices typically differ by less than a tenth of a percentage point. However, concentration ratios and price-cost margins vary greatly. More generally, if an industry experiences a drastic restructuring due to increases in horizontal market size $s$, the new equilibrium is typically one with much
higher concentration and with lower prices.

6 Conclusions

Once one takes entry and technology choice into account, (two-stage) Cournot games allow for a wide range of outcomes. Apart from a unique equilibrium with symmetric firms, co-existence of different types of firms and non-uniqueness of equilibrium may result for different parameter values. An outcome may also be non-existence even though the underlying game is one with linear demand and constant marginal costs. The article characterizes the parameter values for which the various outcomes result and evaluates their respective importance. The characterization of the equilibrium outcomes in this article allows applied papers focusing on technology choice to evaluate the importance of different equilibrium configurations without performing a complete analysis on their own. An example of such an application is Elberfeld, Götz, and Stähler (2002). Technology choice enters their model in the shape of whether production should take place domestically or whether firms should choose a multinational production mode. The formal structure of the problem of vertical foreign direct investment Elberfeld, Götz, and Stähler consider is the same as in my model.

As regards the importance of the different outcomes, the article has shown that non-existence does not occur if vertical market size and the number of firms is large. Existence problems become important if the market supports only a few large firms. The conclusion from this result for the duopoly cases often considered in the literature is the following: Entry should explicitly be taken into account in the respective models. This would ensure that the claimed equilibria can really result from underlying fully specified games.

Compared to non-existence, asymmetric equilibria are much more important in the sense that they apply for a larger set of parameter values. The interesting thing about the equilibria with co-existence of different firm types is that it provides an endogenous explanation of a stylized fact, namely that firms in an industry often differ with respect
to their size. This explanation is based purely on indivisibilities in technology choice and on market size.

The combination of indivisibilities and of technology choice in my model yields two further conclusions. The first regards the integer constraint. In a case in which firms are large compared to the market size, lumpy technology may yield substantial profits due to the integer constraint (see Lambson, 1987). Once one allows for technology choice, the problem of excess profits is partly mitigated by the possibility of small-scale entry, even if the small-scale technology is lumpy as well. The second conclusion concerns the relation between market structure and performance. The discussion of multiple equilibria has shown that the various equilibrium configurations may differ greatly in market structure parameters like the number of firms and industry concentration. These differences map into differences in conduct parameters like price-cost margin in an intuitive and expected way. However, large variations in price-cost margins among industries, for instance, do not imply that these industries differ in performance. On the contrary, the article has shown that rather different industry structures are compatible with nearly identical equilibrium outcomes in terms of prices. Yet another example that policy prescriptions based on determinants of the industry structure only may be quite misleading.

7 Appendix:

Proof of Proposition 1

Case 1: \( T_L < T_S \).

First, I show that a configuration in which \( S \)-firms are active, i.e., \( m > 0 \), cannot be an equilibrium. Entry with the \( L \)-technology would be profitable in this case. To see this note that \( m > 0 \) requires \( \Pi_S(m, n) \geq 0 \). Using (6), this inequality can be rearranged to yield

\[
\frac{a - c - (n - m)c}{(n + 1)} \geq \frac{f_S}{s}
\]  

(14)
Using the definition of $T_S$ this expression can be simplified to

$$\frac{a + mc}{(n + 1)} \geq T_S. \tag{15}$$

Profitable entry of an $L$-firm requires $\Pi_L(m, n + 1) \geq 0$. Similar to the above inequality, using (7) this inequality can be written as

$$\frac{a + mc}{(n + 2)} \geq T_L. \tag{16}$$

Inverting the inequality and rewriting the r.h.s. yields

$$\frac{n + 1}{a + mc} + \frac{1}{a + mc} \leq \frac{1}{T_L}. \tag{17}$$

From equation (15) we know that the first term of (17) is smaller than $1/T_S$. Therefore, inequality (17) is certainly satisfied if

$$\frac{1}{T_S} + \frac{1}{a + mc} \leq \frac{1}{T_L}. \tag{18}$$

holds. Rearranging and multiplying by $a$ yields

$$\frac{a}{a + mc} \leq \frac{a}{T_L} - \frac{a}{T_S}. \tag{19}$$

This inequality is certainly satisfied as the l.h.s. is smaller than 1, while the r.h.s consists of the definition of $D$ which by assumption is greater than 2. This proves that the $S$ technology cannot be used in equilibrium. Additional entry of $L$-firms would occur.

The second step of the proof shows that a firm in a candidate equilibrium with only $L$-type firms does not have an incentive to switch to the $S$-technology. In formal terms a sufficient condition for such a switch not to be profitable is $\Pi_S(1, n^*) < 0$. Here, $n^*$ is defined as the largest integer such that $\Pi_L(0, n) \geq 0$. To prove that switching is unprofitable I calculate the (real) numbers $\bar{n}$ and $\hat{n}$, respectively, for which $\Pi_S(1, n) = 0$ and $\Pi_L(0, n) = 0$, respectively. Using the definitions of $T_S$ and $T_L$ one obtains after some manipulations

$$\bar{n} = \frac{a + c}{T_S} - 1 \tag{20}$$
and
\[ \hat{n} = \frac{a}{T_L} - 1 \]  \hspace{1cm} (21)
respectively. To show that \( \Pi_L(1, n^*) < 0 \) it is sufficient that \( \hat{n} > n + 1 \). Note that 1 must be added due to the integer constraint. Using the definitions of \( \tilde{n} \) and \( \hat{n} \), the condition reads
\[ \frac{a}{T_L} - 1 > \frac{a + c}{T_S}. \]  \hspace{1cm} (22)
Rearranging yields
\[ \frac{a}{T_L} - \frac{a}{T_S} > \frac{c}{T_S} + 1. \]  \hspace{1cm} (23)
This inequality clearly holds. The l.h.s equals \( D \) and is therefore by assumption greater than 2. The expression \( c/T_S \) is smaller than 1. This proves the proposition as concerns the case \( T_L < T_S \).

Case 2: \( T_S \leq T_L \). First, I show that the \( L \)-technology cannot be an equilibrium choice. It is always optimal for a large firm to switch to the \( S \)-technology, irrespective of what the number of large firms is, if
\[ \Pi_S(\hat{m}(i) + 1, \hat{m}(i) + i) - \Pi_L(\hat{m}(i), \hat{m}(i) + i) > 0. \]  \hspace{1cm} (24)
Here \( i \) is an arbitrary number of large firms (of course smaller than the maximum feasible number). \( \hat{m}(i) \) is the maximum viable number of small firms, given there are \( i \) large firms, i.e., it is determined by the condition \( \Pi_S(\hat{m}(i), \hat{m}(i) + i) = 0 \). Using the condition \( T_L = T_S \) tedious calculations yield that the l.h.s. of (24) equals
\[ c_{fs}(2a - c - 2ci)/(a - c - ci)^2. \]  \hspace{1cm} (25)
This expression must be positive if small firms are to be viable at all. Thus, deviation from the \( L \)-technology is always profitable.

It remains to be shown that an \( S \)-firm does not have an incentive to switch to the \( L \)-technology if the zero profit number of \( S \)-firms \( n^* \) is active. That is,
\[ \Pi_S(n^*, n^*) - \Pi_L(n^* - 1, n^*) > 0. \]  \hspace{1cm} (26)
It turns out that the resulting expression is positive both if the integer constraint is taken into account and if it is neglected. Once one employs the condition \( T_L = T_S \) to substitute for \( f_L \), one eventually obtains in the latter case

\[
\frac{(2a - 3c)cf_s + (a - c)2c^2\sqrt{f_s}\sqrt{s}}{(a - c)^2},
\]

which is clearly positive. Condition (26) is also positive if evaluated at \( n^* - 1 \) thus taking the integer constraint into account. It reads

\[
\frac{c\sqrt{f_s}\sqrt{s} \left( 2f_s - 3c\sqrt{f_s}\sqrt{s} + (a - c)2cs \right)}{(\sqrt{f_s} - (a - c)\sqrt{s})^2}.
\]

Omitting the term \( 2f_s \) in the term in brackets in the numerator, the remaining expression is positive once \( n^* > 1.5 \). Thus one obtains that the whole term is positive.\(^\text{17}\)

Only \( S \)-type firms are active is the unique equilibrium if \( T_L \leq T_S \).

**Proof of Proposition 3**

The proof proceeds in several steps.

First, I derive necessary conditions for the non-existence of equilibrium. Suppose that the configuration \((m - 1, n)\) constitutes a candidate equilibrium. Necessary conditions for non-existence are that configurations \((m - 1, n)\) and \((m, n)\) do not constitute an equilibrium. Thus, conditions

\[
\Pi_S(m, n) \geq \Pi_L(m - 1, n)
\]

and

\[
\max\{\Pi_S(m + 1, n + 1), 0\} \leq \Pi_L(m, n + 1),
\]

where \( m = 1, \ldots, n \) must hold. Condition (29) implies that an \( L \)-firm would deviate to the \( S \)-technology, thus \((m - 1, n)\) cannot be an equilibrium. By condition (30) the switch would induce entry of an \( L \)-firm, thus \((m, n)\) cannot be an equilibrium.

Condition (30) implies also that \((m + 1, n + 1)\) cannot be an equilibrium.

\(^{17}\text{For brevity, it is omitted to show that the derivative of condition (26) with respect to } n^* \text{ is monotonous, proving that the the condition must be satisfied for the actual equilibrium number which lies in the interval } [n^* - 1, n^*] \)
Second, I show that a maximum value of \( \tilde{a} \) exists such that the necessary conditions for non-existence, i.e., conditions (29) and (30), can be satisfied simultaneously, given the number of firms. From conditions (29) and (30) we can derive the values of \( s \) (\( s_1 \) and \( s_2 \)) such that both conditions are satisfied with equality. In the case of (30), I drop the 0 and take \( \Pi_S(m + 1, n + 1) \) as the term on the left hand side. This matters only if \( \Pi_S(m + 1, n + 1) < 0 \). In this case the non-existence range is a subset of the set captured by the conditions. Solving (29) and (30) for \( s \) yields

\[
s_1 \equiv \frac{(1 + n)^2 (f_L - f_S)}{2 a c n + 2 c^2 n(m - 1) - c^2 n^2} \quad (31)
\]

and

\[
s_2 \equiv \frac{(2 + n)^2 (f_L - f_S)}{c (1 + n) (2 a - c (n - 2m + 1))} \quad (32)
\]

(29) and (30) are satisfied for values of \( s \) such that \( s_2 < s < s_1 \). This follows from the fact that

\[
\partial \Pi_L / \partial s > \partial \Pi_S / \partial s.
\]

Calculating the value of \( a \) such that \( s_1 = s_2 \) we obtain

\[
\tilde{a} \equiv 2c + c (n - m) + \frac{c (3 + 2n)}{2 (-1 + n + n^2)}. \quad (33)
\]

It is straightforward to show that the derivative of \( s_1 \) with respect to \( a \) evaluated at \( \tilde{a} \) is greater in absolute terms than the respective derivative of \( s_2 \). Thus, \( s_2 < s_1 \) requires \( a < \tilde{a} \). Therefore, \( \tilde{a} \) is the maximum value such that conditions (29) and (30) can be satisfied simultaneously. Note two things about the relation between \( \tilde{a} \) and \( m \), the number of small firms, deriving from (33). First, for given \( n \), \( \tilde{a} \) assumes a maximum for \( m = 1 \), i.e., in a candidate equilibrium with only large firms, in which the pivotal firm may switch from the \( L \)-technology to the \( S \)-technology. Second, for a given number of large firms, i.e. \( n - m \), the case without small firms again yields the maximum value of \( \tilde{a} \). As a consequence of these two properties it is sufficient to consider non-existence for candidate equilibria with only large firms active. Thus, in what follows the analysis assumes \( m = 1 \).
Third, I show that a number of large firms $n^*$ exist such that for $n > n^*$ the necessary conditions for non-existence, i.e., (29) and (30), cannot be satisfied simultaneously. The reason is that, starting from the values of $\bar{a}$ and $s_1$ associated with $n$, an $n$ exists such that $n$ firms are not viable given the underlying values of $a$ and $n$. To see this, calculate the profits of $L$-firms in the candidate equilibrium $(0, n)$ if $a = \bar{a}$ and $s = s_1(\bar{a})$. One obtains

$$\Pi_L(0, n) = \frac{(1 + 4n + 2n^2)^2 (f_L - f_S)}{4n (1 + n)^3 (-1 + n + n^2)} - f_L$$

This expression is decreasing in $n$. Therefore, an $n^*$ must exist such that all firms’ profits are negative for all $n \geq n^*$. Thus, the candidate equilibrium $(0, n)$ with $n \geq n^*$ requires a market size vector $(a, s)$ such that either $a > \bar{a}$ or, in the case of $a < \bar{a}$, that $s > s_1(a)$. The latter statement follows from the fact that profits are an increasing function of $a$ if we use $s = s_1(a)$. The respective profits read

$$\Pi_L(0, n) = \frac{a^2 (f_L - f_S)}{2acn - c^2 n^2} - f_L.$$  

It is straightforward to show that the derivative of this expression with respect to $a$ is positive. Therefore, it is proved that non-existence cannot occur for a number of large firms greater than $n^*$. Values of $a$ and $s$ satisfying the relevant necessary conditions for non-existence do not support the respective candidate equilibrium.

Finally, $a^*$, the threshold value of $a$ which ensures existence is calculated. Substituting $n^*$ into equation (33) and using $m = 1$ yields the respective value (see equation (13) in the main text). It follows from the above reasoning that conditions (29) and (30) cannot be satisfied simultaneously for $a > a^*$. Therefore, non-existence cannot occur for $a > a^*$. This completes the proof.

\[\square\]

**References**


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Figure 1: Condition for the existence of a unique and symmetric equilibrium 

\[ (T_S > T_L). \]
Figure 2: Regions of non-existence of an equilibrium: Candidate equilibria with one large firm.\textsuperscript{18}

\textsuperscript{18}Note that the diagram shows also a part of the non-existence area of the two large firms case.
Figure 3: Regions of non-existence of an equilibrium