A structural base for conditional reasoning

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Abstract. There are several approaches implementing reasoning based on conditional knowledge bases, one of the most popular being System Z [1]. We look at ranking functions [2] in general, conditional structures and c-representations [3] in order to examine the reasoning strength of the different approaches by learning which of the known calculi of nonmonotonic reasoning (System P and R) and *Direct Inference* are applicable to these inference relations. Furthermore we use the recently proposed *Enforcement*-postulate [4] to show dependencies between these approaches.

1 Introduction

Default reasoning is often based on uncertain rules of the form "if A then usually B" representing semantically meaningful relationships between A and B that may serve as guidelines for rational decision making. Such rules are denoted as *conditionals* and formally written as $(B|A)$. Conditionals are different from material implications $A \Rightarrow B$ in that they can not be interpreted truth functionally but need richer epistemic structures to be evaluated. Ordinal conditional functions, or *ranking functions* [2] provide a most convenient way for evaluating conditionals. Here, a conditional $(B|A)$ is accepted if the rank of its verification $A \wedge B$ is more plausible than the rank of its falsification $A \wedge \neg B$. However, it is often not clear where the numerical ranks come from, and people might be reluctant to accept conditionals just due to a comparison of numbers. In this paper, we show how the acceptance of conditionals can be based on structural arguments that emerge from elaborating systematically the three-valued nature of conditionals. More precisely, we assume a knowledge base of conditionals to be explicitly given, and investigate inferences that can be drawn from this knowledge base in a rational way, like in the well-known penguin example: Let $\Delta = \{(f|b), (\overline{f}|p), (b|p)\}$ be the set of conditionals $(f|b) \simeq$ "<u>b</u>irds usually <u>f</u>ly", $(\overline{f}|b) \simeq$ "<u>p</u>enguins usually do not fly" and $(b|p) \simeq$ "penguins are usually birds". Commonsense deliberations tell us that from these conditionals we should be able to infer that birds fly if they are not penguins, but penguin-birds not. The intricacy of this example lies in the nonmonotonic inheritance from a superclass to a subclass: albeit being birds, penguins do not inherit the flight capacity of birds.

We briefly recall the conditional structures approach [3] which allows us to define a preference relation between possible worlds and henceforth a nonmonotonic inference relation between formulas. We prove results on the quality of this inference relation but also illustrate its limits. Fortunately, conditional structures can be linked to ranking functions via c-representations, and together with the novel *Enforcement* postulate adapted from belief revision, we are able to show that rank based inferences may respect structural (i.e., non-numerical) information.

2 Preliminaries

Let $\Sigma = \{V_1, \ldots, V_n\}$ be a set of propositional atoms. A *literal* is a positive or negative atom. The set of formulas $\mathfrak{L}$ over Σ, with the connectives $\wedge$ (*and*), $\vee$ (*or*) and $\neg$ (*not*) shall be defined in the usual way. Let $A, B \in \mathfrak{L}$, we will in the following omit the connective $\wedge$ and write AB instead of $A \wedge B$ as well as indicate negation by overlining, i.e. $\overline{A}$ means $\neg A$; the symbol "$\Rightarrow$" is used as material implication, i.e., $A \Rightarrow B$ is equivalent to $\overline{A} \vee B$. *Interpretations*, or *possible worlds*, are also defined in the usual way; the set of all possible worlds is denoted by Ω. We often use the equivalence between worlds and *complete conjunctions*, i.e. conjunctions of literals where every variable $V_i \in \Sigma$ appears exactly once. A model ω of a propositional formula $A \in \mathfrak{L}$ is a possible world that satisfies A, written as $\omega \models A$. The set of all models of A is denoted by $Mod(A)$. For formulas $A, B \in \mathfrak{L}$, A *entails* B, written as $A \models B$, iff $\mathrm{Mod}(A) \subseteq \mathrm{Mod}(B)$, i.e. iff for all $\omega \in \Omega$, $\omega \models A$ implies $\omega \models B$. For sets of formulas $\mathcal{A} \subseteq \mathfrak{L}$ we have $Mod(\mathcal{A}) = \bigcap_{A \in \mathcal{A}} Mod(A)$. A *conditional* $(B|A)$ encodes a defeasible rule "if A then *usually* B" with the trivalent evaluation $[\![(B|A)]\!]_\omega = true$ if and only if $\omega \models AB$ (verification), $[\![(B|A)]\!]_\omega = false$ if and only if $\omega \models A\overline{B}$ (falsification) and $[\![(B|A)]\!]_\omega = undefined$ if and only if $\omega \models \overline{A}$ (non-applicability). The language of all conditionals over $\mathfrak{L}$ is denoted by $(\mathfrak{L} \mid \mathfrak{L})$. Let $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\} \subseteq (\mathfrak{L} \mid \mathfrak{L})$ be a finite set of conditionals. A conditional $(B|A)$ is *tolerated* by Δ if and only if there is a world $\omega \in \Omega$ such that $\omega \models AB$ and $\omega \models A_i \Rightarrow B_i$ for every $1 \leq i \leq n$. Δ is *consistent* if and only if for every nonempty subset $\Delta' \subseteq \Delta$ there is a conditional $(B|A) \in \Delta'$ that is tolerated by Δ'. We will call such a consistent Δ a *knowledge base* and it shall represent the knowledge an agent uses as a base for reasoning. In this paper, we will only consider Δ that are consistent.

3 Properties of qualitative conditional reasoning

We consider inference relations $\mathrel{|\!\sim}$ between sets of formulas $\mathcal{A} \subseteq \mathfrak{L}$ and single formulas A. $\mathcal{A} \mathrel{|\!\sim} A$ means that A can be inferred defeasibly fom $\mathcal{A}$. Contrary to $\models$, $\mathrel{|\!\sim}$ will usually be nonmonotonic, i.e. we may have $\mathcal{A} \mathrel{|\!\sim} A$ but $\mathcal{A} \cup \{B\} \mathrel{|\!\not\sim} A$. The various possible inference relations can be judged by certain *quality criteria* that have been designed for describing rational human reasoning. From this set of criteria, *calculi* are subsets of the quality criteria canon used to classify inference relations. The usual calculi are System O, C, P and R, where System C and O are included in System P, thus we present the most established systems, Systems P and R.

Definition 1 (System P). *[5]*
System P consists of the following conditions

- *Reflexivity* (REF)*:* *From A defeasibly infer A, resp. $A \mathrel{|\!\sim} A$,*
- *Right Weakening* (RW)*:* $A \mathrel{|\!\sim} B$ *and* $B \models C$ *imply* $A \mathrel{|\!\sim} C$,
- *Left Logical Eqivalence* (LLE)*:* $A \mathrel{|\!\sim} C$ *and* $A \equiv B$ *imply* $B \mathrel{|\!\sim} C$,
- (Cut)*:* $A \mathrel{|\!\sim} B$ *and* $AB \mathrel{|\!\sim} C$ *imply* $A \mathrel{|\!\sim} C$,
- *Cautious Monotony* (CM)*:* $A \mathrel{|\!\sim} B$ *and* $A \mathrel{|\!\sim} C$ *imply* $AB \mathrel{|\!\sim} C$,
- (Or)*:* $A \mathrel{|\!\sim} C$ *and* $B \mathrel{|\!\sim} C$ *imply* $(A \vee B) \mathrel{|\!\sim} C$.

As well as being an important quality criterion for nonmonotonic reasoning systems, empirical studies show that human reasoning makes use of the conditions of System P (c.f. [6]) which renders the inspection of System P especially worthwhile.
The calculi are syntactical and should be based on semantics to be evaluated. A very general one is preferential satisfaction that uses the notion of preferential models which we will introduce with the next two definitions.

Definition 2 ((Classical) Preferential model). *[7]*
Let $M = \{m_1, m_2, \ldots\}$ be an arbitrary set of states, which could be, but is not

limited to, a set of interpretations of a logical language. Let $\vdash$ *be an arbitrary relation* $\vdash \subset M \times \mathfrak{L}$ *called* satisfaction relation *and* $\prec$ *an arbitrary relation* $\prec \subseteq M \times M$ *called* preference relation. *If* $m_1 \prec m_2$ *then* m_1 *is* preferred *to* m_2*. The triple* $\mathcal{M} = \langle M, \vdash, \prec \rangle$ *is called a* preferential model. *A preferential model is called* classical *if* $\prec$ *is transitive, and for all* $m \in M$ *it holds that* $m \vdash \overline{A}$ *iff* $m \not\vdash A$ *and* $m \vdash A \vee B$ *iff* $m \vdash A$ *or* $m \vdash B$*.*

Definition 3 (Preferential satisfaction). *[7]*
Let $\mathcal{A} \subseteq \mathfrak{L}$*,* $\mathcal{M} = \langle M, \vdash, \prec \rangle$ *be a preferential model and* $m \in M$ *be a state. We say that* m *satisfies* $\mathcal{A}$ *(*$m \vdash \mathcal{A}$*) iff* $m \vdash A$ *for every* $A \in \mathcal{A}$*, and* m preferentially *satisfies* $\mathcal{A}$ *(written* $m \vdash_{\prec} \mathcal{A}$*) iff* $m \vdash \mathcal{A}$ *and there is no* $m' \in M$ *such that* $m' \vdash \mathcal{A}$ *and* $m' \prec m$*. We define* $[\mathcal{A}] = \{m \in M \,|\, m \vdash \mathcal{A}\}$ *and say* m *is* $\prec$*-minimal in* $[\mathcal{A}]$ *iff* $m \vdash_{\prec} \mathcal{A}$*.*

Preferential satisfaction is based on a notion of *minimality*. Since $\prec$ is defined to be an arbitrary relation, it is possible for $[A]$ not to have a minimal element (e.g. because $[A]$ is infinite or contains circles $m_1 \prec m_2 \prec \cdots \prec m_1$). The following definition ensures that an associated minimal element exists.

Definition 4 (Stoppered preferential models). *[7]*
We call a preferential model $\mathcal{M} = \langle M, \vdash, \prec \rangle$ stoppered *if and only if for every set* $\mathcal{A} \subseteq \mathfrak{L}$ *and every* $m \in M$ *if* $m \in [\mathcal{A}]$ *then there is a* $\prec$*-minimal element* m' *in* $[\mathcal{A}]$ *such that either* $m' = m$ *or* $m' \prec m$*.*

Having defined preferential models, one can now define an entailment relation on preferential models that facilitates reasoning.

Definition 5 (Preferential entailment). *[7]*
Let $\langle M, \vdash, \prec \rangle$ *be a preferential model,* $m, m' \in M$ *and* $A, B \in \mathfrak{L}$*. We define* B *to be* preferentially entailed *by* A *(written* $A \mid\!\sim B$*) in the following way:*

$$A \mid\!\sim B \qquad \textit{iff} \qquad \forall\, m \in M \;:\qquad m \vdash_{\prec} A \quad \textit{implies} \quad m \vdash B \tag{1}$$

Preferential entailment complies with various properties, [7] has shown that if the underlying preferential model is stoppered, these relations fulfil the properties of System P which we will stress in the following proposition.

Proposition 1. *[7] All preferential entailment operations that are generated by a classical stoppered preferential model comply with System P.*

For classical stoppered preferential models, equation (1) is equivalent to

$$A \mid\!\sim B \quad \text{iff} \quad \forall\, m' :\; m' \vdash A\overline{B} \qquad \exists\, m :\; m \vdash AB \quad \text{with} \quad m \prec m'.$$

The second calculus we announced to inspect was System R which is basically System P with the additional (non-Horn) property *Rational Monotony*.

Definition 6 (System R). *[8]*
System R is composed of (REF)*,* (Cut)*,* (CM)*,* (RW)*,* (LLE)*,* (Or) *and*
– *Rational Monotony* (RM) $\qquad A \mid\!\sim B$ *and* $A \not\mid\!\sim \overline{C}$ *implies* $AC \mid\!\sim B$*.*

As given in section 2, a conditional $(B|A)$ stands for the defeasible rule "A usually entails B" and therefore suggests for each $(B|A) \in \Delta$ that $A \mid\!\sim_{\Delta} B$ holds if $\mid\!\sim_{\Delta}$ is based on Δ. This is claimed by the next property.

Definition 7 (Direct Inference (DI)**).** *[9]*
Let $\Delta \subseteq (\mathfrak{L}|\mathfrak{L})$*, let* $\mid\!\sim_{\Delta}$ *be an inference relation based on* Δ*.* $\mid\!\sim_{\Delta}$ *complies with* (DI) *iff for every* $(B|A) \in \Delta$ *it holds that* $A \mid\!\sim_{\Delta} B$*.*

4 Ranking Functions (OCF)

An *ordinal conditional function* (OCF, [2]), also called *ranking function*, is a function $\kappa : \Omega \to \mathbb{N}_0^{\infty}$ with $\kappa^{-1}(0) \neq \varnothing$ which maps each world $\omega \in \Omega$ to a degree of implausibility $\kappa(\omega)$; ranks of formulas $A \in \mathfrak{L}$ are calculated as $\kappa(A) = \min\{\kappa(\omega) \,|\, \omega \models A\}$. For conditionals $(B|A)$ we have ranks of $\kappa(B|A) = \kappa(AB) - \kappa(A)$ and $\kappa \models (B|A)$ iff. $\kappa(AB) < \kappa(A\overline{B})$, i.e. iff. AB is more plausible than $A\overline{B}$. In this case, we call κ a (ranking) model of $(B|A)$.
A ranking function induces a preference relation $\leq_\kappa$ on worlds such that $\omega \leq_\kappa \omega'$ iff $\kappa(\omega) \leq \kappa(\omega')$. We write $\omega <_\kappa \omega'$ iff $\omega \leq_\kappa \omega'$ and $\omega' \not\leq_\kappa \omega$.
OCF-reasoning uses the $<$-relation on natural numbers and the classical inference relation $\models$ which implies immediately that $\langle \Omega, \models, <_\kappa \rangle$ is a classical stoppered preferential model. The inference relation $|\!\!\sim_\kappa$ turns out to be

$$A |\!\!\sim_\kappa B \qquad \text{iff} \qquad \kappa(AB) < \kappa(A\overline{B}) \qquad \text{iff} \qquad \kappa \models (B|A).$$

For a conditional knowledge base $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\} \subseteq (\mathfrak{L} \mid \mathfrak{L})$ a ranking function κ is Δ*-admissible* iff $\kappa \models (B_i|A_i)$ for every $1 \leq i \leq n$. We write κ_Δ to illustrate that κ is Δ-admissible. Note that for $|\!\!\sim_\kappa$, (DI) is equivalent to Δ-admissability of κ.
Proposition 1 immediately yields the following statement:

Corollary 1. $|\!\!\sim_\kappa$ *complies with System P.*

System R is no consequence of proposition 1. [1] has shown that every κ_Δ complies with (RM), by which the next proposition arises:

Proposition 2. $|\!\!\sim_\kappa$ *complies with System R.*

5 Reasoning with conditional structures

Our intention is to focus on reasoning mechanisms based on the information contained in the knowledge base. By OCF, we have high qualitative reasoning, but the ranks of the worlds need to use the knowledge base as information source. In the following we will examine an approach using the structural information induced by the conditionals in a knowledge base.
Given a conditional knowledge base $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\} \subseteq (\mathfrak{L} \mid \mathfrak{L})$ we assign a pair of abstract symbols $\mathbf{a}_i^+$ and $\mathbf{a}_i^-$ to each $(B_i|A_i) \in \Delta$ to illustrate the effect of conditionals on worlds. With these, we define the free abelian group $\mathfrak{F}_\Delta = \langle \mathbf{a}_1^+, \mathbf{a}_1^-, \ldots, \mathbf{a}_n^+, \mathbf{a}_n^- \rangle$ on Δ with generators $\mathbf{a}_1^+, \mathbf{a}_1^-, \ldots, \mathbf{a}_n^+, \mathbf{a}_n^-$ consisting of all products $(\mathbf{a}_1^+)^{\alpha_1}(\mathbf{a}_1^-)^{\beta_1} \cdot \ldots \cdot (\mathbf{a}_n^+)^{\alpha_n}(\mathbf{a}_n^-)^{\beta_n}$ with $\alpha_i, \beta_i \in \mathbb{Z}$ for all $1 \leq i \leq n$ [3]. We will keep in mind that in abelian groups commutativity holds, e.g. $\mathbf{a}_1^+\mathbf{a}_2^- = \mathbf{a}_2^-\mathbf{a}_1^+$. To connect a world and the effect of a conditional to this world we define the function $\sigma_i : \Omega \to \mathfrak{F}_\Delta$ by $\sigma_i(\omega) := \mathbf{a}_i^+$ iff. $\omega \models AB$, $\sigma_i(\omega) := \mathbf{a}_i^-$ iff. $\omega \models A\overline{B}$ and $\sigma_i(\omega) := 1$ iff. $\omega \models \overline{A}$ for each $1 \leq i \leq n$. So, $\mathbf{a}_i^+$ ($\mathbf{a}_i^-$) indicates that ω verifies (falsifies) $(B_i|A_i)$, and the neutral group element 1 corresponds to non-applicability of the conditional.

Definition 8 (Conditional structure). *[3]*
Let $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\} \subseteq (\mathfrak{L} \mid \mathfrak{L})$*,* $\mathfrak{F}_\Delta = \langle \mathbf{a}_1^+, \mathbf{a}_1^-, \ldots, \mathbf{a}_n^+, \mathbf{a}_n^- \rangle$ *and* σ_i *be as defined above. The conditional structure* σ_Δ *of a world regarding* Δ *is the function* $\sigma_\Delta : \Omega \to \mathfrak{F}_\Delta$ *defined as*

$$\sigma_\Delta(\omega) = \prod_{i=1}^{n} \sigma_i(\omega) = \prod_{\substack{i=1 \\ \omega \models A_i B_i}}^{n} \mathbf{a}_i^+ \cdot \prod_{\substack{i=1 \\ \omega \models A_i \overline{B}_i}}^{n} \mathbf{a}_i^-$$

For every world ω the conditional structure $\sigma_\Delta(\omega)$ indicates formally which conditionals in Δ are verified, or falsified by, or not applicable to this world. Note that the group structure allows an elegant way of encoding this.

Example 1. We use the introductory example $\Delta = \{(f|b), (\overline{f}\,|p), (b|p)\}$. The conditional structures with respect to Δ are shown in the following table.

ω	$\sigma_\Delta(\omega)$	ω	$\sigma_\Delta(\omega)$	ω	$\sigma_\Delta(\omega)$	ω	$\sigma_\Delta(\omega)$
pbf	$\mathbf{a}_1^+\mathbf{a}_2^-\mathbf{a}_3^+$	$pb\overline{f}$	$\mathbf{a}_1^-\mathbf{a}_2^+\mathbf{a}_3^+$	$\overline{p}bf$	$\mathbf{a}_1^+$	$\overline{p}b\overline{f}$	$\mathbf{a}_1^-$
$p\overline{b}f$	$\mathbf{a}_2^-\mathbf{a}_3^-$	$p\overline{b}\,\overline{f}$	$\mathbf{a}_2^+\mathbf{a}_3^-$	$\overline{p}\overline{b}\,f$	1	$\overline{p}\overline{b}\,\overline{f}$	1

With σ_Δ we define a preference relation on worlds based on structural information by σ-preferring a world ω to a world ω' iff ω falsifies less conditionals than ω' and ω' falsifies at least the conditionals falsified by ω.

Definition 9 ($\prec_\sigma$-preference). *[4]*
A world ω shall be σ-preferred to a world ω', in terms $\omega \prec_\sigma \omega'$, if and only if for every $1 \leq i \leq n$, $\sigma_i(\omega) = \mathbf{a}_i^-$ implies $\sigma_i(\omega') = \mathbf{a}_i^-$, and there is at least one i such that $\sigma_i(\omega) \in \{\mathbf{a}_i^+, 1\}$ and $\sigma_i(\omega') = \mathbf{a}_i^-$.

The triple $\langle \Omega, \models, \prec_\sigma \rangle$ is a *preferential model* (cf. definition 2) and hence allows to define nonmonotonic inference of some quality. We show that σ-preferential reasoning follows the lines of System P for which the following preliminaries have to be deployed.
Since it is quite obvious that $\prec_\sigma$ is stoppered and transitive, we have the following lemma:

Lemma 1. *$\langle \Omega, \models, \prec_\sigma \rangle$ is a stoppered classical preferential model.*

Using the preference relation $\prec_\sigma$ we define a structual entailment relation according to definition 5.

Definition 10 (σ-structural inference).
Let A, B be formulas in $\mathfrak{L}$ and $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\} \subseteq (\mathfrak{L} \mid \mathfrak{L})$. B can be structurally inferred*, or σ*-inferred*, from A, written as*

$$A {\mid\!\sim}^{\sigma}_{\Delta} B \quad \textit{iff} \quad \forall\, \omega' : \ \omega' \models A\overline{B} \qquad \exists\, \omega : \ \omega \models AB \qquad \textit{with} \qquad \omega \prec_\sigma \omega'.$$

We see that A σ-infers B if and only if for every world $\omega' \in Mod(A\overline{B})$ there is a σ-preferred world $\omega \in Mod(AB)$.

Example 2. We use the knowledge base Δ from example 1. By σ-structural inference, we find that flying birds are no penguins ($bf {\mid\!\sim}^{\sigma}_{\Delta} \overline{p}$) since for every world ω' which is a model of pbf, namely $\omega' = pbf$, there is a world ω which is a model of $\overline{p}bf$, namely $\omega = \overline{p}bf$, for which we see, that $\sigma_1(\omega') = \mathbf{a}_i^+$, $\sigma_1(\omega) = \mathbf{a}_i^+$, $\sigma_2(\omega') = \mathbf{a}_i^-$, $\sigma_2(\omega) = 1$, $\sigma_3(\omega') = \mathbf{a}_i^+$ and $\sigma_3(\omega) = 1$ and therefore by the above definition it holds that $\omega \prec_\sigma \omega'$.

From lemma 1 and proposition 1, we obtain:

Proposition 3. *${\mid\!\sim}^{\sigma}_{\Delta}$ satisfies System P.*

However, there ist an odd finding, shown by the next example, which suggests that ${\mid\!\sim}^{\sigma}_{\Delta}$ may violate System P.

Example 3 (Counterexample to System P-compliance of ${\mid\!\sim}^{\sigma}_{\Delta}$?). We use the running example with the conditional structures from example 1. We would expect that $p {\mid\!\sim}^{\sigma}_{\Delta} b$ since $(b|p) \in \Delta$, and $p {\mid\!\sim}^{\sigma}_{\Delta} \overline{f}$ since $(\overline{f}|p) \in \Delta$, therefore by (CM) it should follow that $pb {\mid\!\sim}^{\sigma}_{\Delta} \overline{f}$. But $\sigma_\Delta(pbf) = \mathbf{a}_1^+\mathbf{a}_2^-\mathbf{a}_3^+$ and $\sigma_\Delta(pb\overline{f}) = \mathbf{a}_1^-\mathbf{a}_2^+\mathbf{a}_3^+$, so $pb\overline{f} \not\prec_\sigma pbf$, hence $pb \not{\mid\!\sim}^{\sigma}_{\Delta} \overline{f}$.

Example 3 is, of course, no counterexample to the System P compliance of classical stoppered preferential models but to the assumption, that we may always conclude $A {\mid\!\sim} B$ if $(B|A) \in \Delta$, which we formalised as property (DI).

Proposition 4. *${\mid\!\sim}^{\sigma}_{\Delta}$ does not comply with* (DI)

Proof. Example 1 is a counterexample for ${\mid\!\sim}^{\sigma}_{\Delta}$ and DI: We see that for the world $\omega' = pbf$ there is no $\omega \models p\overline{f}$ with $\omega \prec_\sigma \omega'$. So $p {\mid\!\sim}^{\sigma}_{\Delta} \overline{f}$ does not hold.

By this, we see that $p {\mid\!\sim}^{\sigma}_{\Delta} \overline{f}$ does not hold for the running example and because of that, example 3 is no counterexample to proposition 3.
It is apparent that this problem arises due to the incomparability of "alternating symbols" i.e. if for worlds ω, ω' we have $\sigma_i(\omega) = \mathbf{a}_i^+$, $\sigma_i(\omega') = \mathbf{a}_i^-$ and $\sigma_j(\omega) = \mathbf{a}_j^-$, $\sigma_j(\omega') = \mathbf{a}_j^+$ for at least one pair of $1 \leq i, j \leq n$, $i \neq j$. In this case, ω and ω' are *stucturally incomparable.*

6 Reasoning with c-representations

To solve the problem of structurally incomparable worlds that became evident in the previous section, we need to introduce some kind of weights for conditionals to compare the falsification of *different* conditionals.

Definition 11 (c-representations). *A c-representation [3] of a knowledge base* $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\} \subseteq (\mathfrak{L} \mid \mathfrak{L})$ *is defined as an OCF of the form*

$$\kappa^c_\Delta(\omega) = \sum_{\substack{i=1 \\ \omega \models A_i \overline{B}_i}}^{n} \kappa_i^-, \qquad \kappa_i^- \in \mathbb{N}_0 \tag{2}$$

where the values κ_i^- *are penalty points for falsifying conditionals and have to be chosen to make* κ^c_Δ Δ*-admissible, i.e. for all* $1 \leq i \leq n$ *it holds that* $\kappa^c_\Delta \models (B_i|A_i)$ *which is the case if and only if*

$$\kappa_i > \min_{\omega \models A_i B_i} \left\{ \sum_{\substack{i \neq j \\ \omega \models A_j \overline{B}_j}} \kappa_j^- \right\} - \min_{\omega \models A_i \overline{B}_i} \left\{ \sum_{\substack{i \neq j \\ \omega \models A_j \overline{B}_j}} \kappa_j^- \right\}. \tag{3}$$

A minimal *c-representation is obtained by choosing* κ_i^- *minimally according to (3) for all* i*,* $1 \leq i \leq n$*.*

Example 4 (c-represented penguins). We use, from the introductory example, the knowledge base $\Delta = \{(f|b), (\overline{f}\,|p), (b|p)\}$. For the κ_i^- values of a c-representation we get, according to (3), $\kappa_1^- > 0$, $\kappa_2^- > \min\{\kappa_1^-, \kappa_3^-\}$ and $\kappa_3^- > \min\{\kappa_1^-, \kappa_2^-\}$. A minimal c-representation for Δ is calculated with $\kappa_1^- = 1$, $\kappa_2^- = \kappa_3^- = 2$ and the resulting ranking of worlds is shown in the following table.

ω	$\kappa^c_\Delta(\omega)$	ω	$\kappa^c_\Delta(\omega)$	ω	$\kappa^c_\Delta(\omega)$	ω	$\kappa^c_\Delta(\omega)$
pbf	2	$p\overline{b}\,f$	4	$\overline{p}\,bf$	0	$\overline{p}\,\overline{b}\,f$	0
$pb\overline{f}$	1	$p\overline{b}\,\overline{f}$	2	$\overline{p}\,b\overline{f}$	1	$\overline{p}\,\overline{b}\,\overline{f}$	0

For c-representations, we make use of the preference relation $<_\kappa$ and the inference relation $\mid\!\sim_\kappa$ defined for general OCF's κ (see section 4).

Definition 12 (Preference and inference by c-representation). *A world* $\omega \in \Omega$ *is* κ^c_Δ*-preferred to a world* $\omega' \in \Omega$ *(*$\omega <_{\kappa^c_\Delta} \omega'$*) if and only if* $\kappa^c_\Delta(\omega) < \kappa^c_\Delta(\omega')$*. For a knowledge base* Δ*, a formula* B *is* κ^c_Δ*-inferrable from* A *(*$A \mid\!\sim_{\kappa^c_\Delta} B$*) if and only if* $\kappa^c_\Delta(AB) < \kappa^c_\Delta(A\overline{B})$*.*

C-representations elaborate conditional structures in a more sophisticated way than structural inference and provide an inference relation that surpasses, e.g., System Z. For the axiomatic derivation of c-representations from conditional structures, the abelian group property of $\mathfrak{F}_\Delta$ is needed, for further information, please see [3].
Since every c-representation is an OCF, $\mid\!\sim_{\kappa^c_\Delta}$ inherits the properties of $\mid\!\sim_\kappa$.

Corollary 2. $\mid\!\sim_{\kappa^c_\Delta}$ *complies with System P and R.*

We introduced c-representations to solve the problem of structually incomparable worlds which arose in section 5. Indeed, c-representations employ numerical penalty values that may play the roles of weights. However, it is not at all clear that $\mid\!\sim_{\kappa^c_\Delta}$ refines $\mid\!\sim^\sigma_\Delta$. By the following example we see that $A \mid\!\sim^\sigma_\Delta B$ does not necessarily imply $A \mid\!\sim_{\kappa^c_\Delta} B$. So, with c-representations we have a different, high-quality inference, but the relevance for solving the addressed problem of $\mid\!\sim^\sigma_\Delta$is not obvious.

Proposition 5. *There are knowledge bases* Δ *such that* $A \mid\!\sim^\sigma_\Delta B$ *does not imply* $A \mid\!\sim_{\kappa^c_\Delta} B$*.*

Example 5. Let $\Delta = \{(b|a), (bc|a)\}$. A minimal c-representation is obtained from $\kappa_1^- = 0$, $\kappa_2^- = 1$. Ranks and conditional structures are shown below, we see that $a\overline{c} {\mid\!\sim}^{\sigma}_{\Delta} b$ but $a\overline{c} \not{\mid\!\sim}_{\kappa^c_\Delta} b$.

ω	$\kappa^c_\Delta(\omega)$	$\sigma_\Delta(\omega)$	ω	$\kappa^c_\Delta(\omega)$	$\sigma_\Delta(\omega)$	ω	$\kappa^c_\Delta(\omega)$	$\sigma_\Delta(\omega)$	ω	$\kappa^c_\Delta(\omega)$	$\sigma_\Delta(\omega)$
abc	0	$\mathbf{a}_1^+\mathbf{a}_2^+$	$ab\overline{c}$	1	$\mathbf{a}_1^+\mathbf{a}_2^-$	$\overline{a}bc$	0	1	$\overline{a}b\overline{c}$	0	1
$a\overline{b}c$	1	$\mathbf{a}_1^-\mathbf{a}_2^-$	$a\overline{b}\overline{c}$	1	$\mathbf{a}_1^-\mathbf{a}_2^-$	$\overline{a}\overline{b}c$	0	1	$\overline{a}\overline{b}\overline{c}$	0	1

The problem arises because the second rule, $(bc|a)$, also establishes the first rule, $(b|a)$, hence $\kappa_1^- = 0$. To approach this problem we examine the recently proposed postulate of *Enforcement* (ENF) [4]. This was proposed for *revisions* of ranking functions in the belief revision framework. We recall the necessary preliminaries from [4] in the following:
Let κ be a ranking function and $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\} \subseteq (\mathfrak{L} \mid \mathfrak{L})$ be a knowledge base. Let $\kappa^* = \kappa * \Delta$ be the ranking function which results from revising the epistemic state κ by the new information Δ. As a quality criterion, (ENF) postulates that if for two worlds $\omega, \omega' \in \Omega$ it holds that $\omega \prec_\sigma \omega'$, then $\omega \leq_\kappa \omega'$ implies $\omega <_{\kappa^*} \omega'$. To use this postulate for inductive reasoning we revise the uniform ranking function κ_u, that is the ranking function for which $\kappa_u(\omega) = 0$ for all $\omega \in \Omega$, with the knowledge base Δ which we want to rely our reasoning on. For this special case, $\omega \leq_{\kappa_u} \omega'$ is trivially fulfilled for all $\omega, \omega' \in \Omega$ and (ENF) boils down to the following postulate:

Definition 13 (Enforcement for inductive reasoning (Ind-ENF)**).** *Let $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\} \subseteq (\mathfrak{L} \mid \mathfrak{L})$. A Δ-admissible ranking function κ_Δ respects* (Ind-ENF) *if $\omega \prec_\sigma \omega'$ implies $\omega <_{\kappa_\Delta} \omega'$ for all $\omega, \omega' \in \Omega$.*

So a Δ-admissible ranking function κ_Δ respects (Ind-ENF) if the structural preference induced by σ_Δ is respected by κ_Δ. This leads to establishing conditional dependencies more thoroughly: (Ind-ENF) ensures that when learning $(b|a)$, also both conditionals $(b|ac)$ and $(b|a\overline{c})$ are established, as long as no other conditional in the knowledge base inhibits this. So, the problem shown in Example 5 does not occur.

Lemma 2. *If a Δ-admissible ranking function κ_Δ respects* (Ind-ENF)*, then $A {\mid\!\sim}^{\sigma}_{\Delta} B$ implies $A {\mid\!\sim}_{\kappa_\Delta} B$.*

Proof. By definition $A {\mid\!\sim}^{\sigma}_{\Delta} B$ iff for all $\omega' \models A\overline{B}$ there is an $\omega \models AB$ such that $\omega \prec_\sigma \omega'$. If (Ind-ENF) holds, then also $\omega <_{\kappa_\Delta} \omega'$. Hence $\kappa_\Delta(AB) < \kappa_\Delta(A\overline{B})$ and so have $A {\mid\!\sim}_{\kappa_\Delta} B$. ☐

We see that the OCF from example 5 does not respect (Ind-ENF) since $abc \prec_\sigma ab\overline{c}$ but $abc \not<_{\kappa^c_\Delta} ab\overline{c}$. The next proposition gives a simple criterion to check if a given c-representation satisfies (Ind-ENF).

Proposition 6. *Let $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\} \subseteq (\mathfrak{L} \mid \mathfrak{L})$ be a conditional knowledge base. A c-representation κ^c_Δ with $\kappa_i^- > 0$ for all $1 \leq i \leq n$ respects* (Ind-ENF).

Proof. Let $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\} \subseteq (\mathfrak{L} \mid \mathfrak{L})$ with c-representation κ^c_Δ. We define $\mathcal{I}(\omega) = \{i \mid \omega \models A_i\overline{B}_i\} = \{i \mid \sigma_i(\omega) = \mathbf{a}_i^-\}$. If $\omega \prec_\sigma \omega'$, we have $\mathcal{I}(\omega) \subsetneqq \mathcal{I}(\omega')$ and the difference $\kappa^c_\Delta(\omega') - \kappa^c_\Delta(\omega)$ for these worlds is

$$\kappa^c_\Delta(\omega') - \kappa^c_\Delta(\omega) = \sum_{i \in \mathcal{I}(\omega')} \kappa_i^- - \sum_{i \in \mathcal{I}(\omega)} \kappa_i^- = \sum_{i \in (\mathcal{I}(\omega') \setminus \mathcal{I}(\omega))} \kappa_i^- > 0,$$

since $\kappa_i^- > 0$ for all $1 \leq i \leq n$. Hence, (Ind-ENF) holds. ☐

Note, however, that postulating $\kappa_i^- > 0$ for each conditional $(B_i|A_i)$ in the knowledge base Δ is usually too strong, since Δ may contain equivalent conditionals.

Conclusion

In this paper, we presented an approach to base inductive conditional reasoning on structural arguments by observing which conditionals of the knowledge base are verified or falsified, respectively, by possible worlds. This induces a preference relation between possible worlds, and allows us to define a preferential entailment relation with nice properties. We also drew attention to the property of Direct Inference which links inference relations to conditional knowledge bases, and formalized an enforcement postulate for inductive reasoning which claims that structural differences between worlds must be reflected appropriately by the preference relation underlying preferential entailment. We applied these ideas to c-representations which allow for inductive conditional reasoning of high quality. A more thorough evaluation of c-representations that obey (Ind-ENF) is part of our ongoing research.

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