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Spectral Capital Allocation

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INTRODUCTION

Portfolio modelling has two main objectives: the quantification of portfolio risk, which is usually expressed as the economic capital of the portfolio, and its allocation to subportfolios and individual transactions. The standard approach in credit portfolio modelling is to define the economic capital in terms of a quantile of the portfolio loss distribution. The capital charge of an individual transaction is usually based on a covariance technique and called volatility contribution.[1]

Since the work by Artzner *et al* (1997) coherent risk measures are discussed intensively in finance and risk management. More recent is the question of a more coherent capital allocation. Especially the use of expected shortfall allocation as an allocation rule is recommend in Overbeck (1999), Denault (2001), Bluhm, Overbeck and Wagner (2002), Kurth and Tasche (2003), Kalkbrener, Lotter and Overbeck (2004).

Expected shortfall measures are the building blocks of more general coherent risk measures, the spectral risk measure. These are convex mixtures of expected shortfall measures where the mixture can be represented as a probability measure on the confidence level from 0 to 1 or as an increasing weight function for the confidence level. It is known that the only substantial restriction on this weight function is that higher confidence level should have larger weights. Already in the original paper by Acerbi (2002) the interpretation of the weight function in terms of risk aversion is given. In terms of that the requirements just mean that higher losses have higher awareness and aversion.

In this chapter we point out that the axiomatic approach to capital allocation (Kalkbrener, 2002; Kalkbrener *et al*, 2004) can also be carried out easily for spectral risk measures. This approach is strongly related to the capital allocation based on derivatives or sensitives. The only result which is necessary for this is an explicit formula of the density of the probability describing the scenario associated with the weight function. This is given in Theorem 1.

Subsequent research will focus on the study of spectral allocation in a Merton-type credit portfolio model accompined by concrete examples.

SPECTRAL RISK MEASURES AND ALLOCATION

It is well known that the following four conditions define a coherent risk measure, Artzner *et al* (1997, 1999), Delbaen (2000).

Formally, a risk measure is nothing else but a positive real valued function r defined on the set of random variable (potential losses) V. The number $r(X)$ denotes the risk in portfolio X. r is called coherent if it obeys the following four rules.

- subadditivity (Diversification)

$$r(X + Y) < r(X) + r(Y)$$

- positive homogeneous (Scaling)

$$r(aX) = ar(X), \quad a > 0$$

- monotone

$$r(X) < r(Y) \quad \text{if } X < Y \text{ (almost surely)}$$

- translation property

$$r(X + a) = r(X) - a$$

Convex analysis gives already that a sub-additive positive homogeneous function r can be point wise written as the maximal value of all linear functions which are below r (Delbaen, 2000; Kalkbrener, 2002; Kalkbrener *et al*, 2004). For risk measures this means that already axioms 1 and 2 lead to the following representation

$$r(X) = \max\{l(X) | l < r, l \ \text{ linear function}\} \tag{1}$$

The risk measure evaluate at a loss variable X takes the same value as the largest value of all linear function which lies below r on V evaluated on X.

Conceptually, this is similar to the gradient of the function r evaluated at the point X or as the best linear approximation of r which coincides with r at the point X. We will later see that this intuition gives rise to a sensible capital allocation.

A typical linear function for random variable is the expectation operator. Hence the basic result by Artzner *et al* (1997), Delbaen (2000)

$$r(X) = \sup\left\{E_Q[X] \,|\, Q \in \mathcal{Q}\right\} \tag{2}$$

$\mathcal{Q}, = \mathcal{Q}_r$, a suitable set of probability measures of absolutely continuous probability measures $Q \ll P$ with density $\mathrm{d}Q/\mathrm{d}P$, is similar to the representation (1).

The set $\mathcal{Q}$ is called the generalised scenarios associated with r. If the supremum is actually taken at some probability measure, this probability measure or its density with respect to P is called the generalised scenario associated with r. This approach also fits into the intuitive feature of risk measurement, namely scenario or stress analysis. For the interpretation in terms of scenarios the formulation with probability measure is more natural, but for the axiomatic approach to capital allocation the representation (1) is very useful.

The currently most prominent example of a coherent risk measure is expected shortfall (sometimes called conditional VAR/tail conditional expectation). It is denoted by ES_α and measures the average loss above the α-quantile of the loss distribution. The associated generalised scenarios can be explained as follows: To each loss variable Y define the scenario as the "historical" calibrated objective scenario constraint on the condition that the loss variable exceeded its quantile. The expected shortfall coincides with the largest mean loss in these scenarios. Intuitively,

$$E\left[L \,|\, L > q_\alpha(L)\right] = \max\left\{E[L \,|\, Y > q_\alpha(Y)] \,|\, \text{all } Y \in L_\infty\right\}$$

Even if generalised scenarios are defined as a supremum, in the case of expected shortfall we can identify the density of the maximal "scenario". For this we need the formally correct definition of expected shortfall at level α. The problem with the intuitive definition above is the possible positive mass at the quantile itself. The exact definition of the expected shortfall at level α is therefore (see Acerbi and Tasche, 2002; Kalkbrener *et al*, 2004):

Definition 1.

$$\mathrm{ES}_\alpha(L) := (1-\alpha)^{-1}\left[E\left(L\,\mathbf{1}_{\{L>q_\alpha(L)\}}\right) + q_\alpha(L)\cdot\left(\mathbb{P}(L \le q_\alpha(L)) - \alpha\right)\right]$$

Here we take the quantile defined by

$$q_u(L) = \inf\left\{x \,|\, P[L \le x] \ge u\right\}$$

the smallest u-quantile

Since $\mathrm{ES}_\alpha = E[Lg_\alpha(L)]]$ with the function

$$g_\alpha(Y) := (1-\alpha)^{-1}\left(\mathbf{1}_{\{Y>q_\alpha(Y)\}} + \beta_Y\,\mathbf{1}_{\{Y=q_\alpha(Y)\}}\right) \tag{3}$$

where β_Y is a real number and

$$\beta_Y := \frac{\mathbb{P}(Y \le q_\alpha(Y)) - \alpha}{\mathbb{P}(Y = q_\alpha(Y))} \quad \text{if } \mathbb{P}(Y = q_\alpha(Y)) > 0$$

the density of the associated maximal scenario turns out to be the function g_α.[2]

For the interpretation of this density function in terms of risk aversion as outlined in Acerbi (2002), let us reformulate the expected shortfall as an integral over the quantile function, the inverse of the distribution of L. It is well known that

$$\mathrm{ES}_\alpha = (1-\alpha)^{-1}\int_\alpha^1 q_u(L)\,\mathrm{d}u$$

The implicit risk aversion with expected shortfall is, that all quantiles below α or all losses below the α quantile have no weights, ie, there is no risk aversion and all losses above the α-quantile have the same risk aversion. Therefore the risk aversion weight function associated with ES_α turns out to be

$$w_{\mathrm{ES}_\alpha}(u) = (1-\alpha)^{-1}\mathbf{1}_{\{u>\alpha\}} \tag{4}$$

From a risk management point of view there might be many other weights given to some confidence levels u. If the weight function is increasing, which is reasonable since higher losses should have larger risk aversion weight, then we arrive at spectral risk measures.

Definition 2. Let w be an increasing function from $[0,1]$ such that $\int_0^1 w(u)\,\mathrm{d}u = 1$, then the map r_w defined by

$$r_w(L) = \int_0^1 w(u) q_u(L)\,\mathrm{d}u$$

is called a spectral risk measure with weight function w.

The name spectral risk measure comes from the representation

$$r_w(X) = \int_0^1 \mathrm{ES}_\alpha (1-\alpha)\mu_u(\mathrm{d}a) \quad \textbf{(5)}$$

$$\text{with the spectral measure } \mu((0,b]) = w(b) \quad \textbf{(6)}$$

This representation is very useful when we want to find the scenario function representing a spectral risk measure r_w.

Theorem 1. The density of the scenario associated with the risk measure equals

$$L_w := g_w(L) := \int_0^1 g_\alpha(L)(1-\alpha)\mu(\mathrm{d}\alpha) \quad \textbf{(7)}$$

Here $g_\alpha(L)$ is defined in formula (3). In particular

$$r_w(L) = E[LL_w] \quad \textbf{(8)}$$

Proof. We have

$$r_w(L) = \int_0^1 \mathrm{ES}_\alpha(L)(1-\alpha)\mu(\mathrm{d}\alpha)$$
$$= \int_0^1 E[LL_\alpha](1-\alpha)\mu(\mathrm{d}\alpha)$$

$$
\begin{aligned}
&= \int_0^1 \max\left\{E[Lg_\alpha(Y)] \,\middle|\, Y \in L_\infty\right\}(1-\alpha)\mu(d\alpha) \\
&\geq \max\left\{\int_0^1 E\left[L\int_0^1 g_\alpha(Y)(1-\alpha)\mu(d\alpha)\right] \,\middle|\, Y \in L_\infty\right\} \\
&= \max\left\{E[Lg_w(Y)] \,\middle|\, \forall Y \in L_\infty\right\} \\
&\geq E[Lg_w(L)]
\end{aligned}
$$

Hence

$$r_w(L) = \max\left\{E[Lg_w(Y)] \,\middle|\, \forall Y \in L_\infty\right\} = E[Lg_w(L)] \qquad \square$$

Spectral capital allocation

Let us recall the approach in Kalkbrener (2002) and Kalkbrener *et al* (2004). Starting with the representation (1) one can now find for each Y a linear function $h_Y = h_Y^r$ which satisfies

$$r(Y) = h_Y(Y) \quad \text{and} \quad h_Y(X) \leq r(X), \quad \forall X \tag{9}$$

A "diversifying" capital allocation associated with r is given by

$$\Lambda_r(X, Y) = h_Y(X) \tag{10}$$

The function Λ_r is then *linear* in the first variable and *diversifying* in the sense that the capital allocated to a portfolio X is always bounded by the capital of X viewed as its own subportfolio[3]

$$\Lambda(X, Y) \leq \Lambda(X, X) \tag{11}$$

In general we have the following two theorems: A linear and diversifying capital allocation Λ, which is continuous at a portfolio Y, is uniquely determined by its associated risk measure, ie, the diagonal values of Λ.[4] More specifically, given the portfolio Y then the capital allocated to a subportfolio X of Y is the derivative of the associated risk measure ρ at Y in the direction of X.

Theorem 2. Let Λ be a linear, diversifying capital allocation. If Λ is continuous at $Y \in V$ then for all $X \in V$

$$\Lambda(X, Y) = \lim_{\epsilon \to 0} \frac{r(Y + \epsilon X) - \rho(Y)}{\epsilon}$$

The following theorem states the equivalence between positively homogeneous, sub-additive risk measures and linear, diversifying capital allocations.

Theorem 3. (a) If there exists a linear, diversifying capital allocation Λ with associated risk measure r, ie, $r(X) = \Lambda(X, X)$, then r is positively homogeneous and sub-additive.
(b) If r is positively homogeneous and sub-additive then Λ_r as defined in (10) is a linear, diversifying capital allocation with associated risk measure r.

Since in the case of spectral risk measures r_w the maximal linear functional in (9) can be identified as an integration with respect to the probability measure with density (7) from Theorem 1, we obtain $h_Y(X) = E[Xg_w(Y)$ and therefore the following capital allocation

$$\Lambda_w(X, Y) = E[Xg_w(Y)] \qquad \textbf{(12)}$$

Intuitively, the capital allocated to transaction or subportfolio X in a portfolio Y equals its expectation under the generalised maximal scenario associated with w.

Examples

1. as a first step in the application of spectral risk measures one might consider giving different weight to different loss probability levels. This is a straight-forward extension of expected shortfall. One might view expected shortfall at the 99%-level view as a risk aversion which ignores losses below the 99%-quantile and all losses above the 99%-quantile have the same influence. From an investor's point of view this means that only senior debts are cushioned by risk capital. One might on the other hand also be aware of losses which occur more frequently, but of course with a lower aversion than those appearing rarely.

   As a concrete example one might set that losses up to the 50% confidence level should have zero weights, losses between 50% and 99% should have a weight w_0 and losses above the 99%-quantile should have a weight of $k_1 w_0$ and above the 99.9% quantile it should have a weight of $k_2 \mathrm{w}_0$. The first tranch from 50% to 99% correspond to an investor in junior debt, and the tranch from 99% to 99.9% to a senior investor and above the 99.9% a super senior investor or the regulators are concerned. This gives a step function for *w:*

$$w(u) = w_0\mathbf{1}_{\{0.99>u>0.5\}} + k_1 w_0\mathbf{1}_{\{0.999>u>0.99\}} + k_2 w_0\mathbf{1}_{\{1>u>0.999\}}$$

The parameter w_0 should be chosen such that the integral over w is still 1.

2. a more continuous form of this is an exponential function starting at a point u_0 between 0 and 1 and then increasing up to 1

$$w(u) = \mathbf{1}_{\{u>u_0\}}\exp(\kappa u)$$

with some constant κ.

Remarks.

1. expected shortfall allocation which allocates the average loss of transaction i in all cases where the overall portfolio capital exceeds a certain quantile can be interpreted as a causal capital allocation. Literally the actual contribution of the transaction to the overall capital is allocated if the conditional expectation is used. In the same way the spectral allocation – at least when the weight function is a step function – is a causal allocation. Here, of course, the future loss for which the capital is needed gets a different weight from those obtained by a simple conditional expectation, as in the expected shortfall contribution.
2. also from the point of view that all actual losses have different impact or subsequent losses. A large loss which is reported in the press might have consequent losses – due to reputational impacts – exceeding the first actual loss by far, and might even damage the capital basis. On the other hand small losses are directly covered by income and will affect the capital not at all. Therefore a different weighting of different loss sizes might be useful.
3. in the case of a continuous distribution one can rewrite

$$\int_0^1 w(u)qu(L)\,\mathrm{d}u = E[w(U)q_U(L) = E[w(F(L))L]$$

Then the calibration of the weight function can be done in terms of portfolio loss itself instead of the quantiles of the loss distribution.

However, the new weight function, now defined on the range of the loss variable L, has to be transformed $w_F(x) := w(F(x))$

IMPLEMENTATION

There are several ways to implement a spectral contribution in a portfolio model. According to Acerbi (2002) a Monte Carlo-based implementation of the spectral risk measure would work as follows:

Let L^n be the n-th realisation of the portfolio loss. If we have generated N loss distribution scenario, let us denote by $n : N$ index of the n-th largest loss which itself is then denote by $L^{n:N}$, ie, the indices $1 : N, 2 : N, \ldots, N : N \in \mathbb{N}$ are defined by the property that

$$L^{1:N} < L^{2:N} < \cdots < L^{N:N}$$

The approximative spectral risk measure is then defined by

$$\sum_{n=1}^{N} L^{n:N} w(n/N) \Big/ \sum_{k=1}^{N} w(k/N)$$

Therefore a natural way to approximate the spectral contribution of a transaction L_i is

$$\sum_{n=1}^{N} L_i^{n:N} \frac{w(n/N)}{\sum_{k=1}^{N} w(k/N)}$$

It is then expected that

$$E[L_i L_w] = \lim_{N\to\infty} \sum_{n=1}^{N} L_i^{n:N} \frac{w(n/N)}{\sum_{k=1}^{N} w(k/N)}$$

Another possibility is to rely on the approximation of the expected shortfall contribution as in Kalkbrener *et al* (2004) and to integrate over the spectral measure μ:

$$E[L_i L_w] = \lim_{N\to\infty} \int_0^1 \left(\sum_{n=1}^{N} L_i^{n:N} \frac{w_\alpha(i/N)}{\sum_{k=1}^{N} w_\alpha(k/N)} (1-\alpha) \right) \mu(\mathrm{d}a) \qquad \textbf{(13)}$$

If L has a continuous distribution than we have that

$$\begin{aligned} E[L_i L_w] &= E[L_i \int_0^1 L_\alpha \mu(\mathrm{d}\alpha) \\ &= \int_0^1 E[L_i \mathbf{1}_{L>q_\alpha(L)}](1-\alpha)^{-1}\mu(\mathrm{d}\alpha) \\ &= \lim_{N\to\infty} N^{-1} \sum_{n=1}^{N} L_i^n \int_0^1 \mathbf{1}_{L^n>q_\alpha(L)}(1-\alpha)^{-1}\mu(\mathrm{d}\alpha) \end{aligned} \quad \mathbf{(14)}$$

If L has not a continuous distribution we have to use the density function (7) and might approximate the spectral contribution by

$$E[L_i L_w] \sim N^{-1} \sum_{n=1}^{N} L_i^n g_w(L^n) \quad \mathbf{(15)}$$

The actual calculation of the density g_w in (15) might be quite involved. On the other hand the integration with respect to μ in (13) and (14) is also not easy. If w is a step function as in the Example 1 above, then μ is a sum of weighted Dirac-measure and the implementation of spectral risk measure is straightforward.

1 We refer to Bluhm *et al* (2002) and Crouhy *et al* (2000) for a survey on credit portfolio modelling and capital allocation.
2 Note that $\mathrm{ES}_\alpha(Y) = E(Y \cdot g(Y))$ and $\mathrm{ES}_\alpha(X) \geq E(X \cdot g(Y))$ for every $X, Y \in V$.
3 $\Lambda(X, X)$ can be called the standalone capital or risk measure of X.
4 $\lim_{\epsilon\to 0}\Lambda(X, Y + \epsilon X) = \Lambda(X, Y)\forall X$.

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