Modeling Default Dependence with Threshold Models

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Abstract

We investigate the problem of modeling defaults of dependent credits. In the framework of the class of structural default models we study threshold models where for each credit the underling ability-to-pay process is a transformation of a Wiener processes. We propose a model for dependent defaults based on correlated Wiener processes whose time scales are suitably transformed in order to calibrate the model to given marginal default distributions for each underlying credit. At the same time the model allows for a straightforward analytic calibration to dependency information in the form of joint default probabilities.

Key words: credit default, credit derivative, default dependence, structural form models, threshold model

JEL Classification: G12, G13, G24, C69

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1 Introduction

Given the explosive development of the credit derivatives market, the constraints of scarce capital resources, and the rules of regulatory authorities concerning credit risk sophisticated default modeling becomes an indispensable requirement in practice.

The increase in securitization of credit risk via CDO's and the growing popularity of multi-credit derivatives like basket default swaps not only calls for modeling default but, equally importantly, for modeling the dependencies between defaults of different obligors.

Default models can be divided into two mainstreams, structural models and reduced form models. In structural models, as pioneered by the work of Merton [9], the default event is triggered by the value of the firm being below a certain trigger level. Structural models, also called threshold models, are appealing because of their simplicity and intuition linking the credit event to economic fundamentals of the firm. Structural models are relatively straightforward to extend to the situation of multiple credits with correlated defaults. However, the calibration to a given term structure of default probabilities is, depending on the model, often not possible or computationally involved.

The other major class of default models, the reduced form models, follow a completely different approach. Instead of modeling the event of default by some economic primitives, reduced form models rather focus on modeling the (infinitesimal) likelihood of default (cf. [1], [5], [4]). Reduced from models offer a lot of analogy to interest rate term structure models, in particular, they can be easily calibrated to a given term structure of default probabilities. For the multi-credit case, introducing enough default dependence as required in practice is quite involved in the framework of reduced form models ([11], [10]).

In practice, for multi-credit products dependencies between defaults are often modelled using copulas. In the simplest case the joint distribution of default times is generated from the marginal distributions applying a normal copula, see e.g. [8]. In this approach credit spreads are static until the time of the first default (cf. [11]).

In this paper we follow the structural approach to model (dependent) defaults. The model we propose is straightforward to calibrate to any given term structure of defaults. Also, based on a result by Zhou [12], we derive a analytic expression for the probability of joint default, which then allows an efficient calibration of the model to given dependency information. Since we follow a structural approach the credit spreads admit a certain stochastic dynamic contrary to the above mentioned copula approach for dependent defaults.

Recently, Hull and White [3] proposed a model for dependent defaults which achieves the same goals but is computationally much more involved.

The paper is organized as follows. In Section 2 we formulate the problem. The following Section 3 is devoted to a short review of the approach by Hull and White. Our alternative model is introduced in Section 4 where we also provide some example of calibrations and outline how the model is implemented in practice. In Section 5 we given an application of the model to the pricing of basket default swaps and compare the results of our model with the results of the popular normal copula approach. The final Section 6 contains our main conclusions.

2 Formulation of the problem

Our goal is to develop a model for the random times $\tau_1, \ldots, \tau_n$ of default of credits $i = 1, \ldots, n$. What is given, either derived from markets prices of traded defaultable instruments like default swaps or from historical data, is the distribution function F_i of the random variable τ_i:

$$\mathbf{P}(\tau_i < t) = F_i(t), \ t \geq 0, \ i = 1, \ldots, n. \tag{1}$$

We assume that F_i is continuous, strictly increasing and $F_i(0) = 0$.

Information on the dependencies of the random default times involves, in the most general case, some conditions on the joint distribution of $(\tau_1, \ldots, \tau_n)$. As this in unrealistic in practice, we restrict ourselves to given pairwise joint default probabilities for a given fixed time horizon t_0:

$$p_{ij} = \mathbf{P}(\tau_i < t_0, \tau_j < t_0). \tag{2}$$

The event correlation ρ_{ij}^E (for time t_0) is defined as

$$\rho_{ij}^E = \frac{p_{ij} - F_i(t_0)F_j(t_0)}{\sqrt{F_i(t_0)(1 - F_i(t_0))F_j(t_0)(1 - F_j(t_0))}}. \tag{3}$$

So specifying the joint default probability is equivalent to specifying the event correlation. Observe that, in view of $0 \leq \mathbf{P}(\tau_i < t_0, \tau_j < t_0) \leq \min(F_i(t_0), F_j(t_0))$, the event correlation admits the following natural bounds

$$\frac{-F_i(t_0)F_j(t_0)}{\sqrt{F_i(t_0)(1 - F_i(t_0))F_j(t_0)(1 - F_j(t_0))}} \leq \rho_{ij}^E \leq \sqrt{\frac{u(1-v)}{v(1-u)}}, \tag{4}$$

where $u = \min(F_i(t_0), F_j(t_0))$ and $v = \max(F_i(t_0), F_j(t_0))$.

Problem: Given distribution functions F_i and joint default probabilities p_{ij} (or, equivalently, event correlations) we are looking for stochastic processes (Y_t^i) called *ability-to-pay processes* and barriers $K_i(t)$ such that the default time τ_i for credit i can be modelled as the first hitting time of the barrier $K_i(t)$ by the process (Y_t^i):

$$\tau_i = \inf\{t \geq 0 : Y_t^i \leq K_i(t)\}. \tag{5}$$

In other words, the hitting time τ_i defined by (5) satisfies equations (1) and (2).

Given the analytical tractability of the Wiener process and the fact that many other stochastic processes can be transformed into a Wiener process a reasonable approach for finding appropriate processes (Y_t^i) would be to start with correlated Wiener processes (W_t^i) and then to apply suitable transformations G^i:

$$Y_t^i = G^i(W_t^i, t).$$

As we shall see in the next section the recent approach by Hull and White fits into this framework.

3 The Hull and White approach revisited

In their recent paper [3] Hull and White propose a model for dependent defaults where the default time takes values in arbitrary fine discrete time grind $0 = t_0 < t_1 < t_2 \dots, \delta_k = t_k - t_{k-1}$. The default time for credit i is defined as

$$\tau_i = \inf\{t_k : W^i_{t_k} < K_i(t_k),\ k = 1, \dots\}, \tag{6}$$

where (W^i_t) is a Wiener process. The default barriers $K_i(t_k)$ are calibrated successively to match the default distribution function F_i for the time points $t_1, t_2, \dots$ [1]:

$$\begin{aligned} K_i(t_1) &= \sqrt{\delta_1}\mathrm{N}^{(-1)}(F_i(t_1)) \\ F_i(t_k) - F_i(t_{k-1}) &= \int_{K_i(t_{k-1})}^{\infty} f_i(t_{k-1}, u)\ \mathrm{N}\left(\frac{K_i(t_k) - u}{\sqrt{\delta_k}}\right) \mathrm{d}u. \end{aligned}$$

where $f_i(t_k, x)$ is the density of $W^i_{t_k}$ given that $W^i_{t_j} > K_i(t_j)$ for all $j < k$:

$$\begin{aligned} f_i(t_1, x) &= \frac{1}{\sqrt{2\pi\delta_1}} \exp\left(-\frac{x^2}{2\delta_1}\right) \\ f_i(t_k, x) &= \int_{K_i(t_{k-1})}^{\infty} f_i(t_{k-1}, u) \frac{1}{\sqrt{2\pi\delta_k}} \exp\left(-\frac{(x-u)^2}{2\delta_k}\right) \mathrm{d}u. \end{aligned}$$

Numerical procedures have to be employed to evaluate the integrals recursively.

In order to match given default dependencies as in equation (2) we have to calibrate the correlations ρ_{ij} of the Wiener processes W^i, W^j. To this end one has to simulate the default times according to (6), estimate the joint default probabilities from the samples, and calibrate over the results.

Summing up, the Hull & White approach provides an elegant solution to problems (1), (2), at least on a discrete time grid. The major shortcoming of the model is, however, that its calibration is computationally heavily involved.

In the continuous time limit the Hull & White model also allows for the following interpretation. Define

$$\tau_i = \inf\{t \geq 0 : W^i_t < K_i(t)\}.$$

If $K_i(t)$ is absolutely continuous,

$$K_i(t) = K_i(0) + \int_0^t \mu^i_s \mathrm{d}s,$$

[1] N denotes the standard normal distribution function and $\mathrm{N}^{(-1)}$ its inverse.

then the default time τ_i is the first hitting time of the constant barrier $K_i(0)$ for a Wiener process with drift:

$$\begin{aligned} Y_t^i &= W_t^i - \int_0^t \mu_s^i \mathrm{d}s \\ \tau_i &= \inf\{t \geq 0 : Y_t^i < K_i(0)\}. \end{aligned}$$

The drift can be interpreted as *default trend*: the higher μ_s^i the higher is the increase in the likelihood of default.

In general, time dependent default barriers in threshold models can be replaced by constant barriers at the price of some drift in the underlying firm value or ability-to-pay process.

4 Time changed Wiener process

In this section we propose an alternative solution to the problem of modelling dependent defaults which is intuitive and at the same time straightforward to calibrate to both conditions (1) and (2). The idea is to utilize Wiener processes with suitably transformed time scales. As we shall see below this is in principle equivalent to using Wiener processes with time varying but deterministic volatilities.

Given Wiener processes (W_t^i) and strictly increasing time transformations (T_t^i):

$$T^i|[0,\infty) \to [0,\infty),\ T_0^i = 0,$$

we define[2]

$$\begin{aligned} Y_t^i &= W_{T_s^i}^i && (7) \\ \tau_i &= \inf\{s \geq 0 : Y_s^i < K_i\}. && (8) \end{aligned}$$

4.1 Calibrating the term structure of defaults

The following result shows that it is straightforward to chose the time transformation appropriately to match a given term structure of default probabilities (1).

[2] Due to the space-time scaling properties of the Wiener process without restricting the generality we could set the default barrier to be $K_i = -1$. But as we shall see later it is useful to keep this degree of freedom.

Proposition 1 *Let F_i be a continuous distribution function, strictly increasing on $[0,\infty)$ with $F_i(0)=0$. If the time transformation (T^i_t) is given by*

$$T^i_t = \left[\frac{K_i}{\mathrm{N}^{(-1)}\left(\frac{F_i(t)}{2}\right)}\right]^2, \; t \geq 0 \tag{9}$$

then the default time τ_i defined by (8) admits the distribution function F_i, i.e., condition (1) is satisfied.

Proof. The distribution of the hitting time of a Brownian motion W is well-known to be (cf. [6])

$$\mathbf{P}(\min_{s\leq t} W_s < K_i) = 2\,\mathrm{N}(K_i/\sqrt{t}).$$

This yields for τ_i defined by (8)

$$\begin{aligned}\mathbf{P}(\tau_i < t) &= \mathbf{P}(\min_{s\leq t} W^i_{T_s} < K_i)\\ &= \mathbf{P}(\min_{s\leq T^i_t} W^i_s < K_i)\\ &= 2\,\mathrm{N}\left(K_i/\sqrt{T^i_t}\right)\end{aligned}$$

and the assertion follows. ♢

The time transformation (T^i_t) admits an obvious interpretation: the higher the increase of the function T^i_t, the higher is the speed at which the ability-to-pay process $Y^i_t = W^i_{T^i_t}$ passes along the Wiener path thereby increasing the likelihood of default.

If the distribution function F_i admits a density $F'_i = f_i$, then the time transformation (T^i_t) is absolutely continuous,

$$T^i_t = \int_0^t (\sigma^i_s)^2 \mathrm{d}s, \tag{10}$$

and by a well-know representation result the time transformed Wiener process allows for a representation as stochastic integral with *volatility* σ^i_s:

$$W^i_{T^i_t} = \int_0^t \sigma^i_s \mathrm{d}\tilde{W}^i_s, \tag{11}$$

with some new Wiener process $\tilde{W}^i$. The relationship between the volatility σ^i_s and the distribution of τ_i is

$$\sigma^i_s = \sqrt{-\left[\frac{K_i}{\mathrm{N}^{(-1)}\left(\frac{F_i(s)}{2}\right)}\right]^3 \frac{f_i(s)}{K_i\varphi\left(\mathrm{N}^{(-1)}\left(\frac{F_i(s)}{2}\right)\right)}},$$

where φ is the density of the standard normal distribution.

The volatility σ_s^i can be interpreted as *default speed*. The higher the default speed, the higher the volatility of the ability-to-pay process and the higher the likelihood of crossing the default threshold K_i.

4.2 Calibrating joint default probabilities

In this section we show that our time changed Wiener process model can also be easily calibrated to a set of pairwise joint default probabilities for a given fixed time horizon. Based on a result by Zhou [12] we derive an analytic expression for the joint default probability as a function of the correlation of the underlying Wiener processes which then allows for a calibration of these correlations.

Proposition 2 *Let (W_t^1) and (W_t^2) be Wiener processes with correlation ρ. For time changes (T_t^i), $i = 1, 2$, and default times τ_i, $i = 1, 2$, defined by (8) we have the following expression for the joint survival probability*[3]

$$\mathbf{P}(\tau_1 > t_0, \tau_2 > t_0) = \tag{12}$$
$$\begin{cases} \frac{2}{\alpha T}\mathbf{e}^{\frac{-r_0^2}{2T}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \int_{\theta=0}^{\alpha}\int_{r=0}^{\infty} \sin\left(\frac{n\pi\theta}{\alpha}\right) r\mathbf{e}^{\frac{-r^2}{2T}} I_{\frac{n\pi}{\alpha}}\left(\frac{r\, r_0}{T}\right) \\ \qquad \left(-1 + 2\mathrm{N}\left(\frac{r\sin\theta}{\sqrt{\Delta}}\right)\right) \mathrm{d}\theta \mathrm{d}r, & \text{if} \Delta > 0, \\ \frac{2r_0}{\sqrt{2\pi T}}\mathbf{e}^{\frac{-r_0^2}{4T}} \sum_{n=1,3,5,\dots} \frac{1}{n} \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \\ \qquad \left[I_{\frac{1}{2}(\frac{n\pi}{\alpha}+1)}\left(\frac{r_0^2}{4T}\right) + I_{\frac{1}{2}(\frac{n\pi}{\alpha}-1)}\left(\frac{r_0^2}{4T}\right)\right] & \text{if} \Delta = 0, \end{cases}$$

[3] The joint survival probability and the joint default probability are linked by

$$\mathbf{P}(\tau_1 > t, \tau_2 > t) = 1 - \mathbf{P}(\tau_1 \le t) - \mathbf{P}(\tau_2 \le t) + \mathbf{P}(\tau_1 \le t, \tau_2 \le t).$$

where

$$
\begin{aligned}
T &= \min(T^1_{t_0}, T^2_{t_0}) \\
\Delta &= \max(T^1_{t_0}, T^2_{t_0}) - \min(T^1_{t_0}, T^2_{t_0}) \\
\theta_0 &= \begin{cases} \tan^{(-1)}\left(\frac{K_2\sqrt{1-\rho^2}}{K_1-\rho K_2}\right) & \textit{if } \left(\frac{K_2\sqrt{1-\rho^2}}{K_1-\rho K_2}\right) > 0 \\ \pi + \tan^{(-1)}\left(\frac{K_2\sqrt{1-\rho^2}}{K_1-\rho K_2}\right) & \textit{otherwise} \end{cases} \\
r_0 &= \frac{-K_2}{\sin\theta_0} \\
\alpha &= \begin{cases} \tan^{(-1)}\left(\frac{-\sqrt{1-\rho^2}}{\rho}\right) & \rho < 0 \\ \pi + \tan^{(-1)}\left(\frac{-\sqrt{1-\rho^2}}{\rho}\right) & \rho > 0 \end{cases}
\end{aligned}
$$

and I_k denotes the modified Bessel function with order k.

Proof. Assume $T^1_{t_0} \leq T^2_{t_0}$. Using the Markov property of the Wiener process (W^1_t, W^2_t) we obtain

$$
\begin{aligned}
&\mathbf{P}(\tau_1 > t_0, \tau_2 > t_0) \\
&= \mathbf{P}(W^1_s > K_1, s \leq T^1_{t_0}, W^2_u > K_2, u \leq T^2_{t_0}) \\
&= \mathbf{E}\left(\mathbf{1}_{\{W^1_s > K_1, W^2_s > K_2, s \leq T\}} \mathbf{P}(W^2_u > K_2, u \in [T, T+\Delta] | W^2_T)\right) \\
&= \mathbf{E}\left(\mathbf{1}_{\{W^1_s > K_1, W^2_s > K_2, s \leq T\}} \left(1 - 2\,\mathrm{N}\left(\frac{K_2 - W^2_T}{\sqrt{\Delta}}\right)\right)\right) \\
&= \int_{-\infty}^{-K_1}\int_{-\infty}^{-K_2} f(x_1, x_2, T, \rho)\left(1 - 2\mathrm{N}\left(\frac{K_2 + x_2}{\sqrt{\Delta}}\right)\right) \mathrm{d}x_1 \mathrm{d}x_2,
\end{aligned}
$$

where $f(x_1, x_2, T, \rho)$ is the density of $(-W^1_T, -W^2_T)$ given that the barriers $-K_1, -K_2$, respectively, have not been hit by time T. This density is explicitly calculated in the paper [12] by Zhou. Using his result and transforming the integration variables appropriately yields the assertion.

The case $\Delta = 0$, i.e., $T^1_{t_0} = T^2_{t_0}$ follows directly from the result by Zhou, [12]. ♢

4.3 Calibration Examples

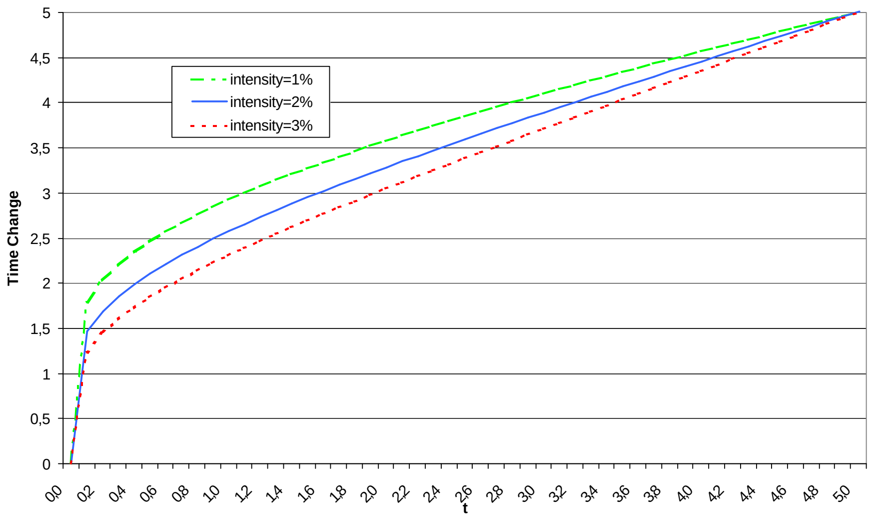
The calibration equation (9) allows some degree of freedom in determining the default threshold K_i. It seems to be natural to choose K_i such that the time change T_t^i is, in some sense, close to the original time scale, more precisely, for a fixed final time horizon $t_0 > 0$ we require

$$T_{t_0}^i = t_0. \tag{13}$$

This implies

$$K_i = \mathrm{N}^{(-1)}\left(\frac{F_i(t_0)}{2}\right)\sqrt{t_0}. \tag{14}$$

The following graph shows calibrated time transformations for default times with $F_i(t) = 1 - \exp(-\lambda_i\, t)$ and default intensities (hazard rates) $\lambda_1 = 1\%, \lambda_2 = 2\%, \lambda_3 = 3\%$. We applied the additional condition (13) with $t_0 = 5$ and obtained the threshold barriers $K_1 = -4,406, K_2 = -3,731, K_3 = -3,306$.



Transformations of time

The sharp increase of the time changes near time zero is due to the fact that for threshold models with continuous ability-to-pay processes the likelihood of default over short time periods near zero is basically vanishing. However, in the example above we have calibrated our model to a flat default rate λ_i per time unit. Due to the time change, the ability-to-pay process, although starting at zero, gets "enough" randomness for very short times to allow for the given non-vanishing default rate.

Remark: An alternative way to overcome the above mentioned problem for threshold models with continuous ability-to-pay processes Y would be to allow for a random initial value Y_0 with some distribution $\mathbf{Q}$. Conditioning on the values of Y_0 and integrating w.r.t. $\mathbf{Q}$ it is straightforward to extend our approach to this situation. We do not follow up with this idea here in more detail.

In the following we present some numerical examples on the calibration of the model to given dependency information (2). We express the joint default probability via its event correlation (3). To simplify the exposition we assume again that the default curves correspond to a flat default rate λ_i

$$F_i(t) = 1 - \exp(-\lambda_i t).$$

Our final time horizon is again $t_0 = 5$ and the event correlation we calibrate to also refers to time t_0, i.e., we calibrate to given pairwise default probabilities for time t_0. The calibration results are the correlations ρ between the underlying Wiener processes.

In the table below the first column contains the event correlations ρ^E ranging from 0% to 90%. The columns to the right show the calibrated correlations ρ for the given pair (τ_1, τ_2) of underlyings with default rates (λ_1, λ_2). Missing results indicate that the event correlation in the left most column is not valid for that combination (cf. (4)).

$\rho^E/(\lambda_1, \lambda_2)$	(1%,1%)	(1%,2%)	(1%,3%)
0,00%	0,00%	0,00%	0,00%
5,00%	18,51%	16,27%	15,28%
10,00%	31,59%	28,82%	27,68%
15,00%	41,96%	39,23%	38,29%
20,00%	50,60%	48,16%	47,63%
25,00%	57,98%	55,99%	55,99%
30,00%	64,40%	62,92%	63,55%
35,00%	70,03%	69,11%	70,46%
40,00%	74,98%	74,66%	76,82%
45,00%	79,35%	79,64%	82,77%
50,00%	83,21%	84,12%	88,49%
55,00%	86,58%	88,15%	
60,00%	89,53%	91,79%	
65,00%	92,07%		
70,00%	94,23%		
75,00%	96,02%		
80,00%	97,47%		
85,00%	98,59%		
90,00%	99,37%		

$\rho^E/(\lambda_1,\lambda_2)$	(2%,2%)	(2%,3%)	(3%,3%)
0,00%	0,00%	0,00%	0,00%
5,00%	13,98%	12,97%	11,94%
10,00%	25,52%	24,07%	22,48%
15,00%	35,43%	33,85%	31,94%
20,00%	44,13%	42,59%	40,52%
25,00%	51,87%	50,47%	48,32%
30,00%	58,78%	57,59%	55,44%
35,00%	64,99%	64,05%	61,92%
40,00%	70,56%	69,91%	67,82%
45,00%	75,55%	75,20%	73,16%
50,00%	80,01%	79,96%	77,97%
55,00%	83,96%	84,22%	82,27%
60,00%	87,43%	88,00%	86,07%
65,00%	90,45%	91,33%	89,40%
70,00%	93,03%	94,23%	92,25%
75,00%	95,19%	96,74%	94,65%
80,00%	96,94%		96,59%
85,00%	98,29%		98,09%
90,00%	99,24%		99,15%

Observe that in the examples above we only calibrate to non-negative event correlations since in practice negative default correlations are rather artificial.

4.4 Implementation

To implement our model for practical applications a Monte Carlo simulation seems to be appropriate for most applications. We first choose a final time horizon t_0, naturally the maturity of the transaction to evaluate, and determine the corresponding threshold level K_i according to (14). The time changes for the underling Wiener processes are calibrated as outlined above. The correlation matrix between the driving Wiener processes is derived from given joint default probabilities or event correlations.

Now choosing a time discretization $0 = s_0 < s_1 < \cdots < s_m = t_0$ for our time interval $[0, t_0]$ we simulate the random variables $W^i_{T^i_{s_j}}, i = 1, \dots, n, j = 1, \dots, m$.

The first time s_j with $W^i_{T^i_{s_j}} < K_i$ we set $\tau_i = s_j$. However, since the first hitting time[4]

$$\min\{s_j : W^i_{T^i_{s_j}} < K_i\}$$

of the discrete path clearly overestimates the true default time τ_i we apply in addition a Brownian bridge technique to capture the probability of possible defaults in between the grid points s_j. For simulation results $\alpha = W^i_{T^i_{s_j}} > K_i$ and $\beta = W^i_{T^i_{s_{j+1}}} > K_i$ the process $W^i_u, u \in [T^i_{s_j}, T^i_{s_{j+1}}]$ follows a so-called Brownian bridge with starting point α and end point β. The probability of crossing the boundary K_i during the time interval $[T^i_{s_j}, T^i_{s_{j+1}}]$ is known to be (cf. [6])

$$\begin{aligned} \mathbf{P}(\min_{u\in[s_j,s_{j+1}]} W^i_{T^i_u} < K_i) &= \mathbf{P}(\min_{u\in[T^i_{s_j},T^i_{s_{j+1}}]} W^i_u < K_i) \\ &= \exp\left(\frac{-2(\beta - K_i)(\alpha - K_i)}{\Delta}\right), \end{aligned}$$

with $\Delta = T^i_{s_{j+1}} - T^i_{s_j}$. For each time interval $[s_j, s_{j+1}]$ given that the threshold K_i has not been crossed before and that $W^i_{T^i_{s_{j+1}}} > K_i$ we draw an additional uniform random variable U^i_j and set

$$\tau_i = (s_j + s_{j+1})/2 \quad \text{if} \quad U^i_j < \exp\left(\frac{-2(\beta - K_i)(\alpha - K_i)}{\Delta}\right)$$

and proceed with our simulations if $U^i_j \geq \exp\left(\frac{-2(\beta-K_i)(\alpha-K_i)}{\Delta}\right)$.

It turns out that this technique allows the simulation results to capture the true theoretical marginal default distribution $F_i(t)$ quite accurately. However, in case of non-vanishing correlations between the underling Wiener processes we may lose some degree of "correlation" in our implementation since, theoretically, the uniforms $U^i_\cdot$ would have to be somehow dependent.

As a result of our simulation we end up with simulated default times $\tau_1, \ldots, \tau_n$ which are the primary input to a Monte Carlo evaluation of any multi-credit product.

[4] As usual we follow the convention $\min(\emptyset) = \infty$.

5 Application example

In this section we apply our model to the pricing of a basket credit default swap.

Credit default swaps are the dominating plain-vanilla credit derivative product which serve also as a building block for many other credit products. A credit default swap offers protection against default of a certain underlying entity over a specified time horizon. A premium (spread) s is paid on a regular basis (e.g., on a quarterly, act/360 basis) and on a certain notional amount N as an insurance fee against the losses from default of a risky position of notional N, e.g., a bond. The payment of the premium s stops at maturity or at default of the underlying credit, whichever comes first. At the time of default before maturity of the trade the protection buyer receives the payment $N(1-R)$, where R is the recovery rate of the underlying credit risky instrument.

A basket credit default swap is an insurance contract that offers protection against the event of the kth default on a basket of n $(n \geq k)$ underlying names. It is quite similar to a plain credit default swap but the credit event to insure against is the event of the kth default. Again a premium (spread) s is paid as insurance fee until maturity or the event of the kth default in return for a compensation for the loss. We denote by s^{kth} the fair spread in a kth-to-default swap, i.e., this is the spread making the value of this swap today equal to 0.

Most popular are first-to-default swaps, i.e., $k = 1$, as they offer highly attractive spreads to a credit investor (protection seller).

We apply our model to a basket with $n = 5$ names trading in the credit default swap market with fair CDS spreads[5] $s_1 = 0,80\%, s_2 = 0,90\%, s_3 = 1,00\%, s_4 = 1,10\%, s_5 = 1,20\%$ for all maturities. We suppose a recovery rate of $R = 15\%$ for all credits. The riskless interest rate curve is assumed to be flat at $5,00\%$. As common in practice (see e.g. [7]) we extract from the given CDS spreads and the recovery assumption the (risk-neutral) distribution functions F_i of the default time τ_i, $i = 1, \ldots, 5$.

For various levels of correlations we price 5 years maturity kth-to-default basket default swaps for $k = 1, \ldots, 5$ and determine their fair spreads. In the model we use a Monte Carlo simulation on a monthly time grid with 10000 simulations as described in section 4.4. The results are compared to alternative valuations of the same transactions using a Monte Carlo implementation of a normal copula model as described in detail e.g. in [8],[11]. In current practice the normal copula model is widely used to value basket credit deriva-

[5] On a quarterly act/360 basis.

tives. The correlations in the normal copula model are calibrated to produce the same pairwise default probabilities for the time horizon $t_0 = 5$ as in our time change model. Here is an example how a fixed correlation of $\rho = 30\%$ between our underling Wiener processes translates into a (asset) correlation matrix for the normal copula model:

$$\begin{pmatrix} 100,00\% & 32,30\% & 31,99\% & 31,73\% & 31,52\% \\ 32,30\% & 100,00\% & 32,06\% & 31,80\% & 31,58\% \\ 31,99\% & 32,06\% & 100,00\% & 31,86\% & 31,63\% \\ 31,73\% & 31,80\% & 31,86\% & 100,00\% & 31,68\% \\ 31,52\% & 31,58\% & 31,63\% & 31,68\% & 100,00\% \end{pmatrix}$$

The two models produced the following numerical results for the basket default swap valuations:

fair basket default swap spreads in the time change model

ρ	10%	20%	30%	40%	50%	60%	70%
s^{first}	4,791%	4,563%	4,296%	3,953%	3,620%	3,252%	2,845%
s^{2nd}	0,625%	0,799%	0,941%	1,055%	1,131%	1,201%	1,259%
s^{3rd}	0,058%	0,130%	0,201%	0,296%	0,398%	0,523%	0,635%
s^{4th}	0,003%	0,015%	0,040%	0,076%	0,126%	0,202%	0,306%
s^{5th}	0,000%	0,003%	0,006%	0,013%	0,030%	0,057%	0,118%

fair basket default swap spreads in the normal copula model

ρ	10%	20%	30%	40%	50%	60%	70%
s^{first}	4,704%	4,442%	4,137%	3,806%	3,486%	3,147%	2,764%
s^{2nd}	0,670%	0,803%	0,941%	1,062%	1,151%	1,215%	1,257%
s^{3rd}	0,074%	0,137%	0,219%	0,320%	0,413%	0,523%	0,640%
s^{4th}	0,005%	0,016%	0,040%	0,084%	0,143%	0,222%	0,334%
s^{5th}	0,001%	0,003%	0,008%	0,016%	0,041%	0,075%	0,135%

For the first-to-default basket compared to the normal copula model the time change model seems to produce slightly higher spreads. For the other cases the results are quite close. This is supported by repeated simulations with different seeds for the random number generator. The seed variance of fair the first-to-default spread is less than 0.10% and much smaller for the other spreads. For the normal copula model we used a variance reduction technique based on stratified sampling whereas for the time change model we applied an in-sample orthogonalization of the increments of the Wiener processes.

In [2] Finger compared the results of the pricing of multi-credit products for various stochastic default rate models. The models were calibrated against each other on a single period time horizon but then applied to a multi-period valuation problem. One of the models in his study was the normal copula model. The comparison results by FINGER show much higher discrepancies between the different models. To a certain extend this can be attributed to the fact that the models were not calibrated for the final time horizon. This might explain why our comparison yields much closer results.

6 Conclusions

We investigated models for dependent defaults based upon so-called structural models where default is triggered by an underlying ability-to-pay process reaching a certain threshold barrier. We have introduced a model which is based on correlated and suitably time changed Wiener processes. This model is flexible and easy to calibrate to any given term structure of default probabilities as backed out from market information, as, for example, CDS spreads or bond prices. An analytic expression for pairwise joint default probabilities is derived which allows to calibrate the model also efficiently to given dependence information.

Potential applications of the model are the pricing of multi-credit derivative products or the valuation of counter-party default risks.

The model has been applied to the problem of pricing a basket credit default swap. A comparison of the results of our model to the widespread used normal copula model, both calibrated to the same dependency information, shows that both models give quite similar results with our model producing slightly higher fair first-to-default premiums.

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