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A model for relational reasoning as verbal reasoning

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Abstract

Deductive reasoning is an essential part of complex cognition. It occurs whenever human beings (or machines) draw conclusions that go beyond what is explicitly provided. Reasoning about spatial relations is an excellent testbed for the assessment of competing reasoning theories. In the present paper we show that such competing theories are often less diverse than one might think. We introduce an approach for how relational reasoning can be conceived as verbal reasoning. We describe a theory of how humans construct a one-dimensional mental representation given spatial relations. In this construction process objects are inserted in a dynamic structure called a “queue” which provides an implicit direction. The spatial interpretation of this direction can theoretically be chosen freely. This implies that choices in the process of constructing a mental representation influence the result of deductive spatial reasoning. To derive the precise rules for the construction process we employ the assumption that humans try to minimize their cognitive effort, and two cost measures are compared to judge the efficiency of the construction process. From this we deduce how the queue should be constructed. We discuss empirical evidence for this approach and provide algorithms for a computational implementation of the construction and reasoning process.

Keywords: Deduction; Relational reasoning; Verbal reasoning; Mental models; Cost function; Computational modeling

1. Introduction

Imagine you are reading a newspaper. In an article about the financial crisis you read that the stock price of bank A is higher than the stock price of bank B. Later in the article you learn that the stock price of B is higher than that of bank C. For most people, it is quite easy to mentally rank the three banks into one single order and to read off from this order that the stock price of bank A is higher than that of bank C. The task is rather easy, but, imagine that you receive the information in a different order e.g., B higher than C, A higher than B, or with other relational expressions, e.g., A higher than B, C lower than B. Or imagine that you have information not just about three banks but many others A, B, C, D, E, etc.? And now imagine that this is not a study in the psychological lab, but you are an actual stock broker and you can really lose a lot of money in just a few milliseconds.

Our example demonstrates that the cognitive process of inferring new information from information that is explicitly provided is a vital and indispensable part of problem-solving and decision-making. Understanding how humans draw inferences is an important field in the area of complex cognition research (Sternberg & Ben-Zeev, 2001) and studies in this field can even help to understand actual problems of our daily life (for a reasoning study on the cognitive aspects of the financial crisis see Knauff, Budeck, Wolf, and Hamburger, 2010). Another point is that relational

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expressions seem to be one of the most essential mental entities (Goodwin & Johnson-Laird, 2005; Halford, Wilson, & Phillips, 1998; Hummel & Holyoak, 2005). In a recent article entitled "Relational knowledge: the foundation of higher cognition" Halford, Wilson, and Phillips (2010) reported accumulating evidence on the nature, function and acquisition of relations and their crucial role in higher cognitive processes. In their review the authors show that relations play a vital role in reasoning, categorization, planning, and language. Relations are omnipresent in our daily life: in the stock prices example, for instance, differences between the banks are represented by spatial relations such as higher and lower. Other examples are: Person A is *smarter* than Person B, X *loves* Y, City A *has more citizens than* City B, marriage *is before* divorce, global warming is *more dangerous than* most people think; computer scientists *are as smart as* psychologists, the earthquake *came earlier than* the Tsunami; Barack Obama is *more popular than* George W. Bush was, Wolfgang Amadeus Mozart was *the son of* the bookbinder Johann Georg Mozart. From a formal point of view, a relation can have many arguments, but in our daily life, they seldom take more than three arguments ("The professor gave the book to the student"), and most often even just two entities (Goodwin & Johnson-Laird, 2005). The important point, however, is that relations are around almost everywhere and that it is difficult to see how we could solve complex daily-life problem, without the use of relations and the comparison of different alternatives. Such relational inferences seem to be quite easy, but, firstly, they often are not, and, secondly, even when they are easy to perform that does not mean that the underlying cognitive processes are easy to understand.

In the following paper we explore human reasoning with binary relations and how the underlying cognitive processes can be algorithmically reconstructed. All of our relations have in common that they are linear ordering relations, which means in particular that they are transitive. This fact allows us to create the underlying linear order between the objects featured in relational expressions. We demonstrate our approach by means of spatial relations between fruits (imagine them lying on a table). This is done just for the sake of easy illustration, but it should be clear, that the postulated mechanisms are much more universal. They are not limited to specific entities or to a specific subset of linear (transitive) relations. The bank stock prices, for instance, can be easily modeled in the framework, too!

So, consider the following two sentences, also called premises.

Example 1:

1. The apple is to the left of the mango.
2. The mango is to the left of the pear.

These premises allow us to create a linear order of the objects named in the premises, apple–mango–pear. This order enables us to draw conclusions about information not directly given in the premises: we can infer that the apple is to the left of the pear. The ability to infer information about relations between objects not explicitly expressed by the premises is the subject of theories about relational reasoning (cf. Goodwin & Johnson-Laird, 2005; Johnson-Laird & Byrne, 1991, chap. 5). The bases of such inferences are mental representations that reflect information conveyed verbally by the premises. There are several theories on how this is accomplished (cf. Goodwin & Johnson-Laird, 2005; Johnson-Laird & Byrne, 1991, chap. 5; Knauff, 2009a,b). They differ in the postulated underlying mental representations and the computational processes that work on these representations. In one theory, it is believed that people think deductively by applying mental rules which are basically similar to rules in computer programs. In the other theory, deductive reasoning is conceived as a process in which the reasoner constructs, inspects, and manipulates mental models. The rule-based theory is usually described as a syntactic theory of reasoning, as it is based on the form of the argument only, whereas the mental models theory is seen as a semantic theory, because it is based on the meaning (the interpretation) of the premises. The rule-based theories are primarily represented by the work of Rips (1994) and Braine and O'Brien (1998). Relational versions of the account have been developed, for instance, by Hagert (1984) and van der Henst (2002). The main claim of this account is that reasoners rely on formal rules of inference akin to those of formal logic, and that inference is a process of proof in which the rules are applied to mental sentences. The formal rules govern sentential connectives such as "if" and quantifiers such as "any", and they can account for relational inferences when they are supplemented with axioms governing transitivity, such as: for any x, y, and z, if x is taller than y and y is taller than z, then x is taller than z. The rules are represented in long-term memory and the sequence of applied rules results in a mental proof or derivation that is seen as analogous to the proofs of formal logic (Rips, 1994).

The theory of mental models has been developed by Johnson-Laird and colleagues (Johnson-Laird, 1983, 2001, 2006; Johnson-Laird & Byrne, 1991). The latest version of the theory of relational reasoning based on mental models has been explicated in Goodwin and Johnson-Laird (2005). According to the mental model theory, human reasoning relies on the construction of integrated mental representations of the information that is given in the reasoning problem's premises. These integrated representations are models in the strict logical sense. It is a mental representation that captures what is common to all the different ways in which the premises can be interpreted. It is a "small scale" representation of how "reality" could be – according to what is stated in the premises of a reasoning problem. Based on the MMT, Knauff, Rauh, Schlieder, and Strube (1998) propose three stages involved in the relational reasoning process: a construction phase, during which reasoners construct a mental model, reflecting the information of the premises, an inspection phase, during which the model is inspected for implicit information of the premises, and a variation phase, during which alternative models are constructed and investigated concerning their compatibility with the information given by the premises, if necessary, resulting into falsification of the preliminary mental model, constructed during the first phase.

The starting point of our paper is that the long-lasting dispute between rule-based and model-based theories is quite unproductive, because no single account can explain all of the experimental findings in reasoning research (e.g., Goel, 2007; Knauff, 2009a, 2009b; Oberauer, 2006; Stenning & van Lambalgen, 2008). Another reason for our research is that the seemingly helpful distinction between models as semantic and rules as syntactic approaches also cannot be upheld from a formal point of view, because Stenning and collaborators have shown the abstract equivalence of all the main psychological competence theories of human reasoning (Stenning, 1998; Stenning & Oberlander, 1995). Their apparently contrasting representations are computationally equivalent for the kind of data presented in the literature (Stenning & van Lambalgen, 2008). In a similar vein we interpret the work by Polk and Newell (1995), who point out that the model-based deduction process, does not necessarily require deduction-specific, non-linguistic mechanisms to operate on internal representations. Especially in reasoners that are not specifically trained on deductive reasoning more general cognitive mechanisms might guide the reasoning process. They introduced an approach, called **verbal reasoning** that assumes the cognitive processes in deductive reasoning to be based upon the same processes as language comprehension and generation. Verbal reasoning describes reasoning as transformation of verbal information provided by the premises of an inference problem. Linguistic skills operate in order to encode and re-encode a reasoning problem until the conclusion becomes obvious or until the reasoner gives up. Polk and Newell (1995) hypothesize that when task-relevant information is provided verbally, the crucial role in reasoning is played by the verbal processes of encoding and re-encoding accordingly and that inferences follow immediately from the encoded information. The computational approach presented by Polk and Newell accounts for many experimental findings in a number of deductive reasoning tasks, among them reasoning with relations.

In the following, we sketch how relational reasoning can be conceived in Polk and Newell's framework of verbal reasoning. In particular, we propose new theoretical assumptions for the special case of reasoning with spatial relations. The key assumption is that the process of constructing a mental representation – a mental model – from the premises influences deductive spatial reasoning. This implies that the process of encoding information is critical for the result of the reasoning process. We discuss empirical evidence as well as a computational implementation of the encoding and reasoning process.

2. A cognitive model

We are proposing a theory on how humans create a mental model from a set of (spatial) relations. The theory consists of two parts: the general structure of models and the most efficient process of constructing these models. The first part lists basic assumptions of what properties the mental model is supposed to have, thereby defining the general structure of the model. The second part is based on the idea of cognitive efficiency. The idea is that humans try to minimize their cognitive effort and thus a computational cost measure can help to estimate the efficiency of an inference. From this approach we derive how a mental model should be constructed within the framework laid out in the first part. This mental model can then be used to reason about (spatial) relations and its properties imply consequences for the reasoning process.

2.1. Basic structural assumptions for the cognitive model

Since we consider arbitrary transitive relations as the basis for the model we assume that models consist of a "*queue*" of objects and an interpretation what this queue represents. The queue describes in which order the objects are aligned but what this order represents depends on the relation that is considered. It can range from stock prices arranged from highest to lowest, over the population of cities from smallest to largest, to alignments of objects in space from left to right. So while the order of a queue is implicit the interpretation of the order is not. The queue is constructed by forming links between objects. The links signify which objects follow each other in that ordered arrangement. These links between the objects are one directional which means that when inspecting the queue we can move from one object to the next object in the queue but not to the preceding object. To access the queue one needs to access the first element of the queue. Therefore the beginning of the queue is marked by a start pointer, marking the starting point.

The queue can be accessed from this starting point which is directed at the first object. From there all other objects in the mental model can be reached by following the links between objects.

This amounts to the following assumptions about the queue

1^{qu} There exists a starting point or first object.
2^{qu} Each object is linked to the next object in the linear order. Only the last object is not linked to other objects.
3^{qu} While this structure has an implicit direction, the interpretation of this direction depends on the context.

The starting point can also be considered a link. This is due to the fact that one has to know how the queue starts in order to access it. Therefore knowing which object is the first constitutes a link, connecting the start of the queue to that object.

This structure is not limited to portraying spatial relations but a model constructed this way can be used to describe any linear order.

2.2. Construction of a queue from spatial information

The question now is how a mental model is constructed from the premises of a reasoning problem. How are objects featured in the premises inserted in the queue? In this process the first premise that is considered has a special function and dominating effect on the construction of the rest of the arrangement. We consider the first premise independently of the following premises and postulate the following two rules for the construction process.

- 1^{fp} The first object inserted in the queue is the starting point of the queue.
- 2^{fp} The second object is linked to the first object. The relation between the first and the second object thereby creates the interpretation of the link and the implicit direction of all the following objects in the queue.

If we know, for example, that the object which has been inserted secondly is supposed to be to the right of the first (starting) object, then the link is interpreted as "to the right".

When we look at our example again from the introduction this results in two options for the first premise: "The apple is to the left of the mango."

We can choose the apple as the starting point (marked by the asterisk) and insert the mango thereafter

$$\text{apple}^{*} \rightarrow \text{mango} \tag{1}$$

The implicit direction of the queue is interpreted as moving from the leftmost object to the right. Theoretically we could also use the mango as a starting point (marked by the asterisk) inserting the apple thereafter. The corresponding model could be depicted as follows:

$$\text{apple} \leftarrow \text{mango}^{*} \tag{2}$$

In this case the implicit direction of the queue is interpreted as moving from the rightmost object to the left. So even though the premise describes only one arrangement of fruits there are two options for representing this arrangement in our queue.

However, with a sentence like: "The apple is to the left of the mango." we can assume that only model (1) will be used with the choice putatively influenced by cultural and/or biological aspects. There is vast evidence for a left to right bias on spatial routines (Chatterjee, Southwood, & Basilico, 1999; Dobel, Diesendruck, & Bölte, 2007; Maass & Russo, 2003; Tversky, Kugelmass, & Winter, 1991). Cross-cultural studies suggest that this bias arises from the scanning habit induced by reading and writing direction predominantly used within a certain culture (e.g., Chan & Bergen, 2005; Dobel et al., 2007; Spalek & Hammad, 2005) and that this cultural bias influences spatial representations of objects. Another view is that the left to right bias arises from aspects fundamentally implemented in the functional architecture of our brains (e.g., Beaumont, 1985; Chatterjee, 2001; Chatterjee et al., 1999; Levy, 1976; for some culture-independent preferences in spatial reasoning, see for instance, Knauff & Ragni, in press).

The results of an early study by De Soto, London, and Handel (1965) indicate that the left end of a linear order is the preferred starting point (in our example the apple). Given a statement about a relation between two people, participants were asked to write down the names in two of four boxes and the results reflect a preference for working from left to right (De Soto et al., 1965).

In a recent experiment conducted in our lab, we asked participants to arrange colored wooden blocks (red, green, blue, yellow) according to a description given by two premises with colors (red, green, blue, yellow) mentioned in the premises representing respective blocks. We found that subjects tended to start with the block first named in the (first) premise. So for both sentences "Red is to the left of blue" and "Red is to the right of blue" the red block was inserted first and then the blue block was placed accordingly. The experiment will be reported in detail elsewhere (Bucher, Krumnack, Nejasmic, & Knauff, in preparation).

For a sentence like: "The apple is to the left of the mango." the order of the objects in the sentence and their left-to-right order in the described spatial arrangement coincide. Therefore, starting with the first object in the sentence, reasoners would presumably build the model from the left to the right meaning model (1) would be constructed.

However, with a sentence such as: "The mango is to the right of the apple." the situation is less clear. Here the object which is named first is supposed to be on the right side of the arrangement while the second object is on the left. Reasoners could either insert the objects in the order in which they appear in the sentence, that is from the right as in model (2) or they could arrange the objects from left to right as in model (1). In both cases they follow a left to right preference.

Once the interpretation of the implicit direction of the queue is fixed by inserting the second object the rest of the objects are inserted according to this interpretation. This amounts to the following options for inserting objects in an existing queue from the second premise:

- 1^{ins} One object (the reference object) of the premise has to be found in the queue.
- 2^{ins}(a) If the new object is to be placed behind this object (with regard to the implicit direction of the queue) it can be either inserted into the queue directly behind the object or at any point further to the end of the queue.
  - (b) If the new object is to be placed in front of the object (with regard to the implicit direction of

the queue) it can be either inserted into the queue directly in front of the object or at any point further to the beginning of the queue.

The question is which of the options outlined in 2^{ins} is used when inserting an object into the queue. Where exactly is the object inserted? To determine this we use the idea of cognitive efficiency which suggests, that humans try to minimize their mental effort. Given e.g., working memory capacity limitations, a reasonable goal would be to minimize usage of available resources in order to maximize performance (e.g., Süß, Oberauer, Wittmann, Wilhelm, & Schulze, 2002).

3. Insertion into the model

There are two kinds of operations to be performed when inserting an object into an existing queue (after the reference object is found):

1. Movement through the queue.
2. Creation of new links.

The cost of the movement through the queue can be calculated as the number of objects that were passed. Also, accessing the starting point from any other object than the first object in the queue counts as movement. For the creation of links we can count the number of new links that need to be created when an object is inserted into the queue.

Let us consider the computational cost that results from inserting a new object into the queue between two objects that are linked. Say, for example, we want to insert pear between apple and mango in the following queue:

$\ldots \rightarrow \text{apple} \rightarrow \text{mango} \rightarrow \ldots$

To insert a new object between two existing objects in the queue the first object, which was linked to the second object before, now has to be linked to the new object. The new object has to be linked to the second object. In our example apple needs to be linked to pear and pear to mango resulting in the following queue:

$\ldots \rightarrow \text{apple} \rightarrow \text{pear} \rightarrow \text{mango} \rightarrow \ldots$

This process of inserting an object between two objects in a queue requires forming two new links. If the object is inserted at the beginning of the queue the starting point needs to be redefined which we will consider as creating a new link. So the process of inserting an object at the beginning of an existing queue also requires forming two new links. Inserting an object at the very end of the queue, following the last object in the queue, only requires creating one new link, no existing links have to be changed. But we have to move through the entire queue to get to the last object.

We assume that the queue is constructed in the most efficient way. So the question to answer is: what is efficient? We will look at two different cost measures that allow us to judge how to create the queue efficiently, that is, where to insert an object into an existing queue and compare the predictions that can be derived from the two cost functions. Firstly we will consider what happens if both kinds of operations are treated the same way, that is, both are assigned the same cost. This amounts to a complexity measure that is used as a standard in computational complexity estimation. Secondly we introduce a cost measure where we assign a higher cost to creating a link than to moving through the queue.

3.1. Computational complexity estimation

In Computer Science the efficiency of algorithms is usually assessed by adding the costs of operations necessary to execute the algorithm. In this context, one often uses a uniform cost measure since the real costs are often not known nor are they important in an asymptotic analysis. We adopt here a similar strategy. In this specific case that implies creating one link and moving one object through the queue produce the same cost. A similar complexity measure has been used by Ragni, Knauff, and Nebel (2005).

We do not imply that this is necessarily a cognitively adequate cost measure but it seems a good default comparison since it does not make any assumptions about different cognitive costs of operations. By comparing measures that make such assumptions with this standardized default it can be seen whether those measures perform better or worse. This should give an indication of how reasonable those assumptions are.

Using this standard computer science cost measures we look at the possibilities from above, 2^{ins}(a) and (b), for insertion into an existing queue. Which of these options is the most cost efficient? When inserting a new object behind an object of the queue as in 2^{ins}(a), and we insert it directly behind the object, we have the cost of creating two links (see above) unless we are at the end of the queue in which case only one new link is required. If we want to insert the object further down the queue we have to move to that point within the queue and the cost of moving through the queue have to be added to the cost of creating new links. And moving one object down the queue costs as much as creating a link. So inserting a new object after an object further down the queue is always at least as expensive as inserting it right behind the reference object even when inserting it at the end of the queue. If the end of the queue is more than one object away, the cost of moving through the queue and creating the link will be even higher than the cost of just inserting the new object right behind the reference object. And since there is no way of knowing how far away the last object is, the cost efficient solution is to insert the new object right behind the reference object.

When inserting a new object in front of an object as in 2^{ins}(b), the same cost results with respect to the links being formed for inserting the new object right in front of the reference object and for inserting it at the beginning of the queue. In both cases two new links need to be created, if we insert the object at the beginning of the queue one of the new links that need to be created is the starting point. However, when we insert the object at the very beginning the starting point of the queue has to be accessed, so we have a movement which means one extra step.

If the object is inserted at any other point of the queue the cost is higher since we first have to move to that point from the beginning of the queue. So using this cost analysis it would be most efficient to insert the object directly in front of the reference object.

Based on this analysis we derive cost efficient rules for inserting nodes into a list:

1^{CC} If the new object has to be placed behind an object of the list it should be inserted into the list *directly behind* the object.

2^{CC} If the new object has to be placed in front of an object of the list it should be inserted into the list *directly in front* of this object.

If we apply these rules to the second premise of example 1 from the introduction (starting with model (1), (2), respectively, from Section 2) we create one of the following two models depending on the direction of the queue.[1]

$$\text{apple}^* \rightarrow \text{mango} \rightarrow \text{pear} \tag{3}$$

$$\text{apple} \leftarrow \text{mango} \leftarrow \text{pear}^* \tag{4}$$

While the results look similar, the costs for building these models differ. The cost for inserting the two objects from the first premise into a new queue are always the same and therefore do not need to be considered. But what are the costs for including the information of the second premise into the model? In case (3) we use rule 1^{CC} but since we insert the object at the end of the list, only one more link needs to be created. In case (4) however, we use rule 2^{CC} which in this case amounts to inserting the object (pear) at the beginning of the queue. So we need to redefine the starting point and create a new link. This results in creating two new links. So the cognitive cost for building the first model is lower. Note that in both cases no movement through the queue is necessary.

Let us look at another example that is not quite as simple:

Example 2:

1 The apple is to the left of the mango.
2 The apple is to the left of the pear.

Here the premises describe an indeterminate order: there are two possible orders of these three fruits: apple–mango–pear and apple–pear–mango.

So the question is, whether one of these orders is preferred over the other? Knauff, Rauh, and Schlieder (1995), Rauh et al. (2005), Jahn, Knauff, and Johnson-Laird (2007) have empirically shown that such preferences exist in human reasoners.

Since the first premise is identical to the one in example (1) with the determinate order we receive the same two options for models when applying the rules for the first premise. If we apply the rules of insertion to the second premise we get one of the following models, using rule 1^{CC} and rule 2^{CC} respectively.

$$\text{apple}^* \rightarrow \text{pear} \rightarrow \text{mango} \tag{5}$$

$$\text{apple} \leftarrow \text{pear} \leftarrow \text{mango}^* \tag{6}$$

Again we see no difference between the models even though the arrangement is indeterminate, so more than one model could be created. However, model (5) was built using rule 1^{CC}, model (6) following rule 2^{CC}. Nevertheless, the insertion of the last object has the same computational cost in both of these models as in both cases two links have to be created.

3.2. Alternative cost measure

We now introduce an alternative cost measure for which the main assumption is that as few new links as possible should be formed to minimize cognitive work. This implies that if it can be avoided, an existing link should not be broken. As a cost measure we therefore use primarily the number of links that need to be formed. If this does not show any difference between the options the required movement through the queue is used as a secondary cost measure. This reflects that forming a link is supposed to require more cognitive effort than moving through the queue, no matter how far we have to move through the queue.

Because of the structure of a model laid out in the assumptions 1^{qu} and 2^{qu} above, at the end of the construction process the complete mental model has as many links as there are objects in the model (including the start pointer as a link). Since the final number of links in a mental model is fixed, costs can only be reduced by altering as few links as possible during the construction process. Therefore when inserting new objects it is most cost efficient to create just one new link and to not change any existing links.

Inserting an object at the very end of the queue, following the last object in the queue, only requires creating one new link, no existing links have to be changed. We have to move through the entire queue to get to the last object but in this cost measure the cost of moving through the queue can be disregarded in this estimation.

Using this information we will now estimate the cost created by the insertion options described in 2^{ins}(a) and (b). As stated before cost will be measured primarily as the number of links that need to be formed. Only if two options require the same number of links to be formed will we use the

[1] As discussed in Section 2, we assume that the queue would generally be constructed from left to right for this specific case. The other direction is included to allow a comparison of cost and a complete theoretical evaluation. Both directions will therefore be covered in this section about cost estimation.

number of steps moved through the queue as a secondary cost measure.

Let us first look at option 2^{ins}(a): if the object is inserted between two objects of the queue two new links need to be formed. If the object is inserted at the end of the queue, only one new link needs to be formed. So in case 2^{ins}(a) it is most cost efficient to insert the object at the very end of the queue. Now we consider 2^{ins}(b): the new object can only be inserted between two objects or at the starting point of the queue. Since we consider the starting point a link to the beginning of the queue both options require two new links to be formed. So it is more cost efficient to not move around the queue but to insert the object directly in front of the found object. Using this analysis we postulate the following rules:

1^{AC} If the new object is to be placed behind an object of the queue it will be inserted *at the end* of the queue.
2^{AC} If the new object is to be placed in front of an object of the queue it will be inserted into the queue *directly in front* of this object.

If we apply these alternative rules of insertion to the second premise from example 2 we get the following models:

$$\text{apple}^{*} \rightarrow \text{mango} \rightarrow \text{pear} \tag{7}$$

$$\text{apple} \leftarrow \text{pear} \leftarrow \text{mango}^{*} \tag{8}$$

Here we see a difference between the models constructed from the indeterminate description depending on the implicit direction of the queue and on the place of insertion. Because the two queues have opposite interpretations of the implicit direction different rules are applied to form the queues. There is also a difference in the cost for building these models. In (7) we were able to apply rule 1^{AC}, again creating only one new link. In (8) we needed to apply rule 2^{AC}, redefining the starting point, creating two new links. So the costs for creating the last model (8) are higher than the ones for creating model (7).

The models illustrate that rules based on a classic computational cost measure produce partly different results than our rules based on the alternative cost measure. Model (7) differs from model (5) above while model (6) is similar to model (8). So if the queue is constructed from left to right, we have different predictions on how the model should be constructed.

For completeness we could also consider assigning higher cost to movement through the queue than to creating a new link. However, this would lead to the same predictions and models as the classical computer science cost measure, which makes it redundant to discuss this case in detail.

3.3. Empirical evidence

Since the two cost measures lead to different rules which result in different models to be constructed the question is which rules predict human behavior better. Or differently phrased: is it justified to assume that forming a link is more cost intensive than moving through the queue? If not, the traditional computational complexity measure should lead to better predictions than our alternative cost measure.

To answer these questions we report an experiment that implies that rules derived from our alternative cost measure predict human behavior better than the rules derived from the traditional computer science cost measure. In this experiment we investigated what kind of mental model participants construct when they are faced with indeterminate problems as in example (2) that allowed more than one model to be constructed. The problems were designed to differentiate between the two discussed options for construction rules.

We only use the relation "left of" because, as discussed in Section 2, we can be relatively certain that the subjects will actually construct the queue from left to right for this relation.

3.3.1. Material and method

Thirty-five participants (three male; age: $M = 22.4$; $SD = 3.2$) from the University of Giessen had to solve 16 determinate (like in example 1) and sixteen indeterminate problems (like in example 2). The three-term problems had two premises each and we used only the relation "left of". The problems were presented to the participants in a random order on a computer screen. Each premise was presented sequentially (in a self-paced manner). Subsequently, after participants had read the premises, a conclusion was presented and the participants were asked if this conclusion was correct or not by indicating their choice by pressing the respective response button ("yes" or "no") with the left or right hand, accordingly. Locations of "yes" and "no" buttons were counterbalanced across participants. For determinate problems the conclusion was either true or false, thus the correct answers were "yes" for determinate/valid items and "no" for determinate/invalid items. For indeterminate problems we used two different types of conclusions which could either hold in a model constructed according to rule 1^{AC} or in a model constructed according to rule 1^{CC} (see Fig. 1 for examples). Please note that indeed both types of conclusions from indeterminate descriptions were valid in a logical sense. However, due to constructing the

1. The apple is to the left of the mango.
2. The mango is to the left of the pear.

C1: The apple is to the left of the pear. (true)

C2: The pear is to the left of the apple. (false)

1. The apple is to the left of the mango.
2. The apple is to the left of the pear.

C1: The mango is to the left of the pear. (true according to rule 1AC)

C2: The pear is to the left of the mango. (true according to rule 1CC)

Fig. 1. Top: A determinate item and with the two possible conclusions C_1 and C_2. Bottom: An indeterminate item with the two possible conclusions C_1 and C_2.

model by applying a preferred rule (1^{AC} or 1^{CC}), participants were expected to accept conclusions that hold in models constructed by the preferred rule and (mistakenly from a logical point of view) reject conclusions that hold in models as would have been constructed by the not preferred rule. The purpose was to gain insight whether rule 1^{AC} or rule 1^{CC} was preferably applied for constructing the model. The correct answer was "yes" for both, indeterminate/1^{AC} and indeterminate/1^{CC} items. Percentage of correct answers and corresponding decision times were recorded. All stimuli were generated and presented using Superlab 4.0 (Cedrus Corporation, San Pedro, CA, 1999) with an RB-530 response pad running on a standard personal computer (2.80 GHz) with a 19" monitor.

3.3.2. Results and discussion

Separate ANOVAs for the percentage of correct responses and decision times for correct responses (determinate/valid, determinate/invalid, indeterminate/1^{AC}/valid and indeterminate/1^{CC}/valid) were calculated. Level of significance was 5%.

ANOVA of the percentage of correct responses yielded a significant main effect [$F(2, 32) = 54.79$, $p < .01$]. Percentage of correct responses of determinate/valid and determinate/invalid items did not differ ($p > .75$). The high percentage of correct responses for the determinate items ($M = 92.19$; $SD = 11.14$) indicate that the participants understood the task and were able to perform well. Because of the reasons discussed in Section 2 and because the determinate items are more efficiently constructed from the left to the right for both cost functions we assume that they were indeed constructed from left to right. We also assume that the indeterminate items were constructed from left to right as well, since the decision has to be made directly after reading the first premise before knowing whether the item is determinate or indeterminate. We find a higher percentage of correct responses for indeterminate/1^{AC} items compared to indeterminate/1^{CC} items (see Fig. 2). Conclusions that held in models constructed according to rule 1^{AC} were significantly more often correctly accepted ($M = 60.22\%$; $SD = 37.40$; $t(34) = 5.49$; $p < .01$) than conclusions that held in models constructed according to rule 1^{CC} ($M = 27.14$; $SD = 5.88$). This indicates that indeed the rules derived from our alternative cost functions are more often applied than the rules derived from the classical computer science cost function.

ANOVA of the decision times of correct responses also yielded a significant main effect [$F(2, 18) = 4.25$, $p < .05$] (see Fig. 3). Decision times for determinate/valid items ($M = 3.62$ s, $SD = 1.43$) were significantly lower compared to determinate/invalid items ($M = 4.89$ s, $SD = 2.69$; $t(34) = -4.67$; $p < .01$). Decision times for indeterminate/rule 1^{AC} items ($M = 4.16$ s, $SD = 3.07$) were significantly lower compared to indeterminate/rule 1^{CC} items ($M = 5.06$ s, $SD = 3.46$; $t(20) = -2.29$; $p < .05$). This implies that conclusions of the determinate/valid items were easier to confirm than the ones of the determinate/invalid items and the conclusions of the indeterminate/rule 1^{AC} items were easier to accept than the ones of the indeterminate/rule 1^{CC} items. The easier items were those where the confirmation could easily be made by following the implicit direction of the queue provided that the queue was indeed constructed from left to right.

3.4. Other evidence

Further evidence for our model comes from the experiments of Jahn et al. (2007). Their participants inserted an object to an existing array, as opposed to adding it to one end of the array, more often for objects that would have been added to the left end of an array than for entities that would have been added to the right end of an array (Jahn et al., 2007, Experiment 2, Table 4). The authors come to the conclusion that: "Given that the participants constructed arrays from left to right, they evidently found it easier to add a new entity to the right-hand end of an array than to the left-hand end of an array [...]."(Jahn et al., 2007, p. 2081).

For a queue that is constructed from left to right our model predicts this behavior: rule 1^{AC} is applied to objects inserted to the right of a reference object and therefore the objects should be inserted at the end of the queue. In

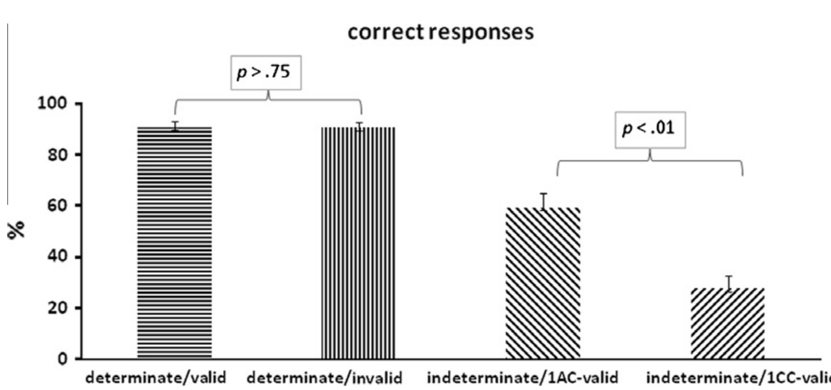

Fig. 2. The left two bars show the percentage of correct responses. For the determinate problems, for half of the problems the correct response was "yes" (hit), for the other half it was "no" (correct rejection). The two bars on the right show how often the participants correctly accepted a conclusion that hold in the model built by rule 1^{AC} or rule 1^{CC}, respectively. Error bars indicate standard errors.

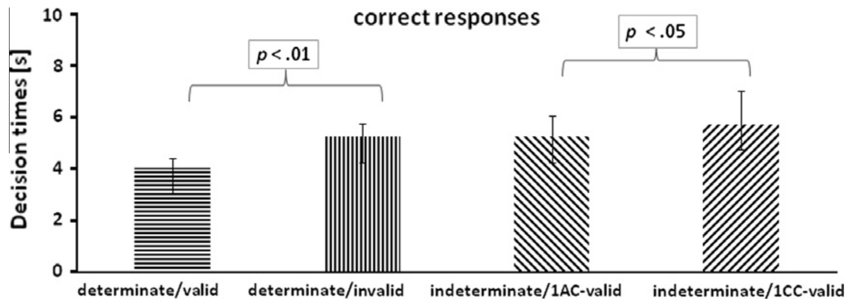

Fig. 3. The two bars on the left show the mean decision times of correct responses. For determinate problems, for half of the problems the correct response was "yes" (hit), for the other half it was "no" (correct rejection). The two bars on the right show reaction times when the participants correctly accepted a conclusion that holds in the model built by rule 1^{AC} or rule 1^{CC}, respectively. Error bars indicate standard errors.

contrast rule 1^{CC} would call for an insertion of an object directly behind the reference objects. Rule 2^{AC} and rule 2^{CC} are identical: if an object is to be inserted to the left of a reference object, it should be inserted directly in front of the reference object. The results of Jahn et al. (2007) confirm that reasoners construct a queue following the rules derived from the alternative cost function.

This evidence combined with our findings suggests that reasoners follow the rules 1^{AC} and 2^{AC} when constructing a mental model. In particular rule 2^{AC} describes the construction process better than rule 2^{CC} if the queue is constructed from left to right, which also implies that if the description of an arrangement is indeterminate (allowing more than one model) the direction of the queue influences which model will be built.

4. Reasoning with the model

Once a model has been constructed it can be used to make inferences. If we build the model

$\text{apple}^{*} \rightarrow \text{mango} \rightarrow \text{pear}$

from the premises of the first example with the implicit direction from left to right we can answer the question "Is the apple to the left of the pear?" by finding the apple in the queue and then moving further down the queue till we find the pear. This search process starts at the beginning of the queue. Since the implicit direction of the queue represents the relation in the question, once the pear is found we can answer the question with yes. That means that the deduction process shares some of the mechanisms with the encoding process: moving through the queue and finding objects.

The deduction process uses the transitivity of the relation "to the left". However, this does not imply that the knowledge of transitivity of the relation "to the left" is not considered until the question "Is the apple to the left of the pear?" is posed. One can argue that the knowledge of transitivity is already used in the encoding process: we only encode the information in a single model and assume that we can read out the relation between objects from the model, because we know the relation is transitive.

The question "Is the apple to the right of the pear?" can be answered in a similar way: find the apple and then move down the queue until the pear is found. However, in this case the implicit direction of the question does not represent the relation in the question. Therefore the answer to the question is No.

If the objects cannot be found in the order in which they appear in the conclusion, for example "Is the pear to the left of the apple?", there are two options: either the process has to be started again with the objects in inverse order and the inverse relation, so here we would have to test "Is the apple to the right of the pear?". The other possibility is that we remember having found the pear, but no apple behind it, and so we search the queue again from the beginning for the apple. In both cases the queue has to be accessed twice.

This illustrates we can also make predictions from the structure of the queue for the reasoning process. It should be easier to infer information that can be obtained following the implicit direction of the queue than to infer information that requires to go in the opposite direction. More specifically: if the objects in a statement are named in the same order in which they appear in the queue, it should be easier to compare this information to the queue than if they are named in the opposite order.

4.1. Empirical evidence

There is empirical evidence from an experiment we recently conducted in our lab. The experiment aims to investigate spatial belief revision. For that purpose it will be reported in detail elsewhere (Bucher et al., in preparation). Here we will report a detailed analysis of the inference tasks participants had to conduct prior to belief revision. This analysis was done specifically for the current context as it is not of particular interest concerning the investigation of belief revision.

4.1.1. Material and method

Sixteen participants (three male, $M = 21.75$; $SD = 1.61$) were individually presented with 32 items, each following the same structure: two premises (presented sequentially

Table 1
Combinations of the relations "left of" and "right of", used in the premises and facts (consistent and inconsistent) of the items.

	Relations "left of" and "right of" in the premises and facts of items in experiment 2			
1st Premise	Left of	Left of	Right of	Right of
2nd Premise	Left of	Right of	Left of	Right of
Fact (consistent or inconsistent)	Left of/right of	Left of/right of	Left of/right of	Left of/right of

in a self-paced manner) containing the relations "left of" and "right of" described a one-dimensional (linear) order of three (small, equal-sized, disyllabic-termed) objects, belonging to either one of two categories (tools or fruits). See Table 1 for the four possible relation orders in premises 1 and 2. Please note, as the last object always had to be inserted at the very right position of the alignment the construction of the linear order always followed a left to right direction. Participants were instructed to choose the correct order from two alternative orders presented on the left and right side of the computer monitor, indicating their choice by pressing a left or right response button with the left or right hand, accordingly. Presentation locations of correct and incorrect (mirrored orders of correct) orders were counterbalanced across the trials. Decision times as well as the number of correct decisions were recorded. Subsequently to the participant's decision, a conclusive fact was presented. The fact was either consistent (for half of the items) or inconsistent (for the other half of the items) with the information provided by the premises and hence with the order of objects. The relation of the fact was either "left of" or "right of" (see Table 1). The Participantś task was to judge whether the conclusive fact was consistent or inconsistent with the order of objects by pressing the respective response button ("yes" for consistent or "no" for inconsistent) with the left or right hand, accordingly. Locations of "yes" and "no" buttons were counterbalanced across participants. Thus, consistency judgments required to infer information from a linear order of three objects after construction of the order given a verbal description. The percentage of correct (consistence and inconsistent) judgments for conclusive facts with the relation "left of" and "right of", respectively, and the corresponding decision times were recorded. Correct "inconsistent"-judgments were followed by revisions of initially constructed orders. This latter part of the experiment is not of interest here. The items were presented in a random order to the participants. All stimuli were generated and presented using Superlab 4.0 (Cedrus Corporation, San Pedro, CA, 1999) with an RB-530 response pad running on a standard personal computer (2.80 GHz) with a 19″ monitor.

4.1.2. Results and discussion

Based on the information provided by the premises the correct order of objects was chosen in 97.27% ($SD = 3.76$) of the cases within 1.67 s ($SD = 0.45$). Erroneous trials were excluded from further analyses. Separate ANOVAs for correct percentages and judgment times with the factors consistency (consistent, inconsistent) × fact (left, right), respectively were conducted. Both ANOVAs revealed a significant interaction of consistency × fact (correct percentages: [$F(1, 15) = 10.368$; $p < .01$]; judgement times: [$F(1, 15) = 11.526$; $p < .01$]. All main effects were non-significant ($ps > .35$). Correct percentages and judgments times for correctly judged consistent and inconsistent conclusive facts with the relation "left of" and "right of", respectively, were compared using paired t-tests. Descriptive statistics can be found in Table 2.

Facts in which the objects were named in the same order as in the described alignment when moving from left to right (consistent/left facts and inconsistent/right facts) led to faster decision times and a higher percentage of correct responses than facts in which objects were named in the inverse order (consistent/right facts and inconsistent/left facts).

Specifically, percentage of correct judgements of consistent/left facts ($M = 98.22\%$; $SD = 4.88$) were significantly higher than consistent/right facts ($M = 90.18\%$; $SD = 11.72$; $t(15) = 2.43$; $p < .05$) as well as significantly higher than inconsistent/left facts ($M = 89.25\%$; $SD = 10.93$; $t(15) = 3.11$; $p < .01$). Also, inconsistent/right facts resulted in significantly higher percentages of correct judgments ($M = 97.66\%$; $SD = 5.04$) than both consistent/right facts ($M = 90.18\%$; $SD = 11.72$; $t(15) = -2.16$; $p < .05$) and inconsistent/left facts ($M = 89.25\%$; $SD = 10.93$; $t(15) = -2.68$; $p < .05$). Differences in percentages of correct judgments between consistent/left facts and inconsistent/right facts were non-significant ($p > .75$) as well as differences in percentages of correct judgments between consistent/right facts and inconsistent/left facts ($p > .75$).

Decision times were significantly lower for consistent/left facts ($M = 5.56$ s; $SD = 3.32$) than for consistent/right facts ($M = 8.01$ s; $SD = 5.90$; $t(15) = -3.11$; $p < .05$) and for inconsistent/left facts ($M = 8.69$ s; $SD = 3.23$; $t(15) = -3.71$; $p < .05$). Also the decision times for inconsistent/right facts ($M = 6.96$; $SD = 4.42$) were marginally

Table 2
Correct judgments [%] and correct judgment times [s] are shown for consistent and inconsistent facts with the relations "left of" and "right of", respectively.

	Correct judgments [%]	Correct judgment times [s]
Consistent fact		
Left	98.22 ($SD = 4.88$)	5.56 ($SD = 3.32$)
Right	90.18 ($SD = 11.72$)	8.01 ($SD = 5.90$)
Inconsistent fact		
Left	89.25 ($SD = 10.93$)	8.69 ($SD = 3.23$)
Right	97.66 ($SD = 5.04$)	6.96 ($SD = 4.42$)

significant lower than decision times for inconsistent/left facts ($M = 8.69$; $SD = 3.23$; $t(15) = 1.98$; $p = .066$). All other differences were non-significant (all $ps > .35$).

There was a clear asymmetry concerning facts with the relation "left of" compared to the relation "right of". For the relation "left of" the processing of consistent facts seems to be easier than for inconsistent facts while for the relation "right of" the opposite is true. The results imply that items with facts in which the order of the named objects corresponded to the left to right order of these objects in the alignment described by the premises were indeed easier to solve than items in which the objects were named in the inverse order.

The results of the experiment suggest that the reactions in the reasoning task are influenced by the encoding process, more specifically by the direction of encoding. This implies that the process of encoding the information is critical for the result of the reasoning process, which qualifies this kind of reasoning as verbal reasoning.

The reasoning process described here can also explain the distance effect, where it takes less time and is more accurate to make an inference about objects that are closer together compared to objects that are further apart (Acuna, Sanes, & Donghue, 2002; Frank, Rudy, Levy, & O'Reilly, 2005; Moyer & Landauer, 1967; Van Opstal, Gevers, de Moor, & Verguts, 2008).

Inferences regarding objects that are further apart in the queue should take longer as more movement through the queue is required than for objects that are close together.

5. Computational implementation

The model construction process can be easily implemented as a computer model using the data structure linked list, consisting of nodes containing data and a pointer to the next node in the list as well as a start pointer pointing to the first node of the list (compare to Fig. 4). If we compare this data structure to our mental model the pointers from one node in the list to the next represent the link between the objects and the data in the nodes represent the objects. It is therefore easy to model a queue such as the one we proposed in a computer program. Algorithms in pseudo-code for moving through the list and inserting new nodes into the list can be seen in Fig. 4.

```
record Node {
  data   // The data being stored in the node
  next   // A reference to the next node, nil for the last node
  }


firstNode   // A reference to the first node of list; nil for an empty list


function InsertAfter(node, newNode) {   // insert newNode after node
     newNode.next:= node.next
     node.next:= newNode
 }


function InsertFirst(firstNode, newNode) {   // insert node before first node
     newNode.next:= firstNode
     firstNode:= newNode
 }


function Move(firstNode, X) {   //move to node containing data X
     node:=firstNode
     while not node.data=X and not node.next=nil
         node:= node.next
 }
```

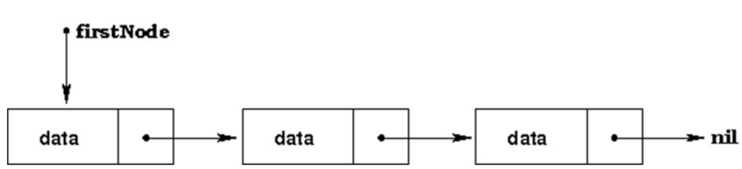

Fig. 4. Top: A pseudo-code definition of a linked list and functions to move through the list, inserting a node after a node, and inserting a node at the beginning of the list. The Node data structure consists of two fields. The variable first Node is a reference that points to the first node in the list, or is nil for an empty list. Since inserting a node at the beginning of a list requires updating first Node, it requires a separate function. Bottom: An illustration of the data type linked list.

It should be noted that technically it is not possible to insert a new node in front of a node, since we only have pointers to the following, but not to the preceding nodes. If a node is to be inserted in front of a node containing certain data, the node in front of that node is found and the new node is inserted behind it. Other than that the rules 1^{AC} and 2^{AC} can be easily implemented for the construction of a linked list. The interpretation of the direction of the links would have to be stored in an extra data structure.

5.1. Algorithmic description of the reasoning process

The reasoning process described in Section 5 can also be transcribed into an algorithm using a linked list as data structure representing the queue. Let r be a transitive binary relation and r^{-1} the inverse relation, for example r is "left of" and r^{-1} "right of". Let M be a model in the form of a linked list with implicit direction D of type r or r^{-1} and X and Y objects featured in that model. Given a sentence of the form XrY, the reasoning process can be described by pseudo-code algorithm in Fig. 5 which returns the value "true" if the sentence is true in the model and the value "false" if it is not.

The Reasoning algorithm starts with a search at the first node of the linked list. It uses the first option listed in Section 4 to verify conclusions in which the objects are not named in the same order as they appear in the queue. The other option could be implemented just as easily. An algorithm for the encoding process is not explicitly listed here. However, it can be easily constructed using rules 1^{fp}, 2^{fp}, 1^{AC}, and 2^{AC}, as well as the functions from Fig. 4. The very compact Reasoning algorithm demonstrates that the information is just read out of the model and no specific reasoning processes are employed.

A linked list is a dynamic data structure, which means that nodes are created and discarded as they are needed. This also implies that a linked list does not exist before the first node is inserted. We believe that this property represents the dynamic character of cognition well.

5.2. Comparison with other computational models and algorithmic descriptions

There are a number of computational models and algorithmic descriptions that also address relational or spatial reasoning. Ragni, Knauff, and Nebel, for instance, introduced a computational model for spatial reasoning by mental models (SRM) which conceptualizes spatial working memory as a two-dimensional array. Within this array models can be build and manipulated using a spatial focus (Ragni et al., 2005). This theory has later been applied to the preferred mental models account (Jahn et al., 2007; Knauff, 1999; Knauff et al., 1995) to model reasoning with indeterminate descriptions in order to determine which mental models are preferably constructed by reasoners (Ragni, Fangmeier, Webber, & Knauff, 2006). While this account would also predict the results of our first experiment, it cannot account for the left–right asymmetry that we found in the second experiment.

The group around Johnson-Laird provided a Lisp program called "Spatial Reasoning" which makes spatial deductions from given premises. The premises may consist of binary statements using objects and the relations right of, left of, in front of, behind, above, and below. The program tries to find a model in which all premises are true by recursively revising possible models. If it succeeds it also tries to find falsifying models to check the conclusion. Models are represented by arrays as well (Spatial Reasoning, unknown year). While the program shows that spatial reasoning can be done using models, it makes no prediction which model a human reasoner would preferably construct.

Van der Henst explicitly describes how (spatial) reasoning problems can be solved through an inference rule approach. He suggests rules to solve two-dimensional spatial reasoning problems for indeterminate as well as determinate problems similar to the ones presented here, using the relations left of, right of, and in front of (Van der Henst, 2002). The main aim of this work was to show that rule-based reasoning can also account for effects

```
function Search(firstNode,X,Y){    \\searches for the objects in the order XY
   node:=firstNode
   if not node.data=X then Move(node,X)
   if node.data=X then Move(node,Y)
   if node.data=X then return true else return false
   }


function Reasoning(firstNode, X,r,Y){   \\tests whether the sentence is true
   if Search(firstNode,X,Y)then
         if D=r return true
         else return false
   else if Search(firstNode,Y,X)then
         if D=r return false
         else return true
   }
```

Fig. 5. A pseudo-code algorithm of the reasoning process using the function Move from Fig. 4. The function Search tests if objects are in the list in a given order. The function Reasoning first tests if the objects are in the list in the order given in the sentence. If that is the case it checks if the implicit direction of the list (stored in D) is equal to the relation used in the sentence. If the objects are not in the list in the order given in the sentence the inverse order is checked.

indeterminacy. However, the rules whose application gives an indeterminate conclusion do not reflect any preference of reasoners for a solution. So, similar to the program “Spatial Reasoning”, no predictions for preferred solutions follow from this approach.

Schlieder proposes an outline for reasoning about the relative position of intervals on a line. For some of these spatial–relational inferences there are several logically equivalent solutions, some of which are preferred by human reasoners (Knauff et al., 1995). Assuming that these problems are solved using mental models that represent the start- and endpoints of the intervals, Schlieder provides an algorithmic description of the model construction process which is able to reproduce most of the preferences by human reasoners (Schlieder, 1999). Schlieder’s model is concerned with the composition of two relations (transitive and intransitive) between three intervals (spatial or temporal) and cannot easily be generalized to other kinds of reasoning. Particularly, as the main point of interest is to explain how two relations of a different nature are connected it cannot be compared to the approach outlined here since we focus on transitive relations that can be easily combined. Also, while Schlieder’s model makes predictions on preferred answers, it does not cover whether an answer is harder or easier to obtain.

Bara and Bucciarelli provide a computational theory of deductive reasoning based on mental models with an implementation in the program UNICORE (UNIfied COmputational REasoner) (Bara & Bucciarelli, 2000; Bara, Bucciarelli, & Lombardo, 2001). Their aim is to unify the main types of deductive reasoning into a single set of basic procedures. They distinguish five phases of deduction: Construction, Integration, Conclusion, Falsification and Response. UNICORE is able to reproduce correct and erroneous performances of human reasoners of different age groups in the three areas syllogistic, propositional and relational reasoning. For relational reasoning Bara, Bucciarelli and Lombardo examine determinate three term series problems. They assume that models are ordered left to right according to the sequential order in which the state of affairs they represent are described in the sentence. In accordance with the mental model theory they claim that two models are created, one from each premise, which are then combined into a single model to make an inference (Bara et al., 2001). While this is not stated explicitly, their approach implies that inferences whose relations correspond to the left to right ordering in their models should be drawn faster than other inferences. However, since they assume that all models are constructed in the order in which objects are named in sentences, independent of the relation used, they would use models constructed from left to right for one half the items and models constructed from right to left for the other half of the items of the experiment reported in Section 4. This would lead to the prediction that on average there should be no difference in reasoning time for conclusions using relation left and conclusions using relation right, which does not match the results we found. For indeterminate problems the theory does not make predictions for preferences of reasoners.

A completely different approach uses the LISA model of analogical reasoning by Hummel and Holyak. This connectionist model employs a neural network to model reasoning. Objects of propositions as well as their relations are represented as patterns of activation distributed over semantic units, which are integrated into representations of propositional structures using synchrony of firing. Through its structure LISA can account for the limits of working memory (Hummel & Holyoak, 2003, 2005). However, LISA focuses on analogical reasoning that is finding correspondences between elements that play parallel roles in two similar situations. While this is an important part of relational thinking, we do not believe that the kind of problems discussed in this work are solved by analogical reasoning. Therefore, there is no straight-forward way how our kind of indeterminate problems can be solved in LISA.

6. Discussion

We introduced an approach about how relational reasoning can be modeled as verbal reasoning. The main idea is that the deduction process does not necessarily require deduction-specific mechanisms to operate on internal representations. Instead we assume that a simple order of objects (represented by words) and some genuine verbal cognitive mechanisms might guide the reasoning process. Following Polk and Newell (1995) we assumed that the cognitive processes in deductive reasoning can be based upon the same processes as language comprehension and generation. Our model satisfies the criteria of verbal reasoning as outlined by Polk and Newell (1995). Verbal in that sense refers to transforming between verbal and semantic representations, that is constructing the queue (encoding) and “reading out” information that is not explicitly provided by verbal descriptions. The most important point is that “reading out” information from the queue does not require mechanisms that are especially dedicated to deduction. Rather, reasoning is accomplished by applying more well-trained linguistic processes that are more likely to be applied by individuals not trained in logic. The approach does not obviate specific mechanisms but provides a more parsimonious explanation how inferences can be drawn from given information without assuming additional mechanisms.

Mechanisms operating on mental models and not reasoning-specific mechanisms are applied during “reading out” information from the queue. The claim is that individuals – especially when untrained in logic – are more likely to apply well-trained linguistic-based, rather than deduction-specific mechanisms to derive implicit information from given information. The algorithms outlined in Section 5 demonstrate how these reasoning processes can be realized by very simple means.

Our empirical work has shown that the approach and the related cost measure leads to good predictions about what kind of model will be created. It predicts behavior better than the standardized computational complexity approach. In addition predictions from the suggested reasoning process were empirically supported.

While the present experiments only used problems with two premises, we believe that the postulated rules also apply for more than two premises and three objects, as long as the premises all contain transitive relations describing the same dimension. It is also possible to mix relations of the same dimension such as left and right, as done in our second experiment and in many other experiments (e.g., Jahn et al., 2007; Ragni et al., 2006). Also the mechanisms described by us are not limited to spatial relation, but are general enough so that they can be employed for all transitive relations. However, although in principle the suggested mechanisms might operate with non-spatial transitive relations in the same way as with spatial relations, the influence of aspects that result from content-inherent factors (e.g., complexity modulating aspects such as lexical marking of adjectives or congruence of relations in the premises and conclusions; Clark, 1969) on the processes have to be investigated and specified.

We provide algorithms for the construction and reasoning process, making it easy to implement the model. The proposed computational implementation is more frugal concerning data-structure and algorithms compared to alternative approaches listed in Section 5 while still containing all aspects of reasoning about linear orders outlined in this study. Also, this is the first model that can account for a left–right asymmetry as found in our second experiment or reported by Jahn et al. (2007).

However, some issues concerning the model still have to be specified. A question that remains is whether the starting point of a queue is really a link like all the other links in the queue. Since this link is different concerning its cognitive nature it might be weaker or stronger than the links between objects in the queue. If this is the case, it would make it either easier or harder to insert an object at the beginning of the queue than between two objects of the queue.

Another point is that if, for some reason, we already know that a reference object is in the queue, and we have to insert an object in front of it, it is more cost efficient to insert a new object at the beginning of the queue instead of first finding the reference object in the queue to insert the object directly in front of it. In this case inserting the object at the beginning of the queue saves moving through the queue. The same is not true for inserting an object behind a reference object. Here we always have to move through the queue, at least to the reference object.

A third problem is that we postulate that the implicit direction of a queue can theoretically be chosen freely, with the choice in our spatial reasoning tasks strongly being influenced by cultural and linguistic factors, as discussed in Section 2. Can this choice be further influenced? In spoken language emphasis can be used. For instance in the sentence: "The apple is to the left of the *mango*" intonation can be used to put the focus on the mango. This might prompt reasoners to change their preferred direction of construction. In addition, to evaluate the influence of culture comparison between participants from a left-to-right reading country with those of a right-to-left reading one would be interesting.

Finally, the validity of the alternative cost measure should be examined in more detail. One problem of our approach results from the assumption that it is easier to move through the queue than to alter existing links, no matter how far we have to move. However, it is possible that, if the queue becomes larger, there might exist a critical distance after which more mental effort is required moving this distance through the queue than altering a link. This would imply that if the queue reaches a certain number of objects new objects would not necessarily be attached to the end of the queue any more. If this is the case one could identify a "break-even-point" and specify how many objects one has to move through the queue to induce the same cognitive effort as creating a link.

We consider our approach to be a helpful addition to the long lasting controversy between models and rules in reasoning (e.g., Hagert, 1984; Johnson-Laird, Byrne, & Schaeken, 1994; Rips, 1994). In fact, models are often identified with visuo-spatial processing and rules with linguistic or sentential mechanisms (e.g., Goel, Buchel, Frith, & Dolan, 2000). Our study, however, shows that this distinction does not reflect the actual differences between the two approaches. In fact, our approach is a model-based approach, because at no time during the inference process rules of inference are used and the new information must be derived from the queue – the model. On the other hand, our results suggest that such models can be the basis of verbal reasoning, and visuo-spatial processes are not necessarily involved in the inference.

Overall, we were able to present some evidence for our assumption that the process of constructing a verbal mental model from premises influences deductive relational reasoning. For indeterminate problems, we can predict which model is preferred over others. For determined problems, we can make predictions on how the conclusion should be phrased so that it can be easily confirmed or invalidated. While our model cannot necessarily be generalized to other domains of reasoning we feel that it can describe some aspects of human reasoning with transitive relations and that it demonstrates that relational reasoning can also be conceived of as verbal reasoning.

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