Some Contributions to the Theory of Buildings Based on the Gate Property

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Contents

Zusammenfassung

Gebäude sind Kammernkomplexe, die die Toreigenschaft haben. Diese ist vor allem deshalb interessant, weil sie sich auf Fixpunktstrukturen vererbt. In der vorliegenden Dissertation werden Kammernkomplexe mit Toreigenschaft betrachtet und die daraus gewonnenen Resultate auf Coxeter-Gruppen und sphärische Moufang-Gebäude angewendet.

Kammernkomplexe mit Toreigenschaft: Zunächst werden elementare Eigenschaften, wie z.B. die Existenz einer Typfunktion, hergeleitet. In einem weiteren Abschnitt werden spezielle Klassen von Kammernkomplexen mit Toreigenschaft untersucht. Das erste Hauptresultat besagt, daß ein schwacher Kammernkomplex mit Toreigenschaft von endlichem Durchmesser 'beinahe' ein sphärisches Gebäude ist. Das zweite ist der schon oben erwähnte Fixpunktsatz für Kammernkomplexe mit Toreigenschaft.

Dünne Gebäude: Dünne Gebäude - oder Coxeter-Komplexe - sind genau die dünnen homogenen Kammernkomplexe mit Toreigenschaft. Mit Hilfe des Fixpunktsatzes lassen sich Coxeter-Komplexe konstruieren, die in einen gegebenen Coxeter-Komplex eingebettet sind. Den eigentlichen Inhalt dieses Kapitels bilden Einbettungen, die nicht unbedingt als Fixpunktstruktur eines Automorphismus' auftreten. Mit Hilfe der Toreigenschaft wird ein lokales Kriterium für die Existenz einer solchen Einbettung hergeleitet. Daraus ergibt sich u. a. ein kurzer Beweis für die Klassifikation der endlichen Coxeter-Gruppen.

Moufang-Gebäude: Das Hauptresultat dieses Kapitels ist ein 'Fixpunktsatz für sphärische Moufang-Gebäude'. Um die Aussage dieses Satzes zu präzisieren, werden Moufang Komplexe eingeführt. Falls der zu Grunde liegende Komplex ein sphärisches Gebäude ist, ist die Definition zur klassischen Definition eines Moufang Gebäudes äquivalent. Für die Richtigkeit des Fixpunktsatzes wird eine zusätzliche Voraussetzung benötigt; dies motiviert eine Version des Satzes von Maschke für sphärische Moufang-Gebäude.

Introduction

In [Ti74] J. Tits defines buildings as chamber complexes endowed with a system of apartments. This definition of buildings is motivated by the properties of the poset of the parabolics in a semisimple algebraic group. The short list of axioms describes these structures in a purely combinatorial set-up.

From a geometrical point of view, buildings are the flag complexes of geometries over a Coxeter diagram (for short: geometries of type M) in the sense of Buekenhout and Tits. Taking into account that geometries of type M had been investigated in the 'prehistory' of buildings in order to find good axioms, this is not surprising.

Originally, J. Tits intended to describe the geometries associated to the simple groups of Lie type as geometries of type M. It turned out that this notion was too general for his purposes; he observed that one has to add one additional axiom to catch the 'good' geometries of type M, namely the intersection property. However, this observation was not pushed further, because he had dicovered a completely different set of axioms; these are the axioms for buildings as chamber complexes. For a more detailed survey on the history of buildings the reader is referred to [Ti78].

The early characterization of buildings as geometries of type M regained interest by the work of F. Buekenhout on the geometric interpretation of the sporadic groups (cf. [Bu79]). The problem how to characterize buildings among the geometries of type M was solved in [Ti81] by J. Tits. Though the following theorem was known to him much earlier, this is the first place where a complete proof of it can be found. This theorem makes reference to the intersection property already mentioned above and points out its importance.

Theorem: *A geometry over a spherical diagram is a building if and only if it has the intersection property.*

The proof of the 'only if'-part is given in [Ti74] Chapter 12. It is an easy consequence of the existence of projection mappings in buildings. Hence in a certain sense, the existence of projection mappings singles out the buildings among the geometries over a Coxeter diagram. The result of R.Scharlau (cf. [Sc85]) which remains valid also for non-spherical diagrams underlines our interpretation of the intersection property.

In [Ti74] the existence of projection mappings is established at an early stage in the theory of abstract buildings. Most of [Ti74] relies on these mappings. The

Theorem 4.1.2. on extensions of locally defined isomorphisms for spherical buildings is perhaps the most beautiful example.

In [DS87] A. Dress and R. Scharlau generalize the concept of projection mappings to arbitrary metric spaces. This leads them to the definition of gated sets in metric spaces. They show that certain properties of projection mappings deduced by J. Tits remain valid in this more general context.

The notion of a 'gated set' is best explained by an example from the Middle Ages. We consider a town A having a town wall and hence also several town gates. The environs of the town will be denoted by M. We endow M with the usual metric. Suppose that someone who lives at a place $p \in M$ wants to go to a certain point $q \in A$. The principal observation is now that this person can always take a shortest way through a gate depending only on p. We denote this particular gate by $proj_A p$. If the person lives in the town A (i.e. $p \in A$), his gate is of course the front-door of his own house, which means $proj_A p = p$. In this way the set of points in the town A becomes a gated subset of M; we say also that A is a gated subset of M. The gate $proj_A p$ is the projection of the place p to the town A.

Buildings are chamber complexes in which all residues are gated. Here we show that this property characterizes the spherical buildings among all thick chamber complexes of finite diameter. This improves the result of R. Scharlau already mentioned above. But it is is not only the characterization of buildings which makes this property interesting. Our investigations of buildings indicate that in some instances it is advantageous to throw away the diagrams and types, and to work only with the gate property.

The present thesis contains three chapters. In the first chapter we consider chamber complexes with the gate property without any further assumptions. In the second chapter we apply a result of Chapter 1 to Coxeter complexes; this motivates the concept of admissible partitions of Coxeter diagrams. Our main results on gated chamber complexes are applied to spherical Moufang buildings in the third chapter; we obtain a general fixed point theorem for these buildings. Each chapter has an own introduction, where the reader can find a detailed description of its contents. The following short abstracts are intended to make the main stream transparent.

Gated chamber complexes: The definition of a gated chamber complex is based on the concept of gated sets in metric spaces, and on the notion of a chamber complex. In order to have good axioms, it is convenient to require strong connectedness and the gate property of the residues of rank smaller or equal to 2. In a further section we consider special classes of gated chamber complexes; here we provide the prerequisites for the proofs of our main results. The first of them asserts the existence of a canonical system of apartments in a weak gated chamber complex of finite diameter (see Theorem 1.7.26). This makes it reasonable to call these complexes spherical; moreover, there is a natural opposition relation. The existence of apartments implies the characterization of the spherical buildings already mentioned above. In the general case, the apartments are thin gated chamber complexes of finite diameter; from those we construct oriented matroids. We show how to extract a building from a spherical gated chamber complex. The results may serve to 'classify' all those complexes. Our second main result is a fixed point theorem for gated chamber complexes (see Theorem 1.8.22). Given a group acting on a gated chamber complex and satisfying a rather weak condition, the set of fixed simplices constitutes a gated chamber complex. This theorem can be improved in the spherical case. The spherical version of the fixed point theorem provides the geometric part of our fixed point theorem for spherical Moufang complexes.

Thin buildings: Thin buildings or Coxeter complexes are precisely the thin gated chamber complexes to which one can associate a Coxeter diagram. Thus we can apply the fixed point theorem in this particular situation. As a consequence we have a proposition which yields several well known facts about Coxeter groups in a uniform set-up. The main observation in this proposition is the following: the structure of the chamber complex fixed by an automorphism acting on a Coxeter complex is already determined by fairly local information. The information is obtained by a partition of a subset of the index set of the Coxeter diagram; the partition may be visualized by a sort of decoration of the Coxeter diagram in question which generalizes the diagrams used in the classification of the semisimple algebraic groups over arbitrary fields. As an easy consequence we obtain a criterion which ensures that the fixed point set is again a Coxeter complex. The main idea of this chapter is to develop a theory which generalizes the diagrams mentioned above in order to produce Coxeter complexes which are embedded in a given Coxeter complex. One important example of such an embedding is the well known inclusion of H_4 in E_8. We introduce admissible partitions of Coxeter diagrams. Our main result is a rank two criterion for the admissibility of an arbitrary partition (see Theorem 2.4.9). Its proof uses the homogenity and the gate property of Coxeter complexes. In a further section we classify all admissible 2-partitions of the diagrams A_n, C_n and D_n. As a first application we will show how to embed a given Coxeter complex over a finite diagram in a Coxeter complex whose diagram is simply-laced. The second application is a proof of the classification of the spherical diagrams which is based on the solution of the word problem in Coxeter groups. The idea to use the solution of the word problem is due to J. Tits (cf. [Ti68]).

Spherical Moufang buildings: The main goal of this chapter is to prove a fixed point theorem for spherical Moufang buildings. The theorem provides a uniform approach to several examples coming from the theory of semisimple algebraic groups and the finite groups of Lie type.

The geometric part of this theorem is an easy consequence of the first chapter. We introduce Moufang complexes. Their definition enables us to prove our theorem with some elementary ingredients from group theory. Our definition of a Moufang complex is a generalization of the notion of a Moufang building; while the classical definition requires the existence of all root elations, our axioms are based on the unipotent radicals at each chamber. The definition makes sense for all spherical gated chamber complexes. We prove the following: Given a group Γ acting on a Moufang complex and satisfying a certain condition (O), the group Γ fixes a Moufang complex (see Theorem 3.4.8).

A further section circles around the condition (O). The content of this section may be viewed as a version of Maschkes theorem for spherical Moufang buildings. In the final section we will state a conjecture, whose validity is known for the most interesting cases. The conjecture is important, if one wants to get control over the associated groups. We do a first step towards a general proof of this conjecture by resuming an idea of J. Tits.

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Chapter 1

Gated Chamber Complexes

1.1 Introduction

In this chapter we intend to elaborate our two main results on gated chamber complexes. The first one is the existence of a system of apartments in spherical gated chamber complexes, the second is a fixed point theorem. In order to do this we first have to develop the theory of gated chamber complexes to some extent. Gated chamber systems have been introduced by R. Scharlau in [Sc85], but - to my knowledge - the notion of a gated chamber complex cannot be found in the literature. So we provide the prerequisites for the proof of the results mentioned above.

The gate property: The concept of gated sets in metric spaces was first investigated by A. Dress and R. Scharlau in [DS87], though the notion of a gate appeared already before. For a brief survey on the history and other applications of the gate property the reader is referred to loc cit.

We will introduce in Section 1.2 gated sets in metric spaces along the line of [DS87] and recall some basic facts concerning this topic. Most of them are immediate. The only exception is the main result of loc cit., which is our Proposition 1.2.3.

We will also introduce parallel gated sets. Though parallelity is already implicit in [DS87], this notion is due to J. Tits (cf. [Ti90]). Parallel panels will play an important role throughout this chapter; in the proof of the fixed point theorem parallelity of arbitrary simplices will come into play.

Chamber complexes: In this section we will fix the notation concerning complexes and chamber complexes. Most of our definitions coincide with those of [Ti74].

In loc cit. J. Tits introduced chamber complexes as a class of simplicial complexes satisfying some additional conditions in order to axiomatize buildings. Buildings are nothing else than the flag complexes of certain geometries over a set I; in other words: they are numbered complexes. Geometries over I, or more specially, geometries over Coxeter diagrams, are the mathematical objects which had been there before buildings were defined for the first time (see for instance [Ti78]). One important fact is that the axioms for buildings given in [Ti74] do not involve types, which can be produced from the axioms. We plan to do a similar thing for gated chamber complexes, i.e. we intend to deduce the existence of a numbering. A numbering allows one to pass from a chamber complex Δ to the chamber system $\mathbf{C}(\Delta)$. However, it is not necessary to have a numbering in order to construct a chamber system from a chamber complex; it suffices to have a system of local numberings which we will introduce as weak numberings. The definition of weak numberings may be seen as a part of our search for good axioms for gated chamber complexes. In Section 1.5 we prove that it is possible to construct a weak numbering of a chamber complex, if the rank one and rank two residues are gated (see Proposition 1.5.3). As a consequence of this observation it turns out that it is no severe restriction to take the strong connectedness as an axiom for gated chamber complexes. This is described in more detail in the subsection about numberings and chamber systems.

As already mentioned a weak numbering allows one to construct a chamber system from the chamber complexes to be considered here; hence, we are not far away from the structures investigated in [Sc85]. It is natural to ask for the motivation for the definition of gated chamber complexes while the gated chamber systems are already there. The main reason for us to consider chamber complexes is perhaps best explained by an example.

Let Σ be a Coxeter complex of type A_n and let s be a reflection in the associated Coxeter group W. The set of simplices fixed by s constitutes a Coxeter complex of type A_{n-1}. However, the restriction of a type function of Σ is of course not a type function of the chamber complex fixed by s.

This example shows that in order to prove a fixed point theorem types are rather disturbing than helpfull. Though it should be possible to state and prove our main results in terms of chamber systems, I believe that they are best understood in the way they are presented here.

The gate property in chamber complexes: In this section we make a first step towards a definition of a gated chamber complex. We define gated simplices in a chamber complex. In the remainder of this section we will be concerned with chamber complexes, in which all simplices of codimension one (called panels) are gated. For those complexes we state some elementary facts. In the rank two case we get control over these chamber complexes, because we can prove that they are numbered. Applying a result of [Sc85] we can deduce that a chamber complex of rank two in which the panels are gated is a generalized n-gon or a chamber subcomplex of a building of type $\tilde{A}_1$.

Chamber complexes in which the panels and copanels are gated: Since we know already the structure of the rank two residues (called copanels) in chamber complexes with gated panels, it is natural to investigate chamber complexes in which the panels and the copanels are gated. In the first part of this section we prove that those chamber complexes are weakly numbered.

If we assume in addition strong connectedness, it will turn out that all simplices are gated (see Proposition 1.5.4). This result is already known for gated chamber complexes defined over Coxeter diagrams (i.e. the homogenous gated chamber complexes) because they are buildings by a result of R. Scharlau [Sc85]. I do not know whether his principal method, namely the deletion condition, can be applied to prove this more general statement. In any case, this result is not surprising, because the restriction "to corank at most 2" is quite usual in the theory of buildings (cf. [RT87], [Ti74] Theorem 4.1.2) and in the theory of oriented matroids ([BLSWZ93]). However, it seems to be new in the generality in which it is proved here.

The two results of this section finish the task of finding convenient axioms for gated chamber complexes. We define a gated chamber complex as a strongly connected chamber complex in which the panels and copanels are gated (see Corollary 1.5.5 and Definition 1.5.6).

Some special classes of gated chamber complexes: In this section we consider gated chamber complexes satisfying some extra conditions. Gated chamber complexes of locally finite type are those in which all rank two residues are generalized n-gons for a finite number n. It is natural to ask whether a gated subset of chambers is already the set of chambers of a star of a simplex. This is not true in an arbitrary gated chamber complex, but it is the case in those of locally finite type. As an easy consequence we get a generalization of Theorem 3.21 in [Ti74] to these complexes. The method of our proof is quite different from the one given in loc cit. and provides perhaps a better insight into the nature of this result. A second consequence of this result is that gated chamber complexes of locally finite type are totally gated.

Totally gated chamber complexes will be considered in another subsection. They are roughly speaking chamber complexes for which the projection of an arbitrary simplex onto another simplex makes sense. The content of this subsection is more or less a first preparation of our fixed point theorem.

Thin gated chamber complexes will be considered in a further subsection. We will define roots, boundaries etc. as natural generalizations of these notions from Coxeter complexes to thin gated chamber complexes. The content of this subsection is relevant for the investigations of spherical gated chamber complexes.

Spherical gated chamber complexes: In Section 1.4 we will deduce that if two chambers in a gated chamber complex of rank two do behave like opposite chambers in a building, then the chamber complex is the flag complex of a generalized n-gon, hence a spherical building. This observation is essentially due to R. Scharlau (cf. [Sc85]).

The result remains almost true in the higher rank case. If two chambers in a gated chamber complex Δ do behave like opposite chambers in a spherical building, then we can show that Δ has already a system of apartments. Therefore, we will define a spherical gated chamber complex as a gated chamber complex in which there exists a pair of 'opposite chambers'.

In the first subsection we show how to associate to such a pair a finite thin gated chamber complex. It is the full convex hull of the two chambers.

The next subsection deals with finite thin gated chamber complexes. It is easily seen that they are spherical and that they admit an opposition involution. The main goal of this subsection is to prove that an isomorphism of roots extends to an isomorphism of the thin gated chamber complexes in question. This extension theorem will be an important tool for the construction of an apartment system in an arbitrary spherical gated chamber complex.

The appended subsection deals again with finite thin chamber complexes. Here we show how to construct an oriented matroid from such a complex. It is easily seen that the oriented matroid constructed in this way is simplicial. Conversely, it is possible to construct from a simplicial oriented matroid a thin gated chamber complex. The gate property of oriented matroids was observed by A. Björner and G. Ziegler (cf. [Zi93]).

In the first subsection we constructed an apartment associated to a pair of opposite chambers. In the fourth subsection we show how to construct an apartment system by starting from a given apartment in a spherical chamber complex. This completes the proof of our first main result.

As a consequence we obtain that the thick spherical gated chamber complexes are precisely the spherical buildings. In a further subsection we indicate what can be done in the case of weak spherical gated chamber complexes. Here we use essentially the content of [Sc87], where a structure theorem for weak spherical buildings is established.

In the last subsection, we will define opposite simplices in spherical gated chamber complexes and state some elementary facts, which are easy generalizations from spherical buildings. The content of this subsection is relevant for the proof of the fixed point theorem.

Since I have not written down a detailed proof of the following statement, I will call it a conjecture. However it should be mentioned that the main ideas of a proof are at least indicated in this section.

Conjecture:

(1) The thin spherical gated chamber complexes are precisely the simplicial oriented matroids

(2) The thick spherical gated chamber complexes are precisely the thick spherical buildings.

(3) *The weak spherical gated chamber complexes are classified up to isomorphism by the triples* $(\Sigma, \bar{W}, \Delta)$, *where*

Σ *is thin spherical gated chamber complex,*

$\bar{W}$ *is the set of conjugates of a group* W *in* $Aut(\Sigma)$ *where* W *is a group generated by reflections in* $Aut(\Sigma)$ *(see Definition 1.6.9 and Proposition 1.6.10),*

Δ *is a thick building of type* M, *where* M *is the Coxeter diagram associated to* W *(see Definition 1.6.9 and Proposition 1.6.10).*

Note that part (2) of the conjecture above is completely proved in this section.

Automorphisms of gated chamber complexes: In the last section we prove a fixed point theorem for totally gated chamber complexes. Our original goal was to prove a theorem which says that the set of simplices fixed by an automorphism group Γ is again a totally gated chamber complex. Though this might be true in general, we restrict ourselves to the case where Γ is a spherical automorphism group. In this case we can apply our results on spherical gated chamber complexes. Our proof relies heavily on projections and opposition in spherical residues.

In the first subsection we will state some facts about this particular subject. Most of them are easy generalizations of the corresponding statements about buildings. For this reason we will prove only the part which is by far the hardest.

In the following subsection we will consider automorphisms of totally gated chamber complexes and introduce spherical automorphisms.

The proof of our main result is divided up into several lemmata. The restriction to spherical automorphisms comes into play in Lemma 1.8.13, which is basic for all what follows.

It is easily seen that the set of fixed simplices of a spherical automorphism group (always denoted by $\tilde{\Delta}$) is a strongly connected chamber complex. Hence the remaining task is to show that the panels and copanels of $\tilde{\Delta}$ are gated. In order to do this, we have to consider first the case where $\tilde{\Delta}$ has rank two.

Combining all our lemmata of this subsection we will state our main result in the last subsection.

1.2 Gated sets in metric spaces

Let R denote the field of the real numbers. A *metric space* is a set M endowed with a distance function $d : M \times M \longrightarrow R$ satisfying

(i) $d(x,y) = d(y,x) \geq 0$ for all $x, y \in M$.

(ii) $d(x,y) = 0$ if and only if $x = y$.

(iii) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in M$.

Let (M, d) be a metric space and let $x, y \in M$. We define the *segment* (denoted by $\sigma(x, y)$) between x and y as follows:

$$\sigma(x, y) = \{z \in M | d(x, z) + d(z, y) = d(x, y)\}$$

A subset C of M is called *convex* if $x, y \in C$ implies $\sigma(x, y) \subseteq C$.

The following lemma is immediate.

Lemma 1.2.1. *Let $x, y, z \in M$ be such that $\sigma(x, y) \subseteq \sigma(x, z)$. Then $d(x, z) \geq d(x, y)$. If $d(x, y) = d(x, z)$, then $y = z$.*

Let $x \in M$ and $A \subseteq M$. A point $y \in A$ is called a *gate* of x to A (or the *projection* of x on A) if $y \in \sigma(x, z)$ for all $z \in A$. It follows immediately that if x has a gate to A then it is unique. We will denote it by $proj_A x$. A subset A is called gated if any $x \in M$ has a gate to A.

We mention an example, which will be applied in the following sections.

Lemma 1.2.2. *Let (V, E) be an undirected graph without loops. Let the distance between two edges be the length of a shortest path joining them. Then the following are equivalent:*

1. *The graph is bipartite.*

2. *If $x, y \in V$ are adjacent vertices, then the set $\{x, y\}$ is gated.*

Let A be a gated set; then $proj_A$ is a mapping from M onto A. In some cases it will be convenient to restrict this mapping to a subset B of M. If we want to emphasize that we consider the restriction of $proj_A$ to B we write $proj_A^B$.

We list some properties of gated sets and projection mappings in metric spaces.

(G1) Gated sets are convex.

(G2) Let $A \subseteq M$ be a gated set. Then $proj_A : M \longrightarrow A$ is a distance decreasing mapping.

(G3) Let A be a gated set in (M, d) and let $B \subseteq A$ be gated in the metric space $(A, d|_{A \times A})$. Then B is gated in M and $proj_B = proj_B^A \circ proj_A$.

(G4) If A is gated and $x \in M$ then $proj_A y = proj_A x$ for any $y \in \sigma(x, proj_A x)$

The following result is due to A.Dress and R.Scharlau (see [DS87])

Proposition 1.2.3. *Let (M, d) be a metric space and let A_1, A_2 be gated subsets of M and put $B_1 = proj_{A_1} A_2, B_2 = proj_{A_2} A_1$. Then B_1, B_2 are gated subsets of M. Moreover, $proj_{B_1}^{B_2}$ and $proj_{B_2}^{B_1}$ are isometries inverse to each other. We have*

$$proj_{A_1}^{A_2} = proj_{B_1}^{A_2} = proj_{B_1}^{B_2} \circ proj_{B_2}^{A_2}.$$

Definition 1.2.4. Two gated sets A, B are called *parallel* if $proj_A B = A$ and $proj_B A = B$.

Remark: It turns out that parallelity is not an equivalence relation in general.

1.3 Chamber complexes

Complexes

Let J be a set and let $P(J)$ be the set of all subsets of J. The inclusion induces a partial ordering on $P(J)$. We will denote this partially ordered set by $(P(J), \subseteq)$.

A partially ordered set $(A, \subseteq)$ is called a *simplex* if there exists a set J such that $(A, \subseteq)$ is isomorphic to $(P(J), \subseteq)$.

Let $(\Delta, \subseteq)$ be a partially ordered set and $A \in \Delta$. Then $\bar{A}$ denotes the set of all $B \in \Delta$ with $B \subseteq A$ endowed with the partial ordering induced by $(\Delta, \subseteq)$.

A *complex* is a partially ordered set $(\Delta, \subseteq)$ such that the following axioms are satisfied:

(1) $\bar{A}$ is a simplex for each $A \in \Delta$.

(2) Two elements $A, B \in \Delta$ have a greatest lower bound.

We will denote the greatest lower bound by $A \cap B$.

Let $(\Delta, \subseteq)$ be a complex. If $A \subseteq B$ for $A, B \in \Delta$ we say that A is a *face* of B or A is *contained* in B. We write $A \subset B$ if $A \subseteq B$ and if we want to emphasize that $A \neq B$.

A complex has always a unique minimal element which we denote by $\emptyset$. If $v \in \Delta$ and $A \subset v$ implies $A = \emptyset$ for every $A \in \Delta$ then v is called a *vertex* of Δ.

If $A \in \Delta$, then the *rank* of A, denoted by $rk(A)$, is the cardinality of the set of all vertices contained in A. The rank of Δ is defined as the supremum of the cardinalities $rk(A)$, where A runs over all elements of Δ.

If $A \in \Delta$ then the set $\{B \in \Delta | A \subseteq B\}$ is also a complex with the partial ordering induced by Δ. We will denote this complex by $St(A)$.

If $A \subseteq B$ the *codimension* or *corank* of A in B, denoted by $codim_B(A)$, is the rank of B in the complex $St(A)$.

A *chamber* of Δ is a maximal element in Δ.

Let Δ, Δ' be two complexes and let α be a mapping from Δ into Δ'. The mapping α is called a *morphism of complexes* if for each simplex $A \in \Delta$ the mapping $\alpha \mid_{\bar{A}}$ maps the simplex $\bar{A}$ isomorphically onto $\overline{\alpha(A)}$.

A *subcomplex* Δ' of Δ is a subset of Δ such that the inclusion $\Delta' \hookrightarrow \Delta$ is a morphism of complexes.

Chamber complexes

A *chamber complex* is a complex Δ satisfying the following two axioms:

(1) Each $A \in \Delta$ is contained in a chamber.

(2) If C, D are chambers in Δ then there exists a sequence $C = C_0, C_1, \ldots, C_k = D$ such that

$$codim_{C_{i-1}}(C_{i-1} \cap C_i) = codim_{C_i}(C_{i-1} \cap C_i) \leq 1$$

for $1 \leq i \leq k$.

It is easily seen that the C_i in the sequence above are also chambers in Δ. We will call such a sequence a *gallery* and k its *length*.

From now on let Δ denote a chamber complex. The *distance* between two chambers $C, D \in \Delta$, denoted by $d(C, D)$, is the minimum taken over all $k \in N_0$ which are the length of a gallery joining C and D. A gallery joining C and D is called *minimal* if its length is equal to $d(C, D)$; in this case we will also say that the gallery is *stretched* from C to D.

If $A, B \in \Delta$, a gallery joining A and B is by definition a gallery starting at a chamber containing A and ending at a chamber containing B. The distance between A and B is the length of a minimal gallery joining them.

Let C and D be chambers in Δ having a common face A. Then $codim_C(A) = codim_D(A)$ (see [Ti74] 1.3.) and we may define the corank of $A \in \Delta$ to be the codimension of A in a chamber containing A.

A *panel* (resp. *copanel*) of Δ is an element of Δ having corank one (resp. two).

A chamber complex is said to be *meager* (resp. *thin* , *weak* , *thick*) if each panel is contained in at most two (resp. exactly two, at least two, at least three) chambers.

The set of chambers in Δ will be denoted by $Cham\Delta$ or $\mathcal{C}$. If Ω is a subset of Δ we put

$$Cham\Omega = Cham\Delta \cap \Omega$$

and

$$\overline{\Omega} = \{A \in \Delta | A \subseteq B \text{ for some } B \in \Omega\}$$

A *morphism of chamber complexes* is a morphism mapping chambers onto chambers.

A *chamber subcomplex* of a chamber complex is a subcomplex which is a chamber complex such that the inclusion is a morphism of chamber complexes.

A chamber complex is called *strongly connected* if StA is a chamber complex for each $A \in \Delta$. Note that this means that we can find for any two chambers in StA gallery in StA joining them.

Numberings of chamber complexes and chamber systems

In this subsection all complexes are meant to be chamber complexes.

Given a set I, the simplex $P(I)$ is by definition a chamber complex consisting of a single chamber.

A *numbering* of an arbitrary chamber complex is a morphism of chamber complexes $\tau : \Delta \longrightarrow P(I)$ for some set I. Recall that this means that $\tau_C := \tau \mid_{\bar{C}}: \bar{C} \longrightarrow P(I)$ is an isomorphism for each chamber $C \in \Delta$.

The next definition is a generalization of numberings. We do not claim the existence of a global morphism, but only the existence of local numberings.

A *weak numbering* of Δ is a system of numberings $(\tau_C : \bar{C} \longrightarrow P(I))_{C \in \mathcal{C}}$ such that $\tau_C \mid_{\bar{C} \cap \bar{D}} = \tau_D \mid_{\bar{C} \cap \bar{D}}$ for any two adjacent chambers $C, D \in \Delta$.

Observe that a numbering provides also a weak numbering. If Δ is strongly connected, then it is easily seen that a weak numbering of Δ is indeed a numbering of Δ using the uniqueness assertion established below.

Let C, D be adjacent chambers and let $\tau_C : \bar{C} \longrightarrow P(I)$ be a numbering of $\bar{C}$. Then there exists a unique numbering $\tau_D : \bar{D} \longrightarrow P(I)$ such that

$$\tau_C \mid_{\bar{C} \cap \bar{D}} = \tau_D \mid_{\bar{C} \cap \bar{D}} .$$

This means that a weak numbering of Δ is already uniquely determined by τ_C for any chamber $C \in \Delta$.

If $(\tau_C)_{C \in \mathcal{C}}$ is a weak numbering of Δ we may associate to Δ a chamber system $\mathbf{C}(\Delta)$ by the following construction. Two chambers C, D are defined to be i-adjacent if $\tau_C(C \cap D) \cup \{i\} = I$.

From an arbitrary chamber system C we may also construct a strongly connected numbered chamber complex $\mathbf{\Delta}(C)$. This means that we may associate to every weak numbered chamber complex a strongly connected numbered chamber complex which has the same chamber graph. This can be done by passing to the chamber system $\mathbf{C}(\Delta)$ and then by going back to its associated chamber complex $\mathbf{\Delta}(\mathbf{C}(\Delta))$.

Note: The idea to pass from chamber complexes to chamber systems and vice versa is due to J. Tits (cf. [Ti81]). While this procedure is only indicated in loc cit., a detailed description of this topic can be found in [Sc93] .

We close this section with a little lemma concernig the rank 2 case. If Δ is a chamber complex we may consider its incidence graph $\Gamma = (V, E)$ which is defined as follows. The set V is the set of vertices, and two vertices are joined by an edge if their union exists in Δ. If Δ has rank 2 then the vertices (resp. edges) of the incidence graph are the panels (resp. chambers) of Δ.

Lemma 1.3.1. *Let Δ be a chamber complex of rank 2. Then the following are equivalent:*

(i) Δ is numbered.

(ii) Δ is weakly numbered.

(iii) The incidence graph of Δ is bipartite.

1.4 The gate property in chamber complexes

Throughout this section Δ always denotes a chamber complex and $\mathcal{C}$ its set of chambers.

Chamber complexes as metric spaces

In Section 1.3 we introduced a distance function on the set of chambers in Δ. It is clear that $(\mathcal{C}, d)$ is a metric space in the sense of section 1.2. In this subsection we fix some further notation concerning the metric d.

Definition 1.4.1. Let C, D be chambers in Δ. We put

$$\sigma(C,D) := \{E \in Cham\Delta | d(C,E) + d(E,D) = d(C,D)\}$$

and

$$\Sigma(C,D) := \overline{\sigma(C,D)}$$

A set of chambers $\mathcal{X}$ is said to be *convex*, if it is convex as a subset of $\mathcal{C}$ in the metric space $(\mathcal{C}, d)$. A chamber subcomplex Δ^* is said to be *convex*, if $Cham\Delta^*$ is convex.

A simplex $A \in \Delta$ is called *gated* if $ChamStA$ is gated in $(\mathcal{C}, d)$. If A is a gated simplex and $C \in \mathcal{C}$, we denote the gate of C to $ChamStA$ by $proj_A C$. Two gated simplices A, B are called *parallel* if $ChamStA$ and $ChamStB$ are parallel.

Chamber Complexes with gated Panels

Throughout this section we assume that Δ is a chamber complex in which all panels are gated. The following lemmas are immediate:

Lemma 1.4.2. *Let $C, D \in Cham\Delta$ and let $C = C_0, C_1, \ldots, C_k = D$ be a gallery stretched from C to D. Then we have for $1 \leq i \leq k$:*

$$C_{i-1} = proj_{C_{i-1} \cap C_i} C$$

$$C_i = proj_{C_{i-1} \cap C_i} D$$

Lemma 1.4.3. *Let C, D be chambers in Δ such that $proj_A D = C$ for each panel A contained in C. Then $C = D$.*

Lemma 1.4.4. *Let Δ^* be a convex chamber subcomplex. Then each panel of Δ^* is gated.*

Lemma 1.4.5. *Let $C, D \in Cham\Delta$ Then*

- *i* $\sigma(C, D) \cap ChamStA = \{proj_A C, proj_A D\}$ *for each panel A contained in $\Sigma(C, D)$. In particular, $\Sigma(C, D)$ is a meager chamber subcomplex.*
- *ii If the rank of Δ is finite, then $\Sigma(C, D)$ is finite.*
- *iii If $E \in \sigma(C, D)$ and A is a panel of C, then $proj_A E \in \sigma(C, D)$.*
- *iv If $E \in \sigma(C, D)$ then $\sigma(C, E) \subseteq \sigma(C, D)$ and equality holds if and only if $E = D$.*
- *v For each panel A of D put $D_A = proj_A C$. Then we have*

$$\sigma(C, D) = \{D\} \cup \bigcup_{D_A \neq D} \sigma(C, D_A)$$

Lemma 1.4.6. *Let A, A' be two parallel panels in Δ and let $C \neq D$ be two chambers containing A. Put $C' = proj_{A'} D$ and $D' = proj_{A'} C$. Then*

$$\sigma(C, C') = \sigma(D, C') \cup \sigma(C, D') = \sigma(D, D')$$

From now on until the end of this subsection Δ is assumed to have rank two.

Lemma 1.4.7. *Let C be a chamber and let A be a panel in Δ. Then there exists a unique gallery $C = C_0, C_1, \ldots, C_k \supset A$ stretched from C to A.*

Proof. Let $C = C'_0, C'_1, \ldots, C'_k \supset A$ be a gallery stretched from C to A. We apply induction on k. For $k = 0$ the assertion is trivial, so assume $k > 0$. It follows $C_k = proj_A C = C'_k$. Let B be the panel of C_k different from A. We have $C_{k-1} \cap C_k = B = C'_{k-1} \cap C'_k$ and therefore $C = C_0, C_1, \ldots, C_{k-1} \supset B$ and $C = C'_0, C'_1, \ldots, C_{k-1} \supset B$ are both galleries stretched from C to B. Now we may apply induction to see $C_i = C'_i$ for $k = 0, 1, \ldots, k-1$.

Lemma 1.4.8. *Δ is a numbered chamber complex.*

Proof. As an immediate consequence of the previous lemma it follows that the edges of the incidence graph of Δ are gated. Combining Lemma 1.3.1 and Lemma 1.2.2 we see that Δ is numbered.

Lemma 1.4.9. *Let C, D be two chambers of Δ and suppose that $proj_A C \neq D$ for both panels of D. Then Δ is the flag complex of a generalized m-gon for finite m.*

Proof. Since Δ is a numbered chamber complex we can pass to the associated chamber system. For chamber systems of rank 2 the assertion is already known by a result of Scharlau (cf. [Sc85]).

If Δ is the flag complex of a generalized m-gon for finite m we will say that Δ is *of finite type*, in the remaining cases we call Δ *of infinite type*. Observe that if Δ is weak and of infinite type, then it is the flag complex of a building of type $\tilde{A}_1$.

If Δ is of finite type, then we may introduce the notion of *opposition*. Observe that two chambers are opposite if they are as in Lemma 1.4.9, while two panels are opposite if they are parallel and distinct.

1.5 Chamber complexes in which the panels and copanels are gated

In this section we always assume that any $A \in \Delta$ with codimension smaller or equal to 2 is gated.

The following observations are immediate from the previous sections:

Lemma 1.5.1.

(1) If $codim A = 2$ then $St A$ is a gated chamber complex.

(2) Suppose $B \in \Delta$ has codimension at most 2 and let $C \in \Delta$ and $D \in Cham St B$. If $proj_A C = D$ for each panel A of D containing B, then $D = proj_B C$.

If $proj_A C \neq D$ for each panel of D containing B, then $St B$ is of finite type and $proj_B C$ is opposite to D in $St B$.

Weak numberings

We will prove now that Δ is weakly numbered.

Lemma 1.5.2. *Let C be a chamber, let B be a face of codimension 2 of C and let $\tau : \bar{C} \longrightarrow P(I)$ be a numbering. Then there exists a unique weak numbering $(\tau_D)_{D \in Cham St B}$ of $\bigcup_{D \in Cham St B} \bar{D}$.*

Proof. This is an immediate consequence of lemma 1.4.9.

Proposition 1.5.3. *Let $C \in Cham\Delta$ and let τ be a numbering of $\bar{C}$. Then there exists a unique weak numbering $(\tau_D)_{D \in Cham\Delta}$ such that $\tau_C = \tau$.*

Proof. Let Ω_k denote the set of all chambers having distance at most k from C. We prove by induction on k that there exists a weak numbering $(\tau_D)_{D\in\Omega_k}$ of the chamber complex $\bigcup_{D\in\Omega_k} \bar{D}$ with $\tau_C = \tau$.

If $k = 0$ put $\tau_C := \tau$ and the induction starts.

Let $(\tau_D)_{D\in\Omega_k}$ be a weak numbering and let D be a chamber at distance $k+1$ from C. For any chamber X adjacent to D with $d(X, C) = k$ let $\tau_{D,X}$ be the unique numbering of $\bar{D}$ which coincides with τ_X on $\bar{X} \cap \bar{D}$. We have to show that $\tau_{D,E} = \tau_{D,F}$ for any two of those chambers X. If $F \neq E$ then $B := F \cap E$ has codimension 2 and $H := proj_B C$ is opposite to D in StB. Let $(\tau'_X)_{X\in ChamStB}$ be the weak numbering of $\bigcup_{X\in ChamStB} \bar{X}$ extending $\tau'_H := \tau_H$. By the uniqueness of the constructions it follows that $\tau'_E = \tau_E, \tau'_F = \tau_F$ and $\tau_{D,E} = \tau'_D = \tau_{D,F}$.

The gate property of arbitrary simplices

Recall first that we assume throughout this section that all panels and copanels of Δ are gated. By the considerations of Section 1.2 and Proposition 1.5.3 it follows now that the chamber graph of Δ is also the chamber graph of a strongly connected chamber complex Δ'. Moreover Δ' has also the property that the elements of codimension smaller or equal to 2 are gated. If Δ is strongly connected we have the following result.

Proposition 1.5.4. *Let Δ be strongly connected. Then each simplex of Δ is gated.*

Proof. Let B be a simplex in Δ. By $\mathcal{B}$ we will denote the set of chambers in StB.

We begin with two remarks:

(1) On $\mathcal{B}$ we have a distance function $d_B : \mathcal{B} \times \mathcal{B} \longrightarrow N_0$ where $d_B(E, F)$ is the length of a minimal gallery in $\mathcal{B}$ joining E and F. We have $d_B(E, F) \geq d(E, F)$.

(2) If A is a simplex containing B having codimension at most 2, then we have for any two chambers $E, F \in ChamStA$ $d_B(E, F) = d(E, F)$. This follows from the convexity of $ChamStA$ in $Cham\Delta$.

Let $C \in Cham\Delta$ and let D be a chamber in $\mathcal{B}$ such that $proj_A C = D$ for each panel A of D containg B. The existence of such a chamber follows easily.

Let E be a chamber in StB. We prove by induction on $d_B(D, E)$, that

(i) $d_B(D, E) = d(D, E)$

(ii) $d(C, D) + d(D, E) = d(C, E)$

(iii) If $B \subseteq A \subseteq E$ such that $codim A \leq 2$ then

  (a) $proj_A D = proj_A C$

  (b) $d_B(D, proj_A D) + d(proj_A D, E) = d_B(D, E)$

For $d_B(D, E) = 0$ (i) and (ii) are obvious. Assertion (iii) follows from Lemma 1.5.1.

Now suppose that the assertion is proved for all chambers $H \in \mathcal{B}$ with $d_B(D, H) < d_B(D, E)$. Let F be a chamber in $\mathcal{B}$ adjacent to E such that $d_B(D, F)+1 = d_B(D, E)$ We have:

$$d_B(D, E) \leq d_B(D, proj_{E\cap F} D) + 1$$

and

$$d_B(D, E) = d_B(D, F) + 1$$

It follows:

(1) $proj_{E\cap F} D = F$

Now we have:

$$\begin{aligned} d(D,E) &= d(D, proj_{F\cap E} D) + d(proj_{F\cap E} D, E) && \\ &= d(D,F) + d(F,E) && \text{by the previous observation} \\ &= d_B(D,F) + 1 && \text{by induction} \\ &= d_B(D,E) && \end{aligned}$$

This establishes (i).

We have also

$$\begin{aligned} d(C,E) &= d(C, proj_{E\cap F} C) + d(proj_{E\cap F} C, E) && \\ &= d(C, proj_{E\cap F} D) + d(proj_{E\cap F} D, E) && \text{by applying induction to the panel } E\cap F \text{ of } F \\ &= d(C,F) + d(F,E) && \text{by (1)} \\ &= d(C,D) + d(D,F) + 1 && \text{by induction applied to } F \\ &= d(C,D) + d(D,E) && \end{aligned}$$

This establishes (ii). It remains to prove (iii). Observe that we have already proved the following:

(2) If Y is a chamber in $\mathcal{B}$ such that $d_B(D,E) = d_B(D,Y) + 1$ then we have $proj_{E\cap Y}D = Y = proj_{E\cap Y}C$.

The second equality follows by applying induction to the panel $F \cap Y$ of Y.

We will prove now:

(3) If A is a panel of E containing B such that $Y := proj_A D \neq E$ then we have $d_B(D,Y) = d_B(D,E) - 1$.

Let F be as above. The assertion holds for the panel $E \cap F$, so let us assume $A \neq E \cap F$. It follows that $P = A \cap F \cap E$ is a face of codimension 2 of F containing B. Applying induction to F we have $d_B(D, proj_P D) + d(proj_P D, F) = d_B(D,F)$. This yields $d_B(D, proj_P D) \leq d_B(D,F)$ and we may apply induction to $proj_P D$ to see that $d(D, proj_P D) = d_B(D, proj_P D)$. Now we have

$$\begin{aligned} d_B(D,Y) &\leq d_B(D, proj_P D) + d_B(proj_P D, Y) \\ &= d(D, proj_P D) + d(proj_P D, proj_A proj_P D) \\ &= d(D, proj_A D) \\ &= d(D,Y) \end{aligned}$$

and therefore $d(D,Y) = d_B(D,Y)$. On the other hand we have $d_B(D,E) = d(D,E) = d(D, proj_A D) + d(proj_A D, E) = d(D,Y) + 1 = d_B(D,Y) + 1$, and (3) is proved.

Now let A be a panel of E. If $proj_A D \neq E$ then (iii) (a) and (b) follow from (2) and (3).

If $proj_A D = E$ (iii) (b) is trivial and it remains to show (iii) (a):

Let F be as before and consider the simplex $Q = E \cap F \cap A$ and observe that

$$\begin{aligned} proj_A C &= proj_A(proj_Q) \\ &= proj_A(proj_Q D) \quad \text{by induction applied to } F \in ChamStQ \\ &= proj_A D \end{aligned}$$

We have to prove (iii) for all faces of codimension 2 of E. Let O be such a face. We distinguish two cases.

1. There exists a panel A of E containing O such that $Y := proj_A D \neq E$. Then $d_B(D,Y) = d(D,E) - 1$ and $Y \in ChamStO$ and we may apply induction to Y to prove the assertion.

2. For each panel A of E containing O we have $proj_A D = E$. Since we have proved (iii) already for each panel of E containing B we have also $proj_A C = E$ for each panel of E containing B. By Lemma 1.5.1 it follows now that $proj_O D = E = proj_O C$ and we are done.

In view of Proposition 1.5.4 we have the following:

Corollary 1.5.5. *Let Δ be a chamber complex in which the panels and copanels are gated. Then the following are equivalent:*

(1) Δ is strongly connected.

(2) Each simplex of Δ is gated.

The Corollary above justifies the following definition.

Definition 1.5.6. A *gated chamber complex* is a strongly connected chamber complex, in which all panels and copanels are gated.

1.6 Some special classes of gated chamber complexes

Throughout this section we will always assume that Δ is a gated chamber complex.

Gated Chamber Complexes of locally finite type

Definition 1.6.1. A gated chamber complex is said to be of locally finite type, if StA is a generalized n-gon for finite n for each copanel $A \in \Delta$.

For a chamber $C \in \Delta$ and $k \in N$ we put

$$E_k(C) := \{D \in Cham\Delta | codim C \cap D \leq k\}$$

Lemma 1.6.2. *Let Δ be the flag complex of a generalized n-gon, let C be a chamber in Δ and let X be a gated subset of $Cham\Delta$ containing $E_1(C)$. Then $X = Cham\Delta$.*

Proof. Easy

Let $C \in Cham\Delta$,let $k \in N$ and let A be a face of C. We put

$$E_{k,A}(C) = \{D \in E_k(C) | A \subseteq C \cap D\}$$

Lemma 1.6.3. *Let $X \subseteq Cham\Delta$ be a gated set and let C be a chamber in X having a face A such that $E_{1,A}(C) \subseteq X$. Then $E_{2,A}(C) \subseteq X$.*

Proof. This is an easy consequence of the previous lemma.

Lemma 1.6.4. *Let X be a gated set in $Cham\Delta$. Then $X = ChamStA$ for a simplex $A \in \Delta$.*

Proof. Let $C \in X$ and put $A = \bigcap_{D \in E_1(C) \cap X} D$. First observe that $E_{1,A}(C) \subseteq X$. By the previous lemma it follows that $E_{2,A}(C) \subseteq X$ and therefore $E_{1,A}(D) \subseteq X$ for each chamber $D \in E_{1,A}(C)$. By induction it follows that $ChamStA \subseteq X$. If there would be a chamber E in X not containing A, a similar argument leads to a contradiction.

Here is an application.

Proposition 1.6.5. *Let Δ, Δ' be two gated chamber complexes of locally finite type and let $\varphi : \Delta \longrightarrow \Delta'$ be an adjacence preserving bijection. Then φ extends to an isomorphism of chamber complexes.*

Thin gated chamber complexes

Throughout this section we always assume that Σ is a thin gated chamber complex.

For each pair (C, A) consisting of a chamber $C \in \Sigma$ and a panel A contained in C we define

$$\phi_{A,C} := \{X \in Cham\Sigma | proj_A X = C\}.$$

and

$$\Phi_{A,C} := \overline{\phi}_{A,C}$$

If A, C are as above we call $\phi_{A,C}$ (resp. $\Phi_{A,C}$) *a root* (resp. *a full root*) of Σ. If A, C are as above and D is the unique chamber contained in StA and different from C, then $\phi_{A,D}$ is called *the root opposite to* $\phi_{A,C}$. The *boundary* of $\phi_{A,C}$ is $\Phi_{A,C} \cap \Phi_{A,D}$.

Here are some properties, which are easy generalizations of the theory of Coxeter complexes to thin gated chamber complexes.

Lemma 1.6.6.

1. *$\phi_{A,C} = \phi_{B,D}$ if and only if A and B are parallel and $proj_A D = C$. In particular, parallel panels determine the same pair of opposite roots and parallelism is an equivalence relation on the set of panels.*

2. *Roots are convex and each convex set of chambers is the intersection of the roots which contain this set.*

Lemma 1.6.7. *Let ϕ^+, ϕ^- be opposite roots in Σ and let $C \in \phi^+$. Then there exists at most one chamber $D \in \phi^-$ adjacent to C. If there exists such a chamber D, then $\phi^+ = \phi_{A,C}$ and $\phi^- = \phi_{A,D}$ where $A = C \cap D$.*

Proof. Suppose $D_1 \neq D_2$ are adjacent to C and contained in ϕ^-. We have $d(D_1, D_2) \leq 2$.

If $d(D_1, D_2) = 1$ then $C \notin StD_1 \cap D_2$ since Σ is thin. But $d(C, D_1) = d(C, D_1) = 1$, which contradicts the gate property of $D_1 \cap D_2$.

If $d(D_1, D_2) = 2$ then $C \in \sigma(D_1, D_2)$ contradicting the convexity of ϕ^-. The remaining assertions are now obvious.

Definition 1.6.8. We say that C is at the boundary of a root ϕ, if $C \in \phi$ and if there exists a chamber D in the root opposite to ϕ which is adjacent to C.

Definition 1.6.9. Let A be a panel. A reflection at the panel A is an involutory automorphism of Σ interchanging the two chambers containing A and being the identity on $\bar{A}$.

The following result is due to M. Dyer ([Dy90]) and V. Deodhar ([De89]) for Coxeter complexes. Its generalization to thin gated chamber complexes is immediate if one applies the technique used by J.Y. Hee in order to give another proof of this result (cf. [He90]).

Proposition 1.6.10. *Let T_1 be a set of reflections of a thin gated chamber complex Σ and put $W = \langle T_1 \rangle \leq Aut(\Sigma)$. Let T be the set of reflections of Σ contained in W. Then there exists a set $S \subseteq T$ such that*

(1) $\langle S \rangle = W$.

(2) (W, S) *is a Coxeter system.*

(3) T *is the set of conjugates of* S *in* W.

In particular, the type M of the Coxeter system in (2) is uniquely determined in view of (3).

Totally gated chamber complexes

So far, we dealt only with projections of chambers onto a simplex. In buildings we can also define the projection of a simplex onto another simplex in a convenient way. The situation in buildings is as follows:

If A, B are simplices in the building, then the set $\{proj_A C | C \in ChamStB\}$ coincides with $ChamStA_1$ for a simplex A_1 containing A, where A_1 is the projection of B onto A.

We are going to define totally gated chamber complexes, in which this observation remains true.

Definition 1.6.11. Let Δ be a gated chamber complex. Then Δ is called totally gated if for any two simplices $A, B \in \Delta$ there exists a simplex A_1 such that $\{proj_A C | C \in ChamStB\} = ChamStA_1$.

Note that the simplex A_1 need not to be unique in general. It is unique if Δ is weak and of finite dimension. However, we can define in a totally gated chamber complex projections of simplices.

Definition 1.6.12. Let Δ be a totally gated chamber complex. For $A, B \in \Delta$ we put

$$proj_A B = \bigcap_{C \in ChamStB} proj_A C.$$

Remark: A gated chamber complex is not totally gated in general. It is easy to construct counterexamples. But I do not know whether each weak chamber complex is totally gated. Note that buildings are totally gated, as well as chamber complexes of locally finite type, as we shall see in the following proposition.

Proposition 1.6.13. *A gated chamber complex of locally finite type is totally gated.*

Proof. Let A, B be two simplices in Δ. Since $ChamStA$ and $ChamStB$ are gated sets in the metric space $(\mathcal{C}, d)$, the set $\{proj_A C | C \in ChamStB\}$ is gated as well by Proposition 1.2.3.

Now we can apply Lemma 1.6 to see that there exists a simplex A_1 with $\{proj_A C | C \in ChamStB\} = ChamStA_1$.

We close this subsection with some easy observations concerning totally gated chamber complexes.

Lemma 1.6.14. *Let A, B, B' be simplices in Δ. If $B \subseteq B'$ then $proj_A B \subseteq proj_A B'$.*

Lemma 1.6.15. *Let A, B be simplices and put $A_1 = proj_A B$ and $B_1 = proj_B A$. Then A_1 and B_1 are parallel, and we have $proj_A B' = proj_{A_1} \circ proj_{B_1} B'$ for each simplex B' containing B.*

Proof. This follows from Proposition 1.2.3.

1.7 Spherical gated chamber complexes

Throughout this section Δ is assumed to be a gated chamber complex.

Definition 1.7.1. Δ is called spherical if there exist two chambers $C, C' \in \Delta$ such that $proj_{A'}C \neq C'$ for each panel A' of C'.

The existence of thin subcomplexes

Let C, C' be as in Definition 1.7.1. Put

$$\sigma = \sigma(C, C') = \{E \in Cham\Delta | d(C, E) + d(E, C') = d(C, C')\}$$

and $\Sigma = \overline{\sigma}$.

Lemma 1.7.2. *Let E be a chamber in σ. If A is a panel or a copanel of E, then $proj_A C$ is opposite to $proj_A C'$ in StA.*

Proof. We proceed by induction on $d(E, C')$. If $d(C', E) = 0$ we have $E = C'$ and the assertion follows from Lemma 1.5.1.

Now let $E \in \sigma$ and suppose that the assertion is proved for all $F \in \sigma$ with $d(F, C') < d(E, C')$. Let $E' \in \sigma$ be adjacent to E such that $d(C', E') = d(C', E) - 1$. Put $A = E \cap E'$. We have $proj_A C = E$ and $proj_A C = E'$. Now let B be a panel of E distinct from A, hence $A \cap B$ is a copanel contained in E and E'. By induction applied to E' we have that $proj_{A\cap B} C$ is opposite to $proj_{A\cap B} C'$ in $St(A \cap B)$. This implies that the assertion is valid for each panel of E and also for each copanel of E, which is contained in a chamber $F \in \sigma$ such that $d(F, C') = d(E, C') - 1$.

Let now B be a copanel of E such that B is not contained in a chamber $F \in \sigma$ with $d(F, C') = d(E, C') - 1$ and let A_1, A_2 be the panels of E containing B. We conclude, that $proj_B C' = proj_{A_1} C' = proj_{A_2} C' = E$ and therefore $proj_{A_1} C \neq E \neq proj_{A_2}$, since we have already proved the assertion for all panels of E. By Lemma 1.5.1 it follows now that $proj_B C$ is opposite to $E = proj_B C'$ and we are done.

Lemma 1.7.3. *Let C, C' and σ be as above. Then:*

(1) $proj_A C' \neq C$ for each panel A of C.

(2) If B is a panel or a copanel contained in a chamber of σ then $proj_B C$ and $proj_B C'$ are also contained in σ.

(3) Σ is thin.

Proof. The first assertion follows from the previous lemma; the second assertion follows directly from the definition of the set σ. Since $proj_A C \neq proj_A C'$ for each panel in Σ, it follows that each panel is contained in at least two chambers. On the other hand Σ is meager by Lemma 1.4.5 i. and we are done.

Lemma 1.7.4. *Let C, C' and σ be as above and let $D \in \sigma$ be adjacent to C. Then there exists a panel A' of C' such that $proj_{A'} D = C'$*

Proof. Suppose the contrary. Then $proj_{A'} D \neq C'$ for each panel of C'. By (1) of the previous lemma this implies that $proj_B C' \neq D$ for each panel of D. Since $D = proj_{C \cap D} C'$, this leads to a contradiction.

Let C, C' and σ be as above and let $D \in \sigma$ be adjacent to C. Put $A = C \cap D$. Let A' be a panel of C' such that $proj_{A'} D = C'$ and put $D' = proj_{A'} C$. Note that $D' \in \sigma$.

Lemma 1.7.5. *We have:*

1. $\sigma(C, C') = \sigma(D, D')$
2. *The panel A' of C' is unique.*
3. $proj_B D' \neq D$ *for each panel B contained in D.*

Proof. We have $proj_{A'} C = D$ and $proj_{A'} D = C'$ and the first assertion follows by Lemma 1.4.6.

Now let A'' be a panel of C' such that $proj_{A''} D = C'$ and put $D'' = proj_A C'$. By 1. we have $\sigma(D, D') = \sigma(C, C') = \sigma(D, D'')$ and hence $D' = D''$. This implies $A' = A''$.

By 1. it follows that $\Sigma(D, D') = \Sigma(C, C')$, which means that $\Sigma(D, D')$ is a thin chamber subcomplex of Δ. This establishes 3..

An easy induction on $d(C, E)$ establishes the following

Lemma 1.7.6. *For each chamber $E \in \sigma$ there exists a unique chamber $E' \in \sigma$ such that $proj_B E' \neq E$ for each panel B contained in E. Moreover, $\sigma(E, E') = \sigma$ and $d(E, E') = d(C, C')$.*

The following proposition is a summary of our previous considerations.

Proposition 1.7.7. *Σ is a thin convex chamber subcomplex of Δ; Σ is strongly connected, locally finite and gated. In particular, Σ is totally gated. If X is a chamber in Σ and if A is a face of codimension smaller or equal to two, then $proj_A X$ means the same in Σ and in Δ. We have an involutory automorphism $opp_\Sigma : \Sigma \longrightarrow \Sigma$ assigning to each chamber its opposite.*

Proof. We have already seen that Σ is thin in 1.7.3. We will show the convexity of Σ. Let E, F be chambers in Σ and let E' be the chamber opposite to E in Σ (cf. Lemma 1.7.6). It follows that $F \in \sigma(E, E')$ and hence $\sigma(E, F) \subseteq \sigma(E, E') = \sigma$, which means that σ is a convex set of chambers.

Our next aim is to prove that Σ is strongly connected. So let A be a simplex in Σ and E, F two chambers in Σ containing A. Since $ChamStA$ is convex in $\mathcal{C}$ it follows that $\sigma(E, F) \subseteq ChamStA$. Since $\sigma(E, F)$ is also contained in σ we are done.

The fact that Σ is of locally finite type follows from Lemma 1.7.2. That Σ is gated follows from its convexity. Since Σ is weak and of locally finite type, it follows that Σ is totally gated by Proposition 1.6.13.

The last assertion follows from the fact that opp_Σ is adjacence preserving on the set of chambers in Σ.

Finite thin gated chamber complexes

Throughout this section Σ always denotes a finite thin gated chamber complex. We put $\sigma = Cham\Sigma$.

Lemma 1.7.8. *Let $C \in \sigma$. Then there exists a unique chamber $C' \in \sigma$ such that $proj_A C' \neq C$ for each panel A contained in C. In particular, Σ is spherical and $\Sigma = \Sigma(C, C')$, and we have an opposition involution as in the previous subsection.*

Proof. Let C' be a chamber at maximal distance from C. Suppose that there exists a panel A' contained in C' such that $proj_{A'} C = C'$. Let D' be the unique chamber in σ such that $A' = D' \cap C'$. Then we have $d(C, D') = d(C, C') + 1$, contradicting our the choice of C'. From Lemma 1.7.3 it follows now that $proj_A C' \neq C$ for each panel A contained in C, and also that $\Sigma(C, C')$ is a thin chamber subcomplex of Σ. It follows that $\Sigma = \Sigma(C, C')$.

Lemma 1.7.9. *Let ϕ^+, ϕ^- be opposite roots in Σ. Then*

1. *If $C \in \phi^+$ then $opp_\Sigma(C) \in \phi^-$.*
2. *The restriction of opp_Σ to ϕ^+ is an adjacence preserving bijection from ϕ^+ onto ϕ^-.*
3. *If C is at the boundary of ϕ^+ and D is the unique chamber in ϕ^- adjacent to C, Then $\phi^+ = \sigma(C, opp_\Sigma(D))$. If D' is a chamber in ϕ^+ such that $\phi^+ = \sigma(C, D')$, then $D' = opp_\Sigma(D)$. This means that $opp_\Sigma(D)$ is the unique chamber at maximal distance from C in ϕ^+.*

Proof. Suppose $opp_\Sigma(C) \in \phi^+$. Then $\sigma = \sigma(C, opp_\Sigma(C)) \subseteq \phi^+$, since ϕ^+ is convex. This is a contradiction and 1. is proved.

Since $opp_\Sigma^2 = id$ it follows that opp_Σ maps ϕ^+ bijectively onto ϕ^-. This mapping is adjacence preserving, because opp_Σ is an automorphism of Σ.

Now let C be at the boundary of ϕ^+ and let D be as in assertion 3.. Then $D_1 = opp_\Sigma(D)$ is contained in ϕ^+ and $C_1 = opp_\Sigma(C)$ is contained in ϕ^- and C_1 and D_1 are adjacent. Put $A = D \cap C$ and $A' = D_1 \cap C_1$. It follows that $proj_{A'}C = D_1$ and $proj_{A'}D = C_1$. By Lemma 1.4.6 we have now

$$\phi^+ \cup \phi^- = \sigma = \sigma(C, C_1) = \sigma(C, D_1) \cup \sigma(D, C_1)$$

.

If D' is a chamber at maximal distance from C in ϕ, it follows that $\sigma = \sigma(C, D_1) \supseteq \sigma(C, D')$. The assertion follows now from Lemma 1.2.1.

Definition 1.7.10. Let ϕ be a root and let C be a chamber at the boundary of ϕ. Then $opp_\phi(D)$ denotes the unique chamber at maximal distance from C in ϕ.

Lemma 1.7.11. *Let C be a chamber in Σ and let A be a panel contained in C. Define $\phi = \phi_{A,C}$.*

a) For a chamber $C' \in \Sigma$ the following are equivalent:

1. $C' = proj_{opp_\Sigma(A)}C$
2. $C' = opp_\phi(C)$
3. $\phi = \sigma(C, C')$

b) If $C' \in \sigma$ satisfies one of the equivalent conditions of a), then $proj_B C' \neq C$ for each panel B of C distinct from A.

Proof. Put $A' = opp_\Sigma(A), C_1 = proj_{A'}C$. Let D be the chamber with $C \cap D = A$, let $\phi^- = \phi_{A,D}$ be the root opposite to ϕ and let $D_1 = proj_{A'}D$.

Observe first that $D_1 = opp_\Sigma(C)$. Since C_1 is adjacent to D_1, it follows that the only chamber, which is at greater distance than C_1 from C is D_1. As D_1 is certainly not in ϕ, it follows that C_1 is at maximal distance from C in ϕ. We have shown that 1. implies 3. by Lemma 1.7.9.

Since A and A' are parallel, it follows that $\sigma(C, C_1) \subseteq \phi$ and $\sigma(D, D_1) \subseteq \phi^-$. Now we use the following identity:

$$\sigma = \phi \dot\cup \phi^- = \sigma(C, C_1) \dot\cup \sigma(D, D_1)$$

The last equality follows from Lemma 1.4.6. This means that in the inequalities above equality holds and hence that 3. implies 1.. The remaining equivalences follow by definition. This completes the proof of a).

Let D be as in the proof of a). Let B be a panel distinct from A. It follows that B is not contained in D and hence $d(D, proj_B D) \geq 1$. This shows that $C = proj_B D$.

On the other hand C' is opposite to D, and therefore $proj_B C' \neq proj_B D = C$.

Proposition 1.7.12. *Let Σ, Σ' be two thin finite gated chamber complexes and let ϕ^+, ϕ^- (resp. ${\phi^+}', {\phi^-}'$) be pairs of opposite roots in Σ (resp. Σ'). Let $\gamma : \phi^+ \longrightarrow {\phi^+}'$ be an adjacence preserving bijection. Then γ extends to an adjacence preserving bijection $\Gamma : \sigma \longrightarrow \sigma'$.*

Proof. Let C be at the boundary of ϕ^+, then $\gamma(C)$ is at the boundary of ${\phi^+}'$ and $\gamma(opp_{\phi^+}(C)) = opp_{{\phi^+}'}(\gamma(C))$. Define now the mapping $\Gamma : \sigma \longrightarrow \sigma'$ by

$$\Gamma(C) = \begin{cases} \gamma(C) & \text{if } C \in \phi^+ \\ opp_{\Sigma'}(\gamma(opp_\Sigma(C))) & \text{if } C \in \phi^- \end{cases}$$

It is clear that Γ is a bijection preserving adjacency on ϕ^+ and ϕ^-. If $C \in \phi^+$ and $D \in \phi^-$ are adjacent, then $\Gamma(D) = opp_{\Sigma'}(\gamma(opp_\Sigma(D))) = opp_{\Sigma'}(\gamma(opp_{\phi^+}(C))) = opp_{\Sigma'}(opp_{{\phi^+}'}(\gamma(C)))$. This proves that $\Gamma(D)$ is adjacent to $\gamma(C) = \Gamma(C)$ and we are done.

Oriented matroids

Finite Coxeter complexes are finite thin gated chamber complexes. The converse is far from being true. Counterexamples are provided by simplicial arrangements of hyperplanes - or more generally - by simplicial arrangements of pseudo-spheres. Combinatorially, those are described by oriented matroids (see [FL78]). In this subsection we indicate how a finite thin gated chamber complex provides an oriented matroid in a natural way. Thus, the gate property may be used to characterize at least the simplicial oriented matroids; however it seems that perhaps more can be done by a definition of thin finite polyhedral chamber complexes in order to characterize the geometric oriented matroids.

We first recall the covector axioms for an oriented matroid. We show how to associate a set of covectors to a finite thin gated chamber complex satisfying those axioms. We do not give detailed proofs; the lemmas stated here are meant to indicate the main steps.

Covector axioms of an oriented matroid: We adopt the notation and definitions of [BLSWZ93].

Let E be a finite set and consider sign vectors $X, Y \in \{-, 0, +\}^E$. The *opposite* of a vector X is the vector $-X$. The *zero vector* is denoted by 0. The *composition* of two vectors X and Y, denoted by $X \circ Y$, is defined as follows: $(X \circ Y)_e = X_e$ if $X_e \neq 0$ and $(X \circ Y)_e = Y_e$ otherwise. The *separation set* of X and Y is $S(X, Y) = \{e \in E | X_e Y_e = -\}$.

Lemma 1.7.13. *A set $\mathcal{L} \subseteq \{-, 0, +\}^E$ is the set of covectors of an oriented matroid if and only if it satisfies:*

(L0) $0 \in \mathcal{L}$.

(L1) $X \in \mathcal{L}$ *implies* $-X \in \mathcal{L}$.

(L2) $X, Y \in \mathcal{L}$ *implies* $X \circ Y \in \mathcal{L}$.

(L3) If $X, Y \in \mathcal{L}$ *and* $e \in S(X, Y)$ *then there exists* $Z \in \mathcal{L}$ *such that* $Z_e = 0$ *and* $Z_f = (X \circ Y)_f = (Y \circ X)_f$ *for all* $f \notin S(X, Y)$.

Parallelity in finite thin gated chamber complexes: For our purposes we need the following lemma.

Lemma 1.7.14. *Let Σ be a finite thin gated chamber complex and let $A, B \in \Sigma$ be two parallel simplices. Then there exists a sequence $A = A_0, A_1, \ldots, A_k = B$ such that*

(i) $A_{i-1} = proj_{A_{i-1} \cap A_i} A$ *for all* $1 \leq i \leq k$.

(ii) $A_i = proj_{A_{i-1} \cap A_i} B$ *for all* $1 \leq i \leq k$.

(iii) A_{i-1} *is opposite to* A_i *in* $St A_{i-1} \cap A_i$.

If $A = A_0, A_1, \ldots, A_k = B$ is such a sequence and $C \in ChamStA$ then there exists a minimal gallery $C = C_0, C_1, \ldots, C_l = D = proj_B C$ such that for each $0 \leq i \leq k$ there exists a $0 \leq j \leq l$ such that $C_j \in ChamStA_i$.

The lemma above is a corollary of the following proposition.

Proposition 1.7.15. *Let Σ be a finite thin gated chamber complex and let $\tilde{C}$ be a simplex in Σ. Put $\tilde{\Sigma} = \{A \in \Sigma | proj_{\tilde{C}} A = \tilde{C}\}$. Then $\tilde{\Sigma}$ is a finite thin gated chamber complex with the induced partial ordering. Its chambers are precisely the simplices of Σ which are parallel to $\tilde{C}$.*

The proof of the above proposition uses arguments, which are quite similar to those used in the proof of the fixed point theorem in the following section. The condition that Σ is thin is essential, whereas the finiteness assumption may be weakened.

The set of covectors associated to a finite thin gated chamber complex: Let Σ be a finite thin gated chamber complex and let Φ denote the set of roots in Σ. For $\phi \in \Phi$ we denote the root opposite to ϕ by ϕ^-.

Fix a chamber C and put $E = \{\phi \in \Phi | C \in \phi\}$. We define a mapping $c : \Sigma \longrightarrow \{-, 0, +\}^E$. For $A \in \Sigma$ we put $(c(A))_\phi = -$ if $A \in \phi^-$ and $A \notin \phi$; $(c(A))_\phi = 0$ if $A \in \phi^- \cap \phi$; $(c(A))_\phi = +$ if $A \notin \phi^-$ and $A \in \phi$.

Observe that if D is a chamber then $c(D) \in \{-, +\}^E$; if A is an arbitrary simplex then $(c(A))_\phi = 0$ if and only if there exist two chambers F, G in StA with $(c(F))_\phi \neq (c(G))_\phi$.

Now we have the following:

Lemma 1.7.16. *Let $A, B \in \Sigma$, then*

(1) $d(A, B) = |S(c(A), c(B))|$.

(2) $S(c(A), c(B)) = S(c(proj_A B), c(proj_B A))$.

(3) $c(proj_A B) = c(A) \circ c(B)$.

Lemma 1.7.17. *Let A, B be parallel simplices and let $A = A_0, A_1, \dots A_k = B$ be a sequence as in Lemma 1.7.14.*

Then $S(c(A), c(B)) = \bigcap_{i=1,\dots,k} S(c(A_{i-1}), c(A_i))$ and $(c(A))_\phi = (c(A_{i-1} \cap A_i))_\phi$ for all $\phi \in E \setminus S(c(A), c(B))$.

Proposition 1.7.18. *Let $A, B \in \Sigma$ and let $\phi \in S(c(A), c(B))$. Then there exists a simplex Z such that $(c(Z))_\phi = 0$ and $(c(Z))_{\phi_1} = (c(proj_A B))_{\phi_1} = (c(proj_B A))_{\phi_1}$ for all $\phi_1 \in E \setminus S(c(A), c(B))$.*

Proof. We put $\bar{A} = proj_A B$ and $\bar{B} = proj_B A$. We have $S(c(A), c(B)) = S(c(\bar{A}), c(\bar{B}))$ by Lemma 1.7.16. Since $\bar{A}$ and $\bar{B}$ are parallel, we may choose a sequence $\bar{A} = \bar{A}_0, \dots, \bar{A}_k = \bar{B}$ as in the previous lemma. It follows that $\phi \in S(c(\bar{A}_{i-1}), c(\bar{A}_i))$ for some $1 \leq i \leq k$ and we may put $Z = \bar{A}_{i-1} \cap \bar{A}_i$. It readily follows that $(c(Z))_\phi = 0$ and by the previous lemma we are done.

The previous proposition provides the validity of axiom (L3). (L2) is provided by Lemma 1.7.16 (3). The covector of the empty simplex is the zero vector and for any simplex A the covector of its opposite in Σ is the opposite of $c(A)$. This yields (L0) and (L1).

Construction of an apartment system

In this section Δ always denotes a gated chamber complex.

Definition 1.7.19. Two chambers C, C' are called Δ-opposite if $proj_A C' \neq C$ for each panel A of C.

Note that the Δ-opposition relation is symmetric by Lemma 1.7.3.

Definition 1.7.20. An apartment is the full convex hull of two Δ-opposite chambers.

We give a brief summary of the results of the previous subsections in the following proposition.

Proposition 1.7.21. *Let Σ be an apartment and let σ denote the set of chambers in Σ. We have:*

1. *Σ is a finite thin chamber subcomplex of Δ.*
2. *σ is convex in the metric space $(\mathcal{C}, d)$.*
3. *Σ is gated. In particular, Σ is strongly connected.*
4. *If $A \in \Sigma$ and $C \in \sigma$ then $proj_A C$ means the same in Σ and in Δ.*
5. *If $C, D \in \sigma$, then $\Sigma(C, D) \subseteq \Sigma$.*
6. *Given $E, E' \in \sigma$, the following are equivalent:*

   a $E' = opp_\Sigma(E)$

   b *E' and E are Δ-opposite.*

   c $\Sigma = \Sigma(E, E')$

Lemma 1.7.22. *Let Σ be an apartment and let $C \in Cham\Sigma$. Let D be a chamber adjacent to C and put $A = C \cap D$. Let Φ be the root of Σ containing C and such that $A \in \delta\Phi$. Then there exists an apartment Σ_1 which contains D and Φ. Moreover there exists an isomorphism $\Theta : \Sigma \longrightarrow \Sigma_1$ which fixes Φ elementwise.*

Proof. Let A' be the panel opposite to A in Σ and put $C' = proj_{A'} C$. By Lemma 1.7.9 we have $C' = opp_\phi(C)$ where $\phi = \phi_{A,C}$ in Σ. Note that $\Phi = \overline{\phi}$. Put $D' = proj_{A'} D$.

We have $proj_B C' \neq C$ for each panel B of C distinct from A. Hence

$$proj_B D' = proj_B^{A'} D' = proj_B^{A'} \circ proj_{A'}^{B} \circ proj_B^{A'} D' = proj_B^{A'} C' \neq C$$

for each panel B of C distinct from A. On the other hand, we have $proj_A D' = D \neq C$ and therefore C and D' are Δ-opposite. Put $\Sigma_1 = \Sigma(C, D')$. Σ_1 is by definition an apartment. Since A' and C are contained in Σ_1, the chamber $C' = proj_{A'} C$ is contained in Σ_1 and therefore $\Phi = \Sigma(C, C') \subseteq \Sigma_1$. Since C' is adjacent to $D' = opp_{\Sigma_1}(C)$ it follows that Φ is also a root in Σ_1 hence $\Phi = \Phi_{A,C}$ in Σ_1.

Now let θ be the identity on ϕ and apply Proposition 1.7.12 to extend it to an adjcence preserving bijection $\Theta : Cham\Sigma \longrightarrow Cham\Sigma_1$. Since apartments are of locally finite type, we have by Proposition 1.6.5 that Θ is in fact an isomorphism of chamber complexes. The last assertion is obvious.

Lemma 1.7.23. *Let C be a chamber and let Σ be an apartment containing C. Let $E \in Cham\Sigma$ and let D be a chamber adjacent to E such that $d(E, C) + 1 = d(D, C)$. Then there exists an apartment Σ' containing D, E and C and an isomorphism $\Theta : \Sigma \longrightarrow \Sigma'$ leaving invariant C and all its faces.*

Proof. Put $A = E \cap D$. By the previous lemma, there exists an apartment Σ' containing $\Phi = (proj_A)^{-1} E$ and an isomorphism $\Theta : \Sigma \longrightarrow \Sigma'$ fixing Φ elementwise. Since $C \in proj_A^{-1} E \cap \Sigma$, the assertion follows

Lemma 1.7.24. *Let C be a chamber, let Σ be an apartment containing C, and let $D \in Cham\Delta$. Then there exists an apartment Σ' containing C and D and an isomorphism $\varphi : \Sigma \longrightarrow \Sigma'$ which fixes C and all its faces.*

Proof. Let $C = E_0, E_1, \ldots, E_k = D$ be a gallery stretched from C to D. By the previous lemma we have inductively a sequence of apartments $\Sigma = \Sigma_0, \Sigma_1, \ldots \Sigma_k = \Sigma'$ and a set of isomorphisms $\Theta_i : \Sigma_{i-1} \longrightarrow \Sigma_i$, where $1 \leq i \leq k$, satisfying the following:

a Σ_i contains C and E_i

b Θ_i leaves C and all its faces invariant.

Now put $\varphi = \Theta_k \circ \Theta_{k-1} \circ \ldots \circ \Theta_1$ and the assertion follows.

Proposition 1.7.25. *Let C, C' be two Δ-opposite chambers. Then:*

1. *Any two chambers $E, F \in \mathcal{C}$ are contained in a common apartment.*

2. *If Σ, Σ' are apartments containing a chamber D, then there exists an isomorphism $\varphi : \Sigma \longrightarrow \Sigma'$ fixing $\Sigma \cap \Sigma'$ elementwise.*

Proof. Since C, C' are Δ-opposite, there exists an apartment containing C. By Lemma 1.7.24, there exists an apartment Σ containing C and E. Now we may apply Lemma 1.7.24 again to the pair E, Σ to conclude that there exists an apartment containing E and F.

Now let Σ, Σ' be two apartments containing a chamber D. Let D' be the unique chamber opposite to D in Σ'. Then Σ' is the unique apartment containing D and D'. So we may apply Lemma 1.7.24 to the pair D, Σ. It follows that there exists an apartment Σ'' containing D and D' and an isomorphism $\varphi : \Sigma \longrightarrow \Sigma''$ fixing D and all its faces. By the previous consideration we have $\Sigma' = \Sigma''$. Thus we have proved the existence of an isomorphism $\varphi : \Sigma \longrightarrow \Sigma'$ fixing D and all its faces.

Now let $A \in \Sigma \cap \Sigma'$, then $H = proj_A D$ is a chamber contained in $\Sigma \cap \Sigma'$ and also each minimal gallery $D = D_0, D_1, \ldots, D_k = H$ is in $\Sigma \cap \Sigma'$, since Σ and Σ' are convex.

Since φ fixes D_0 and all its faces, it fixes D_1 and all its faces, and so on. Now A is a face of D_k and we are done.

The following theorem is a summary of our previous results:

Theorem 1.7.26. *Let Δ be a gated chamber complex and suppose that there exists a pair of Δ-opposite chambers. Define the apartments of Δ as in 1.7.20.*
Then we have:

1. *Δ is of locally finite type and hence totally gated.*

2. *The set of chambers contained in an apartment is convex in $(\mathcal{C}, d)$.*

3. *The apartments are thin finite gated chamber complexes. Moreover, if $A, B \in \Sigma$ then $proj_A B$ is same in Δ and in Σ.*

4. *Any two simplices are contained in a common apartment*

5. *If Σ, Σ' are apartments containing a common chamber, then there exists an isomorphism $\varphi : \Sigma \longrightarrow \Sigma'$ fixing $\Sigma \cap \Sigma'$ elementwise.*

We have now the following consequence:

Theorem 1.7.27. *A thick spherical gated chamber complex is a building.*

Proof. This follows from Theorem 1.7.26 and Theorem 3.7. of [Ti74].

On the structure of weak spherical gated chamber complexes

In this subsection we describe a procedure which associates to each weak spherical chamber complex a thick spherical gated chamber complex, i.e. a building. This procedure is already described in [Sc87] for weak spherical buildings. Therefore we do not give detailed proofs. Our purpose here is to translate the main steps to the slightly more general situation considered here.

Throughout this subsection Δ always denotes a weak spherical gated chamber complex.

Definition 1.7.28. A panel in Δ is called thick (resp. thin), if it is contained in at least three (resp. exactly two) chambers.

A gallery $C = C_0, C_1, \ldots, C_k = D$ is called thin, if $C_{i-1} \cap C_i$ is thin for each $1 \leq i \leq k$.

For each chamber $C \in \Delta$ we define its thin-class C^* to be the set of chambers in Δ which are joined to C by a thin gallery.

Two thin-classes C^*, D^* are called adjacent if they are distinct and if they have adjacent representatives. We will write $C^* \sim D^*$.

Lemma 1.7.29. *Let A be a thick panel and let Σ be an apartment containing A.*

1. *Σ admits a reflection r_A at the panel A.*
2. *The reflection r_A normalizes the set of thick panels in Σ.*

The proof of the first assertion can be done in the same way as the proof of Theorem 3.7 in [Ti74]. The second statement is a natural generalization of Lemma 2 in [Sc87]. Its proof is similar as the proof given there.

Lemma 1.7.30. *Let Σ be an apartment, let C be a chamber in Σ, and let X be the set of thick panels A in Σ with the property that $proj_A C \in C^*$. Put $R^* = \{r_A | A \in X\}$ and $W^* = \langle R^* \rangle$. Then (W^*, R^*) is a Coxeter system of a certain type M^*. Moreover, if t is a reflection in W^* then $t = r_B$ for a thick panel $B \in \Sigma$ and vice versa.*

The idea of the proof is to use Lemma 1.7.29 and the techniques of the proof of Proposition 1.6.10.

Proposition 1.7.31. *Let $\mathcal{C}^*$ be the set of thin classes in Δ. Then $(\mathcal{C}^*, \sim)$ is the chamber graph of a thick building Δ^*. The type of Δ^* is M^*, where M^* is obtained as in the previous lemma.*

The above proposition may be seen as the natural generalization of Theorem 1.7.27.

Opposite simplices in spherical gated chamber complexes

In this subsection we intend to extend the definition of opposition to arbitrary simplices.

Definition 1.7.32. Let Σ be a thin spherical gated chamber complex. A pair (A, A') of simplices is called opposite in Σ, if and only if $A' = opp_\Sigma(A)$.

It readily follows that each simplex in a thin gated chamber complex has a unique opposite and that the opposition relation is symmetric.

Lemma 1.7.33. *Let Σ, Σ' be thin spherical chamber complexes and let $\alpha : \Sigma \longrightarrow \Sigma'$ be an isomorphism. Then we have $\alpha \circ opp_\Sigma = opp_{\Sigma'} \circ \alpha$.*

In view of the previous lemma we have now:

Lemma 1.7.34. *Let Δ be a spherical gated chamber complex and A, A' be two simplices in Δ. If A, A' are opposite in at least one apartment containing them, then they are opposite in each apartment containing both of them.*

This justifies the following definition.

Definition 1.7.35. Two simplices in a spherical gated chamber complex are called opposite, if they are opposite in each apartment containing both.

The next lemma is immediate.

Lemma 1.7.36.

(1) The opposition relation is invariant under isomorphisms of gated spherical chamber complexes.

(2) Two simplices A, A' in a spherical gated chamber complex Δ are opposite if and only if for each chamber $C \in StA$ there exists a chamber $C' \in StA'$ which is opposite to C and vice versa.

(3) Opposite simplices are parallel.

1.8 Automorphisms of gated Chamber Complexes

Throughout this section Δ always denotes a totally gated chamber complex.

Projections and local opposition in totally gated chamber complexes

Definition 1.8.1. A simplex A is called spherical if StA is spherical.

Lemma 1.8.2. *Let C, D be chambers and let $B \subseteq C$. Then $proj_B D = C$ if and only if $Proj_A D = C$ for each panel of C containing B.*

Lemma 1.8.3. *Let C, D be chambers and let A, B be simplices contained in C. Suppose that $proj_A D = C = proj_B D$. Then $C = proj_{A \cap B} D$.*

Lemma 1.8.4. *Let C, D be chambers in Δ and let $B \subseteq C$ be a spherical simplex. Then $proj_B D$ is opposite to C in StB if and only if $proj_A D \neq C$ for each panel A of C containing B.*

Lemma 1.8.5. *Let C, D be chambers and let A, B be spherical faces of C. Suppose that $proj_A D$ (resp. $proj_B D$) is opposite to C in StA (resp. in StB). Then $A \cap B$ is spherical and $proj_{A \cap B} D$ is opposite to C in $StA \cap B$.*

Lemma 1.8.6. *Let A, B be simplices in Δ and let B_1, B_2 be faces of B. If $proj_{B_1} A = B = proj_{B_2} A$, then $proj_{B_1 \cap B_2} A = B$.*

Proof. Let C be a chamber in StA. Then $proj_{B_1} C$ contains B and $proj_{B_2}$ contains B and therefore $proj_{B_1} C = proj_B C = proj_{B_2} C$. This shows that $B \subseteq proj_{B_1 \cap B_2} C$ for each chamber C in StA, and hence $B \subseteq proj_{B_1 \cap B_2} A$.

Lemma 1.8.7. *Let A, B be simplices in Δ and let B_1, B_2 be spherical faces of B. Suppose that $B_2' := proj_{B_1} A$ (resp. $B_2' := proj_{B_2} A$) is opposite to B in StB_1 (resp. in StB_2). Then $B_1 \cap B_2$ is spherical and $proj_{B_1 \cap B_2} A$ is opposite to B in $St(B_1 \cap B_2)$*

Proof. Observe first that B is spherical. Let C be a chamber in StA. We have $C' := proj_B \circ proj_{B_1} C = proj_B \circ proj_{B_2} C$. Put $C_1' = proj_{B_1} C$ and $C_2' = proj_{B_2} C$ and choose a chamber $D \in StB$ which is opposite to C' in StB. Then D is opposite to C_1' in StB_1 and D is opposite to C_2' in StB_2, hence by Lemma 1.8.5 it follows that $B_1 \cap B_2$ is spherical and that $proj_{B_1 \cap B_2} C$ is opposite to D in $St(B_1 \cap B_2)$.

We have proved that $B_1 \cap B_2$ is spherical and that for each chamber C in StA there exists a chamber D in StB, such that $proj_{B_1 \cap B_2} C$ is opposite to D in $St(B_1 \cap B_2)$.

Observe now that $proj_B^A = proj_B^{B_1'} \circ proj_{B_1}^A = proj_B^{B_2'} \circ proj_{B_2}^A$. Since $proj_B^{B_1'}$ and $proj_B^{B_2'}$ are isomorphisms and $proj_{B_1}^A$ and $proj_{B_2'}^A$ are surjective morphisms, it follows that $proj_B^A$ is surjective. Let now $F \in ChamStB$.Let $E' \in ChamStB$ be opposite to F in StB and let $E \in ChamStA$ be such that $proj_B E = E'$. Put $E_1' = proj_{B_1'} E'$

and $E'_2 = proj_{B'_2}E'$. It follows that $E'_1 = proj_{B'_1}E$ is opposite to F in StB_1 and $E'_2 = proj_{B_2}E$ is opposite to F in StB_2. Hence $proj_{B_1 \cap B_2}E$ is opposite to D in $StB_1 \cap B_2$. This establishes the assertion in view of Lemma 1.7.36 (2).

Lemma 1.8.8. *Let V, W be two distinct parallel vertices in Δ which are spherical. Then Δ is spherical and V and W are opposite in Δ.*

Proof. Use Lemma 1.7.36 (2).

Lemma 1.8.9. *Let A, B be parallel and let B_1 be a face of B. Then $proj_{B_1}A$ is parallel to A and B.*

Automorphisms of totally gated chamber complexes

Let $\Gamma \leq Aut(\Delta)$ and put $\tilde{\Delta} = \{A \in \Delta | A^g = A \text{ for all } g \in \Gamma\}$.

Clearly, $(\tilde{\Delta}, \subseteq|_{\tilde{\Delta}})$ is a simplicial complex. Let Δ and Γ be as above. A Γ-chamber is any element A of $\tilde{\Delta}$ which is not contained properly in another simplex fixed by Γ - in other words - a maximal element in $\tilde{\Delta}$. A Γ-vertex is by definition a minimal element in $\tilde{\Delta}$. For $A \in \tilde{\Delta}$ the Γ-rank (denoted by $rk_\Gamma(A)$) is the cardinality of the set of Γ-vertices in A. The following observations are immediate:

Lemma 1.8.10. *Let A, B be simplices in Δ and let $\gamma \in \Gamma$. Then*

$$(proj_A B)^\gamma = proj_{A^\gamma} B^\gamma$$

Lemma 1.8.11. *If $A, B \in \tilde{\Delta}$ then $proj_A B \in \tilde{\Delta}$. In particular, Γ-chambers are parallel in Δ.*

Spherical automorphisms

Definition 1.8.12. Let Δ be as in the previous section and let $\Gamma \leq Aut(\Delta)$. Then Γ is called spherical, if there exists a spherical Γ-chamber.

From now on Γ is always assumed to be spherical.

Lemma 1.8.13. *Let $A \in \tilde{\Delta}$ be at the same time a Γ-vertex and a Γ-chamber. Then:*

1. *Any Γ-chamber is a Γ-vertex.*
2. *If there are at least two Γ-chambers, then Δ is spherical and any two distinct Γ-chambers are opposite.*

Proof. Let A be as in the lemma. Note first, that Γ acts transitively on the vertices of A and hence also on the faces of A having codimension 1 in A. If there is no Γ-chamber distinct from A, we are done.

So let B be a Γ-chamber distinct from A and let A_1 be a face of codimension 1 in A. Suppose first that $proj_{A_1}B = A$. In this case we have for each $\gamma \in \Gamma$ that $proj_{A_1^\gamma}B^\gamma = A^\gamma$ and hence $proj_{A_1^\gamma}B = A$. Since Γ is transitive on the faces of A having codimension 1 in A, it follows that $\bigcup_{\gamma\in\Gamma} A_1^\gamma = \emptyset$ and hence $B = A$ by Lemma 1.8.6. This is a contradiction, and we have $proj_{A_1}B \neq A$. By Lemma 1.8.9 $proj_{A_1}B$ is parallel to A. Note that $proj_{A_1}B$ and A are parallel and distinct spherical vertices in StA_1. By Lemma 1.8.8 it follows that A_1 is a spherical simplex and that A and $proj_{A_1}B$ are opposite in StA_1. Using the same argument as above and Lemma 1.8.7 it follows that Δ is spherical and A and B are opposite in Δ.

Let now $B_1 \in \tilde{\Delta}$ with $\emptyset \neq B_1 \subset B$. Then $proj_{B_1}A$ is a Γ-chamber, which is not opposite to A in Δ. This contradicts our previous considerations and therefore B is also a Γ-vertex.

Lemma 1.8.14. *Let A, B be two distinct Γ-chambers. Then there exist a face A_1 of A such that $A_1 \in \tilde{\Delta}$, $\Gamma - rk(A_1) = \Gamma - rk(A) - 1$ and $proj_{A_1}B \neq A$.*

Proof. Obvious

Lemma 1.8.15. *$\tilde{\Delta}$ is a strongly connected chamber complex.*

Proof. If $A \in \tilde{\Delta}$ and B is a Γ-chamber, then $proj_A B$ is a Γ-chamber, thus any element of $\tilde{\Delta}$ is contained in a Γ-chamber. The fact that any two Γ-chambers A, B are connected by a gallery follows by an easy induction on $d(A, B)$, using the previous lemma. This shows in particular that any two Γ-chambers have the same Γ-rank. If $A \in \tilde{\Delta}$ with $\Gamma - rk(A) \leq rk(\tilde{\Delta}) - 2$ we may apply the above arguments to StA to prove the strong connectedness.

We can now define Γ-panels, Γ-copanels, Γ-faces of elements in $\tilde{\Delta}$ and Γ-galleries. The Γ-distance will be denoted by $\tilde{d}$.

Lemma 1.8.16. *Let A, B be two Γ-panels. Then either A and B are parallel, or $proj_A B$ and $proj_B A$ are Γ-chambers.*

Definition 1.8.17. A Γ-gallery $C = C_0, C_1, \ldots, C_k = D$ is called admissible if $proj_{C_i \cap C_{i+1}} C = C_i$ for $0 \leq i \leq k-1$.

Lemma 1.8.18. *Let $C = C_0, C_1, \ldots, C_k = D$ be an admissible Γ-gallery and let $0 \leq i < k$. Then $proj_{C_i \cap C_{i+1}} C_j = C_i$ for $0 \leq j \leq i$ and $proj_{C_i \cap C_{i+1}} C_l = C_{i+1}$ for $i < l \leq k$.*

Proof. We use induction on k. If $k = 0$ the assertion is trivial. Suppose that $k > 0$ and observe that the gallery $C_0, \ldots, C_{k-1}$ is admissible. By induction we have $proj_{C_0 \cap C_1} C_{k-1} = C_1$. Since $proj_{C_{k-1} \cap C_k} C_0 = C_{k-1}$ it follows that $C_{k-1} \cap C_k$ and $C_0 \cap C_1$ are not parallel and $proj_{C_0 \cap C_1} C_{k-1} \cap C_k = C_1$ and $proj_{C_{k-1} \cap C_k} C_0 \cap C_1 = C_{k-1}$ by the previous lemma and hence $proj_{C_0 \cap C_1} C_k = C_1$ and $proj_{C_{k-1} \cap C_k} C_1 = C_{k-1}$. Now we may apply successively the induction hypothesis to see that the Γ-galleries $C_{k-1}, C_{k-2}, \ldots, C_0$, $C_1, C_2, \ldots, C_k$ and $C_k, C_{k-1}, \ldots, C_1$ are admissible. Now the assertion follows again by induction.

Lemma 1.8.19. *Suppose that $\tilde{\Delta}$ has rank 2 and let C, D be two Γ-chambers.*

1. *If C and D are not opposite in Δ, then there exists a unique admissible gallery joining them.*
2. *i If C and D are opposite in Δ, then there exist exactly two admissible galleries joining them.*

   ii Moreover, these galleries have the same length.

Proof. The assertions 1 and 2i are obvious, and it remains to prove 2ii. Let $C = C_0, C_1, \ldots, C_k = D$ and $C = C'_0, C'_1, \ldots, C'_l = D$ be the two admissible galleries joining C and D. We have $proj_{C_{k-1} \cap C_k} C = C_{k-1}$ and $proj_{C'_0 \cap C'_1} D = C'_1$, which shows that $C_{k-1} \cap C_k$ and $C'_0 \cap C'_1$ are opposite, and also that $C'_1, C'_2, \ldots, C'_l, C_{k-1}$ and $C_{k-1}, C_{k-2}, \ldots, C_0, C'_1$ are admissible galleries. By 1. it follows that C'_1 and C_{k-1} are opposite. Applying this observation inductively, it follows that C_{k-i} is opposite to C'_i for $0 \leq i \leq k$, and hence C is opposite to C'_k, which implies $C'_k = D = C'_l$.

Lemma 1.8.20. *Let C, D be Γ-chambers. For each Γ-face of D having Γ-codimension at most 2 we have*

$$proj_A C \in \tilde{\Delta}$$

and

$$\tilde{d}(C, D) = \tilde{d}(C, proj_A C) + \tilde{d}(proj_A C, D)$$

Proof. The proof uses induction on $\tilde{d}(C, D)$. If $\tilde{d}(C, D) = 0$, the assertion is obvious.

So let us assume that the assertion is true for all Γ-chambers X with $\tilde{d}(C, X) \leq \tilde{d}(C, D)$.

Let E be a Γ-chamber which is Γ-adjacent to D, and such that $\tilde{d}(C,E) = \tilde{d}(C,D)-1$. Put $A = E \cap D$. It follows by induction that $proj_A C = E$ and therefore our assertion is proved for the Γ-panel A and for each Γ-copanel contained in A.

Let now B be a Γ-panel contained in D different from A such that $proj_B C \neq D$. Then $proj_{A\cap B} C$ is opposite to D in $St A \cap B$, and the previous lemma yields $\tilde{d}(proj_B C, proj_{A\cap B} C) = \tilde{d}(proj_A C, proj_{A\cap B} C)$ and hence $\tilde{d}(proj_B C, C) = \tilde{d}(C,D) - 1$.

This shows that our assertion holds for all Γ-panels of D with $proj_B C \neq D$ and by induction also for all Γ-copanels of D which are contained in at least one of those Γ-panels.

Now let B be a face of Γ-rank smaller or equal to 2 with $proj_B C = D$. Suppose that there exists a Γ-chamber X in $St B$ with $\tilde{d}(C,X) < \tilde{d}(C,D)$. Applying the induction hypothesis to X, we have a contradiction. This proves our assertion for all Γ-panels B of D with $proj_B C = D$.

Now let B_1 be a Γ-copanel of D and let B', B'' be the two Γ-panels of D containing B_1. If $proj_{B'} C \neq D$ or $proj_{B''} C \neq D$, then the assertion is proved by our previous observations. If $proj_{B'} C = D = proj_{B''} C$ we have $proj_{B_1} C = D$ and the assertion follows.

Lemma 1.8.21. *$\tilde{\Delta}$ is totally gated.*

Proof. Let $A, B \in \tilde{\Delta}$, then $A_1 = proj_A B$ and $B_1 = proj_B A$ are also in $\tilde{\Delta}$.

Let C be a Γ-chamber in $St B$, then $C' = proj_{B_1} C$ is a Γ-chamber in $St B_1$ with $proj_A C = proj_A C'$.

On the other hand, B_1 and A_1 are parallel. We conclude that the set of Γ-chambers containing A_1 coincides with the set $\{proj_A C | C$ is a Γ-chamber in $St B\}$. This completes the proof.

A fixed point theorem for totally gated chamber complexes

The following theorem is the synthesis of the results in this section.

Theorem 1.8.22. *Let Δ be a totally gated chamber complex and let $\Gamma \leq Aut(\Delta)$ be spherical. Let $\tilde{\Delta}$ be the set of simplices fixed by Γ. Then the following holds:*

1. *$\tilde{\Delta}$ is a strongly connected chamber complex.*
2. *The panels and copanels of $\tilde{\Delta}$ are gated.*

3. *$\tilde{\Delta}$ is a gated chamber complex.*

4. *If C is a Γ-chamber and $A \in \tilde{\Delta}$ is an arbitrary simplex, then $proj_A C$ means the same in $\tilde{\Delta}$ and in Δ.*

5. *$\tilde{\Delta}$ is a totally gated chamber complex.*

Chapter 2

Thin Buildings

2.1 Introduction

The main goal of this chapter is to introduce admissible partitions of Coxeter diagrams. The most important result is a rank two criterion for the admissibility of a partition. We apply this result to give another proof of the classification of the spherical Coxeter diagrams.

Preliminaries: In this section we fix our notation and recall some basic facts about Coxeter groups and Coxeter systems. Most of the definitions are quite standard. A general reference for the results listed here is [Bo68].

Automorphisms of Coxeter complexes: In this section we state Proposition 2.3.1, which is a corollary to the fixed point theorem for gated chamber complexes. As an immediate consequence we show how to find Coxeter complexes which are 'embedded' in a given chamber complex. The content of this section is perhaps well known to the experts; however, it is hard to find it in the literature as it is presented here.

As mentioned in the chapter on gated chamber complexes, I do not know whether the condition (S) in Theorem 1.8.22 is necessary in the case of arbitrary gated chamber complexes; for Coxeter complexes it seems that this condition can be dropped. However, the more general statement will not be needed in this thesis, and its proof would involve more technical details. For instance, in [De82] V. V. Deodhar proves a result (see Proposition 5.5. in loc cit.), which is closely related to our Proposition 2.3.1. There, he decomposes the Coxeter diagram in question into its irreducible components.

We would like to make some comments on the particular statements of Proposition 2.3.1.

ad 2. Consider the geometric representation of the corresponding Coxeter group W in a vector space V over $\mathbf{R}$. The group of automorphisms Γ acts naturally on

V and normalizes the set of the reflection hyperplanes of W. Let U denote the subspace of V consisting of the fixed points of Γ. Then the reflection hyperplanes of W in V induce a simplicial decomposition of the unit sphere in U. Thus we obtain a simplicial arrangement of hyperplanes whose associated chamber complex is gated.

The arrangements occuring in this way are listed in [OT92].

ad 5. The algorithm to determine the combinatorial structure is more or less the same as the one described in [Ri82] and [De82]. This underlines the relations to [De82] already mentioned above.

ad 6. This statement implies immediatly a result due to P. Abramenko (cf [Ab91], Section 2 Corollary). It should not be too difficult to improve his result by applying 6. of our Proposition.

ad 7. Our Lemma 2.3.4, which is a corollary of this statement, is certainly known [Ti] since a long time. But I cannot locate it in the literature.

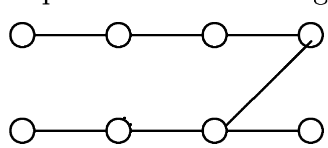
Coxeter groups in Coxeter groups: In Section 2.3 we show how to find Coxeter complexes which are embedded in a given Coxeter complex. In a first step of this section we give a group theoretical interpretation of Lemma 2.3.3. This motivates the definition of admissible partitions.

An interpretation of Proposition 2.3.1, of which Lemma 2.3.3 is a special case, yields the definition of generalized admissible partitions. This is described in Section 2.7. Though the content of Section 2.3 remains valid in this more general context, we restrict ourselves to prove it only for a special case. The main reason for this is that the proofs get more involved and technical, but they do not provide a deeper insight.

Admissible partitions are partitions of the index set of a Coxeter diagram which describe Coxeter subgroups in a given Coxeter group. The Coxeter subgroups obtained in this way behave like those which are fixed by an automorphism group of the diagram. It turns out that the length function associated to such a subgroup is 'compatible' with the length function of the original group. This is a very restrictive condition.

Our main result on admissible partitions is a rank two criterion for the admissibility of an arbitrary partition (see Theorem 2.4.9).

The original motivation to define admissible partitions was the well known example of the embedding of H_4 in E_8.



This embedding had been considered in the literature by several authors (cf [Co81], [Mo87], [Sh88], [MP]). To my knowledge it was first observed by P. McMullen (cf. [Mo87], [Co93]); it seems that he didn't publish his observation. Though the embedding of H_4 in E_8 ocurred in different contexts, no attempt was made to develop a general approach which explains this example.

Admissible partitions describe the Coxeter subgroups of Lemma 2.4.1 and the example described above in a unified and elementary way and generalize this phenomenons to some extent. 'Elementary' means that we need only the solution of the word problem in Coxeter groups (cf. [Ti68]) to establish our results.

Our description of the embeddings of Coxeter groups related to admissible partitons is purely combinatorial. It should be possible to describe them also as projections onto a subspace in the geometric representation of the Coxeter group in question. In the theory of regular polytopes our procedure is a certain kind of a 'mixing operation'.

Admissible 2-partitions: Our rank two criterion for the admissibility reduces the classification of all admissible partitions to the classification of the admissible 2-partitions (i.e. admissible partitions consisting of two subsets of I). The main goal of this section is to provide a complete classification for the diagrams A_n, B_n and D_n. In order to do this we prove some results which will be used also in the appended section about the applications in the theory of Coxeter groups.

Applications: In the first subsection we prove that a Coxeter system satisfying a rather weak finiteness condition can be embedded in a Coxeter system having a simply laced diagram (i.e. $m_{ij} \in \{2, 3\}$). This result is known for Dynkin diagrams [Mo91], whereas it seems to be new in the general case.

In the second subsection we give an example which shows that the theorem becomes false, if the finiteness condition is dropped. Another example shows that this condition is not necessary.

The second application is an alternative proof for the classification of the spherical diagrams. The classification is originally due to H. S. M. Coxeter ([Co35]). The idea of his proof is to show that a diagram is spherical if and only if the associated bilinear form is positive definite. In [Ti68] J. Tits proposes a proof based on the solution of the word problem. The proof given here provides a convenient and short realization of this idea.

The classification of the irreducible spherical diagrams is divided into two parts: show that the diagams listed in Table A are spherical, and show that those in Table B are non-spherical. The considerably harder part is the second one. In [Ti68] J. Tits shows how to use the solution of the word problem to treat some of the diagrams in table B. The arguments are very easy in these cases. Though all cases can be done in this spirit ([Ti]), a complete proof has never been published. The main reason for this is perhaps that the arguments become very involved ([Ti68]).

We have already mentioned that the only ingredient from the theory of Coxeter groups we need to develop our results about admissible partitions is the solution of the word problem. Thus, the proof for the classification may be seen as a systematic version of the procedure proposed in [Ti68]. The main observation is that two Coxeter diagrams which are related via admissible partitions are either both spherical or both non-spherical. This enables us, for instance, to deduce from the non-sphericity of $\tilde{A}_3$ the non-sphericity of $\tilde{E}_7$ (see Lemma 2.6.6). Another observation is that we have only to consider the simply laced diagrams of Table B in view of our first application. However, the case $\tilde{E}_8$ has to be treated seperately, because it is not possible to obtain an admissible partition of $\tilde{E}_8$, but only a generalized admissible partition (see Section 2.7).

We have two further remarks on the classification of the spherical diagrams:
1. In [Ti68] the geometric representation is used to establish the solution of the word problem in Coxeter groups. Therefore, a proof of the classification of the spherical diagrams which does not use the geometric representation is not available. It is however possible to deduce the solution of the word problem using methods from combinatorial group theory ([Ti]).
2. In a sense, the proof of the classification given here is not the best possible. For instance Lemma 2.3.5 may be seen as an exceptional ingredient. Using generalized admissible partitions and twisting operations one can give a 'nicer' proof. More precisely, once we know that $\tilde{A}_2$ is non-spherical the proof can be done by manipulating Coxeter diagrams in the way as it is done here.

Final Remarks: Admissible partitions generalize Lemma 2.3.3, which is a special case of Proposition 2.3.1; one obtains Coxeter complexes in Coxeter complexes without using automorphism groups. For our purposes it was convenient to interpret this lemma group theoretically. Thus our statements and proofs concerning this phenomenon are in terms of Coxeter systems.

It is not difficult to extend the theory of admissible partitions in order to do a similar thing for the whole situation in Proposition 2.3.1 7.. We only give a definition of such generalized admissible partitions. Our main result - namely the rank 2 criterion - remains valid in this more general situation.

As already mentioned above, admissible partitions provide Coxeter subgroups whose length function is 'compatible' with the length function of the Coxeter group in question. A similar observation holds for the Coxeter subgroups obtained by V. V. Deodhar and M. Dyer. We indicate how to see both constructions as a special cases of a certain class of subgroups, which we call pseudo-reflection groups.

2.2 Preliminaries

In this section we fix some notation and recall some results concerning Coxeter groups and Coxeter complexes.

Coxeter diagrams: Let I be a set. A *Coxeter diagram over* I is a mapping $M : I \times I \longrightarrow N \cup \{\infty\}$ such that

1. $M(i,j) = M(j,i) \geq 2$ for $i \neq j$
2. $M(i,i) = 1$ for all $i \in I$

For $M(i,j)$ we will write m_{ij}. For a subset J of I we put $M_J = M \mid_{I \times I}$.

Systems of involutions: Let I be a set. A *system of involutions over* I is a pair (W,S) consisting of a group W and a set S of involutions in W satisfying:

1. $\langle S \rangle = W$
2. There is a bijective mapping $i \longrightarrow s_i$ from I onto S.

The *type of a system of involutions* (W,S) is the Coxeter diagram over I defined by $m_{ij} = |\langle s_i s_j \rangle|$.

Let (W,S) be a system of involutions over I and let F denote the free monoid on I. For $f = i_1 i_2 \ldots i_d \in F$ we put $L(f) = d$ and $s_f = s_{i_1} s_{i_2} \ldots s_{i_d} \in W$.

Let $w \in W$. A *representation of* w *in* (W,S) is a word $f \in F$ with the property that $s_f = w$. The *length* of w in the system (W,S) is the minimum of all $L(f)$ where f runs through all representations of w. We denote the length of w by $l(w)$. A representation $f \in F$ of $w \in W$ is called *reduced* if we have $L(w) = l(w)$.

Now we will fix some notation concerning a system of involutions of a given type: Let I be a set. Let (W,S) be a system of involutions over I and let M be its type. Let F denote the free monoid on I.

If $m_{ij} \neq \infty$, then $p(i,j) \in F$ denotes the word $\ldots ijij$ of length m_{ij}.

Let $J \subseteq I$. We put $S_J = \{s_j | j \in J\}$ and $W_J = \langle S_J \rangle$. It is clear that (W_J, S_J) is a system of involutions of type M_J.

Coxeter systems

Let I be a set and let M be a Coxeter diagram over I. A *Coxeter system* of type M is a system of involutions of type M which satisfies for each $w \in W$ the following axioms:

A For $i \in I$ we have $l(ws_i) = l(w) + 1$ or $l(ws_i) = l(w) - 1$.

B If $i,j \in I$ and $l(ws_i) = l(w) - 1 = l(ws_j)$, then $m_{ij} \neq \infty$ and there is a reduced representation of W ending with $p(i,j)$.

Though this characterization of Coxeter systems is probably well known, we will give a sketch of a proof:
Proposition 4 in [Bo68] chap. 4 no 1.5 provides property A. That property B holds in Coxeter systems may be seen from [Ro89] Theorem 2.16 and from the observation that the elements of maximal length in finite rank 2 Coxeter groups have the reduced representations $p(i,j)$ and $p(j,i)$.
For the other direction one uses induction on $l(w)$ to see that two reduced representations of $w \in W$ can be transformed from into each other by elementary M-operations of type (II). Moreover, one shows by induction on $L(f)$ that a non-reduced word $f \in F$ can be shortened by elementary M-operations. This shows that (W,S) is a Coxeter system. (See [Br89] for the definitions and further details.)

Let I be a set and let M be a Coxeter diagram over I. Up to isomorphism there exists exactly one Coxeter system of type M (see for instance [Br89]).

In the remainder of this subsection we recall some basic definitions and results concerning Coxeter systems. So let from now on until the end of this subsection I be a set, M be a Coxeter diagram over I and (W,S) be the Coxeter system of type M.

Definition 2.2.1.

1. Let $w \in W$. We put $I^+(w) = \{i \in I | l(ws_i) = l(w)+1\}$ and $I^-(w) = \{i \in I | l(ws_i) = l(w)-1\}$.
2. Let $J \subseteq I$ and let $w_0 \in W$. We put $R_J(w) = wW_J$. The set $R_J(w)$ is called the J-residue of w.

Lemma 2.2.2. *Let $w \in W_J$ and let $i_1 i_2 \ldots i_d \in F$ be a reduced representation of w. Then $i_k \in J$ for all $1 \leq k \leq d$. The system of involutions (W_J, S_J) is the Coxeter system of type M_J.*

The first assertion follows from [Ro89] Lemma 2.10, the second follows from Corollary 2.14 in [Ro89].

Lemma 2.2.3. *Let $J \subseteq I$ and $w_0 \in W$, then there exists a unique element $w^\star$ in $R_J(w_0)$ with the property that $l(w^\star) \leq l(w)$ for all $w \in R_J(w_0)$. For each $w_1 \in R_J(w_0)$ we have $l(w_1) = l(w^\star) + l(w^{\star -1} w_1)$.*

For a proof see [Ro89] Theorem 2.9.
Remark: For given $w_0 \in W$ and $J \subseteq I$ we denote this element by $P_J(w_0)$. It is the 'projection' of the unit element onto the J-residue of w_0.

Lemma 2.2.4. *Let $J \subseteq I$ and let $w_0 \in W$. Then the following are equivalent:*

(P1) $P_J(w_0) = w_0$

(P2) $J \subseteq I^+(w_0)$

(P3) $l(w_0 w_1) = l(w_0) + l(w_1)$ *for all* $w_1 \in W_J$.

(P4) $l(w_2) = l(w_0) + l(w_0^{-1} w_2)$ *for all* $w_2 \in R_J(w_0)$.

Proof. $(P1) \Rightarrow (P2)$: $P_J(w_0) = w_0$ implies $l(w_0 s_j) \geq l(w_0)$ for each $j \in J$. From axiom A it is seen that $l(w_0 s_j) = l(w_0) + 1$ for each $j \in J$, hence $J \subseteq I^+(w_0)$.

$(P2) \Rightarrow (P1)$: Let $J \subseteq I^+(w_0)$ and put $w_1 = P_J(w_0)$. From lemma 2.2.3 it follows that $l(w_0) = l(w_1) + l(w_1^{-1} w_0)$ and $l(w_0) + 1 = l(w_0 s_j) = l(w_1) + l(w_1^{-1} w_0 s_j)$ for each $j \in J$. We obtain $l(w_1^{-1} w_0 s_j) = l(w_1^{-1} w_0) + 1$ for each $j \in J$, hence $J \subseteq I^+(w_1^{-1} w_0)$. Since $w_1^{-1} w_0$ lies in W_J it follows that $w_1^{-1} w_0$ is the identity.

$(P1) \Rightarrow (P4)$: This is the second assertion of lemma 2.2.3.

$(P4) \Rightarrow (P1)$: P4 implies $l(w_2) \geq l(w_0)$ for each $w_2 \in R_J(w_0)$, hence $P_J(w_0) = w_0$.

$(P4) \Leftrightarrow (P3)$: This is obvious.

Definition 2.2.5. Let $J \subseteq I$ and let $w \in W$. We say that J is admissible at w if $J \subseteq I^+(w)$ or $J \subseteq I^-(w)$.

Let $\tilde{I}$ be a partition of I and let $w \in W$. The partition $\tilde{I}$ is said to be admissible at w if α is admissible at w for each $\alpha \in \tilde{I}$.

Lemma 2.2.6. *Let* $J_1 \subseteq J \subseteq I$ *and* $w \in W$. *Put* $w' = P_J(w)$ *and* $w'' = w'^{-1} w$. *Then we have:*

$$J_1 \subseteq I^+(w) \Leftrightarrow J_1 \subseteq I^+(w'')$$

and

$$J_1 \subseteq I^-(w) \Leftrightarrow J_1 \subseteq I^-(w'')$$

In particular: J_1 *is admissible at* w *if and only if it is admissible at* w''.

Lemma 2.2.7.

Let $J \subseteq I$. *Then the following are equivalent:*

(S1) W_J *is finite.*

(S2) There exists a unique element $w^\star$ *in* W_J *such that* $l(w^\star) \geq l(w)$ *for all* $w \in W_J$.

Remark: If $J \subseteq I$ satisfies the equivalent conditions of Lemma 2.2.7 then the element $w^\star$ of condition (S2) is an involution. It will be denoted by r_J. If I satisfies the conditions of Lemma 2.2.7, then the diagram M is called *spherical*.

Lemma 2.2.8.
Let $w_0 \in W$ and let $J \subseteq I$. Then the following are equivalent:

(PS1) $J \subseteq I^-(w_0)$

(PS2) W_J *is finite and* $l(w_0) = l(w_0 r_J) + l(r_J)$.

(PS3) W_J *is finite and* $P_J(w_0) = w_0 r_J$.

Proof. $(PS1) \Rightarrow (PS2)$: This follows from [Ro89] Theorem 2.16.

$(PS2) \Rightarrow (PS3)$: Let $w_1 \in R_J(w_0)$ then $|l(w_0) - l(w_1)| \leq l(r_J)$. This shows that $l(w_1) \geq l(w_0) - l(r_J) = l(w_0 r_J)$, hence $w_0 r_J = P_J(w_0)$.

$(PS3) \Rightarrow (PS1)$: Since $J \subseteq I^-(r_J)$ and $P_J(w_0) = w_0 r_J$, it follows from Lemma 2.2.6 that $J \subseteq I^-(w_0 r_J^2) = I^-(w_0)$.

Coxeter complexes

Let M be a Coxeter diagram over a set I. One can associate a chamber complex to M. This can be done in different ways, by using the geometric representation or by considering the parabolic subgroups of the associated Coxeter system (see [Ti61] or [Ti74]). We will call it the *Coxeter complex* of type M.
Let Σ be the Coxeter complex of type M. Then we have:

1. Σ is a thin totally gated chamber complex.

2. We have a natural numbering $\tau : \Sigma \longrightarrow P(I)$.

3. W is the full group of type preserving automorphisms of Σ and if $J \subseteq I$, then the conjugates of W_J in W are the stabilizers of the simplices having cotype J.

4. The stabilizer of a simplex A in W fixes no nontrivial simplex contained in StA.

5. Σ is spherical if and only if M is spherical and the opposition involution opp_Σ induces an involutory permutation opp_M on I.

Recall that a homogeneous gated chamber complex is a chamber complex defined over a Coxeter diagram. The following result is due to R. Scharlau.

Lemma 2.2.9. *The homogenous thin gated chamber complexes are precisely the Coxeter complexes.*

2.3 Automorphisms of Coxeter complexes

Throughout this section let M be a Coxeter diagram over a set I and let Σ be the Coxeter complex of type M. Let $\Gamma \leq Aut(\Sigma)$ and let $\tilde{\Sigma}$ be the set of simplices fixed by Γ endowed with the induced order.

Since Coxeter complexes are thin gated chamber complexes we can apply our fixed point theorem in this special situation. This yields the following proposition.

Proposition 2.3.1. *Suppose, that the following condition (S) is satisfied:*

(S) There exists a simplex $\tilde{A} \in \tilde{\Sigma}$ such that $St\tilde{A}$ is spherical.

Then we have:

1. *$\tilde{\Sigma}$ is a meager gated chamber complex.*

2. *If Σ is spherical, then $\tilde{\Sigma}$ is a thin spherical gated chamber complex.*

3. *If $\tilde{C}$ is a Γ-chamber and $\tilde{A}$ is a Γ-panel contained in $\tilde{C}$, define $\alpha = \tau(\tilde{C}) \setminus \tau(\tilde{A})$. Then $\tilde{A}$ is contained in a Γ-chamber $\tilde{D}$ distinct from $\tilde{C}$ if and only if $St\tilde{A}$ is spherical. In this case $\tilde{D}$ is opposite to $\tilde{C}$ in $St\tilde{A}$.*

4. *$\tilde{\Sigma}$ is uniquely determined by a Γ-chamber $\tilde{C}$ and all Γ-vertices contained in $\tilde{C}$.*

5. *Let $\tilde{C}$ be a Γ-chamber and let $\tilde{V}$ be the set of all Γ-vertices contained in $\tilde{C}$. Put $J = \tau(\tilde{C})$ and $\tilde{J} = \{\tau(\tilde{v}) | \tilde{v} \in \tilde{V}\}$. Then the combinatorial structure of $\tilde{\Sigma}$ is already determined by M, J and $\tilde{J}$.*

6. *Suppose I is finite. Knowing the sets J and $\tilde{J}$ for one Γ-chamber (cf. 5.) it is possible to decide in a finite number of steps whether $\tilde{\Sigma}$ is thin. If this is the case, we can also decide whether it is a Coxeter complex.*

7. *Let $\tilde{C}$ be a Γ-chamber, let $J, \tilde{J}$ be as in 5. and put $\bar{J} = I \setminus J$. Suppose that for each $\alpha \in \tilde{J}$ the following holds:*

   a $M_{\bar{J} \cup \alpha}$ is spherical.

   b $opp_{M_{\bar{J} \cup \alpha}}(\alpha) = \alpha$.

   Then $\tilde{\Sigma}$ is a Coxeter complex and $\tau |_{\tilde{\Sigma}}: \tilde{\Sigma} \longrightarrow P(\tilde{J})$ is a numbering. If $\tilde{M}$ is the diagram of $\tilde{\Sigma}$, the $\tilde{m}_{\alpha\beta}$ can be determined by the diagrams $M_{\bar{J} \cup \alpha \cup \beta}$.

We close this section with some easy corollaries, for which we need some further definitions.

Definition 2.3.2. Let M be a Coxeter diagram over I.

(1) A subset J of I is called spherical, if M_J is spherical.

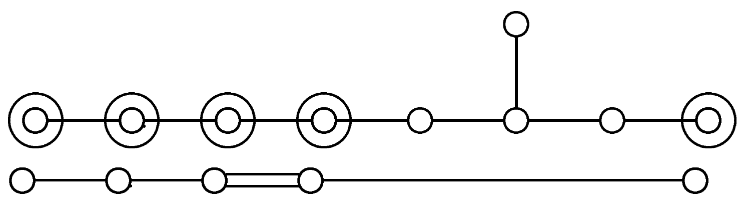
(2) A subset J of I is said to be closed by opposition, if for each $j \in J$ the diagram $M_{j \cup \bar{J}}$ (where $\bar{J} = I \setminus J$) is spherical and $opp_{M_{j \cup \bar{J}}}(j) = j$.

Lemma 2.3.3. *Let Γ_0 be a group of automorphisms of a Coxeter diagram M over I and suppose that all the orbits of Γ_0 in I are spherical. Then Γ_0 can be canonically identified with a subgroup Γ of the stabilizer of one chamber in $Aut(\Sigma)$. If $\Gamma \leq Aut(\Sigma)$ is constructed in this way, it fixes a Coxeter complex.*

Lemma 2.3.4. *Let J be a subset of I which is closed by opposition and put $\bar{J} = I \setminus J$. Then $W_{\bar{J}}$ fixes a Coxeter complex.*

The previous lemma implies the following fact, which will be needed in section 2.6.

Lemma 2.3.5. *Let $M = \tilde{E}_8$ and let J be the subset of the index set corresponding to the D_4 subdiagram. Then W_J fixes a Coxeter complex of type $\tilde{F}_4$.*



2.4 Coxeter groups in Coxeter groups

The content of this section is mainly inspired by the following group theoretical version of Lemma 2.3.3

Lemma 2.4.1. *Let M be a Coxeter diagram over I and let (W, S) be the Coxeter system of type M. Let $\Gamma_0 \leq Aut(M)$ be such that the orbits of Γ_0 in I are spherical. Let $\tilde{I}$ denote the set of orbits. Then Γ_0 acts canonically on W. If $\tilde{W}$ denotes the centralizer of Γ_0 in W, then $\tilde{W}$ is generated by the set $R = \{r_\alpha | \alpha \in \tilde{I}\}$ and $(\tilde{W}, R)$ is a Coxeter system. For the diagram $\tilde{M}$ of diagram $\tilde{M}$ of this Coxeter system we have $\tilde{m}_{\alpha\beta} = o(r_\alpha r_\beta)$.*

Definition 2.4.2. Let M be a Coxeter diagram over I. A partition $\tilde{I}$ of I is said to be a *spherical partition with respect to M* if M_α is spherical for each $\alpha \in \tilde{I}$

We will now fix some notation concerning spherical partitions which will be valid throughout the rest of this section: Let I be a set, let M be a Coxeter diagram over I and let $\tilde{I}$ be a spherical partition of I with respect to M. Let (W, S) be the Coxeter system of type M. We put $R = \{r_\alpha | \alpha \in \tilde{I}\}$ and $\tilde{W} = \langle R \rangle$. Thus $(\tilde{W}, R)$ is a system of involutions over $\tilde{I}$. Let $\tilde{l} : \tilde{W} \longrightarrow N_o$ denote its length function (see section 2.2) and let $\tilde{M} = (\tilde{m}_{\alpha\beta})_{(\alpha,\beta)\in\tilde{I}\times\tilde{I}}$ be its type. Let $\tilde{F}$ denote the free monoid on $\tilde{I}$. If $\tilde{m}_{\alpha\beta} \neq \infty$ let $\tilde{p}(\alpha, \beta) \in \tilde{F}$ be defined to be the word $\ldots\alpha\beta\alpha\beta$ of length $\tilde{m}_{\alpha\beta}$ and let $\tilde{q}(\alpha, \beta)$ be the word $\ldots\alpha\beta\alpha\beta$ of length $\tilde{m}_{\alpha\beta} - 1$.

Definition 2.4.3. Let $\tilde{I}$ be a spherical partition with respect to M.

1. The partition $\tilde{I}$ is said to be admissible if $\tilde{I}$ is *admissible* at each $\tilde{w} \in \tilde{W}$.

2. Let $\tilde{w} \in \tilde{W}$ and let $\tilde{f} = \alpha_1\alpha_2\ldots\alpha_d \in \tilde{F}$ be a representation of $\tilde{w}$ with $\alpha_i \in \tilde{I}$. We say that the representation $\tilde{f}$ is *compatible* if the following holds:

$$l(\tilde{w}) = \sum_{k=1}^{d} l(r_{\alpha_k})$$

The following three lemmas are immediate.

Lemma 2.4.4. *Let $\tilde{I}$ be an admissible partition of I. Let $\tilde{J} \subseteq \tilde{I}$ and put $J = \cup_{\beta\in\tilde{J}}\beta$. Then the partition $\{\beta|\beta \in \tilde{J}\}$ is an admissible partition of J with respect to M_J .*

Lemma 2.4.5. *Assume $|\tilde{I}| = 2$ with $\tilde{I} = \{\alpha, \beta\}$. The following are equivalent:*

a The partition $\{I_\alpha, I_\beta\}$ is admissible.

b If $\tilde{w}$ is in $\tilde{W}$, then a representation $\tilde{f} \in \tilde{F}$ of $\tilde{w}$ is reduced if and only if it is compatible.

Lemma 2.4.6. *Let M be a spherical diagram over I and let $\tilde{I} = \{\alpha, \beta\}$ be a partition of I. Then $\tilde{m}_{\alpha\beta} \neq \infty$. Moreover: The partition $\tilde{I}$ is admissible if and only if the words $\tilde{p}(\alpha, \beta)$ and $\tilde{p}(\beta, \alpha)$ are compatible representations of r_I.*

Proposition 2.4.7. *Let M be a Coxeter diagram over I. Let $\tilde{I}$ be a spherical partition such that for each pair $(\alpha, \beta) \in \tilde{I} \times \tilde{I}$ the partition $\{\alpha, \beta\}$ of $\alpha \cup \beta$ is admissible with respect to $M_{(\alpha\cup\beta)}$. Then we have for all $\tilde{w} \in \tilde{W}$:*

A1 A representation $\tilde{f} \in \tilde{F}$ of $\tilde{w}$ is reduced if and only if it is compatible.

A2 Let $\alpha, \beta \in \tilde{I}$ and put $\tilde{v} = P_{(\alpha\cup\beta)}(\tilde{w})$. Then we have: $\tilde{v} \in \tilde{W}$, $\tilde{l}(\tilde{v}) + \tilde{l}(\tilde{v}^{-1}\tilde{w}) = \tilde{l}(\tilde{w})$ and $\tilde{v}^{-1}\tilde{w} \in \langle r_\alpha, r_\beta \rangle$.

A3 The partition $\tilde{I}$ is admissible at $\tilde{w}$.

A4 $\tilde{l}(\tilde{w}r_\alpha) = \tilde{l}(\tilde{w}) - 1$ if and only if $\alpha \subseteq I^-(\tilde{w})$.

A5 $\tilde{l}(\tilde{w}r_\alpha) = \tilde{l}(\tilde{w}) + 1$ if and only if $\alpha \subseteq I^+(\tilde{w})$.

A6 If $\alpha, \beta \in \tilde{I}$ such that $\alpha \cup \beta \subseteq I^-(\tilde{w})$, then there exists a reduced representation of $\tilde{w}$ in $\tilde{F}$ ending with the word $\tilde{p}(\alpha, \beta)$.

Proof. First observe that if $\alpha, \beta \in \tilde{I}$ are such that $M_{(\alpha\cup\beta)}$ is spherical it follows that $\tilde{p}(\alpha, \beta)$ and $\tilde{p}(\beta, \alpha)$ are compatible representations of $r_{(\alpha\cup\beta)}$. In this case we have also that $\tilde{q}(\alpha, \beta)$ (resp. $\tilde{q}(\beta, \alpha)$) is a compatible representation of $r_{(\alpha\cup\beta)}r_\alpha$ (resp. $r_{(\alpha\cup\beta)}r_\beta$) of length $m_{\alpha\beta} - 1$.

The proof proceeds by induction on $\tilde{l}(\tilde{w})$:

If $\tilde{l}(\tilde{w}) = 0$, then $\tilde{w}$ is the identity. The assertions A1 - A6 are obviously satisfied and the induction starts.

Let $\tilde{w} \in \tilde{W}$ and let $\tilde{f}$ be a reduced representation ending with $\epsilon \in \tilde{I}$. We may write $\tilde{f}$ as $\tilde{f}'\epsilon$. We put $\tilde{w}' = \tilde{w}r_\epsilon$. The word $\tilde{f}'$ is a reduced representation of $\tilde{w}'$. Now $\tilde{l}(\tilde{w}') = \tilde{l}(\tilde{w}) - 1$ and we can apply the induction hypothesis to $\tilde{w}'$.

We have $\tilde{l}(\tilde{w}'r_\epsilon) = \tilde{l}(\tilde{w}') + 1$, so the assertion A5 of the induction hypothesis implies that $\epsilon \subseteq I^+(\tilde{w}')$. From lemma 2.2.4 it is seen that $l(\tilde{w}'r_\epsilon) = l(\tilde{w}') + l(r_\epsilon)$. Now one uses lemma 2.2.8 to see that $\epsilon \subseteq I^-(\tilde{w})$. Since the representation $\tilde{f}'$ of $\tilde{w}'$ is reduced it must be compatible by the assertion A1 of the induction hypothesis. Combining this with the equality $l(\tilde{w}) = l(\tilde{w}'r_\epsilon) = l(\tilde{w}') + l(r_\epsilon)$ we obtain that $\tilde{f}$ is a compatible representation of $\tilde{w}$.

(I) We have shown that every reduced representation of $\tilde{w}$ is compatible. Moreover, if $\alpha \in \tilde{I}$ we have the implication

$$\tilde{l}(\tilde{w}r_\alpha) = \tilde{l}(\tilde{w}) - 1 \Rightarrow \alpha \subseteq I^-(\tilde{w}).$$

Now let $\delta \in \tilde{I}$. First observe that $P_{(\epsilon\cup\delta)}(\tilde{w}) = P_{(\epsilon\cup\delta)}(\tilde{w}')$; we denote this element in $\tilde{W}$ by $\tilde{v}$. The induction hypothesis shows that $\tilde{v} \in \tilde{W}$ and $\tilde{l}(\tilde{w}') = \tilde{l}(\tilde{v}) + \tilde{l}(\tilde{v}^{-1}\tilde{w}')$. Since $\tilde{l}(\tilde{w}) = \tilde{l}(\tilde{w}') + 1$ and $\tilde{l}(\tilde{v}^{-1}\tilde{w}) = \tilde{l}(\tilde{v}^{-1}\tilde{w}'r_\epsilon) \leq \tilde{l}(\tilde{v}^{-1}\tilde{w}') + 1$ it follows:

$$\tilde{l}(\tilde{w}) \leq \tilde{l}(\tilde{v}) + \tilde{l}(\tilde{v}^{-1}\tilde{w}) \leq \tilde{l}(\tilde{v}) + \tilde{l}(\tilde{v}^{-1}\tilde{w}') + 1 = \tilde{l}(\tilde{w}') + 1 = \tilde{l}(\tilde{w})$$

and equality must hold. Since $\tilde{v}^{-1}\tilde{w}' \in \langle r_\epsilon, r_\delta \rangle$ it follows that $\tilde{v}^{-1}\tilde{w} = \tilde{v}^{-1}\tilde{w}'r_\epsilon \in \langle r_\epsilon, r_\delta \rangle$.

Since $\tilde{v}^{-1}\tilde{w} \in \langle r_\epsilon, r_\delta \rangle$ it follows that ϵ and δ are admissible at $\tilde{v}^{-1}\tilde{w}$. Lemma 2.2.6 provides now the admissibility of ϵ and δ at $\tilde{w}$.

(II) This proves A3. Moreover, we have shown A2 for all pairs $(\alpha, \beta) \in \tilde{I} \times \tilde{I}$ having the property that there is a reduced representation of $\tilde{w}$ ending with α or β.

Now let $\mu \in \tilde{I}$ be such that $\mu \subseteq I^-(\tilde{w})$. We put $\tilde{z} = P_{(\epsilon \cup \mu)}$. Since we have already shown that $\epsilon \subseteq I^-(\tilde{w})$, it follows that $\epsilon \cup \mu \subseteq I^-(\tilde{w})$. From Lemma 2.2.8 it is seen that $M_{(\epsilon \cup \mu)}$ is spherical and that $\tilde{w} = \tilde{z} r_{(\epsilon \cup \mu)}$. Now let $\tilde{f}_z$ be a reduced representation of $\tilde{z}$. Since $\tilde{z} \neq \tilde{w}$ the induction hypothesis shows that $\tilde{f}_z$ is a compatible representation of $\tilde{z}$. On the other hand our first remark above shows that $\tilde{q}(\epsilon, \mu)$ is a compatible representation of $r_{(\epsilon \cup \mu)} r_\epsilon = \tilde{z}^{-1} \tilde{w}'$ of length $\tilde{m}_{\epsilon\mu} - 1$. Combining these considerations we obtain that $\tilde{f}_z \tilde{q}(\epsilon, \mu)$ is a compatible representation of $\tilde{w}'$ of length $\tilde{l}(\tilde{z}) + \tilde{m}_{\epsilon,\mu} - 1$. By the induction hypothesis the representation $\tilde{f}_z \tilde{q}(\epsilon, \mu)$ is reduced. We obtain that $\tilde{f}_z \tilde{p}(\mu, \epsilon)$ is a reduced representation of $\tilde{w}$ and that $\tilde{l}(\tilde{w}) = \tilde{l}(\tilde{z}) + \tilde{m}_{\epsilon\mu}$. We have also that $\tilde{f}_z \tilde{q}(\mu, \epsilon)$ is a representation of $\tilde{w} r_\mu$ of length $\tilde{l}(\tilde{z}) + \tilde{m}_{\epsilon\mu} - 1 = \tilde{l}(\tilde{w}) - 1$. This shows $\tilde{l}(\tilde{w} r_\mu) = \tilde{l}(\tilde{w}) - 1$.

(III) For $\alpha \in \tilde{I}$ we have proved the implication

$$\alpha \subseteq I^-(\tilde{w}) \Rightarrow \tilde{l}(\tilde{w} r_\alpha) = \tilde{l}(\tilde{w}) - 1$$

which accomplishes A4. Moreover we have also shown that A6 is valid.

Let now $\tilde{c}$ be a compatible representation of $\tilde{w}$ ending with γ. So we may write $\tilde{c}$ as $\tilde{c}'\gamma$. Since $\tilde{c}$ is compatible it follows that $\gamma \subseteq I^-(\tilde{w})$. Since we have already shown A4 we can deduce $\tilde{l}(\tilde{w} r_\gamma) = \tilde{l}(\tilde{w}) - 1$. So we may apply the induction hypothesis to $\tilde{w} r_\gamma$. Since $\tilde{c}'$ is a compatible representation of $\tilde{w} r_\gamma$ we can use the induction hypothesis to see that $\tilde{c}'$ is a reduced representation of $\tilde{w} r_\gamma$, hence its length is $\tilde{l}(\tilde{w}) - 1$. This shows that the representation $\tilde{c}$ of $\tilde{w}$ has length $\tilde{l}(\tilde{w})$, so $\tilde{c}$ is a reduced representation of $\tilde{w}$.

(IV) The proof of A1 is now complete.

Let now $\alpha, \beta \in \tilde{I}$. Assume first that $\alpha \cup \beta \subseteq I^+(\tilde{w})$. Then we have $\tilde{v} = P_{(\alpha \cup \beta)}(\tilde{w}) = \tilde{w}$ by lemma 2.2.4 and the assertions made in A2 are obviously satisfied. Now assume that $\alpha \cup \beta \not\subseteq I^+(\tilde{w})$. Since we have already shown A3 we may assume w.l.o.g. that $\alpha \subseteq I^-(\tilde{w})$. Now, since we have already shown A4, we obtain $\tilde{l}(\tilde{w} r_\alpha) = \tilde{l}(\tilde{w}) - 1$ and A2 follows by (II).

(V) This accomplishes the proof of A2

Now let $\alpha \in \tilde{I}$ be such that $\tilde{l}(\tilde{w} r_\alpha) = \tilde{l}(\tilde{w}) + 1$. From A4 applied to $\tilde{w}$ it follows $\alpha \not\subseteq I^-(\tilde{w})$; now it follows from A3 applied to $\tilde{w}$ that $\alpha \subseteq I^+(\tilde{w})$.

(VI) For $\alpha \in \tilde{I}$ we have shown the implication

$$\tilde{l}(\tilde{w} r_\alpha) = \tilde{l}(\tilde{w}) + 1 \Rightarrow \alpha \subseteq I^+(\tilde{w}).$$

Let $\pi \in \tilde{I}$ be such that $\pi \subseteq I^+(\tilde{w})$. By Lemma 2.2.4 we have $\tilde{w} = P_\pi(\tilde{w})$. Assume that $\tilde{l}(\tilde{w}r_\pi) \neq \tilde{l}(\tilde{w})$. It follows from A4 applied to $\tilde{w}$ that the case $\tilde{l}(\tilde{w}r_\pi) = \tilde{l}(\tilde{w}) - 1$ is not possible since $\pi \subseteq I^+(\tilde{w})$. So we obtain $\tilde{l}(\tilde{w}r_\pi) = \tilde{l}(\tilde{w})$. Note that we can now apply A1 - A4 also to $\tilde{w}_1 = \tilde{w}r_\pi$.

We have $\tilde{l}(\tilde{w}_1 r_\pi) = \tilde{l}(\tilde{w}) \neq \tilde{l}(\tilde{w}_1) - 1$. Applying A4 to $\tilde{w}_1$ we obtain $\pi \not\subseteq I^-(\tilde{w}_1)$; applying now A3 to $\tilde{w}_1$ yields $\pi \subseteq I^+(\tilde{w}_1)$. Lemma 2.2.4 provides $\tilde{w}_1 = P_\pi(\tilde{w}_1)$.

Observe that $R_\pi(\tilde{w}_1) = R_\pi(\tilde{w})$ and therefore $P_\pi(\tilde{w}_1) = P_\pi(\tilde{w})$. Combining all these considerations we obtain

$$\tilde{w}r_\pi = \tilde{w}_1 = P_\pi(\tilde{w}_1) = P_\pi(\tilde{w}) = \tilde{w}$$

which is a contradiction. It follows that $\tilde{l}(\tilde{w}r_\pi) = \tilde{l}(\tilde{w}) + 1$.

(VII) For $\alpha \in \tilde{I}$ we have shown the implication

$$\alpha \subseteq I^+(\tilde{w}) \Rightarrow \tilde{l}(\tilde{w}r_\alpha) = \tilde{l}(\tilde{w}) + 1.$$

This completes the proof of A5 and we are done.

Using the above Proposition we can now prove the following theorems.

Theorem 2.4.8. *Let I be a set, let M be a Coxeter diagram over I. Let $\tilde{I}$ be an admissible partition of I with respect to M. Then the pair $(\tilde{W}, R)$ is a Coxeter system, where R and $\tilde{W}$ are as above. The type of $(\tilde{W}, R)$ is the Coxeter diagram $\tilde{M}$ over $\tilde{I}$, where $\tilde{m}_{\alpha\beta} = |\langle r_\alpha r_\beta \rangle|$ for each pair $(\alpha, \beta) \in \tilde{I} \times \tilde{I}$.*

Proof. It is obvious that $(\tilde{W}, R)$ is a system of involutions of type $\tilde{M}$. Since $\tilde{I}$ is an admissible partition of I with respect to M it follows from Lemma 2.4.4 that $\{\alpha, \beta\}$ is an admissible partition of $\alpha \cup \beta$ with respect to $M_{(\alpha\cup\beta)}$ for each pair $(\alpha, \beta) \in \tilde{I} \times \tilde{I}$. So we can apply the Proposition. Axiom A of our characterisation of Coxeter systems follows now from A3, A4 and A5 of the Proposition. The assertion A6 provides the validity of axiom B.

Theorem 2.4.9. *Let I be a set and let M be a Coxeter diagram over I. Let $\tilde{I}$ be a spherical partition of I with respect to M. The following are equivalent:*

(1) The partition $\tilde{I}$ is admissible with respect to M.

(2) For each pair $(\alpha, \beta) \in \tilde{I} \times \tilde{I}$ the partition $\{\alpha, \beta\}$ of $\alpha \cup \beta$ is admissible with respect to $M_{(\alpha\cup\beta)}$.

Proof. The implication (1) $\Rightarrow$ (2) is provided by Lemma 2.4.4. The assertion A3 of Proposition 2.4.7 shows the other implication.

Remark: The situation of Lemma 2.4.1 has now a natural meaning in terms of admissible partitions:

Lemma 2.4.10. *Let $\Gamma_0 \leq Aut(M)$ be such that its orbits in I are spherical. Then the orbits constitute an admissible partition. In particular, the partitions of Table C and those of Table D are admissible.*

2.5 Admissible 2-partitions

Preliminaries

Definition 2.5.1.

a Let G be a group, let $x, y \in G$ and let $m \in N_0$. Then $\Pi_m(x,y)$ denotes the product $\ldots xyxy$ with m factors.

b Let $x, y \in Z$ and let $m \in N_0$. Then $\Sigma_m(x,y)$ denotes the sum $\ldots + x + y + x + y$ with m summands.

Lemma 2.5.2. *Let M be a spherical diagram and let $\tilde{I} = \{\alpha, \beta\}$ be a 2-partition of I. Put $m = |\langle r_\alpha r_\beta \rangle|$. Then:*

a $m \neq \infty$

b The 2-partition $\tilde{I}$ is admissible if and only if $\Pi_m(r_\alpha, r_\beta) = \Pi_m(r_\beta, r_\alpha) = r_I$ and $\Sigma_m(l(r_\alpha), l(r_\beta)) = \Sigma_m(l(r_\beta), l(r_\alpha)) = l(r_I)$.

Proof. This is a reformulation of Lemma 2.4.6.

Lemma 2.5.3. *Let I_1, I_2 be disjoint sets and let M_1 and M_2 be spherical diagrams over I_1 and I_2 respectively. For $l = 1, 2$ let $\tilde{I}_l = \{\alpha_l, \beta_l\}$ be a 2-partition of I_l and put $m_l = |\langle r_\alpha r_\beta \rangle|$.*

Put $I = I_1 \cup I_2$ and $M = M_1 \times M_2$ and let $\tilde{I} = \{\alpha, \beta\}$ be the 2-partition of I defined by $\alpha = \alpha_1 \cup \alpha_2$ and $\beta = \beta_1 \cup \beta_2$. Put $m = |\langle r_\alpha r_\beta \rangle|$.

Then the following are equivalent:

1 The partition $\tilde{I} = \{\alpha, \beta\}$ of I is admissible with respect to M.

2 For $l = 1, 2$ the partition $\tilde{I}_l$ of I_l is admissible with respect to M_l and $m_1 = m = m_2$.

Proof. We have $W = W_{I_1} \times W_{I_2}$. In particular we have for $x, x' \in W_{I_1}$ and $y, y' \in W_{I_2}$:

a $xy = yx$

b $xy = x'y' \Rightarrow x = x'$ and $y = y'$

c $l(xy) = l(x) + l(y)$

Assume now that $\tilde{I}$ is admissible. We have

$$r_{I_1} r_{I_2} = r_I = \Pi_m(r_\alpha, r_\beta) = \Pi_m(r_{\alpha_1}, r_{\beta_1}) \Pi_m(r_{\alpha_2}, r_{\beta_2})$$

The last equality above follows from a. Now it is seen from b that $\Pi_m(r_{\alpha_1}, r_{\beta_1}) = r_{I_1}$ and $\Pi_m(r_{\alpha_2}, r_{\beta_2}) = r_{I_2}$. Similarly one deduces that $r_{I_1} = \Pi_m(r_{\alpha_1}, r_{\beta_1})$ and $r_{I_2} = \Pi_m(r_{\alpha_2}, r_{\beta_2})$.

Since we have for $l = 1, 2$ that $r_{I_l} = \Pi_m(r_{\alpha_l}, r_{\beta_l})$ it follows that

$$(1)\ \Sigma_m(l(r_{I_l}), l(r_{I_2})) \geq l(r_I) \text{ for } l = 1, 2.$$

Since $l(r_{I_1}) + l(r_{I_2}) = l(r_I) = \Sigma_m(l(r_\alpha), l(r_\beta)) = \Sigma_m(l(r_{\alpha_1}) + l(r_{\alpha_2}), l(r_{\beta_1}) + l(r_{\beta_2})) = \Sigma_m(l(r_{\alpha_1}), l(r_{\beta_1})) + \Sigma_m(l(r_{\alpha_2}), l(r_{\beta_2}))$ it follows that equality holds in (1). Similarly one shows the equality $\Sigma_m(l(r_{\beta_l}), l(r_{\alpha_l})) = r_{I_l}$ for $l = 1, 2$.

We have proved that 1 implies 2. The other direction is obvious.

Lemma 2.5.4. *Let $k \in N$ and for $l = 1, 2 \ldots k$ let I_l be pairwise disjoint sets, let M_l be a spherical diagram over I_l and let $\tilde{I}_l = \{\alpha_l, \beta_l\}$ be a 2-partition of I. For $l = 1, 2, \ldots, k$ put $m_l = |\langle r_{\alpha_l} r_{\beta_l} \rangle|$. Let I be the union of the sets I_l and let M be the direct product of the M_l. Let α be the union of the α_l and let β be the union of the β_l. Put $m = |\langle r_\alpha r_\beta \rangle|$.*

Then the following are equivalent:

a *The partition $\tilde{I} = \{\alpha, \beta\}$ is admissible with respect to M.*

b *For $l = 1, 2 \ldots k$ the partition $\tilde{I}_l$ of I_l is admissible with respect to M_α and $m = m_1 = m_2 = \ldots = m_k$.*

Proof. Use induction on k and the previous lemma.

Admissible partitions of admissible partitions

Dealing with admissible partitions of admissible partitions it is useful to change our usual notation in this subsection.

Let I be a set and let M be a Coxeter diagram over I. Let (W, S) be the Coxeter system of type M. Let I' be an admissible partition of I with respect to M. We put $S' = \{r_\alpha | \alpha \in I'\}$ and we put $W' = \langle S' \rangle$. The pair (W', S') is a Coxeter system of a type M', where M' is a certain Coxeter diagram over I'. Let $l' : W' \longrightarrow N_0$ denote the length function associated to this Coxeter system.
We define for $w' \in W'$:

$$I'^-(w') = \{\alpha \in I' | l'(w' r_\alpha) = l(w') - 1\}$$

$$I'^+(w') = \{\alpha \in I' | l'(w' r_\alpha) = l(w') + 1\}$$

Lemma 2.5.5.

1. *The diagram M' is spherical if and only if M is spherical.*
2. *Assume that M' is spherical. Let $r'_{I'}$ denote the unique element in W' of maximal length with respect to l'. Then $r_I = r'_{I'}$.*

Proof. Assume that M' is spherical and let $r'_{I'}$ be as defined above. We have $l'(r'_{I'} r_\alpha) = l'(r'_{I'}) - 1$ for each $\alpha \in I'$. It follows from (R2) that $\alpha \subseteq I^-(r'_{I'})$ for each $\alpha \in I'$, hence $I \subseteq I^-(r'_{I'})$. This shows that M is spherical and that $r'_{I'} = r_I$. The other direction of 1. is obvious.

Let now I'' be a spherical partition of I' with respect to M'. For $\delta \in I''$ let r'_δ be the longest element in W'_δ with respect to the length function l'. Put $S'' = \{r'_\delta | \delta \in I''\}$ and $W'' = \langle S'' \rangle$.
For $\delta \in I''$ put $I_\delta = \bigcup_{\alpha \in \delta} \alpha$ and $\bar{I} = \{I_\delta | \delta \in I''\}$.
By the previous lemma it follows that $\bar{I}$ is a spherical partition of I with respect to M. Moreover: if we put $\bar{S} = \{r_{I_\delta} | \delta \in I''\}$ and $\bar{W} = \langle \bar{S} \rangle$ it follows that the pairs $(\bar{W}, \bar{S})$ and (W'', S'') are equal.
Now let $w'' \in W''$ and let $\delta \in I''$. By (R2) we have the following equalities:

$$\delta \subseteq I'^-(w'') \Leftrightarrow I_\delta \subseteq I^-(w'')$$

$$\delta \subseteq I'^+(w'') \Leftrightarrow I_\delta \subseteq I^+(w'')$$

These considerations prove

Lemma 2.5.6. *The partition I'' is admissible with respect to M' if and only if the partition $\bar{I}$ is admissible with respect to M.*

The Coxeter system of type A_n

Let $k \in N$ and let S_k denote the group of permutations of the set $\{1, 2, \ldots, k\}$. We represent the permutation $\pi \in S_k$ by its cycle notation (see [As86]). The following proposition is well known:

Proposition 2.5.7. *Let $n \in N$, and for $1 \leq i \leq n$ let $s_i \in S_{n+1}$ be the transposition $(i, i+1)$ and let $S = \{s_i | 1 \leq i \leq n\}$. Then the pair (S_{n+1}, S) is the Coxeter system of type A_n where the diagram is labelled by:*

Let (S_{n+1}, S) be the Coxeter system described in the proposition and let $l : S_{n+1} \longrightarrow N_0$ be its associated length function. We can describe the length function more concretely. In order to do this we introduce inversions.

Definition 2.5.8. Let $\pi \in S_{n+1}$. An inversion of π is a subset $\{i, j\}$ of $\{1, 2, \ldots, n+1\}$ such that $i \neq j$ and $(i^\pi - j^\pi)(i - j) \leq 0$. For $\pi \in S_{n+1}$ we define $I(\pi)$ to be the set of all inversions of π and $i(\pi) = |I(\pi)|$.

Proposition 2.5.9. *We have for $\pi \in S_{n+1}$: $l(\pi) = i(\pi)$*

Proof. See [Ai79] Proposition 1.8.

Lemma 2.5.10. *Let $\Omega \in S_{n+1}$ denote the element of maximal length and let $\pi \in S_{n+1}$. We have:*

1. $l(\pi) \leq (n+1)n/2$
2. $\Omega = (1, n+1)(2, n)(3, n-1) \ldots ((n+1)/2, (n+1)/2 + 1)$ *if n is odd.*
3. $\Omega = (1, n+1)(2, n)(3, n-1) \ldots (n/2, n/2 + 2)$ *if n is even.*
4. $l(\Omega) = (n+1)n/2$.

Proof. The first assertion follows from the fact that there are only $(n+1)n/2$ subsets of cardinality 2 contained in $\{1, 2, \ldots, n+1\}$. Moreover it is easy to check that each subset of cardinality 2 lies in $I(\Omega)$.

Straightforward calculations in the group S_{n+1} yield

Lemma 2.5.11.

A *Let $n \in N$ be odd and put*

$$\pi_1 = (1,2)(3,4)\dots(n,n+1)$$

and

$$\pi_2 = (2,3)(4,5)\dots(n-1,n).$$

Then $(\pi_1\pi_2)^{(n+1)/2} = (\pi_2\pi_1)^{(n+1)/2} = \Omega.$

B *Let $n \in N$ be even and put*

$$\pi_1 = (1,2)(3,4)\dots(n-1,n)$$

and

$$\pi_2 = (2,3)(4,5)\dots(n,n+1).$$

Then $(\pi_1\pi_2)^{n/2}\pi_1 = \pi_2(\pi_1\pi_2)^{n/2} = \Omega.$

Lemma 2.5.12. *Let I be the set $\{1,2,\dots,n\}$ and let ω (resp. ϵ) be the set of odd (resp. even) numbers in I. Then $m = |\langle r_\omega r_\epsilon \rangle| = n+1$ and the partition $\{\omega,\epsilon\}$ is admissible with respect to the diagram A_n.*

Proof. With the previous lemma it is easy to check that the conditions of Lemma 2.5.2 are satisfied.

Admissible 2-partitions of irreducible spherical diagrams

If M is an irreducible spherical diagram, then its graph is a tree and we may therefore speak of the bipartite partition of M.

Lemma 2.5.13. *The bipartite partition of an irreducible spherical diagram is admissible, and $\tilde{m}_{\alpha\beta}$ is the Coxeter number.*

Proof. If the Coxeter number is even use Lemma 2.4.6 and Exercise 2 of 3.19 in [Hu90]. If M has type A_n this is Lemma 2.5.12.

Lemma 2.5.14. *The 2-partitions of Table E are admissible.*

Proof. The admissibility of the bipartite partitions was shown in the previous lemma. The admissibility of E X follows from Lemma 2.4.10. The admissibility of E II follows from the admissibility of the partitions C II and E III by applying Lemma 2.5.6. The admissibility of E VI is deduced from the admissibility of the partitions C IV and E X in the same way.

Classification of admissible 2-partitions for $M = A_n, C_n, D_n$

Lemma 2.5.15. *Let M be a diagram over a set I and let $\tilde{I} = \{\alpha, \beta\}$ be an admissible partition of I with respect to M. Let $i_0 \in \alpha$ and suppose that $m_{i_0 j} = 2$ for all $j \in \beta$. Then we have $M = M_\alpha \times M_\beta$.*

Proof. If $a \in W_\alpha$ and $b \in W_\beta$ then we have $l(ab) = l(ba) = l(a) + l(b)$.

Our assumptions imply $s_{i_0} r_\beta = r_\beta s_{i_0}$ and $l(r_\alpha s_{i_0}) = l(r_\alpha) - 1$. Now we have $l(r_\alpha r_\beta s_{i_0}) = l(r_\alpha s_{i_0} r_\beta) = l(r_\alpha s_{i_0}) + l(r_\beta) = l(r_\alpha) - 1 + l(r_\beta) = l(r_\alpha r_\beta) - 1$.

Since α is admissible at $r_\alpha r_\beta$ it follows that $\alpha \subseteq I^-(r_\alpha r_\beta)$, hence $I \subseteq I^-(r_\alpha r_\beta)$. Therefore we have $r_I = r_\alpha r_\beta = r_\beta r_\alpha$ and we are done.

Lemma 2.5.16. *Let $I = \{1, 2, \ldots, n\}$ and let M be the diagram A_n labelled in the natural way. Let $\tilde{I} = \{\alpha, \beta\}$ be an admissible partition of I and assume that $\{i, i+1\} \subseteq \alpha$.*

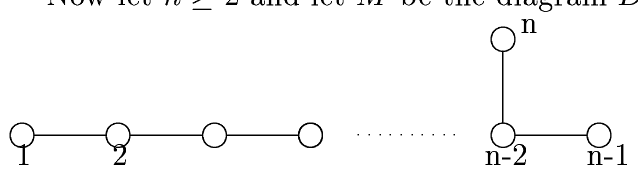
Then n is even and $i = n/2$.

Proof. We identify the Coxeter system (W, S) with the pair (S_{n+1}, S) as in Proposition 2.5.7.

By Lemma 2.5.15 we have that $i - 1$ and $i + 2$ are not contained in α. This implies that r_α and r_β leave $i+1$ invariant and therefore $r_I = \Omega$ leaves $i+1$ invariant. Now, our assertions follow from Lemma 2.5.10 2. and 3..

Lemmas 2.5.15 and 2.5.16 show that E I and E II are the only possible 2-partitions for the diagram A_n. Since every admissible 2-partition of the diagram C_n provides an admissible 2-partition of the diagram A_{2n-1} (in view of Lemma 2.5.6), the partition III is the only possible one in this case.

Now let $n \geq 2$ and let M be the diagram D_{n+1} labelled as follows:



If $\tilde{I} = \{\alpha, \beta\}$ is an admissible 2-partition of M then $\{n, n+1\}$ cannot be a subset of α or β by Lemma 2.5.15. Therefore it provides an admissible 2-partition of C_n by applying Lemma 2.5.6 to the partition C II. This shows that the partition E IV is the only possible.

2.6 Applications

Embeddings of Coxeter groups having a simply laced diagram

The purpose of this section is to prove the following theorem.

Theorem 2.6.1. *Let $\tilde{M}$ be a Coxeter diagram over a set $\tilde{I}$ and assume that the set*

$$\{\tilde{m}_{\alpha\beta} | (\alpha, \beta) \in \tilde{I} \times \tilde{I}\}$$

is finite. Then there exists a set I and a Coxeter diagram M over I having only single bonds (i.e. $m_{ij} \in \{1, 2, 3\}$ for $i, j \in I$) such that the following holds:
There exists an admissible partition $\{I_\alpha | \alpha \in \tilde{I}\}$ of I with respect to M and the Coxeter system obtained by this partition has type $\tilde{M}$.

We start with some preliminary considerations.
We define the function $\delta : \{2, 3, \ldots\} \cup \{\infty\} \longrightarrow N$ by

$$\delta(x) = \begin{cases} x - 1 & \text{if } x \text{ is even} \\ (x-1)/2 & \text{if } x \text{ is odd} \\ 2 & \text{if } x = \infty \end{cases}$$

Lemma 2.6.2. *Let $\tilde{m} \in \{2, 3, \ldots\} \cup \{\infty\}$. Then there exist disjoint sets I_α and I_β of cardinality $\delta(\tilde{m})$ and a Coxeter diagram M over the set $I = I_\alpha \cup I_\beta$ such that the following holds:*

1. *The diagram M has only single bonds.*
2. *If $i \neq j$, and if $\{i, j\} \subseteq I_\alpha$ or $\{i, j\} \subseteq I_\beta$, then $m_{ij} = 2$.*
3. *The partition $\{I_\alpha, I_\beta\}$ of I is admissible with respect to M.*
4. *$\tilde{m} = |\langle r_\alpha r_\beta \rangle|$.*

Proof. If $\tilde{m}$ is odd we take $M = A_{\tilde{m}-1}$. If $\tilde{m}$ is even we take the diagram $M = A_{\tilde{m}-1} \times A_{\tilde{m}-1}$. If $\tilde{m} = \infty$ then we take $M = \tilde{A}_3$. The partitions $\{I_\alpha, I_\beta\}$ are the bipartite partitions with $|\alpha| = |\beta|$.

From Lemma 2.5.12 and Lemma 2.5.4 it follows that this construction is valid in the case $\tilde{m} \neq \infty$. The admissibility of the partition of $\tilde{A}_3$ follows from Lemma 2.4.10.

From Lemma 2.5.4 the following is now immediate:

Lemma 2.6.3. *Let $d \in N$, let $\tilde{m} \in \{2, 3, \ldots\} \cup \{\infty\}$ and assume that $\delta(\tilde{m})$ divides d. Then there exist disjoint sets I_α and I_β of cardinality d and a Coxeter diagram M over the set $I = I_\alpha \cup I_\beta$ such that the following holds:*

1. *The diagram M has only single bonds.*

2. *If $i \neq j$, and if $\{i,j\} \subseteq I_\alpha$ or $\{i,j\} \subseteq I_\beta$, then $m_{ij} = 2$.*

3. *The partition $\{I_\alpha, I_\beta\}$ of I is admissible with respect to M.*

4. $\tilde{m} = |\langle r_\alpha r_\beta \rangle|$.

Proof of Theorem 2.6.1: Let $\tilde{I}$ be a set and let $\tilde{M}$ be a Coxeter diagram over $\tilde{I}$ such that the set $\{\tilde{m}_{\alpha\beta} | \alpha, \beta \in \tilde{I}\}$ is finite. Let d denote the lowest common multiple of the $\tilde{m}_{\alpha\beta}$.
For each $\alpha \in \tilde{I}$ let I_α be a set of cardinality d such that the I_α are pairwise disjoint. If $\alpha \neq \beta$ put $I_{\alpha\beta} = I_\alpha \cup I_\beta$ and let $M_{\alpha\beta}$ be a diagram over $I_{\alpha\beta}$ as described in lemma 2.6.3. Put now $I = \bigcup_{\alpha \in \tilde{I}} I_\alpha$ and let M be the diagram over I whose restriction to each $I_{\alpha\beta}$ is $M_{\alpha\beta}$. Property 2. of Lemma 2.6.3 shows that M is well defined. From (R3) and 3. of Lemma 2.6.3 it follows that the partition $\{I_\alpha | \alpha \in \tilde{I}\}$ is admissible. From Lemma 2.6.3 4. we have $\tilde{m}_{\alpha\beta} = |\langle r_\alpha r_\beta \rangle|$ and we are done.

A remark on the finiteness hypothesis in Theorem 2.6.1

The classification of the admissible 2-partitions for the diagrams A_n, C_n and D_n and Lemma 2.5.3 implies

Lemma 2.6.4. *There exists $0 < \lambda \in \mathbf{R}$ such that the following holds: Let M be a diagram over I, let $\{\alpha, \beta\}$ be an admissible partition of I with respect to M and let $|\langle r_\alpha r_\beta \rangle| \in \{3, 4, 5, \ldots\}$. Then the cardinalities of α and β are greater than $\lambda |\langle r_\alpha r_\beta \rangle|$.*

Consider the diagram $\tilde{M}$ over $\tilde{I} = \{2, 3, 4, \ldots\}$ defined by:

$$\tilde{m}_{ij} = \begin{cases} 1 & \text{if } i = j \\ 2 & \text{if } i \neq j \text{ and } 2 \notin \{i,j\} \\ k & \text{if } i \neq j \text{ and } \{i,j\} = \{2,k\} \end{cases}$$

Let M be a diagram over a set I having only single bonds and suppose that there exists an admissible partition $\{I_\alpha | \alpha \in \tilde{I}\}$ such that $\tilde{m}_{\alpha\beta} = |\langle r_{I_\alpha} r_{I_\beta} \rangle|$. Lemma 2.6.4 implies that the cardinality of I_2 is bigger than λk for each $k \in \{3, 4, 5, \ldots\}$. This is a contradiction.

Here is an example of a diagram $\tilde{M}$ which does not satisfy the finiteness condition of the Theorem and for which there is a diagram M as described in the Theorem.

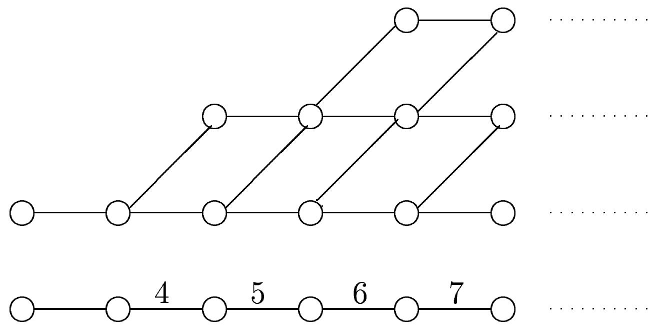


Figure 6

An alternative proof for the classification of the spherical diagrams

In this subsection we will give a proof of the spherical diagrams based on the solution of the word problem. The solution of the word problem has the following immediate conseqence (cf [Ti68])

Lemma 2.6.5. *Let M be a Coxeter diagram and suppose that the graph associated to M has a circuit or at least two distinct points of valency 3. Then M is not spherical.*

Lemma 2.6.6. *$\tilde{A}_3$ is spherical iff $\tilde{C}_2$ is spherical iff $\tilde{B}_2$ is spherical iff $\tilde{D}_4$ is spherical iff $\tilde{G}_2$ is spherical iff $\tilde{E}_6$ is spherical iff $\tilde{F}_4$ is spherical iff $\tilde{E}_7$ is spherical.*

Proof. Use Lemma 2.5.5 and Table D.

Lemma 2.6.7. *Let M be an irreducible simply laced diagram, which is not of type A_n, D_n or E_n $(n = 6, 7, 8)$. Then M is not spherical.*

Proof. By Lemma 2.6.5 the diagram $\tilde{A}_3$ is not spherical. Lemma 2.6.6 implies now that the diagrams $\tilde{E}_6, \tilde{E}_7$ and $\tilde{F}_4$ are not spherical. Since $\tilde{F}_4$ is not spherical, we have by lemma 2.3.5 that $\tilde{E}_8$ is not spherical.

If M is an arbitrary irreducible simply laced diagram which is not of type A_n, D_n or E_n $(n = 6, 7, 8)$, then its graph contains a circuit or two points of valency at least 3 or a subgraph of isomorphic to the graph of $\tilde{E}_n$ $(n = 6, 7, 8)$. This concludes the proof.

Once we have shown that the diagrams of type A_n, D_n or E_n $(n = 6, 7, 8)$ are spherical, the classification of the simply laced diagrams is complete. In view of Lemma 2.5.5 and Theorem 2.6.1 this completes the classification of the spherical diagrams.

2.7 Final Remarks

Generalized admissible partitions

The main goal of this subsection is to indicate how one can generalize admissible partitions in order to include the general situation in Theorem 2.3.1. We get in this way all diagrams used by Tits in [Ti65] for the classification of the semi-simple algebraic groups. Most of the results about admissible partitions remain valid in this more general context.

Definition 2.7.1. Let M be a spherical diagram over I. A subset J of I is called self-opposite if $opp_M(J) = J$.

Lemma 2.7.2. *Let M be a spherical diagram over I and let $J \subseteq I$. The following are equivalent:*

a J is self-opposite with respect to M.

b $r_I r_J = r_J r_I$.

Definition 2.7.3. Let M be a Coxeter diagram over I, let J be a subset of I and let $\tilde{J}$ be a partition of J. Put $\bar{J} = I \setminus J$. Then the pair $(J, \tilde{J})$ is called a *generalized spherical partition* if the following condition is satisfied for each $\alpha \in \tilde{J}$:

1 The diagram $M_{\alpha \cup \bar{J}}$ is spherical.

2 α is self-opposite with respect to $M_{\alpha \cup \bar{J}}$.

If $(J, \tilde{J})$ is a generalized spherical partition, we define $\tilde{s}_\alpha = r_{(\bar{J} \cup \alpha)} r_\alpha$ for each $\alpha \in \tilde{J}$, $\tilde{S} = \{\tilde{s}_\alpha | \alpha \in \tilde{J}\}$ $\tilde{m}_{\alpha\beta} = o(\tilde{s}_\alpha \tilde{s}_\beta)$ for each pair $(\alpha, \beta) \in \tilde{J} \times \tilde{J}$, $\tilde{M} = (\tilde{m}_{\alpha\beta})_{\alpha\beta \in \tilde{J}}$ and $\tilde{W} = \langle \tilde{S} \rangle$.

Generalized admissible partitions Let $(J, \tilde{J})$ be a generalized spherical partition. The partition $(J, \tilde{J})$ is called a *generalized admissible partition* if for each $\tilde{w} \in \tilde{W}$ and each $\alpha \in \tilde{J}$ the following is satisfied: $l(\tilde{w}\tilde{s}_\alpha) = l(\tilde{w}) + l(\tilde{s}_\alpha)$ or $l(\tilde{w}\tilde{s}_\alpha) = l(\tilde{w}) - l(\tilde{s}_\alpha)$.

Pseudo-reflection subgroups

One may wonder in general about subgroups of Coxeter groups which are again Coxeter groups. This is perhaps far too general. However, the subgroups arising from (generalized) admissible partitions have much more structure than just being Coxeter subgroups. They are related to Coxeter complexes, which may be found quite naturally in the original Coxeter complex.

There is another canonical way to produce Coxeter subgroups in a given Coxeter group:

Theorem 2.7.4. *A subgroup generated by reflections in a Coxeter group W is again a Coxeter group whose reflections are reflections in the group W.*

This result was found by V.Deodhar [De89] and M.Dyer [Dy90] independently.

In [He90] J.Y.Hée gives a proof of this theorem, which provides a geometric insight into the meaning of this result. In particular it shows how our Coxeter subgroups and the reflection subgroups may be seen as special cases of a class of subgroups, which will be called *pseudo-reflection groups*.

We will give a sufficient criterion for a set of involutions in a Coxeter group to be the set of reflections of a Coxeter subgroup. This criterion is satisfied by the groups obtained from our construction and also by the subgroups generated by reflections.

Definition 2.7.5.

A pair (A, A') of simplices in $\Sigma(M)$ is called *special* if A and A' are of the same type and opposite in $St(A \cap A')$. Let $i(A, A')$ denote the unique involution interchanging A and A' and which sends each chamber $C \in StA$ onto its projection on A'.

Lemma 2.7.6. *Let Ω be a set of special pairs and define $\tilde{T} = \{i(A, A')|(A, A') \in \Omega\}$. Suppose the following for each $(A, A'), (B, B') \in \Omega$.*

1 $i(A, A')^{i(B,B')} \in T$.

2 $proj_{A \cap A'} B$ and $proj_{A,A'} B'$ are contained in $\{A, A'\}$.

Then $\tilde{T}$ is a set of reflections of a Coxeter subgroup $\tilde{W}$.

The proof can be done in the way of J.Y.Hee's proof for reflection subgroups of Coxeter groups.

2.8 Tables

A The irreducible spherical diagrams

B Non-spherical diagrams

C Admissible partitions of spherical diagrams

D Admissible partitions of affine diagrams

E 2-Partitions of spherical diagrams

Chapter 3

Spherical Moufang Buildings

3.1 Introduction

In this chapter we prove a fixed point theorem for spherical Moufang complexes. It is motivated by several examples from the theory of semisimple algebraic groups and finite groups of Lie type. The first examples we have in mind are provided by the Galois descent in the classification theory of the semisimple algebraic groups over arbitrary fields (see [Ti65]). From a combinatorial point of view, a theorem (Theorem 8.2 in [St68]) due to R. Steinberg extends the Galois descent in the quasi split case. For the finite groups of Lie type the following is known:

Theorem: *Let $G = G(q)$ be a connected semisimple group of Lie type over F_q, where q is a power of the prime p. Let τ be an automorphism of G whose order is not divisible by p and let G_0 be the group of fixed points of τ. Let K be the subgroup of G_0 which is generated by all p-elements in G_0. Then K is a connected semisimple group of Lie type over a finite extension of F_p.*

This theorem is stated in [Fe93]. It is due to G. Seitz and R. Steinberg. Unfortunately, there is no proof available in the literature.

Our fixed point theorem for Moufang complexes provides a uniform approach to all these examples and indicates that they are best understood from a geometrical point of view.

It should be mentioned that it can be generalized to Moufang twin buildings. A theorem which points in that direction exists already for the particular situation of Galois descents. It was proved by G. Rousseau in order to study Galois descents in groups of Kac-Moody type (see [Ro88]).

Moufang complexes: In [Ti74] the irreducible buildings of spherical type having rank at least 3 are classified. Roughly speaking, they turn out to be the buildings associated to semisimple algebraic groups. In the rank 2 case such a classification is not feasible. In order to describe the generalized n-gons coming from algebraic groups J. Tits introduced the Moufang condition, which requires the existence of all root elations. In the case of projective planes this condition is satisfied if they are defined over a skew field or if they are the classical Moufang planes, which explains this notation.

For a building of spherical type this definition is convenient, if the diagram has no direct component of type A_1. Moufang buildings of type A_1 had been introduced in [Ti90], where they are needed to define a Moufang foundation. They are called Moufang lines; they generalize the concept of a Desarguesian projective line (cf. [Bu75]). The axioms of a Moufang line are similar to those for a split BN-pair given in [HKS72]. In the case of a Moufang line, the unipotent radicals of the Borel groups are axiomatized. Our definition of a Moufang building has its origin in this observation. We will require the existence of the unipotent radicals of the Borel groups.

While this definition is equivalent to the usual definitions of Moufang buildings in the particular situations considered above, its main advantage is that it makes sense also for weak gated chamber complexes. Thus, a Moufang complex is defined to be a weak gated chamber complex endowed with a system of automorphism groups U_C where C runs through the chambers of the complex.

The group G generated by the U_C will be called the group associated to the Moufang complex. An element $u \in G$ will be called an elation, if it is contained in some U_C; our elations are precisely the good unipotent elements in the semisimple algebraic groups.

Preliminaries: In this section we deduce several elementary consequences of our definition of a Moufang complex. We are interested in the properties of the unipotent radicals of the parabolics. In an appended subsection we indicate how to establish the equivalence of our definition of a Moufang building with the definition given in [Ti74]. In a further remark we show how to extract a Moufang building from a Moufang complex.

Automorphisms of Moufang complexes: We consider the situation in which a group Γ acts on a Moufang complex. Thus Γ acts on the underlying spherical gated chamber complex Δ and the associated group. We denote the set of fixed simplices by $\tilde{\Delta}$, which is known to be a chamber complex in general. If we assume in addition that $\tilde{\Delta}$ contains two Γ-chambers which are opposite in Δ, then we know by the results of Chapter 1 that $\tilde{\Delta}$ is spherical. This is our assumption (O). We define a canonical Moufang system for $\tilde{\Delta}$. The verification of our axioms is immediate.

The condition (O): The assumption (O) is equivalent to the assumption that for each simplex in $\tilde{\Delta}$ there exists a simplex in $\tilde{\Delta}$ such that both are opposite in Δ. In projective spaces this means that each subspace fixed by Γ has a complement which is fixed as well; by the theorem of Maschke this is always the case if $|\Gamma|$ is finite and not divisible by the characteristic of the projective space. We will generalize

Maschkes theorem to spherical Moufang complexes. If Δ has characteristic p and Γ is a finite p'-group, then the condition (O) is satisfied.

Elation groups in the centralizer of Γ: Given a group Γ acting on a Moufang complex, it acts canonically on the associated group G. We will state a conjecture, which says that the elations centralized by Γ are contained in the unipotent radical of a fixed parabolic. By [BT71] and [BW76] the conjecture is known to be true, if the Moufang building arises from an algebraic group defined over a perfect field, and for the buildings associated to finite groups of Lie type. Probably more is true for Moufang buildings arising from semisimple algebraic groups, but I cannot extract a precise statement. Nevertheless, it seems unavoidable to approach this conjecture from a geometrical point of view, if one wants to prove it in general.

In [Ti62] J. Tits indicates a geometric proof of a similar statement. Unfortunately, these ideas have never been published, because he has found a more efficient procedure for his purposes. We introduce characteristic functions of Moufang complexes; this resumes perhaps what he might had in mind. The existence of a characteristic function for each irreducible Moufang building implies the validity of the conjecture. We prove the existence of characteristic functions for all irreducible Moufang buildings, whose type is not F_4 or $E_n (n = 6, 7, 8)$. For the diagrams A_n, C_n and D_n we use the point line characterizations of the projective spaces and the polar spaces. Though it should be possible to do a similar thing for the exceptional diagram by using the root group geometries, I have not found a proof so far.

3.2 Moufang Complexes

Throughout this section Δ always denotes a spherical gated chamber complex and $\mathcal{C}$ denotes the set of chambers in Δ. The group of automorphisms of Δ will be denoted by $Aut(\Delta)$, the group of automorphisms preserving a numbering will be denoted by $Spe(\Delta)$.

Definition 3.2.1. A *Moufang system of* Δ is a collection of groups $(U_C)_{C \in \mathcal{C}}$ satisfying the following axioms for each chamber $C_0 \in \mathcal{C}$:

(1) $U_{C_0} \leq Aut(\Delta)$.

(2) U_{C_0} fixes C_0 and all its faces.

(3) U_{C_0} acts transitively on the chambers opposite to C_0.

(4) U_{C_0} normalizes the set $\{U_C | C \in \mathcal{C}\}$ in $Aut(\Delta)$.

(5) If $D \in \mathcal{C} \setminus \{C_0\}$ is adjacent to C_0, and if $u \in U_{C_0}$ fixes D, then $u \in U_X$ for any chamber $X \in St(C_0 \cap D)$.

A *Moufang complex* $\mathcal{M}$ is a pair $(\Delta, (U_C)_{C\in\mathcal{C}})$ consisting of a spherical gated chamber complex Δ and a Moufang system $(U_C)_{C\in\mathcal{C}}$ of Δ.

A *spherical Moufang building* is a thick Moufang complex.

We will introduce some notation and further definitions concerning Moufang complexes.

Let $\mathcal{M} = (\Delta, (U_C)_{C\in\mathcal{C}})$ be a Moufang complex. We put $G = \langle U_C | C \in \mathcal{C}\rangle$ and we will call it the group associated to $\mathcal{M}$.

If $u \in G$ (resp. $U \leq G$) is contained in U_C for at least one $C \in \mathcal{C}$, it will be called an *elation* (resp. *elation group*). The set of all elations (resp. elation groups) contained in G will be denoted by $\mathcal{E}$ (resp. $\mathcal{U}$).

For a simplex $A \in \Delta$ we put:

1. $P_A = Stab_G(A)$.

2. $G_A = \langle u | u \in \mathcal{E} \cap P_A\rangle$.

3. $U_A = \langle u | u \in \mathcal{E}, X^u = X$ for all $X \in StA\rangle$.

We will show in 3.3.5 that for a chamber C, the group defined in 3. coincides with the group U_C defined earlier.

Let $\mathcal{M}' = (\Delta', (U'_{C'})_{C'\in\mathcal{C}'})$ be a Moufang complex. An isomorphism $\alpha : \mathcal{M} \longrightarrow \mathcal{M}'$ is a pair $(\alpha_\Delta, (\alpha_C)_{C\in\mathcal{C}})$ consisting of an isomorphism $\alpha_\Delta : \Delta \longrightarrow \Delta'$ of chamber complexes and a system $(\alpha_C)_{C\in\mathcal{C}}$ of group isomorphisms $\alpha_C : U_C \longrightarrow U'_{C^{\alpha_\Delta}}$ satisfying the following condition.

For any triple (C, u, A) consisting of a chamber $C \in \mathcal{C}$, an elation $u \in U_C$ and a simplex $A \in \Delta$ we have

$$(A^{\alpha_\Delta})^{(u^{\alpha_C})} = (A^u)^{\alpha_\Delta}$$

An automorphism is an isomorphism with $\mathcal{M}' = \mathcal{M}$. The group of automorphisms of $\mathcal{M}$ will be denoted by $Aut(\mathcal{M})$. In view of axiom (4) we have a natural injection from G in $Aut(\mathcal{M})$.

3.3 Preliminaries

We start with two observations for spherical gated chamber complexes.

Lemma 3.3.1. *Let Δ be a spherical gated chamber complex and $\alpha \in Aut(\Delta)$. Suppose that α fixes an apartment Σ and all chambers containing a panel in Σ. Then α is the identity.*

Lemma 3.3.2. *Let Δ be a spherical gated chamber complex. Let C, D be chambers in Δ and let Σ, Σ' be two apartments containing C and D. Then there exists a sequence of apartments $\Sigma = \Sigma_0, \Sigma_1, \ldots, \Sigma_k = \Sigma'$ such that*

(1) $\Sigma_{i-1} \cap \Sigma_i$ *is a root* ϕ_i *for each* $1 \leq i \leq k$.

(2) $C, D \in \Sigma_i$ *for each* $0 \leq i \leq k$.

From now on until the end of this section $\mathcal{M} = (\Delta, (U_C)_{C \in \mathcal{C}})$ always denotes a Moufang complex.

Lemma 3.3.3. *Let* $C, D \in \mathcal{C}$*, let* $u \in U_C$ *be such that* $D^u = D$ *and let* $C = C_0, C_1, \ldots, C_k = D$ *be a gallery stretched from* C *to* D*. Then* $u \in U_Y$ *for each chamber*

$$Y \in \bigcup_{i=1}^{k} ChamSt(C_{i-1} \cap C_i).$$

In particular we have $u \in U_D$.

Proof. First observe that $proj_{C_0 \cap C_1} D = C_1$. Since u fixes $C_0 \cap C_1$ (by axiom (2)) and D, the elation u fixes also C_1. Now apply axiom (5) and induction.

Lemma 3.3.4. *Let* $A \in \Delta$ *and let* $u \in P_A$ *(resp.* $U \leq P_A$*) be an elation (resp. elation group.). Then* $u \in U_C$ *(resp.* $U \leq U_C$*) for some chamber* $C \in StA$.

Proof. By definition $u \in U_D$ for some chamber $D \in \mathcal{C}$. Put $C = proj_A D$ and apply the previous lemma.

Corollary 3.3.5. *Let* $A \in \Delta$ *be a simplex.*

(1) U_A *is an elation group.*

(2) $U_A \leq U_C$ *for each* $C \in ChamStA$.

(3) $U_A = \bigcap_{C \in ChamStA} U_C$.

We say that two simplices are locally opposite if they are opposite in the star of their intersection.

Lemma 3.3.6.

(1) *Let* C, D *be two opposite chambers in* Δ*, then* $U_C \cap P_D = \{id\}$.

(2) *Let* C, D *be two locally opposite chambers. Then* $U_C \cap P_D \leq U_{C \cap D}$.

(3) *Let* A, B *be two opposite simplices in* Δ*. Then* $U_A \cap P_B = \{id\}$.

(4) *Let* A, B *be two locally opposite chambers. Then* $U_A \cap P_B \leq U_{A \cap B}$.

Proof. The assertion (1) follows from Lemma 3.3.1 and Lemma 3.3.3. Assertion (2) follows by the same arguments applied to the star of $C \cap D$. To verify (3) consider two chambers C, D which are opposite in StA. If $u \in U_A \cap P_B$, then u fixes C and $proj_B D$ and the assertion follows by (1). To establish (4) one uses again the same arguments in $St(A \cap B)$.

Lemma 3.3.7. *For every simplex A the group U_A acts sharply transitively on the simplices opposite to A in Δ.*

Proof. Let A', A'' be two simplices which are opposite to A in Δ. Let C be a chamber in StA and let $D \in ChamStA$ be opposite to C in StA. Put $D' = proj_{A'} D, D'' = proj_{A''} D$ and let $u \in U_C$ be an elation sending D' onto D''. It follows that u fixes D and by Lemma 3.3.6 (2) it follows that $u \in U_A$. Hence U_A is transitive on the simplices opposite to A in Δ. The assertion follows now from Lemma 3.3.6 (3).

Remarks

Moufang Buildings: Let Δ be a spherical building whose diagram has no irreducible component of type A_1. The building is said to be Moufang (in the sense of [Ti74]) if all root elations exist. In such a building we can associate to each root ϕ a root group U_ϕ acting regularly on the set of apartments containing ϕ.

In a Moufang building (in the sense of [Ti74]) one can show the following easy lemma.

Lemma 3.3.8. *Let C be a chamber and Σ an apartment containing C. Let $\Psi_{C,\Sigma}$ (resp. Ψ_C) be the set of all roots in Σ (resp. Δ) containing C. Then $\langle U_\phi | \phi \in \Psi_{C,\Sigma} \rangle = \langle U_\phi | \phi \in \Psi_C \rangle$.*

For each chamber C define the group $U_C = \langle U_\phi | \phi \in \Psi_C \rangle$. With this definition it is immediate that our axioms (1), (2) and (4) are satisfied. Moreover, U_C acts sharply transitively on the apartments containing C (use Lemma 3.3.8 and Theorem 6.15 in [Ro89]). To establish (5) one uses the regularity of U_C (resp. U_D) on the set of the apartments containing C (resp. D) and Lemma 3.3.2.

The existence of all root elations in a Moufang building (in the sense of our definition) is an easy consequence of Lemma 3.3.3 above.

Moufang buildings from Moufang complexes In Section 1.7 we indicated how to extract a spherical building Δ^* from a spherical gated chamber complex Δ. The chambers of Δ^* are defined to be the thin classes of the chambers in Δ. If two chambers C, D in a Moufang complex $\mathcal{M}$ belong to the same thin class, it readily follows that $U_C = U_D$ and hence we may define U_{C^*} in an obvious way. With this definition it can be verified that $(\Delta^*, (U_{C^*})_{C^* in \mathcal{C}^*})$ becomes a Moufang complex.

3.4 Automorphisms of Moufang complexes

Throughout this section $\mathcal{M} = (\Delta, (U_C)_{C \in \mathcal{C}})$ is assumed to be a Moufang complex, G its associated group and Γ a subgroup of $Aut(\mathcal{M})$. Hence the group Γ acts on Δ and on G as a group of automorphisms. We put

$$\tilde{\Delta} = \{A \in \Delta | A^\gamma = A \text{ for all } \gamma \in \Gamma\}$$

and

$$\tilde{G} = \{g \in G | g^\gamma = g \text{ for all } \gamma \in \Gamma\}.$$

The set of Γ-chambers is denoted by $\tilde{\mathcal{C}}$.

We start with a preliminary observation:

Lemma 3.4.1. *Let $A \in \Delta$ and $\gamma \in \Gamma$. Then $U_A^\gamma = U_{A^\gamma}$.*

Proof. If A is a chamber, the assertion follows by definition. If A is arbitrary one can use Corollary 3.3.5.

Let $\tilde{C}$ be a Γ-chamber. In view of the previous lemma that $U_{\tilde{C}}$ is normalized by Γ. We define

$$\tilde{U}_{\tilde{C}} = \tilde{G} \cap U_{\tilde{C}}.$$

From now on until the end of this section we make the following additional assumption.

(O) Γ fixes a pair of opposite Γ-chambers.

Lemma 3.4.2. *The set of fixed simplices $\tilde{\Delta}$ constitutes a spherical gated chamber complex with the ordering induced by Δ.*

Proof. This follows from (O) and the main results of Chapter 1.

Lemma 3.4.3. *$\tilde{U}_{\tilde{C}}$ fixes $\tilde{C}$ and all its Γ-faces.*

Proof. Since $\tilde{U}_{\tilde{C}} \leq U_{\tilde{C}}$ it follows that $\tilde{U}_{\tilde{C}}$ fixes $\tilde{C}$ and all faces of $\tilde{C}$, hence it fixes also all Γ-faces of $\tilde{C}$.

Lemma 3.4.4. *Let $\tilde{C} \in \tilde{\mathcal{C}}$; then $\tilde{U}_{\tilde{C}} \leq Aut(\tilde{\Delta})$.*

Proof. Since $\tilde{U}_{\tilde{C}}$ centralizes Γ in $Aut(\Delta)$, it acts on $\tilde{\Delta}$.

Lemma 3.4.5. *Let $\tilde{C} \in \tilde{\mathcal{C}}$; then $\tilde{U}_{\tilde{C}}$ acts sharply transitively on the set of Γ-chambers opposite to $\tilde{C}$ in $\tilde{\Delta}$.*

Proof. Let $\tilde{D}, \tilde{E}$ be two Γ-chambers opposite to $\tilde{C}$. By Lemma 3.3.7 there exists a unique $u \in U_{\tilde{C}}$ such that $\tilde{D}^u = \tilde{E}$. Let $\gamma \in \Gamma$, then $\gamma^{-1}u\gamma$ is in $U_{\tilde{C}}$ and sends $\tilde{D}$ onto $\tilde{E}$. We conclude that $\gamma^{-1}u\gamma = u$ and hence $u \in \tilde{U}_{\tilde{C}}$.

Lemma 3.4.6. *Let $\tilde{C}_0 \in \tilde{\mathcal{C}}$; then $\tilde{U}_{\tilde{C}_0}$ normalizes $\{\tilde{U}_{\tilde{C}} | \tilde{C} \in \tilde{\mathcal{C}}\}$ in $Aut(\tilde{\Delta})$.*

Proof. Let $\tilde{D}$ be a Γ-chamber and $u \in \tilde{U}_{\tilde{C}}$. It follows that $U_{\tilde{D}}^u = U_{\tilde{D}^u}$ and hence $\tilde{U}_{\tilde{D}}^u \leq U_{\tilde{D}^u}$. Since $u \in \tilde{G}$ and $\tilde{U}_{\tilde{D}} \leq \tilde{G}$ it follows that $\tilde{U}_{\tilde{D}}^u \leq \tilde{G}$ and hence $\tilde{U}_{\tilde{D}^u} \leq \tilde{U}_{\tilde{D}^u}$. Applying the same argument to u^{-1} one shows the other inclusion and hence the claim.

Lemma 3.4.7. *Let $\tilde{C} \in \tilde{\mathcal{C}}$ and let $\tilde{D}$ be a Γ-chamber which is Γ-adjacent to $\tilde{C}$. Suppose $u \in \tilde{U}_{\tilde{C}}$ fixes $\tilde{D}$. Then $u \in \tilde{U}_{\tilde{X}}$ for all Γ-chambers $\tilde{X}$ containing $\tilde{C} \cap \tilde{D}$.*

Proof. Since $\tilde{D}$ and $\tilde{C}$ are locally opposite, it follows by Lemma 3.3.6 (4) that $u \in U_{\tilde{D}\cap\tilde{C}}$ and hence $u \in U_{\tilde{X}}$ for each Γ-chamber $\tilde{X}$ containig $\tilde{C} \cap \tilde{D}$. Since u centralizes Γ, the assertion follows.

Theorem 3.4.8. *$\tilde{\mathcal{M}} = (\tilde{\Delta}, (\tilde{U}_{\tilde{C}})_{\tilde{C}\in\tilde{\mathcal{C}}})$ is a Moufang complex.*

Proof. The previous lemmas provide a verification of the axioms.

3.5 The condition (O)

Double representations

Let $\Pi = (\Omega, G, \pi_G)$ be a permutation representation of G on a set Ω. Let $\Gamma \leq Aut(\Pi)$. Hence we have a permutation representation π_Γ of Γ on Ω and a representation δ of Γ as group of automorphisms of G, such that the following is satisfied for each $\omega \in \Omega, g \in G, \gamma \in \Gamma$:

$$(\omega^{\pi_G(g)})^{\pi_\Gamma(\gamma)} = (\omega^{\pi_\Gamma(\gamma)})^{\pi_G(g^{\delta(\gamma)})}.$$

We call such a tuple $(\Omega, G, \Gamma, \pi_G, \pi_\Gamma, \delta)$ a double representation.

A double representation will be called regular if Π is regular.

Lemma 3.5.1. *Let* $(\Omega, G, \Gamma, \pi_G, \pi_\Gamma, \delta)$ *a regular double representation and let* G_1 *be a characteristic subgroup of* G*. Let* $\overline{\Omega}$ *be the set of orbits of* G_1 *in* Ω*. Put* $\overline{G} = G/G_1$*. Then we get a regular double representaion* $(\overline{\Omega}, \overline{G}, \Gamma, \pi_{\overline{G}}, \overline{\pi}_\Gamma, \overline{\delta})$ *in a natural way. If* Γ *fixes an element* Ω_1 *in* $\overline{\Omega}$*, then we have also regular double representation* $(\Omega_1, G_1, \Gamma, \pi_{G_1}, \pi_{1\Gamma}, \delta_1)$.

We need also the following observation:

Lemma 3.5.2. *Let* $(\Omega, G, \Gamma, \pi_G, \pi_\Gamma, \delta)$ *be a regular double representation and suppose that* G *is the additive group of a vector space over a prime field. Then* Γ *acts on* Ω *as a group of affine transformations of* V.

Maschke's Theorem for affine spaces over prime fields

Throughout this section p denotes the characteristic of a field and F_p denotes the prime field of characteristic p. A positive integer n is said to be prime to p if $p = 0$ or p does not divide n.

We shall need the following version of Maschke's Theorem:

Theorem 3.5.3. *Let* V *be a vector space over* F_p *and let* Γ *be a finite group of affine transformations of* V *such that* $|\Gamma|$ *is prime to* p*. Then* Γ *has a fixed point.*

A group G is said to be an F_p-group if there exists a sequence $G = G_0, G_1, \ldots, G_k = \{id\}$ such that

1. G_i is a characteristic subgroup of G.
2. G_i/G_{i+1} is isomorphic to a vector space over F_p.

Combining the lemmata of the previous subsection and Maschke's Theorem for affine spaces we get the following

Lemma 3.5.4. *Let* $(\Omega, G, \Gamma, \pi_G, \pi_\Gamma, \delta)$ *be a regular double representation and suppose that* G *is an* F_p*-group,* Γ *is finite and* $|\Gamma|$ *is prime to* p*. Then* Γ *has a fixed point in* Ω.

Maschke's Theorem for Moufang buildings

We will say that a Moufang building $\mathcal{M} = (\Delta, (U_C)_{C\in\mathcal{C}})$ has characteristic p if U_A is an F_p-group for each $A \in \Delta$. We have now the following

Theorem 3.5.5. *Let* $\mathcal{M}$ *be a Moufang building of characteristic* p*, and let* Γ *be a finite group of automorphisms of* $\mathcal{M}$*, such that* $|\Gamma|$ *is prime to* p*. Let* $\tilde{C}$ *be a* Γ*-chamber. Then there exists a* Γ*-chamber opposite to* $\tilde{C}$.

Proof. Let Ω be the set of all simplices opposite to $\tilde{C}$ in Δ. By Lemma 3.3.7 we get a regular double representation, if we put $G = U_{\tilde{C}}$. We can apply Lemma 3.5.4 to finish the proof.

3.6 Elation groups in the centralizer of Γ

Throughout this section $\mathcal{M}$ denotes a Moufang building. In view of the theorems stated below, it seems reasonable to make the following conjecture.

Conjecture: *Let $\Gamma \leq Aut(\mathcal{M})$. Let U be an elation subgroup of G which is normalized by Γ. Then there exists a simplex $A \in \tilde{\Delta}$ such that $U \leq U_A$.*

The following theorem is an easy consequence of the results due to A. Borel and J. Tits in [BT71]

Theorem 3.6.1. *The conjecture is true for all Moufang buildings associated to semisimple algebraic groups defined over perfect fields.*

In view of [BW76] we have also

Theorem 3.6.2. *The conjecture is true for all finite Moufang buildings.*

The conjecture above is of importance in our context, because it describes the relations between the groups associated to $\mathcal{M}$ and $\tilde{\mathcal{M}}$ in the situation of Theorem 3.4.8.

Theorem 3.6.3. *Let $\mathcal{M}$ be a Moufang building and G its associated group. Suppose, that the conjecture is true for $\mathcal{M}$. Let $\Gamma \leq Aut(\mathcal{M})$ and suppose that there exists a pair of two opposite Γ-chambers. Let $\tilde{G}_0$ be the group which is generated by the elations in G which are centralized by Γ. Then $\tilde{G}_0$ is the group associated to $\tilde{\mathcal{M}}$.*

The rest of this section makes a first step towards a proof of the conjecture above. We refer to [Ti62], where J. Tits indicates a proof of a similar statement. The kind of arguments proposed there are applied here. We introduce characteristic functions in order to make his ideas transparent.

Definition 3.6.4. Let H be a subgroup of G. A set $S \subseteq \Delta$ is called H-characteristic, if $N_{Aut\mathcal{M}}(H)$ leaves S invariant. A simplex $A \in \Delta$ is called H-characteristic if $\{A\}$ is H-characteristic.

We recall that $\mathcal{U}$ denotes the set of all elation subgroups.

Definition 3.6.5. Let $\mathcal{M}$ be a Moufang complex. A function $\mathcal{U} \longrightarrow \Delta$ is called a characteristic function for $\mathcal{M}$ if the following is satisfied:

1. If $U \neq \{id\}$ then $F(U) \neq \emptyset$.

2. F is invariant under automorphisms, i.e. for each $U \in \mathcal{U}$ and each $\gamma \in Aut\mathcal{M}$ we have
$$F(U^\gamma) = F(U)^\gamma.$$
In particular $F(U)$ is a U-characteristic simplex.

Characteristic functions are of particular interest in connection with the conjecture above, because of the following proposition, which can be proved by standard reduction arguments.

Proposition 3.6.6. *The conjecture is true if each irreducible Moufang building admits a characteristic function.*

Characteristic functions for Moufang buildings over certain diagrams

In this subsection $\mathcal{M}$ denotes a Moufang building of type M, where M is a Coxeter diagram.

Lemma 3.6.7. *If $M = A_1$, then there exists a characteristic function for $\mathcal{M}$.*

Proof. Let U be an elation group. There exists a unique chamber C in Δ with $U \leq U_C$. Hence, we may put $F(U) = C$.

To treat the case of generalized n-gons we begin with a preliminary definition.

Definition 3.6.8. Let $\mathcal{Y}$ be a connected set of chambers. Then $B(\mathcal{Y})$ denotes the set of chambers in $\mathcal{Y}$ sharing only one panel with another chamber in $\mathcal{Y}$.

The following is easy

Lemma 3.6.9. *Let $\mathcal{Y}$ be a convex set of chambers in a generalized n-gon. Then $\mathcal{Y} \setminus B(\mathcal{Y})$ is a convex set of chambers.*

We can now prove:

Lemma 3.6.10. *If M has rank 2, then there exists a characteristic function.*

Proof. Let U be an elation group and let $\mathcal{X}$ be the set of chambers fixed by U. Put $\mathcal{X}_0 = \mathcal{X}$ and $\mathcal{X}_{i+1} = \mathcal{X}_i \setminus B(\mathcal{X}_i)$. It is clear that all $\mathcal{X}_i$ are U-characteristic and since $\mathcal{X}_0$ is convex all $\mathcal{X}_i$ are convex by the previous lemma. There exists a $k \in N$ such that $\mathcal{X}_k = \mathcal{X}_{k+1}$. If $\mathcal{X}_k \neq \emptyset$, then $\mathcal{X}_k$ is a convex set of chambers such that $\overline{\mathcal{X}_k}$ is a weak gated chamber subcomplex. But this means that there are two opposite chambers in $\mathcal{X}_k$ and hence also in $\mathcal{X}$. By Lemma 3.3.6 1. this implies that U is the trivial group. So if $U \neq \{id\}$, then we have $\mathcal{X}_k = \emptyset$. It follows that $\bigcap_{C \in \mathcal{X}_{k-1}} C$ is a non-trivial U-characteristic simplex. Now we can choose the function which assigns to each non-trivial elation group the characteristic simplex constructed above.

In order to attack the diagrams A_n and C_n for $n \geq 2$ we consider the associated point-line-spaces. The following observation is an easy consequence of the axioms of a Moufang system:

Lemma 3.6.11. *Let $\mathcal{M}$ be of type A_n or C_n with $n \geq 2$ and let U be an elation group. Then*

(1) U acts as group of automorphisms of the associated point-line-space.

(2) If U fixes two points on a line l, then it fixes each point on l.

(3) If U fixes a subspace (resp. singular subspace) S, then S contains a point fixed by U and S is contained in a hyperplane (resp. maximal singular subspace) fixed by U.

Lemma 3.6.12. *If $\mathcal{M}$ has type A_n, then there exists a characteristic function.*

Proof. By [Ti74] the associated point-line-space is a projective space; we denote its set of points by $\mathcal{P}$. Let U be an elation group and let $\mathcal{H}$ denote the set of all hyperplanes in the projective space under consideration. Put

$$A = \{p \in \mathcal{P} | p^U = p\}$$

and

$$C = \{h \in \mathcal{H} | h^U = h\}$$

Clearly, A and C are non-trivial subspaces fixed by U, if $U \neq \{id\}$. The previous lemma implies that A is contained in a hyperplane fixed by U, and C contains a point fixed by U. Hence we have $A \cap C \neq \emptyset$ and $\langle A, C \rangle \neq \mathcal{P}$. Now $\{A \cap C, \langle A, C \rangle\}$ is a simplex in Δ which is fixed by any automorphism of $\mathcal{M}$ which normalises U. Note that this is also true for dualities. So we define F to be the function which assigns to each elation group the simplex constructed above.

Lemma 3.6.13. *If $\mathcal{M}$ has type C_n, then there exists a characteristic function.*

Proof. By [Ti74] the point line space under consideration is a polar space. Let $\mathcal{P}$ denote its set of points and $\mathcal{L}$ its set of lines. Put $X = \{p \in \mathcal{P} | p^U = p\}$ and $Y = \{l \in \mathcal{L} | |l \cap X| \geq 2\}$. Note first that X and Y are not empty and that for each

$l \in Y$ it follows by the lemma above that $l \subseteq X$. Now it is easily seen that the point line space (X, Y) satisfies the one-or-all axiom of Buekenhout-Shult (see [BS74]). So (X, Y) is a polar space. Suppose that its radical is empty. Then it follows by the third statement of our lemma above that this polar space has rank n, that $X = \mathcal{P}$ and finally that $U = \{id\}$. So if U is a non-trivial elation group, then the radical of its fixed point set is not empty, and it is clearly a singular subspace, which means that it is a vertex of Δ. Hence we may define our characteristic function as the function which assigns to each $U \in \mathcal{U}$ the radical of its fixed point set.

The case of D_n with $n \geq 5$ can be treated similarly to the case C_n, whereas for D_4 one has to be careful because of the possible trialities. Here one argues as in the case of polarities in the case of A_n. Combining the results of this subsection the following proposition can be shown by reduction arguments

Proposition 3.6.14. *The conjecture is true for all spherical Moufang buildings whose diagram contains no subdiagram of type E_6 or F_4.*

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