Poincaré-Hopf type formulas on convex sets of Banach spaces

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Abstract

We consider locally Lipschitz and completely continuous maps $A : C \to C$ defined on a closed convex subset $C \subset X$ of a Banach space X. The main interest lies in the case when C has empty interior. We establish Poincaré-Hopf type formulas relating fixed point index information about A with homology Conley index information about the semiflow on C induced by $-\mathrm{id} + A$. If A is a gradient we also obtain results on the critical groups of isolated fixed points of A in C.

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1 Introduction

Let X be a Banach space, $C \subset X$ a closed and convex subset. It is allowed that C has empty interior as is the case for order intervals in Sobolev spaces, for instance. We consider completely continuous maps $A : C \to C$ which are locally Lipschitz continuous. Then the vector field $-\mathrm{id} + A$ induces a semiflow $\varphi : \mathcal{D}(\varphi) \subset [0, \infty) \times C \to C$ on C; see Section 2. For such kind of flow Conley index theory as developed in [12] applies. The goal of this paper is to relate homology Conley index information about φ with fixed point index information about A in the spirit of the Poincaré-Hopf formula. In applications, C has empty interior so the classical Poincaré-Hopf formula on manifolds with boundary and the generalizations we are aware of do not apply. If $\dim X < \infty$ and $\operatorname{int} C \neq \emptyset$ then our results can be deduced from [13]. In that situation we present however a rather simple proof of the Poincaré-Hopf formula which we haven't seen in the literature. Our results are applicable to a variety of problems and save calculations in

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each individual case.

We use the following standard notation. Given a set $N \subset C$ its *invariant set* is defined by

$$\text{inv}(N) = \text{inv}(N, \varphi) := \{x \in N : \varphi(t, x) \in N \text{ for all } t \in \mathbb{R}\}.$$

Here we write $\varphi(t, x)$ for $t < 0$, $x \in C$, if there exists $y \in C$ with $\varphi(-t, y) = x$. There exists at most one such y, so we may set $\varphi(t, x) := y$ if it exists. Thus inv(N) consists of all $x \in N$such that $\varphi(t, x)$ exists for all $t \in \mathbb{R}$ and lies in N. N is said to be an *isolating neighbourhood* of inv(N) if N is closed and bounded and inv$(N) \subset$ int(N). Here and in the sequel all topological notions refer to the topology of C induced from X, in particular int$(N) =$ int$_C(N)$. A set $S \subset C$ is then said to be *isolated invariant* if an isolating neighbourhood N exists with $S =$ inv(N). In that case S is compact and one can define the Conley index $\mathcal{C}_C(N, \varphi) = \mathcal{C}_C(S, \varphi)$; see Section 2. Moreover, since A cannot have any fixed points on ∂N, its fixed point index ind$_C(A, N) \in \mathbb{Z}$ is defined. We refer to [1, 6, 9] for its definition and properties.

For our first result let H_* denote singular homology with coefficients in a commutative ring R, e.g. $\mathbb{Z}$ or a field. For a pair (X, Y) of topological spaces such that $H_*(X, Y)$ is finitely generated, $\chi(X, Y) = \sum_{i=0}^{\infty}(-1)^i \text{rank}\, H_i(X, Y)$ denotes its Euler characteristic.

Theorem 1.1. *Given an isolating neighbourhood $N \subset C$ then its Conley index $\mathcal{C}_C(N, \varphi)$ has the homotopy type of a finite pointed CW-complex, hence the homology Conley index $H_*(\mathcal{C}_C(N, \varphi))$ is finitely generated. There holds the Poincaré-Hopf formula:* ind$_C(A, N) = \chi(\mathcal{C}_C(N, \varphi))$.

We now specialize to the case where X is a Hilbert space and $A = \nabla g$ is the gradient of a C^1-function $g : D \to \mathbb{R}$ defined on an open neighbourhood $D \subset X$ of C. Then $-\text{id} + A$ is the negative gradient of the functional $f(x) = \frac{1}{2}\|x\|^2 - g(x)$. We say that $a \in \mathbb{R}$ is a *regular value of f in C* if f does not have any critical points in $C \cap f^{-1}(a)$, i.e., A does not have any fixed points in $C \cap f^{-1}(a)$. In this setting we use some standard notation. For $a \in \mathbb{R}$ we set $f^a := \{x \in D : f(x) \le a\}$ and for $a < b$ we set $f_a^b := \{x \in D : a \le f(x) \le b\}$. f is said to satisfy the *Palais-Smale condition $(PS)_c$ in a set $M \subset D$* if every sequence $(x_n)_n$ in M with $f(x_n) \to c$ and $f'(x_n) \to 0$ has a convergent subsequence.

Theorem 1.2. *Let $a < b$ be regular values of f in C such that f satisfies $(PS)_c$ in $C \cap f_a^b$ for $c \in [a, b]$. Then* ind$_C(A, f_a^b \cap C) = \chi(f^b \cap C, f^a \cap C)$.

Observe that ind$_C(A, f_a^b \cap C) \in \mathbb{Z}$ is well defined because Fix$A \cap f_a^b$ is compact as a consequence of the Palais-Smale condition in $f_a^b \cap C$, and Fix$A \cap f_a^b \subset$ int$(f_a^b \cap C)$.

Next we state a result relating the local fixed point index of an isolated fixed point $x_0 \in C$ of A and its *critical groups* $H_*\big(f^c \cap C, f^c \cap C \setminus \{x_0\}\big)$ as a critical point of f in C; here $c = f(x_0)$.

Theorem 1.3. *If $x_0 \in C$ is an isolated fixed point of A in C and $c = f(x_0)$ then*

$$H_*\big(\mathcal{C}_C(\{x_0\},\varphi)\big) \cong H_*\big(f^c \cap C, f^c \cap C \setminus \{x_0\}\big)$$

and therefore $\operatorname{ind}_C(A,x_0) = \chi\big(\mathcal{C}_C(\{x_0\},\varphi)\big) = \chi\big(f^c \cap C, f^c \cap C \setminus \{x_0\}\big)$.

As a corollary of Theorem 1.3 we obtain that the critical groups $H_*\big(f^c \cap C, f^c \cap C \setminus \{x_0\}\big)$ are homotopy invariant.

Corollary 1.4. *Let $g_\lambda : D \to \mathbb{R}$, $0 \le \lambda \le 1$, be a continuous family of C^1-functions, such that $A_\lambda := \nabla g_\lambda$ induces a continuous family of locally Lipschitz and completely continuous maps $A_\lambda : C \to C$. Set $f_\lambda(x) = \frac{1}{2}\|x\|^2 - g_\lambda(x)$. If A_λ has a continuous family of fixed points $x_\lambda \in C$, $\lambda \in [0,1]$, which are isolated in C then the critical groups $H_*\big(f_\lambda^{c_\lambda} \cap C, f_\lambda^{c_\lambda} \cap C \setminus \{x_\lambda\}\big)$, $c_\lambda := f_\lambda(x_\lambda)$, are independent of $\lambda \in [0,1]$.*

In our last result we compute the critical groups in Theorem 1.3 of an isolated fixed point of A in C provided a nondegeneracy condition holds. We first need to recall some concepts from [4,5,7]. The *tangent wedge* of a point $x \in C$ is defined by

$$W_x := \bigcup_{t>0} t \cdot (C - x) = \{y \in X : x + \varepsilon y \in C \text{ for some } \varepsilon > 0\}.$$

Clearly W_x is a wedge, i.e., $ty \in W_x$ for every $y \in W_x$ and $t > 0$. The *tangent space*

$$T_x := \overline{W}_x \cap (-\overline{W}_x)$$

is a closed linear subspace of X. If A is differentiable at x with derivative $L = DA(x) : X \to X$ then $L(W_x) \subset W_x$ and $L(T_x) \subset T_x$. A fixed point $x \in C$ of A is said to be a *nondegenerate fixed point of A in C* if A is differentiable at x and $Ly \neq y$ for all $y \in \overline{W}_x \setminus \{0\}$. We say that L *repels* $y \in \overline{W}_x \setminus T_x$ if there exists $t > 1$ with $Ly - ty \in T_x$.

Theorem 1.5. *Let $x_0 \in C$ be a nondegenerate fixed point of A in C, $c = f(x_0)$, m the sum of the multiplicities of the eigenvalues in $(1,\infty)$ of $L = DA(x_0)$ restricted to T_{x_0}. Then*

$$H_k\big(f^c \cap C, f^c \cap C \setminus \{x_0\}\big) \cong \begin{cases} 0 & \text{if } L \text{ repels at least one point in } \overline{W}_{x_0} \setminus T_{x_0}; \\ \delta_{km} R & \text{else.} \end{cases}$$

Remark 1.6. a) *An important special case is when $\overline{W}_{x_0} = T_{x_0} = X$. Then Theorem* 1.5 *says that the critical groups of f in C are isomorphic to the full critical groups of f in X:*

$$H_k\big(f^c \cap C, f^c \cap C \setminus \{x_0\}\big) \cong H_k\big(f^c, f^c \setminus \{x_0\}\big).$$

This happens for instance if $X = H_0^1(\Omega)$, $C = \{x \in X : x \ge 0$ a.e.$\}$, and $x_0 > 0$ in Ω. Observe that $\operatorname{int} C = \emptyset$.
b) *Theorem* 1.5 *can frequently be used to compute critical groups even if x_0 is a degenerate fixed point of A in C by looking at perturbations and using homotopy invariance.*

We conclude this section by mentioning one application where Theorem 1.5 simplifies arguments and leads to a conceptually more satisfying proof.

Remark 1.7. *In [2] we were interested in positive solutions $u, v > 0$ of the system*

$$\begin{cases} -\Delta u + u = \mu_1 u^3 + \beta v^2 u & \text{in } \Omega \\ -\Delta v + v = \mu_2 v^3 + \beta u^2 v & \text{in } \Omega \end{cases} \tag{1.1}$$

of nonlinear Schrödinger (or Gross-Pitaevskii) type equations on a bounded (or radially symmetric unbounded) domain $\Omega \subset \mathbb{R}^N$, $N \le 3$. Here $\mu_1, \mu_2 > 0$ are fixed and β is taken as bifurcation parameter. For each positive solution $w \in H_0^1(\Omega)$ of the equation $-\Delta w + w = w^3$ there exists a trivial branch

$$\mathcal{T}_w = \big\{(\beta, u_\beta, v_\beta) : -\sqrt{\mu_1\mu_2} < \beta < \min\{\mu_1, \mu_2\}\big\}$$

of positive solutions of (1.1). *[2, Theorem 2.1] states the existence of a sequence of bifurcation points $(\beta_k, u_{\beta_k}, v_{\beta_k})$ on $\mathcal{T}_w$ for positive solutions of* (1.1). *Solutions of* (1.1) *are obtained as critical points of an associated functional $J_\beta : X = H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$. In the proof we showed that the critical groups $H_k\big(J_\beta^c, J_\beta^c \setminus \{(u_\beta, v_\beta)\}\big)$ of the trivial solutions in $\mathcal{T}_w$ change infinitely often along the trivial branch at parameter values β_k. The homotopy invariance of the critical groups of isolated critical points (see [8, Theorem 8.8], for instance) implies the bifurcation of critical points of J_β near $(\beta_k, u_{\beta_k}, v_{\beta_k})$. In order to show that these bifurcating critical points are actually positive solutions we considered a modified functional J_β^+ whose critical points are positive solutions of* (1.1). *Since J_β^+ is only of class C^{2-0} the computation of the critical groups $H_k\big((J_\beta^+)^c, (J_\beta^+)^c \setminus \{(u_\beta, v_\beta)\}\big) \cong H_k\big(J_\beta^c, J_\beta^c \setminus \{(u_\beta, v_\beta)\}\big)$ required an ad-hoc argument based on some nontrivial results. Using Theorem* 1.5 *one can instead directly compute the critical groups $H_k\big(J_\beta^c \cap C, J_\beta^c \cap C \setminus \{(u_\beta, v_\beta)\}\big)$ of J_β in the cone $C = \{(u, v) \in X : u, v \ge 0 \text{ a.e.}\}$. Since these change infinitely often (at β_k) Corollary* 1.4 *yields the existence of the bifurcation points with bifurcation into the cone. Theorem* 1.5 *can also be applied to compute the critical groups of J_β in C at isolated "semitrivial" solutions $(u, 0)$ or $(0, v)$ of* (1.1). *This can be used to prove bifurcation of positive solutions from the set of semitrivial solutions. One can then deduce information on the critical groups of the bifurcating solutions except when the bifurcation is vertical. See [3] where the fixed point index in C has been applied.*

2 Some Conley index theory

As in the introduction X is a Banach space, $C \subset X$ a closed convex subset, and $A : C \to C$ is locally Lipschitz and completely continuous. The vector field $-\mathrm{id} + A$ then induces a semiflow

$$\varphi : \mathcal{D}(\varphi) = \{(t, x) \in [0, \infty) \times C : 0 \le t < T(x)\} \to C.$$

To see this one rewrites the initial value problem

$$
\begin{cases} \dot{x} = -x + A(x) \\ x(0) = x_0 \end{cases} \tag{2.1}
$$

as integral equation using the variation-of-constant formula

$$
x(t) = e^{-t}x_0 + \int_0^t e^{s-t}A(x(s))ds. \tag{2.2}
$$

Observe that given a continuous function $x : [0,T] \to C$ then

$$
\int_0^t e^{s-t}A(x(s))ds \in (1-e^{-t})C \quad \text{for } t \in [0,T]
$$

because $\int_0^t e^{s-t}ds = 1-e^{-t}$. Therefore

$$
e^{-t}x_0 + \int_0^t e^{s-t}A(x(s))ds \in C \quad \text{for } t \in [0,T]
$$

so that one can apply a standard iteration method to construct $\varphi(t,x_0)$ as unique solution of (2.1). In general the solution of (2.1) cannot be extended to $t < 0$. However, if for some $x_0 \in C$ and some $t < 0$ there exists $y_0 \in C$ with $\varphi(-t,y_0) = x_0$ then this y_0 is uniquely determined and we define $\varphi(t,x_0) := y_0$. In fact, suppose $x,y : [-\delta,\delta] \to C$ are solutions of (2.1). Then (2.2) holds for $t \in [-\delta,\delta]$, and also with y instead of x. This implies for $t \in [-\delta,\delta]$:

$$
\begin{aligned}
\|x(t)-y(t)\| &= \left\| \int_0^t e^{s-t}\big(A(x(s)) - A(y(s))\big)ds \right\| \\
&\le |t|e^{|t|}K \max\{\|x(s)-y(s)\| : |s| \le t\}
\end{aligned}
$$

where K is a Lipschitz constant for A. We deduce $x(t) = y(t)$ for all t with $|t|e^{|t|}K < 1$.

For the semiflow φ we recall a few basic concepts from Conley index theory on metric spaces (which are not necessarily locally compact) due to Rybakowski [12]. A closed subset $N \subset C$ is said to be *strongly admissible*, if the following two conditions hold:

(A_1) if $x \in N$ is such that $\varphi(t,x) \in N$ for all $0 \le t < T(x)$ then $T(x) = \infty$;

(A_2) given sequences $x_n \in N$, $t_n \to \infty$, such that $\varphi([0,t_n],x_n) \subset N$ for all $n \in \mathbb{N}$, then $\varphi(t_n,x_n)$, $n \in \mathbb{N}$, has a convergent subsequence.

In our situation we have the following simple result concerning admissibility.

Lemma 2.1. *Every bounded set is strongly admissible.*

Proof. Let $N \subset C$ be bounded and recall the variation-of-constant formula:

$$\varphi(t,x) = e^{-t}x + \int_0^t e^{s-t}A(\varphi(s,x))ds.$$

Given $x \in N$ with $\varphi(t,x) \in N$ for all $0 \le t < T(x)$, there holds:

$$\int_0^t e^{s-t}A(\varphi(s,x))ds \in \operatorname{clos}\operatorname{conv}(A(N) \cup \{0\}) =: M \quad \text{for every } 0 \le t < T^+(x).$$

M is compact because N is bounded and A is completely continuous. Consequently $T^+(x) = \infty$, and (A_1) follows. Similarly, if $x_n \in N$, $t_n \to \infty$, are as in (A_2) then $e^{-t_n}x_n \to 0$, and $\varphi(t_n,x_n) - e^{-t_n}x_n \in M$ has a convergent subsequence. Therefore $\varphi(t_n,x_n)$ has a convergent subsequence, and (A_2) follows. $\square$

Given a strongly admissible isolating neighbourhood $N \subset C$ there exist (quasi-)index pairs (N_1,N_2) in N, and the pointed homotopy type of the quotient space N_1/N_2 is independent of the choice of the (quasi-)index pair. This homotopy type is the Conley index of N which we denote by $\mathcal{C}(N,\varphi)$; see [12, Chapter 1] for details.

We need the following weak version of the continuation invariance. It is a consequence of the more general continuation invariance [12, Chapter 1, Theorem 12.2].

Theorem 2.2. *Let $A_\lambda : C \to C$, $0 \le \lambda \le 1$, be a continuous family of locally Lipschitz and completely continuous maps. Let φ_λ be the associated family of semiflows on C satisfying*

$$\frac{d}{dt}\varphi_\lambda(t,x) = A_\lambda(\varphi(t,x)).$$

Suppose $N \subset C$ is an isolating neighbourhood for every φ_λ. Then the Conley indices $\mathcal{C}(N,\varphi_\lambda)$ are independent of $\lambda \in [0,1]$.

In the proof of Theorem 1.5 we also need the following reduction property of the homology Conley index.

Theorem 2.3. *Let $C_0 \subset C$ be closed convex, and suppose $A(C) \subset C_0$, so C_0 is positively invariant under φ and there is an induced semiflow $\varphi|_{C_0}$. If $N \subset C$ is an isolating neighbourhood for φ then $N \cap C_0$ is an isolating neighbourhood for $\varphi|_{C_0}$ and the homology Conley indices $H_*\big(\mathcal{C}_C(N,\varphi)\big)$ and $H_*\big(\mathcal{C}_{C_0}(N \cap C_0, \varphi|_{C_0})\big)$ are isomorphic.*

One might expect that not only the homology Conley indices are isomorphic but that even the Conley indices are the same: $\mathcal{C}_C(N,\varphi) = \mathcal{C}_{C_0}(N \cap C_0, \varphi|_{C_0})$. This does not seem to be easy to prove, though.

Proof. Consider $x \in C$ such that $\varphi(t,x)$ exists for all $t < 0$ and remains bounded. For $t < 0$ the variation-of-constant formula yields

$$x - e^t\varphi(t,x) = \int_t^0 e^s A(\varphi(s,x))ds \in (1-e^t)C_0$$

because $A(C) \subset C_0$ and $\int_t^0 e^s ds = 1 - e^t$. Letting $t \to -\infty$ we deduce that $x \in C_0$. It follows that

$$N^- := \{x \in N : \ \varphi(t,x) \in N \text{ for all } t < 0\} \subset C_0, \tag{2.3}$$

that $\mathrm{inv}(N,\varphi) = \mathrm{inv}(N \cap C_0, \varphi|_{C_0})$, and that $N \cap C_0$ is an isolating neighbourhood for $\varphi|_{C_0}$. Given an index pair (N_1, N_2) for φ in N, [11, Theorem 4.6] implies that the inclusion

$$(N_1 \cap N^-, N_2 \cap N^-) \hookrightarrow (N_1, N_2)$$

induces an isomorphism in Alexander-Spanier cohomology:

$$H^*(N_1/N_2, \{N_2\}) \cong H^*(N_1, N_2) \cong H^*(N_1 \cap N^-, N_2 \cap N^-).$$

One easily checks that $(N_1 \cap C_0, N_2 \cap C_0)$ is an index pair for $\varphi|_{C_0}$ in $N \cap C_0$. Hence, using (2.3) and [11, Theorem 4.6] once more we also have an isomorphism

$$H^*(N_1 \cap C_0/N_2 \cap C_0, \{N_2 \cap C_0\}) \cong H^*(N_1 \cap C_0, N_2 \cap C_0) \cong H^*(N_1 \cap N^-, N_2 \cap N^-).$$

It follows that the Conley indices $\mathcal{C}_C(N,\varphi)$ and $\mathcal{C}_{C_0}(N \cap C_0, \varphi|_{C_0})$ have isomorphic Alexander-Spanier cohomology groups with arbitrary coefficients. Then also the (co-)homology groups are isomorphic because the Conley indices have the homotopy type of finite CW-complexes by Theorem 1.1. (The proof of Theorem 1.1 in the next section does not require Theorem 2.3.) $\square$

3 Proof of Theorem 1.1

The closure $\overline{A(N)}$ is compact because N is bounded and A is completely continuous. Hence, for fixed $\varepsilon > 0$ there exist points $x_1, \dots, x_n \in C$ such that $\operatorname{clos} A(N) \subset \bigcup_{i=1}^n U_\varepsilon(x_i)$. We consider the partition of unity subordinated to this covering given by

$$\pi_i : \operatorname{clos} A(N) \to [0,1], \quad \pi_i(x) := \frac{\operatorname{dist}(x, N \setminus U_\varepsilon(x_i))}{\sum_{j=1}^n \operatorname{dist}(x, N \setminus U_\varepsilon(x_j))}.$$

Now we define

$$Y := \left\{ \sum_{i=1}^n \alpha_i x_i : \alpha_i \in \mathbb{R}, \ \sum_{i=1}^n \alpha_i = 1 \right\}$$

and approximate A by the finite-dimensional map

$$A_\varepsilon : C \to C \cap Y, \quad A_\varepsilon(x) := \sum_{i=1}^{n} \pi_i(A(x))x_i.$$

We may assume that $0 \in C$ and $x_1 = 0$, so that Y is a finite-dimensional linear subspace of X. Observe that $\mathrm{int}_Y(C \cap Y) \neq \emptyset$. Clearly we have $\|A_\varepsilon(x) - A(x)\| \leq \varepsilon$ for every $x \in N$. Since the maps $\pi_1, \dots, \pi_n$, hence A_ε, are locally Lipschitz continuous the vector field $-\mathrm{id} + \lambda A_\varepsilon + (1-\lambda)A$, with $\lambda \in [0,1]$, induces a semiflow $\varphi_{\varepsilon,\lambda}$ on C. For $0 < \varepsilon \ll 1$, N is an isolating neighbourhood for $\varphi_{\varepsilon,\lambda}$ for every $\lambda \in [0,1]$. We fix such an ε from now on. The continuation property of the Conley index yields

$$\mathcal{C}_C(N,\varphi) = \mathcal{C}_C(N,\varphi_{\varepsilon,0}) = \mathcal{C}_C(N,\varphi_{\varepsilon,1}).$$

We choose $y_0 \in \mathrm{int}_Y(C \cap Y)$ and define

$$B_\delta(x) := (1-\delta)A_\varepsilon(x) + \delta y_0.$$

Let ψ_δ be the semiflow on C induced by $-\mathrm{id} + B_\delta$. As before, for $0 < \delta \ll 1$, N is an isolating neighbourhood for ψ_δ, hence

$$\mathcal{C}_C(N,\varphi_{\varepsilon,1}) = \mathcal{C}_C(N,\psi_0) = \mathcal{C}_C(N,\psi_\delta).$$

We fix such a δ now and set $\psi = \psi_\delta$. We claim that

$$\mathcal{C}_C(N,\psi) = \mathcal{C}_Y(N \cap Y,\psi). \tag{3.1}$$

In order to see (3.1) we first observe that $\mathrm{inv}(N \cap Y, \psi) \subset \mathrm{int}_Y(C \cap Y)$ because $B_\delta(C) \subset \mathrm{int}_Y(C \cap Y)$ by our choice of y_0. In fact, $C \cap Y$ is strongly positive invariant for ψ_δ, i.e., $\psi_\delta(t,x) \in \mathrm{int}_Y(C \cap Y)$ for every $y \in C$, $t > 0$. It follows that ψ induces a flow in the open subset $\mathrm{int}_Y(C \cap Y)$ of the finite-dimensional space Y. According to [14, Theorem 2.4] there exists a $\mathcal{C}^\infty$ isolating block with corners (M, M^-) for the isolated invariant set

$$S := \mathrm{inv}(N \cap Y, \psi) \subset M \subset N \cap Y.$$

This means that $M \subset N \cap Y$ is a compact isolating neighbourhood of S with exit set M^-, $\partial M = M^- \cup M^+$ is a union of $\mathcal{C}^\infty$-manifolds $M^\pm$ with boundaries $\partial M^- = \partial M^+ = M^- \cap M^+$, so M is a ∂-manifold with corners. This implies that the Conley index $\mathcal{C}_Y(N \cap Y, \psi) = M/M^-$ has the homotopy type of a finite pointed CW-complex and that $H_*(\mathcal{C}(N,\varphi)) \cong H_*(M/M^-, \{M^-\}) \cong H_*(M, M^-)$ is finitely generated. Moreover, ψ is transverse to $M^\pm$, i.e., at a point $y \in M^\pm$ the vector $-y + B_\delta(y)$ is transverse to $T_yM^\pm$. It follows that for such y there exists $\rho_y > 0$ so that the vector $-y + B_\delta(x)$ is transverse to $T_yM^\pm$ for every $x \in C$ with $\|x - y\| \leq \rho_y$.

We choose a closed complement Z of Y in X, and write the elements of $X \cong Y \times Z$ as $x = (y,z) \in Y \times Z$. Using the compactness of ∂M it follows that there exists $\rho > 0$ such that for $y = (y,0) \in M^\pm$ and $\|z\| \le \rho$, the vector $-(y,z) + (B_\delta(y,z),0)$ is transverse to $M^\pm \times N_\rho(0,Z)$ at (y,z); here $N_\rho(0,Z) = \{z \in Z : \|z\| \le \rho\}$. Consequently, $\big((M \times N_\rho(0,Z)) \cap C, (M^- \times N_\rho(0,Z)) \cap C\big)$ is an isolating block for the invariant set $\mathrm{inv}(N,\psi)$ of the semiflow ψ in C. Now (3.1) follows because

$$\begin{aligned}\mathcal{C}_C(N,\psi) &= \big((M \times N_\rho(0,Z)) \cap C\big) / \big((M^- \times N_\rho(0,Z)) \cap C\big) \\ &\simeq M/M^- = \mathcal{C}_Y(N \cap Y, \psi)\end{aligned}$$

Now we make the same reduction process for the fixed point index. The homotopy invariance and the commutativity property of the fixed point index imply

$$\begin{aligned}\mathrm{ind}_C(A,N) &= \mathrm{ind}_C(A_\varepsilon,N) = \mathrm{ind}_C(B_\delta,N) = \mathrm{ind}_{C\cap Y}(B_\delta, N \cap Y) \\ &= \mathrm{ind}_Y(B_\delta, N \cap Y).\end{aligned} \tag{3.2}$$

Here we use the same choices of ε and δ as above. In fact, it suffices that $\varepsilon < \mathrm{dist}(S, C \setminus N)$. By (3.1) and (3.2) it remains to prove

$$\mathrm{ind}_Y(B_\delta, N \cap Y) = \chi\big(\mathcal{C}_Y(N \cap Y, \psi)\big) \tag{3.3}$$

This is the Poincaré-Hopf formula in the finite-dimensional setting essentially going back to Morse [10]. A version which applies here can be found in [12, Chapter 3, Theorem 3.8].

In our setting the Poincaré-Hopf formula (3.3) is actually very easy to prove. Since this formula is the core of our paper and since we haven't seen the following proof in the literature we present it here. Recall the isolating block with corners (M, M^-) from above. The homotopy

$$h : [0,1] \times (N \cap Y) \to Y, \quad h(t,x) := \begin{cases} B_\delta(x) & t = 0 \\ \dfrac{1}{t}(\psi(t,x) - x) + x & t > 0 \end{cases}$$

is continuous and there exists $t_0 > 0$ such that

$$h(t,x) = x,\ 0 \le t \le t_0 \qquad \Longrightarrow \qquad x \in S \subset \mathrm{int}\, M.$$

It follows that

$$\mathrm{ind}(B_\delta, N \cap Y) = \mathrm{ind}(B_\delta, M) = \mathrm{ind}(h(0,\,.\,), M) = \mathrm{ind}(h(t_0,\,.\,), M). \tag{3.4}$$

Next the homotopy

$$[0,1] \times (N \cap Y) \to Y, \quad (s,x) \mapsto \left(\frac{1-s}{t_0} + s\right)\big(\psi(t_0,x) - x\big) + x$$

shows that

$$\text{ind}(h(t_0, \,.\,), M) = \text{ind}(\psi(t_0, \,.\,), M).$$

Now consider the map

$$\tau : M \to [0, \infty), \quad \tau(x) := \min \big\{ t_0, \sup\{t \geq 0 : \psi(s, x) \in M \setminus M^- \text{ for all } s \in [0, t]\} \big\}$$

which is continuous because (M, M^-) is an isolating block. If $x \in M^-$ then $\tau(x) = \sup \emptyset = 0$. Consequently the map

$$f : M \to M, \quad f(x) := \psi(\tau(x), x),$$

is well defined and continuous. Clearly $\text{Fix}(f) = \text{Fix}(\psi^{t_0}) \cup M^-$ and there are disjoint neighbourhoods V, W of $\text{Fix}(\psi^{t_0})$ and M^-, respectively, such that $f(x) = \psi(t_0, x) = \psi^{t_0}(x)$ for $x \in V$, and $f(x) \in M^-$ for $x \in W$. It follows that

$$\begin{aligned}
\chi(M) &= \text{ind}(\text{id}, M) = \text{ind}(f, M) = \text{ind}(f, V) + \text{ind}(f, W) \\
&= \text{ind}(\psi^{t_0}, V) + \text{ind}(f, W) = \text{ind}(\psi^{t_0}, V) + \text{ind}(f, M^-) \\
&= \text{ind}(\psi^{t_0}, V) + \text{ind}(\text{id}, M^-) = \text{ind}(\psi^{t_0}, V) + \chi(M^-) \\
&= \text{ind}(\psi^{t_0}, M) + \chi(M^-).
\end{aligned} \tag{3.5}$$

Here the first and the second to last equalities are consequences of the Lefschetz index formula, which applies because M and M^- are compact ENR's. The second holds because f is homotopic to the identity using the homotopy

$$H : [0, 1] \times M \to M, \quad H(t, x) = \psi(t\tau(x), x).$$

The third equality is a consequence of the additivity property of the fixed point index, the fourth is obvious by our choice of V. The fifth follows from the commutativity property of the fixed point index because f retracts W onto M^-, the sixth is trivial because f is the identity on M^-, and finally, the last uses the excision property of the fixed point index. The Poincaré-Hopf formula (3.3) follows from (3.4) and (3.5) immediately:

$$\begin{aligned}
\text{ind}_Y(B_\delta, N \cap Y) &= \text{ind}(\psi^{t_0}, M) = \chi(M) - \chi(M^-) = \chi(M, M^-) = \chi(M/M^-, [M^-]) \\
&= \chi\big(\mathcal{C}_Y(N \cap Y, \psi)\big).
\end{aligned}$$

Here the fourth equality holds because the inclusion $M^- \subset M$ is a cofibration (M^- is a deformation retract of a neighbourhood in M).

4 Proof of Theorems 1.2, 1.3, and Corollary 1.4

Proof of Theorem 1.2. We set $F := \text{Fix}(A) \cap f_a^b \subset \text{int}\, f_a^b$ and $S := \text{inv}(\varphi, f_a^b)$. The proof consists of several steps.

STEP 1: S is bounded.
Suppose to the contrary that there exist $x_n \in S$, $n \in \mathbb{N}$, with $\|x_n\| \to \infty$. Then we define

$$t_n := \inf\{t \geq 0 : \text{dist}(\varphi(s, x_n), F) \geq 1 \text{ for all } s \in [0, t]\}.$$

Clearly $t_n < \infty$ if $\text{dist}(x_n, F) > 1$ because $\text{dist}(\varphi(s, x_n), F) \to 0$ as $t \to \infty$. Since f satisfies the Palais-Smale condition in $C \cap f_a^b$ we have

$$\delta := \inf\{\|\nabla f(x)\| : x \in C \cap f_a^b,\ \text{dist}(x, F) \geq 1\} > 0.$$

We obtain a contradiction as follows:

$$\begin{aligned} b - a \geq f(x_n) - f(\varphi(t_n, x_n)) &= -\int_0^{t_n} \frac{d}{dt} f(\varphi(t, x_n)) dt = \int_0^{t_n} \|\nabla f(\varphi(t, x_n))\|^2 dt \\ &\geq \delta \int_0^{t_n} \|\nabla f(\varphi(t, x_n))\| dt = \delta \int_0^{t_n} \left\| \frac{d}{dt} \varphi(t, x_n) \right\| dt \geq \delta \|x_n - \varphi(t_n, x_n)\| \\ &\geq \delta(\|x_n\| - \|\varphi(t_n, x_n)\|) \geq \delta(\|x_n\| - 1 - \sup\{\|x\| : x \in F\}) \to \infty \end{aligned}$$

STEP 2: S is compact
Since S is closed it suffices to prove that S is precompact. This follows from the fact that $S = \varphi^t(S)$ for $t \geq 0$ and the variation-of-constant formula:

$$\varphi^t(x) = e^{-t}x + \int_0^t e^{s-t} A(\varphi^s(x)) ds.$$

The first summand $e^{-t}x \in e^{-t}S$ lies in an arbitrarily small ball for $t \to \infty$ because S is bounded. Concerning the second summand, $A(\varphi^s(x)) \in A(S)$ for $x \in S$, $s \geq 0$, and $A(S)$ is precompact. Then $M := \{\lambda y : y \in A(S),\ 0 \leq \lambda \leq 1\}$ is precompact and $\int_0^t e^{s-t} A(\varphi^s(x)) ds \in t \cdot \overline{\text{conv}} M$ where $\overline{\text{conv}} M$ denotes the closed convex hull of M, which is compact. Thus the integral lies in a compact set and STEP 2 follows easily.

Now we choose a bounded neighbourhood $U \subset \{x \in C : a < f(x) < b\}$ of S and set

$$N := \{\varphi^t(x) : x \in U,\ t \geq 0,\ f(\varphi^t(x)) \geq a\}, \quad N^- := N \cap f^{-1}(a).$$

STEP 3: (N, N^-) is an index pair for S and

$$\text{ind}_C(A, C \cap f_a^b) = \chi(N/N^-, [N^-]) = \chi(\mathcal{C}_C(S, \varphi)).$$

That N is bounded can be proved in the same way as the boundedness of S in STEP 1. The Palais-Smale condition implies that N is closed. Clearly $S \subset U \subset \text{int}\,(N \setminus N^-)$. That N^- is an exit set is also obvious.

We fix some $T > 0$ and define

$$\tau = \tau_T : C \cap f^b \to [0, \infty], \quad \tau(x) = \min \{T, \sup\{t \geq 0 : f(\varphi^t(x)) \geq a\}\},$$

where $\tau(x) := T$ if $f(\varphi^t(x)) > a$ for all $t \geq 0$. We also consider the deformation

$$h = h_T : [0, 1] \times (C \cap f^b) \to C \cap f^b, \quad h(t, x) := \varphi(t\tau(x), x). \tag{4.1}$$

STEP 4: τ, h are continuous and $h(1, C \cap f^b) \subset \text{int}\,(N) \cup (C \cap f^a)$ for T large.
The continuity of τ, hence of h, is easy to prove. In order to see the inclusion suppose to the contrary that there exists $T_n \to \infty$ and $x_n \in C \cap F^b$ with $\varphi(T_n, x_n) \notin \text{int}\,(N) \cup f^a$. Since $\text{int}\,(N) \cup f^a$ is positive invariant by the construction of N, it follows that $\varphi(t, x_n) \notin \text{int}\,(N) \cup f^a$ for $0 \leq t \leq T_n$, hence,

$$\|\nabla f(\varphi(t, x_n))\| \geq \delta := \inf\{\|\nabla f(x)\| : x \in f_a^b \setminus \text{int}\,(N)\} > 0$$

for every $t \in [0, T_n]$. This yields the contradiction

$$b - a \geq f(x) - f(\varphi(t, x_n)) = \int_0^{T_n} \|\nabla f(\varphi(t, x_n))\|^2 dt \geq T_n \delta^2 \to \infty$$

STEP 5: $\chi(\mathcal{C}_C(S, \varphi)) = \chi(C \cap f^b, C \cap f^a)$
Choose T as in STEP 4 and consider $h = h_T$ as in (4.1). Then we have:

$$\begin{aligned} \chi(C \cap f^b, C \cap f^a) &= \chi(h(1, C \cap f^b), C \cap f^a) = \chi(h(1, C \cap f^b) \cap N, C \cap f^a \cap N) \\ &= \chi(\mathcal{C}_C(S, \varphi)) \end{aligned} \tag{4.2}$$

The first equality is clear because the two pairs of topological spaces are homotopy equivalent. The second equality is a consequence of the excision property of homology. The pair $(h(1, C \cap f^b) \cap N, C \cap f^a \cap N)$ is a regular index pair for S which implies the third equality.

Theorem 1.2 is a consequence of STEP 3 and STEP 5. □

Proof of Theorem 1.3. Let

$$W^\pm := \{x \in C : \varphi^t(x) \to x_0 \text{ as } t \to \pm\infty\}$$

be the positive (negative) invariant set of x_0 with respect to φ. For $\varepsilon > 0$ we define

$$\begin{aligned} N_\varepsilon := &\{\varphi(t, x) : x \in C \cap f^{c+\varepsilon},\ \text{dist}(x, W^+) \leq \varepsilon,\ t \geq 0,\ f(\varphi(t, x)) \geq c - \varepsilon\} \\ &\cup \{x \in W^- : f(x) \geq c - \varepsilon\}. \end{aligned}$$

and

$$N_\varepsilon^- := N_\varepsilon \cap f^{-1}(c - \varepsilon).$$

It is easy to check that $(N_\varepsilon, N_\varepsilon^-)$ is an index pair for $S = \{x_0\}$ in C provided $0 < \varepsilon \ll 1$. Since $\nabla f = \mathrm{id} - A$ and A is completely continuous, f satisfies the Palais-Smale condition in bounded sets, in particular in N_ε. It follows easily that N_ε^- is a deformation retract of $N_\varepsilon \cap f^c \setminus \{x_0\}$. Therefore $N_\varepsilon^- \subset N_\varepsilon$ is a cofibration and

$$H_*\big(\mathcal{C}_C(\{x_0\}, \varphi)\big) \cong H_*\big(\mathcal{C}_C(N_\varepsilon, \varphi)\big) \cong H_*(N_\varepsilon, N_\varepsilon^-). \tag{4.3}$$

The excision property of homology yields

$$H_*\left(C \cap f^c, C \cap f^c \setminus \{x_0\}\right) \cong H_*\left(N_\varepsilon \cap f^c, N_\varepsilon \cap f^c \setminus \{x_0\}\right). \tag{4.4}$$

Again by the Palais-Smale condition the map

$$\tau : N_\varepsilon \setminus W^+ \to [0, \infty), \quad \tau(x) := \sup\{t \geq 0 : f(\varphi(t, x)) < c\},$$

is well defined and continuous, and it satisfies $\tau(x) \to \infty$ as $\mathrm{dist}(x, W^+) \to 0$. Therefore the map

$$h : [0,1] \times N_\varepsilon \to N_\varepsilon, \quad h(t,x) := \begin{cases} \varphi(t/(1-t), x) & \text{if } x \in W^+,\ t < 1; \\ x_0 & \text{if } x \in W^+,\ t = 1; \\ \varphi(t\tau(x)/((1-t)\tau(x)+1), x) & \text{if } x \in N_\varepsilon \setminus W^+. \end{cases}$$

is well defined and continuous. h shows that $N_\varepsilon \cap f^c$ is a deformation retract of N_ε, hence

$$H_*\left(N_\varepsilon \cap f^c, N_\varepsilon \cap f^c \setminus \{x_0\}\right) \cong H_*\left(N_\varepsilon, N_\varepsilon \cap f^c \setminus \{x_0\}\right). \tag{4.5}$$

Since N_ε^- is a deformation retract of $N_\varepsilon \cap f^c \setminus \{x_0\}$, we have

$$H_*\left(N_\varepsilon, N_\varepsilon \cap f^c \setminus \{x_0\}\right) \cong H_*(N_\varepsilon, N_\varepsilon^-). \tag{4.6}$$

Theorem 1.3 follows from (4.3) – (4.6). $\square$

Proof of Corollary 1.4. This follows immediately from Theorem 1.3 and the homotopy invariance of the Conley index as stated in Theorem 2.2. $\square$

5 Proof of Theorem 1.5

The proof owes a lot to the proof of [7, Theorem 1]. We may assume without loss of generality that $x_0 = 0$ and $f(x_0) = 0$. We consider first the case where L repels a point $y \in \overline{W}_0 \setminus T_0$. In that case, by [7, Lemma 1] there exists $y_0 \in C$ such that $x - Lx \neq y_0$ for all $x \in \overline{W}_0$. This implies in particular $y_0 \neq 0$. Now we define

$$f_t(x) := \frac{1}{2}\|x\|^2 - (1-t)g(x) - t\langle y_0, x\rangle.$$

Since $\nabla f_t(x) = x - ((1-t)A(x) + ty_0)$ and $(1-t)A(x) + ty_0 \in C$ for $x \in C$, $0 \le t \le 1$, it follows that C is positively invariant with respect to the negative gradient flow of f_t for $0 \le t \le 1$. The proof of [7, Theorem 1(a)] shows that 0 is an isolated critical point of f_t for $0 \le t \le 1$. Moreover, $\nabla f_1(x) = x - y_0 \ne 0$ for x close to 0. Consequently,

$$C_*(f|_C, 0) \cong C_*(f_1|_C, 0) \cong 0.$$

It remains to consider the case where L does not repel a point in $\overline{W}_0 \setminus T_0$. Let $E_0 \subset X$ be the finite-dimensional eigenspace of L associated to $\sigma(L) \cap (1, \infty)$, so that $m = \dim(E_0 \cap T_0)$. Let $P : X \to E_0 \cap T_0$ denote the orthogonal projection and set $L_0 := P \circ L : X \to E_0 \cap T_0$. According to [7, Lemma 2] there exist $\varepsilon_0, \delta > 0$ such that

(5.1) $\|x - (1-\lambda)A(x) - \lambda L_0(x)\| \ge \delta\|x\|$ for all $\lambda \in [0,1]$, $x \in C$, $0 < \|x\| \le \varepsilon_0$.

It is here that the condition "L does not repel a point in $\overline{W}_0 \setminus T_0$" enters. Since the set

$$M := \{L_0 x : x \in \overline{W}_0,\ \|x\| = 1\} \subset E_0 \cap T_0 \subset \overline{W}_0$$

is precompact as a bounded subset of the finite-dimensional space $E_0 \cap T_0$ there exist $y_1, \dots, y_j \in W_0 = \bigcup_{t>0} tC$ such that $M \subset \bigcup_{i=1}^{j} U_{\delta/3}(y_i)$. Setting $\mu_i(y) := \max\left\{0, \frac{\delta}{3} - \|y - y_i\|\right\}$ we consider the finite-dimensional map

$$Q_\delta : M \to \mathrm{conv}\{y_1, \dots, y_j\}, \quad Q_\delta(y) := \frac{1}{\sum_{i=1}^{j} \mu_i(y)} \sum_{i=1}^{j} \mu_i(y) y_i.$$

This is a $\frac{\delta}{3}$-approximation of the identity on M:

$$\|Q_\delta(y) - y\| \le \frac{1}{\sum_{i=1}^{j} \mu_i(y)} \sum_{i=1}^{j} \mu_i(y)\|y_i - y\| \le \frac{\delta}{3} \qquad \text{for all } y \in M.$$

We choose $t_i > 0$ and $x_i \in C$ with $y_i = t_i x_i$. Observe that for $s \le 1/t_i$ we have $sy_i = st_i x_i + (1 - st_i) \cdot 0 \in C$. Therefore, setting $s_0 := \min\{1/t_i : i = 1, \dots, j\}$ we have

$$sQ_\delta(y) \in C \qquad \text{for all } y \in M,\ 0 \le s \le s_0.$$

Now we we define $A_\delta : \overline{W}_0 \to C$ by

$$A_\delta(x) := \begin{cases} 0 & x = 0; \\ \|x\| Q_\delta\big(L_0(x/\|x\|)\big) & x \in \overline{W}_0,\ 0 < \|x\| \le s_0; \\ s_0 Q_\delta\big(L_0(x/\|x\|)\big) & x \in \overline{W}_0,\ \|x\| \ge s_0. \end{cases}$$

A_δ is completely continuous and satisfies:

$$\|A_\delta(x) - L_0 x\| = \|x\| \cdot \|Q_\delta(L_0(x/\|x\|)) - L_0(x/\|x\|)\| \leq \frac{\delta}{3}\|x\| \tag{5.2}$$

for all $x \in \overline{W}_0$ with $\|x\| \leq s_0$. Clearly Q_δ and A_δ are locally Lipschitz continuous. Consequently, for $0 \leq \lambda \leq 1$ the map

$$g_\lambda := -\mathrm{id} + (1-\lambda)A + \lambda A_\delta : C \to C$$

is locally Lipschitz continuous and induces a semiflow $\varphi_\lambda : \mathcal{D}_\lambda \subset [0,\infty) \times C \to C$:

$$\begin{cases} \dfrac{d}{dt}\varphi_\lambda(t,x) = g_\lambda(\varphi_\lambda(t,x)) \\ \varphi_\lambda(0,x) = x \end{cases}$$

Lemma 5.1. $\{0\}$ *is an isolated invariant set for* φ_λ*,* $0 \leq \lambda \leq 1$*; here* δ *is from* (5.1).

Proof. Recall δ, ε_0 from (5.1) and choose $\varepsilon \leq \varepsilon_0$ such that

$$\|A(x) - L_0 x\| \leq \frac{\delta}{3}\|x\| \qquad \text{for all } x \in C \text{ with } \|x\| \leq \varepsilon. \tag{5.3}$$

We consider the family of functions

$$f_\lambda(x) := \frac{1}{2}\|x\|^2 - (1-\lambda)g(x) - \frac{\lambda}{2}\langle L_0 x, x\rangle.$$

and show that $f_\lambda(\varphi_\lambda(t,x))$ is strictly decreasing in t for every $x \in C$ with $0 < \|x\| \leq \varepsilon$. Observe that (5.1) says that

$$\|\nabla f_\lambda(x)\| \geq \delta\|x\| \quad \text{for all } \lambda \in [0,1],\ x \in C,\ 0 < \|x\| \leq \varepsilon_0,$$

so using this and (5.3) we have for $\lambda \in [0,1]$, $x \in C$, $0 < \|x\| \leq \varepsilon_0$:

$$\begin{aligned} \langle \nabla f_\lambda(x), g_\lambda(x)\rangle &= -\langle x - (1-\lambda)A(x) - \lambda L_0 x, x - (1-\lambda)A(x) - \lambda A_\delta(x)\rangle \\ &= -\|\nabla f_\lambda(x)\|^2 + \lambda\langle \nabla f_\lambda(x), A_\delta(x) - L_0 x\rangle \\ &\leq -\delta^2\|x\|^2 + \delta\|x\| \cdot \|A_\delta(x) - L_0 x\| \\ &\leq -\delta^2\|x\|^2 + \delta\|x\| \cdot \frac{\delta}{3}\|x\| \\ &< 0 \end{aligned}$$

The lemma follows immediately. □

Lemma 5.1 and Theorem 2.2 imply

$$H_k(f^c \cap C, f^c \cap C \setminus \{0\}) \cong H_k\big(\mathcal{C}_C(\{0\}, \varphi_{\delta,0})\big) \cong H_k\big(\mathcal{C}_C(\{0\}, \varphi_{\delta,1})\big). \tag{5.4}$$

Since A_δ is defined on $\overline{W}_0$, the vector field $-\mathrm{id} + A_\delta$ induces a semiflow ψ on $\overline{W}_0$ which satisfies $\psi|_C = \varphi_{\delta,1}$. And since $A_\delta(\overline{W}_0) \subset C$, Theorem 2.3 implies

$$H_k\big(\mathcal{C}_C(\{0\}, \varphi_{\delta,1})\big) \cong H_k\big(\mathcal{C}_{\overline{W}_0}(\{0\}, \psi)\big). \tag{5.5}$$

Now we consider the homotopy

$$h_\lambda := -\mathrm{id} + (1-\lambda)A_\delta + \lambda L_0 : \overline{W}_0 \to \overline{W}_0, \quad 0 \le \lambda \le 1,$$

which induces semiflows ψ_λ on $\overline{W}_0$ satisfying

$$\frac{d}{dt}\psi_\lambda(t,x) = h_\lambda(\psi_\lambda(t,x)).$$

Clearly we have $\psi_0 = \psi$. Observe that (5.1) implies $\|x - L_0x\| \ge \delta\|x\|$ for all $c \in C$ close to 0, hence for all $x \in \overline{W}_0$. Using this and (5.2) we see that the function $f(x) = \langle x - L_0x, x\rangle$ satisfies $\langle \nabla f(x), h_\lambda(x)\rangle < 0$ for $x \in \overline{W}_0$ with $0 < \|x\| \le \varepsilon_0$, hence f serves as a strict Lyapunov function for ψ_λ near 0. It follows that $\{0\}$ is an isolated invariant set for ψ_λ, and the continuation invariance Theorem 2.2 yields

$$H_k\big(\mathcal{C}_{\overline{W}_0}(\{0\}, \psi)\big) \cong H_k\big(\mathcal{C}_{\overline{W}_0}(\{0\}, \psi_1)\big). \tag{5.6}$$

Moreover, since $L_0(\overline{W}_0) \subset E_0 \cap T_0$ Theorem 2.3 implies

$$H_k\big(\mathcal{C}_{\overline{W}_0}(\{0\}, \psi_1)\big) \cong H_k\big(\mathcal{C}_{E_0\cap T_0}(\{0\}, \psi_1|_{E_0\cap T_0})\big) \cong \delta_{km}R. \tag{5.7}$$

The last isomorphism is obvious because 0 is a repeller for ψ_1 in $E_0 \cap T_0$ and $m = \dim(E_0 \cap T_0)$.

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