Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org
pp. **X–XX**

MULTIBUMP SOLUTIONS OF NONLINEAR SCHRÖDINGER EQUATIONS WITH STEEP POTENTIAL WELL AND INDEFINITE POTENTIAL

Thomas Bartsch

Mathematisches Institut
University of Giessen
Arndtstr. 2 35392 Giessen Germany

Zhongwei Tang

School of Mathematical Sciences, Beijing Normal University
Laboratory of Mathematics and Complex Systems, Ministry of Education
Beijing 100875 P. R. of China

(Communicated by the associate editor name)

Dedicated to Jean Mawhin on the occasion of his 70th birthday.

Abstract. We are concerned with the existence of single- and multi-bump solutions of the equation $-\Delta u + (\lambda a(x) + a_0(x))u = |u|^{p-2}u$, $x \in \mathbb{R}^N$; here $p > 2$, and $p < \frac{2N}{N-2}$ if $N \geq 3$. We require that $a \geq 0$ is in $L^\infty_{loc}(\mathbb{R}^N)$ and has a bounded potential well Ω, i.e. $a(x) = 0$ for $x \in \Omega$ and $a(x) > 0$ for $x \in \mathbb{R}^N \setminus \bar{\Omega}$. Unlike most other papers on this problem we allow that $a_0 \in L^\infty(\mathbb{R}^N)$ changes sign. Using variational methods we prove the existence of multibump solutions u_λ which localize, as $\lambda \to \infty$, near prescribed isolated open subsets $\Omega_1, \dots, \Omega_k \subset \Omega$. The operator $L_0 := -\Delta + a_0$ may have negative eigenvalues in Ω_j, each bump of u_λ may be sign-changing.

1. **Introduction and main result.** We are concerned with the stationary nonlinear Schrödinger equation

$$\begin{cases} -\Delta u + (\lambda a(x) + a_0(x))u = |u|^{p-2}u & x \in \mathbb{R}^N; \\ u(x) \to 0 & \text{as } |x| \to \infty; \end{cases} \tag{S_λ}$$

here $p < 2^* = 2N/(N-2)^+$. We require that $a \geq 0$ and $\Omega := \operatorname{int} a^{-1}(0) \neq \emptyset$. Thus for $\lambda > 0$ large the potential $\lambda a + a_0$ develops a steep potential well and one expects to find solutions which localize near its bottom Ω. This problem has found much interest after being first considered in [3]–[1]; see the papers [10, 12] for recent results and references to the literature.

2000 *Mathematics Subject Classification.* Primary: 35J60; Secondary:35B33.
Key words and phrases. Nonlinear Schrödinger equation; single-bump standing waves; multi-bump solutions; potential well; variational methods..
The second author is supported by Alexander von Humboldt foundation and NSFC(11171028).

Fixing disjoint isolated open subsets $\Omega_1, \dots, \Omega_k \subset \Omega$ we develop a method of constructing solutions u_λ for $\lambda > 0$ large such that the restrictions $u_\lambda|_{\Omega_j}$ converge as $\lambda \to \infty$ towards a least energy solution of

$$-\Delta u + a_0(x)u = |u|^{p-2}u, \qquad u \in H_0^1(\Omega_j), \tag{P_j}$$

$j = 1, \dots, k$. If $-\Delta + a_0$ is positive such a result has been proved in [5]. In that case, the trivial solution $u = 0$ is a nondegenerate local minimum of the variational functional associated to (P_j), and the least energy solution is positive and of mountain pass type. More recently, Sato and Tanaka [10] considered the case where $a_0 \equiv 1$, so again $-\Delta + a_0$ is positive. It is well known that (P_j) has an unbounded sequence $u_i^{(j)}$, $i \in \mathbb{N}$, of critical points. This uses the oddness of the nonlinearity in an essential way. Assuming $\Omega = \Omega_1 + \Omega_2$, Sato and Tanaka constructed for λ large solutions $u_\lambda \in H^1(\mathbb{R}^N)$ of (S_λ) such that $u_\lambda|_{\Omega_1}$ converges towards $u_1^{(1)}$, the mountain solution of (P_1), and $u_\lambda|_{\Omega_2}$ converges towards $u_j^{(2)}$, some $j \geq 1$.

In this paper we allow that $-\Delta + a_0$ is indefinite. As a consequence, the least energy solution of (P_j) may change sign and will not be of mountain pass type in general. It is obtained via a higher dimensional linking argument, or via a minimization on a certain submanifold of $H_0^1(\Omega_j)$ of higher codimension. Our method is quite different from those of [5] and [10]. It does not use the oddness of the nonlinearity and can therefore be extended to deal with more general nonlinearities $f(u)$ instead of $|u|^{p-2}u$; see Remark 1.2.

Let us fix our hypotheses on a and a_0:

(V_1) $a \in L^\infty_{loc}(\mathbb{R}^N)$, $a \geq 0$, $\Omega := \operatorname{int} a^{-1}(0) \neq \emptyset$ is bounded with $\partial\Omega$ smooth, $\liminf_{|x|\to\infty} a(x) > 0$;

(V_2) $a_0 \in L^\infty(\mathbb{R}^N)$;

(V_3) there exist nonempty disjoint open sets $\Omega_1, \dots, \Omega_m \subset \Omega$ such that $\Omega = \bigcup_{1\leq j\leq m} \Omega_j$. For each $j = 1, \dots, m$ there holds $\overline{\Omega_j} \cap \overline{\Omega \setminus \Omega_j} = \emptyset$ and $-\Delta + a_0$ is nondegenerate in $H_0^1(\Omega_j)$.

It is well known that under assumptions (V_2) and (V_3) problem (P_j) has a solution obtained via a linking argument applied to the energy functional

$$I_j(u) = \frac{1}{2}\int_{\Omega_j} \left(|\nabla u|^2 + a_0 u^2\right) - \frac{1}{p}\int_{\Omega_j} |u|^p.$$

In fact, the solution can also be obtained by minimizing I_j on the Nehari-Pankov manifold; see Section 2. It is a least energy solution, i.e. it lies on the level

$$c_j := \inf\{I_j(u) : u \in H_0^1(\Omega_j),\ u \neq 0 \text{ solves } (P)\},$$

and may be considered as ground state solution (see [11]. If 0 is a local minimum of I_j then this solution is positive and of mountain pass type; otherwise it changes sign and has higher Morse index.

Theorem 1.1. *Fix a subset $J \subset \{1, 2, \cdots, m\}$ and set $\Omega_J := \bigcup_{j\in J} \Omega_j$. Then for any $\varepsilon > 0$, there exists $\Lambda(\varepsilon) > 0$ such that for any $\lambda \geq \Lambda(\varepsilon)$, ($S_\lambda$) has a solution u_λ satisfying:*

(i) For $j \in J$ there holds

$$\left|\int_{\Omega_j} \left(\frac{1}{2}(|\nabla u_\lambda|^2 + a_0 u_\lambda^2) - \frac{1}{p}|u_\lambda|^p\right) dx - c_j\right| \leq \varepsilon.$$

(ii) $\displaystyle\int_{\mathbb{R}^N\setminus\Omega_J} \left(|\nabla u_\lambda|^2 + (\lambda a + a_0)u_\lambda^2\right) \le \varepsilon$

(iii) *Every sequence* $\lambda_n \to \infty$ *has a subsequence* (λ_{n_i}) *such that* $u_{\lambda_{n_i}} \to \bar u$ *as* $i \to \infty$. *The restriction* $\bar u|_{\Omega_j}$ *is a least energy solution of* (P_j) *for* $j \in J$. *Moreover,* $\bar u(x) = 0$ *for* $x \in \mathbb{R}^N \setminus \Omega_J$.

This is a generalization of the result from [5] who considered the case where $-\Delta + a_0$ is positive definite, so that I_j has mountain pass structure. A new feature in the proof of our result is a combination of a global linking applied in each $H_0^1(\Omega_j)$, $j \in J$, and a local linking near $0 \in H_0^1(\Omega_j)$, $j \notin J$. These are extended to $H^1(\mathbb{R}^N)$ and "added". We believe that this technique can be used in a variety of other singular limit problems.

Remark 1.2. The results continue to hold for $-\Delta u + (\lambda a(x) + a_0(x))u = f(u)$ provided the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following conditions:

- (f_1) $f(u) = o(u)$ as $u \to 0$.
- (f_2) $|f(u)| \le \gamma(1 + |u|^{p-1})$ for some $\gamma > 0$.
- (f_3) $F(u)/u^2 \to \infty$ as $|u| \to \infty$ where $F(u) = \int_0^u f$.
- (f_4) The map $u \mapsto f(u)/|u|$ is strictly increasing in $\mathbb{R} \setminus \{0\}$.

Also the hypotheses on the potential can be weakened. In (V_1) the assumption $\liminf_{|x|\to\infty} a(x) > 0$ can be replaced by the following one: There exists $M > 0$ such that the measure of the set $\{x \in \mathbb{R}^N : a(x) \le M\}$ is finite; see [1]. In (V_2) it suffices to assume that $a_0 \in L^\infty_{loc}(\mathbb{R}^N)$ and $\operatorname{ess\,inf} a_0 > -\infty$. In order to keep the presentation readable we refrained from treating the most general situation.

Remark 1.3. If the least energy solutions $\bar u_j$ of (P_j) are isolated then Theorem 1.1 follows from [2]. In fact, one can show that they have nontrivial critical groups, hence [2, Theorem 1.4] applies. If they have nontrivial degree then according to [2, Theorem 1.2] there exists a connected set $\mathcal{S} \subset \{(\lambda, u) \in \mathbb{R}^+ \times H^1(\mathbb{R}^N) : (\lambda, u) \text{ solves } (S_\lambda)\}$ of solutions such that for any sequence $(\lambda_n, u_n) \in \mathcal{S}$ with $\lambda_n \to \infty$ there holds $u_n \to \sum_{j\in J} \bar u_j$ as $n \to \infty$. If they are even nondegenerate, then [2, Theorem 1.3] yields a smooth function $\lambda \mapsto u_\lambda$ satisfying $u_\lambda \to \sum_{j\in J} \bar u_j$ as $\lambda \to \infty$.

Our paper is organized as follows: In section 2 we recall the Nehari-Pankov manifold and study the properties of the least energy solutions. Since the standard functional associated to (S_λ) does not satisfy the Palais-Smale condition under our hypotheses, in Section 3 we construct and investigate a penalized functional J_λ. This does satisfy the (PS)-condition for λ large and its critical points in a certain energy range are solutions of (S_λ). In Section 4, we study the behavior of the eigenvalues and eigenspaces of $-\Delta + \lambda a + a_0$ when $\lambda \to \infty$. Based on this we construct a new linking and define a possible critical value for J_λ, $\lambda > 0$ large, in Section 5. This is based on an intersection lemma which we prove in Section 6. Sections 5 and 6 are the new key ingredients of our work. Finally, Section 7 contains the proof of Theorem 1.1.

We will use C to denote various generic positive constants which are independent of λ and n, and we will write $o(1)$ and $o_n(1)$ to denote quantities that tend to 0 as $\lambda \to \infty$, resp. $n \to \infty$.

2. **The Nehari-Pankov manifold.** We consider an open subset $\mathcal{O} \subset \mathbb{R}^N$ and a potential $b \in L^\infty_{loc}(\mathcal{O})$ which is bounded below. The functional

$$J(u) = \frac{1}{2}\int_{\mathcal{O}} \left(|\nabla u|^2 + b(x)u^2\right) - \frac{1}{p}\int_{\mathcal{O}} |u|^p$$

is defined for $u \in H^1(\mathcal{O})$ satisfying $\int_{\mathcal{O}} |b|u^2 < \infty$. We write E for either of the energy spaces $\left\{u \in H^1(\mathcal{O}) : \int_{\mathcal{O}} |b|u^2 < \infty\right\}$ or $\left\{u \in H^1_0(\mathcal{O}) : \int_{\mathcal{O}} |b|u^2 < \infty\right\}$. In this paper the operator $-\Delta + b(x)$ has finite Morse index and is nondegenerate on E. Then E splits as an orthogonal sum $E = E^- \oplus E^+$ of the negative and positive eigenspace of $-\Delta + b(x)$, and $\dim E^- < \infty$. Let $P^- : E \to E^-$ denote the orthogonal projection.

The Nehari-Pankov manifold is defined as

$$\mathcal{N} := \{u \in E \setminus \{0\} : \ P^-\nabla J(u) = 0,\ DJ(u)[u] = 0\} \subset E \setminus E^-.$$

It has been introduced by Pankov [8] in a situation where $\dim E^- = \infty$, and coincides with the Nehari manifold if $E^- = \{0\}$. In order to formulate certain geometric properties of $\mathcal{N}$ we need some notation. For $w \in E \setminus E^-$ and $R > r > 0$ set

$$H_w := \{v + tw : v \in E^-,\ t > 0\} \tag{2.1}$$

and

$$A_{w,r,R} := \{v + tw : v \in E^-,\ \|v\| < R,\ t \in (r, R)\} \subset H_w. \tag{2.2}$$

Then we have

$$\mathcal{N} = \{w \in E \setminus E^- : \nabla(J|H_w) = 0\}.$$

Proposition 2.1. a) *For every $w \in E^+ \setminus \{0\}$ there exist $t_w > 0$ and $\varphi(w) \in E^-$ such that $H_w \cap \mathcal{N} = \{\varphi(w) + t_w \cdot w\}$.*

b) *For every $w \in \mathcal{N}$ and every $u \in H_w \setminus \{w\}$ there holds $J(u) < J(w)$.*

c) $c_0 := \inf_{u\in\mathcal{N}} J(u) > 0$

d) *For every $w \in \mathcal{N}$ there holds $\|P^+w\| > \max\{\|P^-w\|, \sqrt{2c_0}\}$.*

e) *For $w \in \mathcal{N}$ and $0 < r < \|w\| < R$ the map*

$$f : H_w \to E^- \times \mathbb{R}, \quad f(u) := \left(P^-\nabla J(u), DJ(u)[u]\right),$$

has degree $\deg(f, A_{w,r,R}, 0) = 1$. Here we identify $H_w \subset E^- \oplus \mathbb{R}w$ and $E^- \times \mathbb{R}^+ \subset E^- \times \mathbb{R}$.

Proof. The proof of a) – d) can be found in [11]. For the proof of e) observe that f is homotopic to $\nabla(J|H_w) : H_w \to E^- \oplus \mathbb{R}w \cong E^- \times \mathbb{R}$. By a) and b) the constrained functional $J|H_w$ has a unique critical point, namely w, which is the global maximum. Since the local degree of a global maximum is $+1$ we deduce

$$\deg(f, A_{w,r,R}, 0) = \deg(\nabla(J|H_w), A_{w,r,R}, 0) = 1.$$

□

Remark 2.2. Set $d := \dim E^-$ and let $e_1, \dots, e_d$ be an orthonormal basis of E^-. We also need the sets $A := \{(s,t) \in \mathbb{R}^d \times \mathbb{R} : |s| \le 1,\ 0 \le t \le 1\}$ and $B := \partial A \subset \mathbb{R}^{d+1}$. Given $w \in \mathcal{N}$ and $0 < r < \|w\| < R$ the map

$$h_{w,r,R} : (A, B) \to (E, E \setminus \mathcal{N})), \quad h_{w,r,R}(s,t) := R\sum_{i=1}^d s_i e_i + ((1-t)r + tR)w.$$

is well defined. It is not difficult to see that all maps $h_{w,r,R}$ are homotopic. As a consequence of Proposition 2.1 we have

$$c_0 = \inf_{u\in\mathcal{N}} J(u) = \inf_{\substack{w\in\mathcal{N}\\ 0<r<\|w\|<R}} \max_{u\in A_{w,r,R}} J(u) = \inf_{\gamma\in\Gamma}\max_{(s,t)\in A} J\circ\gamma(s,t)$$

where

$$\Gamma = \{\gamma : (A,B) \to (E, E\setminus\mathcal{N}) \mid \gamma|_B \text{ is homotopic to some } h_{w,r,R}\}.$$

The proof of the following result is standard.

Proposition 2.3. *If J satisfies the Palais-Smale condition at the level $c_0 = \inf_{u\in\mathcal{N}} J(u)$ then c_0 is achieved by a least energy solution $u_0 \in \mathcal{N}$.*

3. **The penalized functional.** We first construct a variational functional whose critical points (in a certain energy range) will be solutions of (S_λ) and which satisfies the Palais-Smale condition. By assumption (V_3) there exist smoothly bounded open sets $\Omega'_1,\dots,\Omega'_m \subset \mathbb{R}^N$ such that

$$\overline{\Omega_j} \subset \Omega'_j, \quad \overline{\Omega'_i}\cap\overline{\Omega'_j} = \emptyset \text{ for } i\neq j, \quad \text{and} \quad \overline{\Omega'_j}\cap\overline{\Omega\setminus\Omega_j} = \emptyset.$$

Using $(V_1)-(V_3)$, we may choose $\Lambda_0 > 0$ such that

$$\Lambda_0 a(x) + a_0(x) \ge 1 \quad \text{if } x \notin \Omega' := \bigcup_{j=1}^m \Omega'_j. \tag{3.1}$$

Setting $V_\lambda := \lambda a + a_0$ we look for solutions lying in the energy space

$$E := \left\{ u \in D^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V_{\Lambda_0}^+ u^2 < \infty \right\} \subset H^1(\mathbb{R}^N). \tag{3.2}$$

As a consequence of (3.1) the norms

$$\|u\|_\lambda := \left(\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V_\lambda^+ u^2\right) \right)^{1/2}$$

are equivalent for $\lambda \ge \Lambda_0$, and satisfy $\|\cdot\|_\lambda \le \|\cdot\|_{\lambda'}$ for $\lambda \le \lambda'$. Occasionally we write E_λ for $(E, \|\cdot\|_\lambda)$, and we observe that

$$\|\cdot\|_{H^1} \le C\|\cdot\|_\lambda \quad \text{for all } \lambda \ge \Lambda_0 \tag{3.3}$$

with embedding constant $C > 1$ independent of λ. The functional

$$I_\lambda : E \to \mathbb{R}, \quad I_\lambda(u) := \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V_\lambda u^2\right) - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p,$$

is of class $\mathcal{C}^2$, and critical points of I_λ are solutions of (S_λ). I_λ is the standard functional associated to (S_λ).

Since I_λ does not need to satisfy the Palais-Smale condition we shall now modify it. We first define for $t\in\mathbb{R}$ and $\delta > 0$:

$$f_\delta(t) := \begin{cases} |t|^{p-2}t & \text{if } |t| \le \delta \\ \delta^{p-2}t & \text{if } |t| > \delta \end{cases}$$

and set $F_\delta(t) := \int_0^t f_\delta(s)ds$. Let $\chi : \mathbb{R}^N \to [0,1]$ denote the characteristic function of Ω'. We consider the penalized nonlinearity

$$g_\delta(x,t) := \chi(x)|t|^{p-2}t + (1-\chi(x))f_\delta(t).$$

Setting $G_\delta(x,t) := \int_0^t g_\delta(x,s)ds$ we can now define the functional

$$J_\lambda : E \to \mathbb{R}, \quad J_\lambda(u) := \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V_\lambda(x)u^2\right) - \int_{\mathbb{R}^N} G_\delta(x,u),$$

The constant δ is suppressed in the notation because it will be fixed. We only require that $3C\delta^{p-2} < 1$ with C from (3.3). This implies in particular that $G_\delta(x,t) \le t^2/2$ for $x \in \mathbb{R}^N \setminus \Omega'$. It is standard to check that J_λ is of class $\mathcal{C}^1$ and that its nontrivial critical points are solutions of

$$-\Delta u + (\lambda a(x) + a_0(x))u = g_\delta(x,u) \quad \text{in} \quad \mathbb{R}^N.$$

If moreover u satisfies $|u(x)| < \delta$ for all $x \in \mathbb{R}^N \setminus \Omega'$, then u solves the original problem (S_λ).

Proposition 3.1. *J_λ satisfies the Palais-Smale condition for $\lambda \ge \Lambda_0$. More precisely, any sequence (u_n) in E with*

$$J_\lambda(u_n) \le c, \quad \nabla J_\lambda(u_n) \to 0 \text{ strongly in } E_\lambda, \tag{3.4}$$

contains a strongly convergent subsequence in E.

For the proof we need the following

Lemma 3.2. *Suppose that a sequence (u_n) in E satisfies (3.4). Then there exists a constant $M(c)$ which is independent of λ such that*

$$\limsup_{n\to\infty} \|u_n\|_\lambda^2 \le M(c). \tag{3.5}$$

Proof. Setting $\varepsilon_n := \|\nabla J_\lambda(u_n)\|$ it follows from (3.4) that

$$\begin{aligned} \int_{\Omega'} \left(\frac{1}{2} - \frac{1}{p}\right)|u_n|^p + \int_{\mathbb{R}^N\setminus\Omega'} \left(\frac{1}{2} f_\delta(u_n)u_n - F_\delta(u_n)\right) \\ = \frac{1}{2}\int_{\mathbb{R}^N} g_\delta(x,u_n)u_n - \int_{\mathbb{R}^N} G_\delta(x,u_n) \\ = J_\lambda(u_n) - \frac{1}{2}J'_\lambda(u_n)u_n \le c + \varepsilon_n\|u_n\|_\lambda. \end{aligned} \tag{3.6}$$

Observe that for $|t| \in (\delta, \infty)$,

$$\frac{1}{2}f_\delta(t)t - F_\delta(t) = \frac{1}{2}\delta^{p-2}t^2 - \frac{1}{2}\delta^{p-2}t^2 + \frac{p-2}{2p}\delta^p = \frac{p-2}{2p}\delta^p \ge 0, \tag{3.7}$$

and for $|t| \le \delta$,

$$\frac{1}{2}f(t)t - F(t) = \left(\frac{1}{2} - \frac{1}{p}\right)|t|^p. \tag{3.8}$$

Combining (3.6)-(3.8) we obtain

$$\left(\frac{1}{2} - \frac{1}{p}\right)\int_{\Omega'} |u_n|^p \le c + o(1) + \varepsilon_n\|u_n\|_\lambda.$$

Since V_λ^- is non-increasing with respect to λ and $\operatorname{supp} V_\lambda^- \subset \Omega'$ for $\lambda \ge \Lambda_0$ we deduce for $\lambda \ge \Lambda_0$:

$$\begin{aligned} \int_{\mathbb{R}^N} V_\lambda^- u_n^2 = \int_{\Omega'} V_\lambda^- u_n^2 \le \int_{\Omega'} V_{\Lambda_0}^- u_n^2 \le C + \int_{\Omega'} |u_n|^p \\ \le C\big(1 + c + (\varepsilon_n)\|u_n\|_\lambda\big), \end{aligned} \tag{3.9}$$

where C is a positive constant which is independent of λ and n.

Using (3.4) once more, we obtain

$$\begin{aligned}\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^N}(|\nabla u_n|^2+V_\lambda^+u_n^2)-\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^N}V_\lambda^-u_n^2&\\ +\frac{1}{p}\int_{\mathbb{R}^N}g_\delta(x,u_n)u_n-\int_{\mathbb{R}^N}G(x,u_n)&\\ =J_\lambda(u_n)-\frac{1}{p}J_\lambda'(u_n)u_n\le c+\varepsilon_n\|u_n\|_\lambda.&\end{aligned}\tag{3.10}$$

A similar argument yields

$$\frac{1}{p}\int_{\mathbb{R}^N}g_\delta(x,u_n)u_n-\int_{\mathbb{R}^N}G_\delta(x,u_n)\ge-\left(\frac{1}{2}-\frac{1}{p}\right)\delta^{p-2}\int_{\mathbb{R}^N\setminus\mathcal{O}'}u_n^2\tag{3.11}$$

Combining (3.10) and (3.11) gives

$$\begin{aligned}\left(\frac{1}{2}-\frac{1}{p}\right)(1-\delta^{p-2})\|u_n\|_\lambda^2&=\left(\frac{1}{2}-\frac{1}{p}\right)(1-\delta^{p-2})\int_{\mathbb{R}^N}[|\nabla u_n|^2+V_\lambda^+u_n^2]\\ &\le C\big(1+c+\varepsilon_n\|u_n\|_\lambda\big).\end{aligned}$$

Since $\delta^{p-2}<1$ it easily follows that there exists $M(c)$ which is independent of $\lambda\ge\Lambda_0$ such that (3.5) holds. This completes the proof of Lemma 3.2. □

Now we can give the

Proof of Proposition 3.1. From Lemma 3.2, we know that (u_n) is bounded in E_λ, so after passing to a subsequence there holds

$$\begin{aligned}&u_n\rightharpoonup u \text{ weakly in } E_\lambda,\\ &u_n\to u \text{ strongly in } L^q_{loc}(\mathbb{R}^N) \text{ for } 2\le q<2^*,\\ &u_n\to u \text{ a.e in } \mathbb{R}^N.\end{aligned}$$

Now we prove that $u_n\to u$ in E_λ. First of all, it is easy to check that u is a critical point of $J_\lambda(u)$, that is,

$$\int_{\mathbb{R}^N}(\nabla u\nabla\psi+V_\lambda(x)u\psi)=\int_{\mathbb{R}^N}g_\delta(x,u)\psi\quad\text{for every }\psi\in E_\lambda.$$

It follows from (3.4) that

$$\begin{aligned}o_n(1)&=(J_\lambda'(u_n)-J_\lambda'(u))(u_n-u)\\ &=\int_{\mathbb{R}^N}(|\nabla(u_n-u)|^2+V_\lambda(x)|u_n-u|^2)-\int_{\mathbb{R}^N}g_\delta(x,u_n)(u_n-u)\\ &\qquad+\int_{\mathbb{R}^N}g_\delta(x,u)(u_n-u)\\ &=\|u_n-u\|_\lambda^2-\int_{\Omega'}V_\lambda^-(x)|u_n-u|^2-\int_{\Omega'}|u_n|^{p-2}u_n(u_n-u)\\ &\quad-\int_{\mathbb{R}^N\setminus\Omega'}f_\delta(u_n)(u_n-u)+\int_{\Omega'}|u|^{p-2}u(u_n-u)+\int_{\mathbb{R}^N\setminus\Omega'}f_\delta(u)(u_n-u)\end{aligned}$$

By the definition of $f_\delta(t)$ we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N\setminus\Omega'} f_\delta(u_n)(u_n-u)\right| \\
&\qquad \le \left|\int_{\mathbb{R}^N\setminus\Omega'} (f_\delta(u_n)-\delta^{p-2}u_n)(u_n-u)\right| + \delta^{p-2}\left|\int_{\mathbb{R}^N\setminus\Omega'} u_n(u_n-u)\right| \\
&\qquad \le 3\delta^{p-2}\|u_n-u\|_{L^2}^2 + \delta^{p-2}\left|\int_{\mathbb{R}^N\setminus\Omega'} u(u_n-u)\right|,
\end{aligned}$$

Now $u_n \rightharpoonup u$ in E_λ implies

$$\left|\int_{\mathbb{R}^N\setminus\Omega'} u(u_n-u)\right| \to 0 \qquad \text{and} \qquad \left|\int_{\mathbb{R}^N\setminus\Omega'} f_\delta(u)(u_n-u)\right| \to 0.$$

Finally, since $u_n \to u$ strongly in $L^p(\Omega')$, and since $\|\cdot\|_{L^2} \le C\|\cdot\|_\lambda^2$, see (3.3), we deduce:

$$\begin{aligned}
&(1-3C\delta^{p-2})\|u_n-u\|_\lambda^2 \le \|u_n-u\|_\lambda^2 - 3\delta^{p-2}\|u_n-u\|_{L^2}^2 \\
&\quad \le \int_{\Omega'} |u_n|^{p-2}u_n(u_n-u) - \int_{\Omega'} |u|^{p-2}u(u_n-u) + \int_{\Omega'} V_\lambda^-(x)|u_n-u|^2 + o_n(1) \\
&\quad \to 0
\end{aligned}$$

as $n\to\infty$. Therefore $u_n \to u$ in E_λ because $3C\delta^{p-2} < 1$. $\square$

Proposition 3.3. *Suppose the sequences $\lambda_n \to \infty$ and (u_n) in E satisfy*

$$J_{\lambda_n}(u_n) \le c, \quad \|\nabla J_{\lambda_n}(u_n)\|_{\lambda_n} \to 0. \tag{3.12}$$

Then, after passing to a subsequence, we have:

a) *$u_n \rightharpoonup u$ weakly in E for some $u \in E$.*

b) *$u \equiv 0$ in $\mathbb{R}^N \setminus \Omega$, and $u|_{\Omega_j}$ solves* $\begin{cases} -\Delta u + a_0 u = |u|^{p-2}u & \text{in } \Omega_j \\ u \in H_0^1(\Omega_j) \end{cases}$ *for $j = 1,\dots,m$.*

c) *$\|u_n - u\|_{\lambda_n} \to 0$, consequently $u_n \to u$ in $H^1(\mathbb{R}^N)$.*

d) *(u_n) also satisfies for $n\to\infty$:*

(i) $\displaystyle\int_{\mathbb{R}^N} \lambda_n a(x) u_n^2 \to 0$

(ii) $\displaystyle\int_{\mathbb{R}^N\setminus\Omega} \left(|\nabla u_n|^2 + V_{\lambda_n} u_n^2\right) \to 0$

(iii) $\displaystyle\int_{\Omega_j'} \left(|\nabla u_n|^2 + V_{\lambda_n} u_n^2\right) \to \int_{\Omega_j} \left(|\nabla u|^2 + a_0(x) u^2\right)$ *for $j = 1,\dots,m$.*

Proof. As in the proof of Lemma 3.2, one shows that $\limsup_{n\to\infty} \|u_n\|_{\lambda_n}^2 \le M(c)$. Thus (u_n) stays bounded as $n\to\infty$ in E, so we may assume that for some $u \in E$:

$$\begin{aligned}
&u_n \rightharpoonup u \text{ weakly in } E, \\
&u_n \to u \text{ a.e. in } \mathbb{R}^N, \\
&u_n \to u \text{ strongly in } L^q_{loc}(\mathbb{R}^N) \text{ for } 2 \le q < 2^*.
\end{aligned}$$

Now we prove b). Setting $C_k := \{x \in \mathbb{R}^N : a(x) \geq \frac{1}{k}\}$, we have for n large:

$$\begin{aligned}\int_{C_k} u_n^2 &\leq \frac{k}{\lambda_n}\int_{\mathbb{R}^N} \lambda_n a(x) u_n^2 = \frac{k}{\lambda_n}\int_{\mathbb{R}^N} (\lambda_n a(x) + a_0(x)) u_n^2 - \frac{k}{\lambda_n}\int_{\mathbb{R}^N} a_0(x) u_n^2 \\ &\leq \frac{k}{\lambda_n}\|u_n\|_{\lambda_n}^2 + \frac{k}{\lambda_n}\|a_0\|_{L^\infty}\|u_n\|_{L^2}^2 \to 0.\end{aligned}$$

It follows that $u(x) = 0$ in $\bigcup_{k=1}^\infty C_k = \mathbb{R}^N \setminus \Omega$.

Next we have for any test function $\varphi \in C_0^\infty(\Omega_j)$, $j = 1, 2, \dots, m$:

$$|J'_{\lambda_n}(u_n)\varphi| \leq \|\nabla J_{\lambda_n}(u_n)\|_{\lambda_n}\|\varphi\|_{\lambda_n} \to 0.$$

Here we use the fact that $\|\varphi\|_{\lambda_n}$ does not depend on λ_n. It follows that

$$\int_{\Omega_j} (\nabla u \nabla \varphi + a_0 u \varphi) = \int_{\Omega_j} g(x, u)\varphi.$$

This implies b).

In order to prove c) we observe that

$$\begin{aligned}&J'_{\lambda_n}(u_n)(u_n - u) - J'_{\lambda_n}(u)(u_n - u) \\ &\qquad = \|u_n - u\|_{\lambda_n}^2 - \int_{\mathbb{R}^N\setminus\Omega'} f_\delta(u_n)(u_n - u) + \int_{\mathbb{R}^N\setminus\Omega'} f_\delta(u)(u_n - u) \\ &\qquad = -\int_{\Omega'} V_{\lambda_n}^-(u_n - u)^2 - \int_{\Omega'} |u_n|^{p-2}u_n(u_n - u) + \int_{\Omega'} |u|^{p-2}u(u_n - u).\end{aligned}$$

Here we have used the fact that supp $V_{\lambda_n}^- \subset \Omega'$ for n large. Since $u_n \to u$ in $L^p(\Omega')$, we have

$$\int_{\Omega'} (|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u) \to 0 \quad \text{and} \quad \int_{\Omega'} V_{\lambda_n}^-(u_n - u)^2 \to 0 \ \text{ as } n \to \infty.$$

On the other hand

$$\begin{aligned}|J'_{\lambda_n}(u_n)(u_n - u)| &\leq \|\nabla J_{\lambda_n}(u_n)\|_{\lambda_n}\|u_n - u\|_{\lambda_n} \\ &\leq \|\nabla J_{\lambda_n}(u_n)\|_{\lambda_n}(\|u_n\|_{\lambda_n} + \|u\|_{\lambda_n}) \to 0.\end{aligned}$$

This implies

$$\|u_n - u\|_{\lambda_n}^2 - \int_{\mathbb{R}^N\setminus\Omega'} (f_\delta(u_n) - f_\delta(u))(u_n - u) \to 0.$$

We obtain $(1 - 3C\delta^{p-2})\|u_n - u\|_{\lambda_n}^2 \to 0$ as in the proof of Proposition 3.1, hence c) holds.

It remains to prove d). Using c) we see that

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^N} \lambda_n a(x) u_n^2 &= \frac{1}{2}\int_{\mathbb{R}^N\setminus\Omega} \lambda_n a(x) u_n^2 = \frac{1}{2}\int_{\mathbb{R}^N\setminus\Omega} \lambda_n a(x)|u_n - u|^2 \\ &\leq \|u_n - u\|_{\lambda_n}^2 \to 0\end{aligned}$$

which proves (i); (ii) and (iii) also follow immediately from c) □

Proposition 3.4. *Given $c > 0$ there exists $\Lambda_c > \Lambda_0$ such that for $\lambda \geq \Lambda_c$ a critical point u_λ of J_λ with $|J_\lambda(u_\lambda)| \leq c$ satisfies $|u_\lambda| \leq \delta$ for $x \in \mathbb{R}^N \setminus \Omega'$.*

Proof. Since $u_\lambda \in E_\lambda$ is a critical point of $J_\lambda(u)$ it satisfies the equation

$$-\Delta u_\lambda + (\lambda a(x) + a_0(x))u_\lambda = g_\delta(x, u_\lambda), \quad \text{in } \mathbb{R}^N.$$

Using that u_λ is bounded in E independent of λ, an argument as in the proof of [1, Lemma 5.1] shows that $\|u_\lambda\|_{L^\infty}$ is bounded independent of λ. On the other hand, by the definition of g_δ, we know that $A_\delta(x) := g_\delta(x, u_\lambda(x))/u_\lambda(x)$ is bounded in $L^\infty(\mathbb{R}^N)$. Moreover, (V_1) implies that the negative part of $W_\lambda := \lambda a + a_0 - A_\delta$ is bounded uniformly in λ. It follows from [9, A.2.1] that the norm of W_λ^- in the Kato class K_N is bounded uniformly in λ. Thus by the subsolution estimate [9, Theorem C.1.2] there exists a constant C which is independent of λ such that

$$|u_\lambda(x)| \le C(r) \int_{B_r(x)} |u_\lambda|\,; \tag{3.13}$$

here $B_r(x) = \{y \in \mathbb{R}^N : |x - y| < r\}$. Proposition 3.3 implies that for any sequence $\lambda_n \to \infty$, after passing to a subsequence there holds $u_{\lambda_n} \to u_0 \in H_0^1(\Omega)$ strongly in E, and therefore $u_{\lambda_n} \to 0$ strongly in $L^2(\mathbb{R}^N \setminus \overline{\Omega})$. Since $\lambda_n \to \infty$ was arbitrary, we have

$$u_\lambda \to 0 \ \text{ strongly in } L^2(\mathbb{R}^N \setminus \overline{\Omega}) \text{ as } \lambda \to \infty.$$

Thus, choosing $r = \frac{1}{2}\operatorname{dist}(\Omega, \mathbb{R}^N \setminus \Omega')$, we have uniformly in $x \in \mathbb{R}^N \setminus \Omega'$ that

$$\begin{aligned}|u_\lambda(x)| \le C(r) \int_{B_r(x)} |u_\lambda(x)| &\le C(r)(\operatorname{meas} B_r(x))^{1/2} \|u_\lambda\|^{1/2}_{L^2(B_r(x))}\\ &\le C(r)(\operatorname{meas} B_r(x))^{1/2} \|u_\lambda\|^{1/2}_{L^2(\mathbb{R}^N \setminus \Omega)} \to 0.\end{aligned}$$

This completes the proof. □

4. **Behavior of eigenvalues and eigenspaces.** Recall the smoothly bounded open neighborhoods Ω'_j of Ω_j from the definition of the penalized functional in Section 3, and denote $X_j := H^1(\Omega'_j)$. Let $\mu^\lambda_{j,1} < \mu^\lambda_{j,2} < \mu^\lambda_{j,3} < \dots$ be the distinct eigenvalues of L_λ in X_j and let $V^\lambda_{j,n}$, $n \in \mathbb{N}$, be the corresponding eigenspaces. Similarly, let $\mu_{j.1} < \mu_{j,2} < \mu_{j,3} < \dots$ denote the distinct eigenvalues of $L_0 = -\Delta + a_0$ in $E_j = H_0^1(\Omega_j)$ with eigenspaces $V_{j,n}$. Then we have:

Lemma 4.1. *$\mu^\lambda_{j,n} \to \mu_{j,n}$ and $V^\lambda_{j,n} \to V_{j,n}$ as $\lambda \to \infty$.*

Here $V^\lambda_{j,n} \to V_{j,n}$ means that, given any sequence $\lambda_i \to \infty$ and normalized eigenfunctions $\psi_i \in V^{\lambda_i}_{j,n}$, there exists a normalized eigenfunction $\psi \in V_{j,n}$ such that $\psi_i \to \psi$ strongly in X_j along a subsequence.

Corollary 4.2. *For λ large the operator $-\Delta + \lambda a + a_0$ on $X_j = H^1(\Omega'_j)$ is nondegenerate and has finite Morse index $d_j := \dim E_j^-$ uniformly in λ.*

Proof of Lemma 4.1. Since $j \in \{1, \dots, m\}$ is fixed, to simplify notation we denote $\mu^\lambda_{j,n}$ by μ^λ_n, $\mu_{j,n}$ by μ_n, $V^\lambda_{j,n}$ by V^λ_n, and $V_{j,n}$ by V_n. For $n = 1$ the result has been proved by Ding and Tanaka [5, Lemma 1.2]). Now suppose $n \ge 2$ and the result holds up to $n - 1$. Set

$$d := \dim V_1 + \dots + \dim V_{n-1} = \dim V^\lambda_1 + \dots + \dim V^\lambda_{n-1}.$$

By the minmax description of the eigenvalues, see Reed and Simon [9, XIII.1], for instance, there holds:

$$\begin{aligned}\mu_n^\lambda &= \inf\big\{(L_\lambda\psi,\psi) : \psi\in H^1(\Omega_j'),\ \|\psi\|_{L^2(\Omega_j')}=1,\\ &\qquad\qquad \psi\perp V_m^\lambda = 0 \text{ for } m=1,\dots,n-1\big\}\\ &= \max_{\phi_1,\dots,\phi_d\in H^1(\Omega_j')}\inf\big\{(L_\lambda\psi,\psi) : \psi\in H^1(\Omega_j'),\ \|\psi\|_{L^2(\Omega_j')}=1,\\ &\qquad\qquad (\psi,\phi_i)=0 \text{ for } i=1,\dots,d\big\}\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}\mu_n &= \inf\big\{(L_0\psi,\psi) : \psi\in H_0^1(\Omega_j),\ \|\psi\|_{L^2(\Omega_j)}=1,\\ &\qquad\qquad \psi\perp V_m \text{ for } m=1,\dots,n-1\big\}\\ &= \max_{\phi_1,\cdots,\phi_{d-1}\in H_0^1(\Omega_j)}\inf\big\{(L_0\psi,\psi) : \psi\in H_0^1(\Omega_j),\ \|\psi\|_{L^2(\Omega_j)}=1,\\ &\qquad\qquad (\psi,\phi_i)=0 \text{ for } i=1,\dots,d-1\big\}.\end{aligned} \tag{4.2}$$

Since $V_m^\lambda\to V_m$ for $1\le m\le n-1$ as $\lambda\to\infty$, and since $(L_\lambda\psi,\psi)=(L_0\psi,\psi)$, for every $\psi\in H_0^1(\Omega_j)$, (4.1) and (4.2) imply:

$$\limsup_{\lambda\to\infty}\mu_n^\lambda\le\mu_n. \tag{4.3}$$

In order to prove equality consider a sequence $\lambda_i\to\infty$ and normalized eigenfunctions ψ_i corresponding to $\mu_n^{\lambda_i}$. Then we have:

$$\int_{\Omega_j'}\psi_i^2=1,\quad \int_{\Omega_j'}\big(|\nabla\psi_i|^2+(\lambda_i a(x)+a_0(x))\psi_i^2\big)=\mu_n^{\lambda_i},$$

and

$$\psi_i\perp V_m^{\lambda_i}\qquad\text{for } m=1,2,\dots,n-1.$$

By (4.3), ψ_i is bounded in $H^1(\Omega_j')$, so we may assume that $\psi_i\rightharpoonup\psi\in H^1(\Omega_j')$ and $\psi_i\to\psi$ in $L^2(\Omega_j')$. It is easy to see that $\psi=0$ in $\Omega_j'\setminus\Omega_j$, because $a(x)>0$ in $\Omega_j'\setminus\Omega_j$. Since $\partial\Omega_j$ is smooth it follows that $\psi\in H_0^1(\Omega_j)$. Strong convergence in $L^2(\Omega_j')$ implies $\int_{\Omega_j}\psi^2=\int_{\Omega_j'}\psi^2=1$. Since by our induction assumption, $V_m^{\lambda_i}\to V_m$, $m=1,\dots,n-1$, we obtain

$$\psi\perp V_m,\quad m=1,\dots,n-1. \tag{4.4}$$

By the minmax description of the nth-eigenvalue there holds:

$$\begin{aligned}\mu_n &\le \int_{\Omega_j}\big(|\nabla\psi|^2+a_0(x)\psi^2\big)\\ &\le \liminf_{i\to\infty}\int_{\Omega_j'}\big(|\nabla\psi_i|^2+(\lambda_i a(x)+a_0(x))\psi_i^2\big)=\liminf_{i\to\infty}\mu_n^{\lambda_i}\le\mu_n.\end{aligned} \tag{4.5}$$

This and (4.3) show that $\mu_n^\lambda\to\mu_n$ as $\lambda\to\infty$. It also follows from (4.5) that $\psi_i\to\psi\in V_n$ strongly in X_j, hence $V_n^\lambda\to V_n$. $\square$

5. **Definition of the critical value.** For $j = 1, \dots, m$, we set $E_j := H_0^1(\Omega_j) \subset E$, where E is defined in (3.2), and consider the functional

$$I_j : E_j \to \mathbb{R}, \quad I_j(u) = \frac{1}{2}\int_{\Omega_j} \left(|\nabla u|^2 + a_0 u^2\right) - \frac{1}{p}\int_{\Omega_j} |u|^p.$$

By assumption (V_3), E_j splits as the orthogonal sum $E_j = E_j^- \oplus E_j^+$ of the negative and positive eigenspace of $-\Delta + a_0$. As in Section 2 let $P_j^- : E_j \to E_j^-$ denote the orthogonal projection. Since Ω_j is bounded, $p < 2N/(N-2)$ if $N > 2$, I_j satisfies the Palais-Smale condition, hence the infimum of I_j on the Nehari-Pankov manifold

$$\mathcal{N}_j = \{u \in E_j \setminus \{0\} : \ P_j^-(\nabla I_j(u)) = 0, \ DI_j(u)[u] = 0\}$$

is achieved by some $w_j \in \mathcal{N}_j$,

$$c_j := \inf_{u \in \mathcal{N}_j} I_j(u) = I_j(w_j) > 0. \tag{5.1}$$

We fix a subset $J \subset \{1, 2, \dots, m\}$, set $d_j := \dim E_j^-$, and let e_{ji}, $i = 1, \dots, d_j$, be an orthonormal basis of E_j^-, $j = 1, \dots, m$. We also need the sets

$$\begin{aligned} A := \big\{(s_1, \dots, s_m, t) \in \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_m} \times \mathbb{R}^J : \|s_i\|_\infty \le 1, \ i = 1, \dots, m,& \\ 0 \le t_j \le 1, \ j \in J\big\}& \end{aligned}$$

and $B := \partial A$. For $R > \max_{j \in J} \|w_j\|$ large and $0 < r < \min_{j \in J} \|w_j\|$ small, to be determined below, we define the map $\gamma_0 : A \to E$ by

$$\gamma_0(s,t) := \sum_{j \in J}\left(R\sum_{i=1}^{d_j} s_{ji}e_{ji} + ((1-t_j)r + t_j R)w_j\right) + \sum_{j \notin J}\left(r\sum_{i=1}^{d_j} s_{ji}e_{ji}\right).$$

Observe that $I_j(u) \le 0$ for $u \in E_j^-$, and therefore

$$\sum_{j \notin J} I_j\left(r\sum_{i=1}^{d_j} s_{ji}e_{ji}\right) \le 0 \quad \text{for all } s_{ji}.$$

Hence if some $s_{ji} \neq 0$ or some $t_j \neq 0$ then

$$\begin{aligned} J_\lambda\big(\gamma_0(s,t)\big) &= \sum_{j \in J} I_j\left(R\sum_{i=1}^{d_j} s_{ji}e_{ji} + ((1-t_j)r + t_j R)w_j\right) + \sum_{j \notin J} I_j\left(r\sum_{i=1}^{d_j} s_{ji}e_{ji}\right) \\ &\to -\infty \end{aligned}$$

as $R \to \infty$. Also, if $t_j = 0$ for $j \in J$ and $r = 0$ then $J_\lambda\big(\gamma_0(s,t)\big) \le 0$. It follows that for $R > 0$ large and $r > 0$ small there holds

$$J_\lambda\big(\gamma_0(s,t)\big) < \sum_{j \in J} c_j \quad \text{for all } \ (s,t) \in B, \ \lambda \ge 0. \tag{5.2}$$

If r is small enough there exists $\alpha > 0$ such that

$$I_j(u_j) \ge \alpha \|u_j\|_{E_j}^2 \quad \text{for } u_j \in E_j^+, \ \ \|u_j\|_{E_j} \le r. \tag{5.3}$$

We fix r, R satisfying (5.2) and (5.3). Now we define the sets

$$\begin{aligned} \mathcal{H}_\lambda := \{h : A \times [0,1] \to E : h \in C^0, \ h(s,t,0) = \gamma_0(s,t),& \\ J_\lambda(h(s,t,\tau)) \text{ is nonincreasing with respect to } \tau\}& \end{aligned}$$

and

$$\Gamma_\lambda := \{\gamma : A \to E \mid \exists\, h \in \mathcal{H}_\lambda\ \forall (s,t) \in A : \gamma(s,t) = h(s,t,1)\}\,.$$

Finally we arrive at a minmax description of a possible critical value:

$$c_\lambda := \inf_{\gamma\in\Gamma_\lambda} \max_{(s,t)\in A} J_\lambda(\gamma(s,t)). \tag{5.4}$$

Lemma 5.1. $c_\lambda \le \sum_{j\in J} c_j$

Proof. This follows from $\gamma_0 \in \Gamma_\lambda$, the choice of the w_j, and Proposition 2.1. □

In order to obtain a lower bound for c_λ we need the smoothly bounded open neighborhoods Ω'_j of Ω_j from the definition of the penalized functional in Section 3. We consider the functional $I_j^\lambda : X_j = H^1(\Omega'_j) \to \mathbb{R}$ defined by

$$I_j^\lambda(u) := \frac{1}{2}\int_{\Omega'_j} \left(|\nabla u|^2 + (\lambda a + a_0)u^2\right) - \frac{1}{p}\int_{\Omega'_j} |u|^p,$$

and its associated Nehari-Pankov manifold

$$\mathcal{N}_j^\lambda := \{u \in X_j \setminus \{0\} : Q_j^{\lambda,-}(\nabla I_j^\lambda(u)) = 0\ , DI_j^\lambda(u)[u] = 0\}.$$

Here $Q_j^{\lambda,-} : X_j \to X_j^{\lambda,-}$ is the orthogonal projection on the negative eigenspace associated to $L_\lambda := -\Delta + \lambda a + a_0$ in X_j. As a consequence of Corollary 4.2 the results from Section 2 apply and the infimum

$$c_j^\lambda := \inf_{u\in\mathcal{N}_j^\lambda} I_j^\lambda(u) > 0$$

is achieved. We have the following asymptotic behavior for c_j^λ as $\lambda \to \infty$.

Lemma 5.2. *$c_j^\lambda \to c_j$ as $\lambda \to \infty$.*

Proof. Clearly $\mathcal{N}_j \subset \mathcal{N}_j^\lambda$ because

$$Q_j^{\lambda-}\left(\nabla I_j^\lambda(u_j)\right) = P_j^-\left(\nabla I_j(u_j)\right) \quad \text{and} \quad DI_j^\lambda(u_j)[u_j] = DI_j(u_j)[u_j]$$

for every $u \in H_0^1(\Omega_j)$. It follows that

$$c_j^\lambda \le c_j. \tag{5.5}$$

On the other hand, it is easy to see that c_j^λ is nondecreasing with respect to λ. Thus (5.5) implies that the limit $\lim_{\lambda\to\infty} c_j^\lambda$ exists and

$$\lim_{\lambda\to\infty} c_j^\lambda \le c_j. \tag{5.6}$$

Now we prove the inverse of (5.6). Indeed, since I_j^λ satisfies the Palais-Smale condition, c_j^λ is achieved by a critical point w^λ of I_j^λ. Given a sequence $\lambda_i \to \infty$, we deduce from (5.6) that w^{λ_i} is uniformly bounded in $H^1(\Omega'_j)$, so we may assume $w^{\lambda_i} \rightharpoonup w$ in $H^1(\Omega'_j)$. As in the proof of Proposition 3.3 one sees that $w^{\lambda_i} \to w$ strongly in $H^1(\Omega'_j)$, $w \in H_0^1(\Omega_j)$, and $c_j^{\lambda_i} = I_j^{\lambda_i}(w^{\lambda_i}) \to I_j(w)$; in particular $w \neq 0$. Moreover,

$$DI_{\lambda_i}(w^{\lambda_i})[w^{\lambda_i}] \to DI_j(w)[w]$$

and

$$Q_j^{\lambda_i}\nabla I_j^{\lambda_i}(w^{\lambda_i}) \to P_j\nabla I_j(w);$$

here we also used Lemma 4.1. Thus $w \in \mathcal{N}_j$ and

$$c_j \le I_j(w) = \lim_{\lambda\to\infty} c_j^\lambda. \tag{5.7}$$

The lemma follows from (5.6) and (5.7). $\square$

Let $\Omega_0 := \bigcup_{j\notin J}\Omega_j$ and $\Omega_0' := \bigcup_{j\notin J}\Omega_j'$. We denote $X_0 := H^1(\Omega_0') = \bigoplus_{j\notin J} X_j$ and $E_0 := H_0^1(\Omega_0) = \bigoplus_{j\notin J} E_j$. Let $X_0^{\lambda-}$ be the negative eigenspace associated to $-\Delta+\lambda a+a_0$ in X_0, and let E_0^- be the negative eigenspace associated to $-\Delta+a_0$ in E_0. Clearly $X_0^{\lambda-} = \bigoplus_{j\notin J} X_j^{\lambda-}$ and $E_0^- = \bigoplus_{j\notin J} E_j^-$. Finally, let $Q_0^{\lambda-} : X_0 \to X_0^{\lambda-}$ and $P_0^- : E_0 \to E_0^-$ be the orthogonal projections.

The following linking property for $\gamma \in \Gamma_\lambda$ is the key to the proof of the lower bound of c_λ. It will be proved in the next section.

Lemma 5.3. *If λ is sufficiently large, then for any $\gamma\in\Gamma_\lambda$, there exists $(s,t)\in A$ such that $u := \gamma(s,t)$ satisfies*

$$u_j := u|_{\Omega_j'} \in \mathcal{N}_j^\lambda \quad \textit{for } j \in J, \tag{5.8}$$

and

$$u_0 \perp X_0^{\lambda-},\ \ \|u_0\| < r. \tag{5.9}$$

Lemma 5.4. $c_\lambda \ge \sum_{j\in J} c_j^\lambda$.

Proof. Lemma 5.3 yields that, given $\gamma\in\Gamma_\lambda$ there exists $(s,t)\in A$ such that $u := \gamma(s,t)$ satisfies (5.8) and (5.9). Using (5.3) this implies $I_0^\lambda(u_0)\ge 0$, hence

$$\max_A J_\lambda\circ\gamma \ge J_\lambda(u) \ge \sum_{j\in J} I_j^\lambda(u_j) \ge \sum_{j\in J} c_j^\lambda.$$

$\square$

As a consequence of the lemmas 5.1, 5.4 and 5.2, we deduce:

Corollary 5.5. *There holds* $\lim\limits_{\lambda\to\infty} c_\lambda = \sum\limits_{j\in J} c_j$ *and for λ large, c_λ is achieved by a critical point u_λ of J_λ.*

Proof. In fact, for λ large enough (5.2) implies

$$c_\lambda > \max_{(s,t)\in B} J_\lambda(\gamma_0(s,t)).$$

A standard argument now yields that c_λ is achieved by a critical point u_λ of J_λ provided $\lambda\ge\Lambda_0$ as in Proposition 3.1. As a consequence of Proposition 3.4, u_λ is a solution of (S_λ) for λ large. $\square$

6. **Proof of Lemma 5.3.** For $u\in E$ we write $u_j := u|_{\Omega_j'}$, $j\in J_0 := J\cup\{0\}$. We need the map

$$f_\lambda : E \to X_0^{\lambda-} \times \prod_{j\in J}\left(X_j^{\lambda-}\times\mathbb{R}\right)$$

defined by

$$f_{\lambda,0} := Q_0^{\lambda-} : E \to X_0^{\lambda-}$$

and for $j\in J$:

$$f_{\lambda,j} : E \to X_j^{\lambda-}\times\mathbb{R}, \quad f_{\lambda,j}(u) := \left(Q_j^{\lambda-}(\nabla I_j^\lambda(u_j)), DI_j^\lambda(u_j)[u_j]\right).$$

Clearly we have:

$$f_\lambda(u) = 0 \quad\Longleftrightarrow\quad u_0\perp X_0^{\lambda-}, \ \text{ and } \ u_j\in\mathcal{N}_j^\lambda \ \text{ for } j\in J \tag{6.1}$$

Consider $\gamma \in \Gamma_\lambda$ and let $h \in \mathcal{H}_\lambda$ be a homotopy from γ_0 to γ. We have to show that for λ large there exists $(s,t) \in A$ such that $u = \gamma(s,t)$ satisfies $f_\lambda(u) = 0$ and $\|u_0\| < r$. This will be done with a degree argument.

First we claim that for $(s,t,\tau) \in A \times [0,1]$, $u := h(s,t,\tau)$, and λ large the following holds:

$$f_\lambda(u) = 0 \quad \Longrightarrow \quad \|u_0\|_{X_0} \neq r. \tag{6.2}$$

In order to see this we observe that Lemma 4.1 and (5.3) imply the existence of $\beta > 0$ such that

$$I_0^\lambda(v) \geq \beta \quad \text{for all } v \in X_0^+,\ \|v\|_{X_0} = r,$$

and

$$I_0^\lambda(v) \geq 0 \quad \text{for all } v \in X_0^+, \|v\|_{X_0} \leq r,$$

hold for λ large. Moreover, Lemma 5.2 shows that

$$\sum_{j\in J} c_j < \sum_{j\in J} c_j^\lambda + \beta$$

for λ large. Now suppose that

$$\|u_0\|_{X_0} = r. \tag{6.3}$$

Our choice of δ implies for $v \in E$ and $\lambda \geq \Lambda_0$ that

$$\begin{aligned}
J_\lambda(v) &= \frac{1}{2}\int_{\mathbb{R}^N\setminus\Omega'} \left(|\nabla v|^2 + (\lambda a + a_0)v^2\right) - \int_{\mathbb{R}^N\setminus\Omega'} G_\delta(x,v) \\
&\qquad + \sum_{j\in J_0}\left(\frac{1}{2}\int_{\Omega_j'} \left(|\nabla v|^2 + (\lambda a + a_0)v^2\right) - \int_{\Omega_j'} G_\delta(x,v)\right) \\
&\geq \sum_{j\in J_0}\left(\frac{1}{2}\int_{\Omega_j'} \left(|\nabla u|^2 + (\lambda a + a_0)v^2\right) - \frac{1}{p}\int_{\Omega_j'} |v|^p\right) \\
&= \sum_{j\in J_0} I_j^\lambda(v|\Omega_j').
\end{aligned}$$

Thus we get for $u = h(s,t,r)$

$$J_\lambda(u) \geq \sum_{j\in J_0} I_j^\lambda(u_j) \geq \beta + \sum_{j\in J} c_j^\lambda > \sum_{j\in J} c_j. \tag{6.4}$$

On the other hand, using that $J_\lambda(h(s,t,\tau))$ is nonincreasing with respect to $\tau \in [0,1]$ we have

$$J_\lambda(u) = J_\lambda(h(s,t,\tau)) \leq J_\lambda(h(s,t,0)) = J_\lambda(\gamma_0(s,t)) \leq \sum_{j\in J} c_j$$

which contradicts with (6.4). This contradiction implies that (6.3) is impossible, which proves (6.2).

Now we consider the sets

$$\mathcal{G}_\lambda := \{(s,t,\tau) \in A \times [0,1] : f_\lambda(h(s,t,\tau)) = 0\}$$

and

$$\mathcal{G}_\lambda^0 := \{(s,t,\tau) \in \mathcal{G}_\lambda : u = h(s,t,\tau) \text{ satisfies } \|u_0\|_{X_0} < r\}.$$

By (6.2), for λ large there exists a neighborhood U_λ of $\mathcal{G}_\lambda^0$ in $A \times [0,1]$ such that $\overline{U_\lambda} \cap (\mathcal{G}_\lambda \setminus \mathcal{G}_\lambda^0) = \emptyset$. We define $U_\lambda^\tau := \{(s,t) \in A : (s,t,\tau) \in U_\lambda\}$. The lemma is proved if we can find $(s,t) \in U_\lambda^1$ such that $f_\lambda(\gamma(s,t)) = 0$. By the homotopy invariance of the degree we have

$$\deg(f_\lambda \circ \gamma, U_\lambda^1, 0) = \deg(f_\lambda \circ \gamma_0, U_\lambda^0, 0). \tag{6.5}$$

Setting

$$s^* = (0,\dots,0) \in \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_m} \quad \text{and} \quad t^* = \left(\frac{1-r}{R-r},\dots,\frac{1-r}{R-r}\right) \in \mathbb{R}^J \tag{6.6}$$

we have

$$\mathcal{G} \cap (A \times \{0\}) = \{(s^*, t^*, 0)\},$$

and therefore

$$\deg(f_\lambda \circ \gamma_0, U_\lambda^0, 0) = \deg(f_\lambda \circ \gamma_0, A, 0). \tag{6.7}$$

Clearly γ_0 is linear in (s,t) and defines a homeomorphism

$$\gamma_0 : A \to A' := B_{0,r} \times \prod_{j\in J} A_{w_j,r,R} \subset E_0^- \times \prod_{j\in J} H_{w_j} \subset H_0^1(\Omega).$$

Here $A_{w_j,r,R} \subset H_{w_j} \subset E_j^- \oplus \mathbb{R}w_j$ is defined as in (2.1) and (2.2), and

$$B_{0,r} := \left\{ u \in E_0^- : u = r \sum_{j\notin J} \sum_{i=1}^{d_j} s_{ji} e_{ji},\ |s_{ji}| \le 1 \right\}.$$

It follows that

$$\deg(f_\lambda \circ \gamma_0, A, 0) = \pm \deg(f_\lambda, A', 0). \tag{6.8}$$

Moreover, since $A' \subset H_0^1(\Omega)$ we have for $u \in A'$ that $u_j = u|_{\Omega_j'} \in H_0^1(\Omega_j)$. This implies

$$Q_0^-(u_0) = P_0^-(u_0),$$

and for $j \in J$:

$$Q_j^{\lambda-}(\nabla I_j^\lambda(u_j)) = P_j^-(\nabla I_j(u_j)), \quad DI_j^\lambda(u_j)[u_j] = DI_j(u_j)[u_j].$$

Thus for $u \in A$ we have $f_\lambda(u) = (g_j(u_j))_{j\in J_0}$ with $g_0(u) = P_0^-(u)$ and

$$g_j(u_j) = \left(P_j^-(\nabla I_j(u_j)), DI_j(u_j)[u_j]\right), \quad j \in J.$$

Now Proposition 2.1 e) yields

$$\deg(f_\lambda, A', 0) = \deg(g_0, B_{0,R}, 0) \cdot \prod_{j\in J} \deg(g_j, A_{w_j,r,R}, 0) = 1. \tag{6.9}$$

The equations (6.5)-(6.9) imply the existence of $(s,t) \in U_\lambda^1$ with $f_\lambda(\gamma(s,t)) = 0$. It follows that $u = \gamma(s,t)$ satisfies $\|u_0\|_{X_0} < r$, in addition to $f_\lambda(u) = 0$. This proves Lemma 5.3.

7. **Proof of Theorem 1.1.**

For $u \in E$ and $M \subset \mathbb{R}^N$ measurable we use the notation

$$\|u\|_{\lambda,M} := \left(\int_M \left(|\nabla u|^2 + (\lambda a(x) + a_0(x))u^2\right)\right)^{1/2}.$$

We choose $\varepsilon > 0$ small so that $B_\varepsilon(0, E_j)$ contains only $0 \in E_j$ as critical point of I_j, for all $j \notin J$. We also require that $\varepsilon < \sqrt{2pc_j/(p-2)}$ for $j \in J$. Now we define

$$D_\lambda^\varepsilon = \Big\{ u \in E_\lambda : \|u\|_{\lambda,\mathbb{R}^N\setminus\Omega_J'} \le \varepsilon/3$$
$$\left| \|u\|_{\lambda,\Omega_j'} - \sqrt{2pc_j/(p-2)} \right| \le \varepsilon/3 \text{ for all } j \in J \Big\}.$$

Setting $c^* := \sum_{j\in J} c_j$, it is easy to check that $D_\lambda^\varepsilon \cap J_\lambda^{c^*}$ contains all functions of the form

$$w(x) = \begin{cases} v_j(x) & x \in \Omega_j,\ j \in J,\\ 0 & x \in \mathbb{R}^N \setminus \Omega_J; \end{cases}$$

where v_j minimizes I_j in $\mathcal{N}_j$; see Section 5.

Lemma 7.1. *There exists $\sigma_0 > 0$ and $\Lambda_1 \ge \Lambda_0$ such that*

$$\|\nabla J_\lambda(u)\|_\lambda \ge \sigma_0 \quad \text{for } \lambda \ge \Lambda_1 \text{ and } u \in \big(D_\lambda^{2\varepsilon} \setminus D_\lambda^\varepsilon\big) \cap J_\lambda^{c^*} \tag{7.1}$$

Proof. We argue by contradiction. Suppose there exist $\lambda_n \to \infty$ and $u_n \in (D_{\lambda_n}^{2\varepsilon} \setminus D_{\lambda_n}^\varepsilon) \cap J_{\lambda_n}^{c^*}$ such that $\|\nabla J_{\lambda_n}(u)\|_{\lambda_n} \to 0$. Since $D_{\lambda_n}^{2\varepsilon}$ is bounded we can apply Proposition 3.3, so up to a subsequence $u_n \to u$ in E and $u|_{\Omega_j}$ is a critical point of I_j. In addition we have:

$$\lim_{n\to\infty} \|u_n\|_{\lambda_n,\Omega_j'} = \int_{\Omega_j} (|\nabla u|^2 + a_0(x)u^2) \quad \text{for } 1 \le j \le m, \tag{7.2}$$

and

$$\lim_{n\to\infty} \|u_n\|_{\lambda_n,\mathbb{R}^N\setminus\Omega'} = 0. \tag{7.3}$$

This implies that $u \equiv 0$ in $\mathbb{R}^N \setminus \Omega$. Since $\|u|_{\Omega_j}\| < \varepsilon$ for $j \notin J$ we also have $u \equiv 0$ in $\mathbb{R}^N \setminus \Omega_J$. On the other hand, (7.2) and our choice of ε imply $u|_{\Omega_j} \ne 0$ for $j \in J$, hence $I_j\big(u|_{\Omega_j}\big) \ge c_j$ for $j \in J$. Then $J_{\lambda_n}(u_n) \le c^*$ yields $I_j\big(u|_{\Omega_j}\big) = c_j$ for $j \in J$. From this we deduce

$$\int_{\Omega_j} (|\nabla u|^2 + a_0 u^2) = \left(\frac{1}{2} - \frac{1}{p}\right)^{-1} c_j = 2pc_j/(p-2) \quad \text{for } j \in J,$$

hence $u_n \in D_{\lambda_n}^\varepsilon$ for large n by (7.2) and (7.3), contradicting $u_n \in D_{\lambda_n}^{2\varepsilon} \setminus D_{\lambda_n}^\varepsilon$. □

The following proposition is the key of the proof of our main result.

Proposition 7.2. *Let Λ_1 be the constant given in Lemma 7.1 and Λ_{c^*} the constant from Proposition 3.4. Then for $\lambda \ge \max\{\Lambda_1, \Lambda_{c^*}\}$ there exists a solution u_λ of (S_λ) satisfying $u_\lambda \in D_\lambda^\varepsilon \cap J_\lambda^{c^*}$.*

Proof. We argue indirectly and assume that J_λ has no critical points in $D_\lambda^\varepsilon \cap J_\lambda^{c^*}$. Since J_λ satisfies the Palais-Smale condition, there exists a constant $d_\lambda > 0$ such that

$$\|\nabla J_\lambda(u)\|_\lambda \ge d_\lambda \quad \text{for all } u \in D_\lambda^\varepsilon \cap J_\lambda^{c^*}. \tag{7.4}$$

By Lemma 7.1 there holds

$$\|\nabla J_\lambda(u)\|_\lambda \ge \sigma_0 \quad \text{for all } u \in (D_\lambda^{2\varepsilon} \setminus D_\lambda^\varepsilon) \cap J_\lambda^{c^*}$$

Let $\varphi : E \to \mathbb{R}$ be a Lipschitz continuous function such that

$$\varphi(u) = \begin{cases} 1 & \text{for } u \in D_\lambda^{3\varepsilon/2},\\ 0 & \text{for } u \notin D_\lambda^{2\varepsilon} \end{cases}$$

and $0 \le \varphi(u) \le 1$ for every $u \in E$. Then the vector field

$$V : J_\lambda^{c^*} \to E, \quad V(u) = -\varphi(u)\frac{\nabla J_\lambda(u)}{\|\nabla J_\lambda(u)\|_\lambda},$$

is well defined, Lipschitz continuous and satisfies

$$\|V(u)\|_\lambda \le 1 \text{ for all } u. \tag{7.5}$$

We consider the associated flow $\eta : [0,\infty) \times J_\lambda^{c^*} \to J_\lambda^{c^*}$ defined by

$$\dot\eta(\tau,u) = \frac{d\eta}{d\tau}(\tau,u) = V(\eta(\tau,u)), \quad \eta(0,u) = u.$$

Obviously η satisfies

$$\frac{d}{d\tau} J_\lambda(\eta(\tau,u)) = -\varphi(u)\|\nabla J_\lambda(u)\|_\lambda \le 0, \tag{7.6}$$

and

$$\eta(\tau,u) = u \quad \text{for all } \tau \ge 0,\ u \in J_\lambda^{c^*} \setminus D_\lambda^{2\varepsilon}. \tag{7.7}$$

We consider $\eta(\tau,\gamma_0)$ for large τ. Since $\gamma_0(s,t) \notin D_\lambda^{2\varepsilon}$ for $(s,t) \in B$, (7.7) implies

$$\eta(\tau,\gamma_0(s,t)) = \gamma_0(s,t) \quad \text{ for } (s,t) \in B,\ \tau \ge 0. \tag{7.8}$$

Recall that $\operatorname{supp}\gamma_0(s,t) \subset \bigcup_{j\in J}\overline{\Omega_j}$ for every $(s,t) \in A$, hence $J_\lambda(\gamma_0(s,t))$ and $\|\gamma_0(s,t)\|_{\lambda,\Omega'}$ etc. do not depend on $\lambda \ge 0$. On the other hand

$$J_\lambda(\gamma_0(s,t)) \le c^* \quad \text{ for } (s,t) \in A$$

and there exists a unique $(s^*,t^*) \in A$, see (6.6), with $J_\lambda(\gamma_0(s^*,t^*)) = c^*$, that is, $\gamma_0(s^*,t^*))|_{\Omega_j} = w_j$ for $j \in J$ and $\gamma_0(s^*,t^*))(x)|_{\Omega_j} = 0$ for $j \notin J$. Thus we have

$$m_0 := \max\{J_\lambda(u) : u \in \gamma_0(A) \setminus D_\lambda^\varepsilon\} < c^* \tag{7.9}$$

is independent of λ.

Now we claim that for large $\bar\tau$,

$$\max_{(s,t)\in A} J_\lambda(\eta(\bar\tau,\gamma_0(s,t)) \le \max\{m_0, c^* - \sigma_0\varepsilon/6\} \tag{7.10}$$

with σ_0, m_0 from (7.1), (7.9), respectively. In fact, (7.9) yields $J_\lambda(\eta(\tau,\gamma_0(s,t))) \le m_0$ if $\gamma_0(s,t) \notin D_\lambda^\varepsilon$, $\tau \ge 0$. In the case $\gamma_0(s,t) \in D_\lambda^\varepsilon$ we consider the behavior of $\tilde\eta(\tau) := \eta(\tau,\gamma_0(s,t))$. We set $\tilde d_\lambda := \min\{d_\lambda,\sigma_0\}$ and $\bar\tau = \sigma_0\mu/6\tilde d_\lambda$, where d_λ is from (7.4). We consider two cases:

1) $\tilde\eta(\tau) \in D_\lambda^{3\varepsilon/2}$ for all $\tau \in [0,\bar\tau]$.
2) $\tilde\eta(\tau_0) \in \partial D_\lambda^{3\varepsilon/2}$ for some $\tau_0 \in [0,\bar\tau]$.

In case 1) we have $\varphi(\tilde\eta(\tau)) \equiv 1$ and $\|\nabla J_\lambda(\tilde\eta(\tau))\|_\lambda \ge \tilde d_\lambda$ for all $\tau \in [0,\bar\tau]$. Then (7.1) implies

$$\begin{aligned} J_\lambda(\tilde\eta(\tau)) &= J_\lambda(\gamma_0(s,t)) + \int_0^{\bar\tau} \frac{d}{ds} J_\lambda(\tilde\eta(\tau)) \\ &= J_\lambda(\gamma_0(s,t)) - \int_0^{\bar\tau} \varphi(\tilde\eta(s)))\|\nabla J_\lambda(\tilde\eta(s))\|_\lambda\, ds \\ &\le c^* - \int_0^{\bar\tau} \tilde d_\lambda ds = c^* - \tilde d_\lambda\bar\tau = c^* - \sigma_0\varepsilon/6. \end{aligned}$$

In case 2) there exist $0 \le \tau_1 < \tau_2 \le \bar\tau$ such that

$$\tilde\eta(\tau_1) \in \partial D_\lambda^\varepsilon\,, \quad \tilde\eta(\tau_2) \in \partial D_\lambda^{3\varepsilon/2}, \tag{7.11}$$

and

$$\tilde\eta(\tau) \in D_\lambda^{3\varepsilon/2} \setminus D_\lambda^\varepsilon \quad \text{ for all } \tau \in [\tau_1,\tau_2]. \tag{7.12}$$

It follows from (7.11) that

$$\|\tilde\eta(\tau_1)\|_{\lambda,\mathbb{R}^N\setminus\Omega'_J} \le \varepsilon/3 \quad \text{and} \quad \left|\|\tilde\eta(\tau_1)\|_{\lambda,\Omega'_j} - \sqrt{2pc_j/(p-2)}\right| \le \varepsilon/3 \text{ for all } j \in J$$

and

$$\|\tilde{\eta}(\tau_2)\|_{\lambda,\mathbb{R}^N\setminus\Omega'_J} = \frac{\varepsilon}{2} \quad \text{or} \quad \left|\|\tilde{\eta}(\tau_2)\|_{\lambda,\Omega'_j} - \sqrt{2pc_j/(p-2)}\right| = \frac{\varepsilon}{2} \quad \text{for some } j \in J.$$

This immediately implies

$$\|\tilde{\eta}(\tau_1) - \tilde{\eta}(\tau_2)\|_\lambda \geq \varepsilon/6. \tag{7.13}$$

Now (7.5), (7.13) and the mean value theorem imply $\tau_2 - \tau_1 \geq \varepsilon/6$. Using (7.1) we deduce

$$\begin{aligned} J_\lambda(\tilde{\eta}(\bar{\tau})) &= J_\lambda(\gamma_0(s,t)) - \int_0^{\bar{\tau}} \varphi(\tilde{\eta}(s))\|\nabla J_\lambda(\tilde{\eta}(s))\|_\lambda \, ds \\ &\leq c^* - \int_{\tau_1}^{\tau_2} \sigma_0 ds = c^* - \sigma_0(\tau_2 - \tau_1) \leq c^* - \sigma_0\mu/6 \end{aligned}$$

and thus (7.10) is proved.

Now we define $\tilde{h}(s,t,r) := \eta(r\bar{\tau}, \gamma_0(s,t))$ and $\tilde{\gamma}(s,t) := \tilde{h}(s,t,1) = \eta(\bar{\tau}, \gamma_0(s,t))$. Observe that $\tilde{h} \in \mathcal{H}_\lambda$ due to (7.6), (7.8), hence $\gamma \in \Gamma_\lambda$. Thus we have

$$c_\lambda \leq J_\lambda(\tilde{\gamma}(s,t)) \leq \max\{m_0, c^* - \sigma_0\mu/6\} \tag{7.14}$$

However by Corollary 5.5 we have $c_\lambda \to c^*$ as $\lambda \to \infty$. This contradicts (7.10), and thus J_λ has a critical point $u_\lambda \in D_\lambda^\varepsilon$. By Proposition 3.4, u_λ is a solution of the original problem (S_λ). □

Finally we easily prove the main result.

Proof of Theorem 1.1. Let u_λ be a solution of (S_λ) obtained in Proposition 7.2. Applying Proposition 3.3, for any given sequence $\lambda_n \to \infty$ we can extract a subsequence, which satisfies the conclusion of Proposition 3.3. With the same argument as in the proof of Lemma 7.1, we can extract a subsequence of u_{λ_n} such that $u_{\lambda_n} \to u$ in E along this subsequence, and $u|_{\mathbb{R}^N\setminus\Omega_J} \equiv 0$. Furthermore

$$\lim_{n\to\infty} \int_{\Omega_j} \left(\frac{1}{2}(|\nabla u_{\lambda_n}|^2 + a_0(x)u_{\lambda_n}^2) - \frac{1}{p}|u_{\lambda_n}|^p) \right) = c_j \quad \text{for } j \in J \tag{7.15}$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N\setminus\Omega_J} (|\nabla u_{\lambda_n}|^2 + (\lambda_n a(x) + a_0(x))u_{\lambda_n}^2) = 0. \tag{7.16}$$

Since the limits in (7.15) and (7.16) do not depend on the choice of the sequence $\lambda_n \to \infty$ Theorem 1.1 is proved. □

Acknowledgements. This paper was partially written while the second author visited the Mathematical Institute of the University of Giessen as an Alexander von Humboldt Fellow. He wishes to express his gratitude to the Alexander von Humboldt Foundation for the financial support and also to the Mathematical Institute of the University of Giessen for the hospitality during his stay.

REFERENCES

[1] T. Bartsch, A. Pankov and Z.-Q. Wang: *Nonlinear Schrödinger equations with steep potential well,* Commun. Contemp. Math. **3** (2001), 549569

[2] T. Bartsch and M. Parnet, *Nonlinear Schrödinger equations near an infinite well potential,* Preprint.

[3] T. Bartsch and Z.-Q. Wang, *Existence and multiplicity results for some superlinear elliptic equation* $\mathbb{R}^N$*,* Comm. Part. Diff. Eq. **20** (1995), 1725-1741.

[4] T. Bartsch and Z.-Q. Wang, *Multiple positive solutions for a nonlinear Schrödinger equation,* Z. angew. Math. Phys. **51** (2000), 366-384.

[5] Y. Ding and K. Tanaka, *Multiplicity of positive solutions of a nonlinear Schrödinger equation,* Manuscripta Math. **112** (2003), 109-135.

[6] Y. Ding and A. Szulkin, *Existence and number of solutions for a class of semilinear Schrödinger equation,* Progr. Nonlin. Diff. Equ. Appl. **66** (2006), 221-231.

[7] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order. Second edition,* Springer-Verlag, New York, 1983.

[8] A. Pankov, *Periodic nonlinear Schrödinger equation with application to photonic crystals,* Milan J.Math. **73** (2005), 563 - 574.

[9] M. Reed and B. Simon, *Schrödinger semigroups,* Bull. Amer. Math. Soc. (N.S) **7**(1982),447-526.

[10] Y. Sato and K. Tanaka, *Sign-changing multi-bump solutions for nonlinear Schrödinger equations with steep potential wells,* Trans. Amer. Math. Soc. **361** (2009), 6205-6253.

[11] A. Szulkin and T. Weth, *Ground state solutions for some indefinite variational problems,* J. Funct. Anal. **257** (2009), 3802-3822.

[12] Z.-P. Wang and H.-S. Zhou, *Positive solutions for nonlinear Schrödinger equations with deepening potential well,* J. Europ. Math. Soc. **11** (2009), 545-573.

E-mail address: `Thomas.Bartsch@math.uni-giessen.de`

E-mail address: `tangzw@bnu.edu.cn`