MULTI-BUBBLE NODAL SOLUTIONS FOR SLIGHTLY SUBCRITICAL ELLIPTIC PROBLEMS IN DOMAINS WITH SYMMETRIES

THOMAS BARTSCH & TERESA D'APRILE & ANGELA PISTOIA

Abstract. We study the existence of sign-changing solutions with multiple *bubbles* to the slightly subcritical problem

$$-\Delta u = |u|^{2^*-2-\varepsilon}u \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega,$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$, $2^* = \frac{2N}{N-2}$ and $\varepsilon > 0$ is a small parameter. In particular we prove that if Ω is convex and satisfies a certain symmetry, then a nodal four-bubble solution exists with two positive and two negative bubbles.

Mathematics Subject Classification 2000: 35B40, 35J20, 35J65
Keywords: slightly subcritical problem, sign-changing solutions, finite-dimensional reduction, max-min argument

1. Introduction

We are concerned with the slightly subcritical elliptic problem

$$\begin{cases} -\Delta u = |u|^{2^*-2-\varepsilon}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth and bounded domain in $\mathbb{R}^N$, $N \geq 3$, $\varepsilon > 0$ is a small parameter. Here 2^* denotes the critical exponent in the Sobolev embeddings, i.e. $2^* = \frac{2N}{N-2}$.

In [21] Pohožaev proved that the problem (1.1) does not admit a nontrivial solution if Ω is star-shaped and $\varepsilon \leq 0$. On the other hand problem (1.1) has a positive solution if $\varepsilon \leq 0$ and Ω is an annulus, see Kazdan and Warner [18]. In [2] Bahri and Coron found a positive solution to (1.1) with $\varepsilon = 0$ provided that the domain Ω has a *nontrivial topology*. Moreover in [12, 13, 14, 20] the authors considered the slightly supercritical case where $\varepsilon < 0$ is close to 0 and proved solvability of (1.1) in Coron's situation of a domain with one or more small holes.

In the subcritical case $\varepsilon > 0$ the problem (1.1) is always solvable, since a positive solution u_ε can be found by solving the variational problem

$$\inf\left\{ \int_\Omega |\nabla u|^2 \,\middle|\, u \in H_0^1(\Omega),\ \|u\|_{2^*-\varepsilon} = 1 \right\}.$$

In [9, 16, 17, 23, 24] it was proved that, as $\varepsilon \to 0^+$, u_ε blows up and concentrates at a point ξ which is a critical point of the Robin's function of Ω. In addition to the one-peak solution u_ε, several papers have studied concentration phenomena for positive solutions of (1.1) with

T.B. has been supported by the Vigoni Project 50766047.
T.D. and A.P. have been supported by the Italian PRIN Research Project 2009 *Metodi variazionali e topologici nello studio dei fenomeni non lineari*.

multiple blow-up points ([3, 22]). In a convex domain such a phenomenon cannot occur. Grossi and Takahashi [15] proved the nonexistence of positive solutions for the problem (1.1) blowing up at more than one point. On the other hand, multi-peak nodal solutions always exist for problem (1.1) in a general bounded and smooth domain Ω. Indeed, in [6] a solution with exactly one positive and one negative blow-up point is constructed for the problem (1.1) if $\varepsilon > 0$ is sufficiently small. The location of the two concentration points is also characterized and depends on the geometry of the domain. Moreover the presence of sign-changing solutions with a multiple blow-up at a single point has been proved in [19, 25] for problem (1.1); such solutions have the shape of towers of alternating-sign bubbles, i.e. they are superpositions of positive bubbles and negative bubbles blowing-up at the same point with a different velocity. We also quote the paper [8], where the authors study the blow up of the low energy sign-changing solutions of problem (1.1) and they classify these solutions according to the concentration speeds of the positive and negative part. Finally, we mention the papers [4] and [7] where, by a different approach, the authors provide existence and multiplicity of sign-changing solutions for more general problems than (1.1). These papers are however not concerned with the profile of the solutions.

In this paper we deal with the construction of sign-changing solutions which develop a spike-shape as $\varepsilon \to 0^+$, blowing up positively at some points and negatively at other points, generalizing the double blowing up obtained in [6]. We are able to prove that on certain domains Ω, (1.1) admits solutions with exactly two positive and two negative blow-up points. Moreover, the asymptotic profile of the blow-up of these solutions resembles a *bubble*, namely a solution of the equation at the critical exponent in the entire $\mathbb{R}^N$. It is natural to ask about the existence of solutions with k blow-up points, also for $k \neq 2, 4$, and in more general domains. We shall discuss this difficult problem below.

In order to formulate the conditions on the domain Ω, we need to introduce some notation. Let us denote by $G(x, y)$ the Green's function of $-\Delta$ over Ω under Dirichlet boundary conditions; so G satisfies

$$\begin{cases} -\Delta_y G(x,y) = \delta_x(y) & y \in \Omega, \\ G(x,y) = 0 & y \in \partial\Omega, \end{cases}$$

where δ_x is the Dirac mass at x. We denote by $H(x, y)$ its regular part, namely

$$H(x,y) = \frac{1}{(N-2)\sigma_N |x-y|^{N-2}} - G(x,y),$$

where σ_N is the surface measure of the unit sphere in $\mathbb{R}^N$. The diagonal $H(x, x)$ is called the Robin's function of the domain Ω.

Here are our assumptions on Ω.

(A1) $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded domain with a $\mathcal{C}^2$-boundary.

(A2) Ω is invariant under the reflection $(x_1, x') \mapsto (x_1, -x')$ where $x_1 \in \mathbb{R}$, $x' \in \mathbb{R}^{N-1}$.

For simplicity of notation we write the restrictions of G and H to the x_1-axis as g and h respectively, i.e.

$$g(t,s) = G((t,0,\dots,0),(s,0,\dots,0)) \quad \text{and} \quad h(t,s) = H((t,0,\dots,0),(s,0,\dots,0)).$$

Our last assumption concerning the domain is:

(A3) There exists a connected component (a, b) of the set $\{t \mid (t, 0, \dots, 0) \in \Omega\} \subset \mathbb{R}$ such that

$$\text{the function } (a,b) \ni t \mapsto h(t,t) \text{ is convex} \tag{1.2}$$

and

$$\text{for any } t, s \in (a,b),\ t \neq s:\ (t-s)\frac{\partial g}{\partial t}(t,s) < 0. \tag{1.3}$$

We can now state our main result.

Theorem 1.1. *If Ω satisfies (A1), (A2), (A3), then for $\varepsilon > 0$ sufficiently small problem* (1.1) *has a solution u_ε with the following property. There exist numbers $\lambda_i^\varepsilon > 0$ and points $\xi_i^\varepsilon = (t_i^\varepsilon, 0, \dots, 0) \in \Omega$ with $t_i^\varepsilon \in (a,b)$, $i = 1,2,3,4$, such that*

$$u_\varepsilon(x) = \alpha_N \sum_{i=1}^{4} (-1)^{i+1} \left(\frac{\lambda_i^\varepsilon \varepsilon^{\frac{1}{N-2}}}{\varepsilon^{\frac{2}{N-2}} \lambda_i^\varepsilon + |x - \xi_i^\varepsilon|^2} \right)^{\frac{N-2}{2}} + o(1) \text{ uniformly in } \overline{\Omega};$$

here $\alpha_N = (N(N-2))^{(N-2)/4}$. Moreover, the numbers λ_i^ε are bounded above and below away from zero, and the numbers t_i^ε are aligned on (a,b) and remain uniformly away from the boundary and from one another, i.e.

$$\delta < \lambda_i^\varepsilon < \frac{1}{\delta} \quad \forall i = 1,2,3,4,$$

and

$$a + \delta < t_1^\varepsilon < t_2^\varepsilon < t_3^\varepsilon < t_4^\varepsilon < b - \delta, \quad t_{i+1}^\varepsilon - t_i^\varepsilon > \delta \quad \forall i = 1,2,3,$$

for some $\delta > 0$.

Let us observe that the assumption (A3) is satisfied for a (not necessarily strictly) convex domain Ω as a consequence of some properties of the Green's and the Robin's functions. Indeed, (1.2) follows from the result in [11] according to which the Robin's function of a convex domain is strictly convex. Moreover in a convex domain the function $G(\cdot, y)$ is strictly decreasing (with non-zero derivative) along the half-lines starting from y (see Lemma A.2), hence (1.3) holds true. Assumption (A3) is also satisfied for some non-convex domains, for instance those which are $\mathcal{C}^2$-close to convex domains. It seems to be an open problem whether (A3) holds, for instance, on annuli.

The proof of Theorem 1.1 relies on a Lyapunov-Schmidt reduction scheme. This reduces the problem of finding multi-bubble solutions for (1.1) to the problem of finding critical points of a functional which depends on points ξ_i and scaling parameters λ_i. The leading part of the reduced functional is explicitly given in terms of the Green's and Robin's functions. The reduced functional has a quite involved behaviour, due to the different interactions among the bubbles (which depends on their respective sign). The symmetry of the domain plays a crucial role: indeed, the validity of the hypothesis (A2) allows us to place the positive and negative bubbles alternating along the one-dimensional interval (a,b). Then we use a variational approach and we obtain the existence of a saddle point by applying a max-min argument. An important step is the proof of a compactness condition which ensures that the max-min level actually is a critical value, and this is the most technical and difficult part of the proof.

As remarked above, it is natural to ask about other types of multibump solutions, and to consider more general domains. First of all, the Lyapunov-Schmidt reduction scheme works in a very general setting. In particular, (A2) and (A3) are not required for this. The problem lies in finding critical points of the reduced functional. This problem seem to be very subtle. In the paper [5] we consider the case of a ball and we show the existence of two three-bubble solutions having different nodal properties. However, these solutions are not found via a global variational argument and the proof strongly depends on the explicit formula of the Green's and the Robin's function in a ball. It also seems very hard to weaken the assumptions on the domain. In our argument we use the symmetry condition (A2) in order to localize and order the peaks on the x_1-axis. Together with (A3) this allows comparison arguments involving the Green's and Robin's functions which do not hold in general.

The paper is organized as follows. In Section 2 we sketch the finite-dimensional reduction method. Section 3 is devoted to solving the reduced problem by the max-min procedure. Finally in the Appendix A we collect some properties of the Green's function which are usually referred to throughout the paper.

2. The reduced functional

The proof of Theorem 1.1 is based on the *finite dimensional reduction* procedure which has been used for a wide class of singularly perturbed problems. We sketch the procedure here and refer to [6] for details. Related methods have been developed in [12]-[13]-[14] where the almost critical problem (1.1) was studied from the supercritical side. In this section the assumptions (A2) and (A3) are not required.

For any $\varepsilon > 0$ let us introduce the functions

$$U_{\varepsilon,\lambda,\xi}(x) = \alpha_N \left(\frac{\lambda \varepsilon^{\frac{1}{N-2}}}{\lambda^2 \varepsilon^{\frac{2}{N-2}} + |x-\xi|^2} \right)^{\frac{N-2}{2}}, \quad \alpha_N = (N(N-2))^{(N-2)/4},$$

with $\lambda > 0$ and $\xi \in \mathbb{R}^N$. These are actually all positive solutions of the limiting equation

$$-\Delta U = U^{2^*-1} \text{ in } \mathbb{R}^N,$$

and constitute the extremals for the Sobolev's critical embedding (see [1], [10], [26]). Fixing $k \geq 1$, we define the configuration space

$$\mathcal{O}_k := \left\{ (\boldsymbol{\lambda}, \boldsymbol{\xi}) = (\lambda_1, \ldots, \lambda_k, \xi_1, \ldots, \xi_k) \;\middle|\; \begin{array}{l} \delta < \lambda_i < \delta^{-1},\ \xi_i \in \Omega,\ \mathrm{dist}(\xi_i, \partial\Omega) > \delta \ \ \forall i \\ |\xi_i - \xi_j| > \delta \ \text{ if } i \neq j \end{array} \right\}$$

where $\delta > 0$ is a sufficiently small number. For fixed integers $a_1, \ldots, a_k \in \{-1, 1\}$, we seek suitable scalars λ_i and points ξ_i such that a solution u exists for (1.1) with $u \approx \sum_{i=1}^k a_i U_{\varepsilon,\lambda_i,\xi_i}$. In order to obtain a better first approximation, which satisfies the boundary condition, we consider the projections $\mathcal{P}_\Omega U_{\varepsilon,\lambda,\xi}$ onto the space $H_0^1(\Omega)$ of $U_{\varepsilon,\lambda,\xi}$, where the projection $\mathcal{P}_\Omega : H^1(\mathbb{R}^N) \to H_0^1(\Omega)$ is defined as the unique solution of the problem

$$\begin{cases} \Delta \mathcal{P}_\Omega u = \Delta u & \text{in } \Omega, \\ \mathcal{P}_\Omega u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then the following estimate holds

$$\mathcal{P}_\Omega U_{\varepsilon,\lambda_i,\xi_i} = U_{\varepsilon,\lambda_i,\xi_i} + O(\sqrt{\varepsilon}) \tag{2.4}$$

uniformly with respect to $(\boldsymbol{\lambda}, \boldsymbol{\xi}) \in \mathcal{O}_k$. We look for a solution to (1.1) in a small neighbourhood of the first approximation, i.e. a solution of the form

$$u := \sum_{i=1}^k a_i \mathcal{P}_\Omega U_{\varepsilon,\lambda_i,\xi_i} + \phi,$$

where the rest term ϕ is small. To carry out the construction of a solution of this type, we first introduce an intermediate problem as follows.

We consider the spaces

$$\mathcal{K}_{\varepsilon,\boldsymbol{\lambda},\boldsymbol{\xi}} = \mathrm{span}\left\{ \mathcal{P}_\Omega\left(\frac{\partial U_{\varepsilon,\lambda_i,\xi_i}}{\partial \xi_i^j}\right), \mathcal{P}_\Omega\left(\frac{\partial U_{\varepsilon,\lambda,\xi}}{\partial \lambda_i}\right) \,\middle|\, i=1,\dots,k,\ j=1,\dots,N \right\} \subset H_0^1(\Omega),$$

and

$$\mathcal{K}^\perp_{\varepsilon,\boldsymbol{\lambda},\boldsymbol{\xi}} = \left\{ \phi \in H_0^1(\Omega) \,\middle|\, \langle \phi,\psi\rangle := \int_\Omega \nabla\phi\nabla\psi = 0 \ \ \forall \psi \in \mathcal{K}_{\varepsilon,\boldsymbol{\lambda},\boldsymbol{\xi}} \right\} \subset H_0^1(\Omega);$$

here we denote by ξ_i^j the j-th component of ξ_i. Then it is convenient to solve as a first step the problem for ϕ as a function of ε, $\boldsymbol{\lambda}$, $\boldsymbol{\xi}$. This turns out to be solvable for any choice of points ξ_i and scalars λ_i, provided that ε is sufficiently small. The following result was established in [6].

Lemma 2.1. *There exists $\varepsilon_0 > 0$ and a constant $C > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ and each $(\boldsymbol{\lambda},\boldsymbol{\xi}) \in \mathcal{O}_k$ there exists a unique $\phi_{\varepsilon,\boldsymbol{\lambda},\boldsymbol{\xi}} \in \mathcal{K}^\perp_{\varepsilon,\boldsymbol{\lambda},\boldsymbol{\xi}}$ satisfying*

$$\Delta(V_{\varepsilon,\boldsymbol{\lambda},\boldsymbol{\xi}} + \phi) + |V_{\varepsilon,\boldsymbol{\lambda},\boldsymbol{\xi}} + \phi|^{2^*-2-\varepsilon}(V_{\varepsilon,\boldsymbol{\lambda},\boldsymbol{\xi}} + \phi) \in \mathcal{K}_{\varepsilon,\boldsymbol{\lambda},\boldsymbol{\xi}} \tag{2.5}$$

and

$$\|\phi\| := \left(\int_\Omega |\nabla\phi|^2\right)^{1/2} < C\varepsilon. \tag{2.6}$$

Here $V_{\varepsilon,\boldsymbol{\lambda},\boldsymbol{\xi}} = \sum_{i=1}^k a_i \mathcal{P}_\Omega U_{\varepsilon,\lambda_i,\xi_i}$. Moreover the map $\mathcal{O}_k \to H_0^1(\Omega)$, $(\boldsymbol{\lambda},\boldsymbol{\xi}) \mapsto \phi_{\varepsilon,\boldsymbol{\lambda},\boldsymbol{\xi}}$ is $\mathcal{C}^1$.

After this result, let us consider the following energy functional associated with problem (1.1):

$$I_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{2^*-\varepsilon}\int_\Omega |u|^{2^*-\varepsilon} dx, \quad u \in H_0^1(\Omega). \tag{2.7}$$

Solutions of (1.1) correspond to critical points of I_ε. Now we introduce the new functional

$$J_\varepsilon : \mathcal{O}_k \to \mathbb{R}, \quad J_\varepsilon(\boldsymbol{\lambda},\boldsymbol{\xi}) = I_\varepsilon(V_{\varepsilon,\boldsymbol{\lambda},\boldsymbol{\xi}} + \phi_{\varepsilon,\boldsymbol{\lambda},\boldsymbol{\xi}}) \tag{2.8}$$

where $\phi_{\varepsilon,\boldsymbol{\lambda},\boldsymbol{\xi}}$ has been constructed in Lemma 2.1. The next lemma has been proved in [3] and reduces the original problem (1.1) to the one of finding critical points of the functional J_ε.

Lemma 2.2. *The pair $(\boldsymbol{\lambda},\boldsymbol{\xi}) \in \mathcal{O}_k$ is a critical point of J_ε if and only if the corresponding function $u_\varepsilon = V_{\varepsilon,\boldsymbol{\lambda},\boldsymbol{\xi}} + \phi_{\varepsilon,\boldsymbol{\lambda},\boldsymbol{\xi}}$ is a solution of (1.1).*

Finally we describe an expansion for J_ε which can be obtained as in [13]-[14].

Proposition 2.3. *With the change of variables $\lambda_i = (c_N\Lambda_i)^{\frac{1}{N-2}}$ the following asymptotic expansion holds:*

$$J_\varepsilon(\boldsymbol{\lambda},\boldsymbol{\xi}) = kC_N + \frac{k}{2}\omega_N\varepsilon\log\varepsilon + k\gamma_N\varepsilon + \omega_N\varepsilon\Psi_k(\boldsymbol{\Lambda},\boldsymbol{\xi}) + o(\varepsilon) \tag{2.9}$$

$\mathcal{C}^1$-uniformly with respect to $(\boldsymbol{\lambda},\boldsymbol{\xi}) \in \mathcal{O}_k$. Here:

$$\Psi_k(\boldsymbol{\Lambda},\boldsymbol{\xi}) = \frac{1}{2}\sum_{i=1}^k \Lambda_i^2 H(\xi_i,\xi_i) - \sum_{i<j} a_i a_j \Lambda_i\Lambda_j G(\xi_i,\xi_j) - \log(\Lambda_1\cdot\ldots\cdot\Lambda_k),$$

and, setting $U = U_{1,1,0}$, the constants C_N, c_N, ω_N, and γ_N are given by

$$C_N = \int_{\mathbb{R}^N} |\nabla U|^2 - \frac{1}{2^*}\int_{\mathbb{R}^N} U^{2^*}, \quad c_N = \frac{1}{2^*}\frac{\int_{\mathbb{R}^N} U^{2^*}}{(\int_{\mathbb{R}^N} U^{2^*-1})^2}, \quad \omega_N = \frac{1}{2^*}\int_{\mathbb{R}^N} U^{2^*},$$

and

$$\gamma_N = \frac{1}{(2^*)^2}\int_{\mathbb{R}^N} U^{2^*} - \frac{1}{2^*}\int_{\mathbb{R}^N} U^{2^*}\log U + \frac{1}{2}\omega_N \log c_N.$$

Thus in order to construct a solution of problem (1.1) such as the one predicted in Theorem 1.1 it remains to find a critical point of J_ε. This will be accomplished in the next two sections.

We finish this section with a symmetry property of the reduction process.

Lemma 2.4. *Suppose Ω is invariant under the action of an orthogonal transformation $T \in O(N)$. Let $\mathcal{O}_k^T := \{(\mathbf{\Lambda}, \boldsymbol{\xi}) \in \mathcal{O}_k : T\xi_i = \xi_i \quad \forall i\}$ denote the fixed point set of T in $\mathcal{O}_k$. Then a point $(\mathbf{\Lambda}, \boldsymbol{\xi}) \in \mathcal{O}_k^T$ is a critical point of J_ε if it is a critical point of the constrained functional $J_\varepsilon|\mathcal{O}_k^T$.*

Proof. We first investigate the symmetry inherited by the function $\phi_{\varepsilon,\boldsymbol{\lambda},\boldsymbol{\xi}}$ obtained in Lemma 2.1. Setting $T\boldsymbol{\xi} := (T\xi_1, \ldots, T\xi_k)$ for $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_k) \in \Omega^k$, we claim that

$$\phi_{\varepsilon,\boldsymbol{\lambda},\boldsymbol{\xi}} = \phi_{\varepsilon,\boldsymbol{\lambda},T\boldsymbol{\xi}} \circ T \quad \forall(\boldsymbol{\lambda}, \boldsymbol{\xi}) \in \mathcal{O}_k. \tag{2.10}$$

Indeed, because of the symmetry of the domain, we see that

$$\mathcal{P}_\Omega U_{\varepsilon,\lambda_i,\xi_i} = (\mathcal{P}_\Omega U_{\varepsilon,\lambda_i,T\xi_i}) \circ T$$

and

$$\mathcal{K}_{\varepsilon,\boldsymbol{\lambda},\boldsymbol{\xi}} = \{f \circ T \mid f \in \mathcal{K}_{\varepsilon,\boldsymbol{\lambda},T\boldsymbol{\xi}}\}, \qquad \mathcal{K}^\perp_{\varepsilon,\boldsymbol{\lambda},\boldsymbol{\xi}} = \{f \circ T \mid f \in \mathcal{K}^\perp_{\varepsilon,\boldsymbol{\lambda},T\boldsymbol{\xi}}\}.$$

Then the function $\phi_{\varepsilon,\boldsymbol{\lambda},T\boldsymbol{\xi}} \circ T$ belongs to $\mathcal{K}^\perp_{\varepsilon,\boldsymbol{\lambda},\boldsymbol{\xi}}$ and satisfies (2.5) and (2.6). The uniqueness of the solution ϕ implies (2.10). Therefore the functional J_ε satisfies

$$J_\varepsilon(\boldsymbol{\lambda}, \boldsymbol{\xi}) = J_\varepsilon(\boldsymbol{\lambda}, T\boldsymbol{\xi}).$$

The lemma follows immediately. □

3. A max-min argument: proof of Theorem 1.1

In this section we will employ the reduction approach to construct the solutions stated in Theorem 1.1. The results obtained in the previous section imply that our problem reduces to the study of critical points of the functional J_ε defined in (2.8). In what follows, we assume (A1), (A2), (A3). For $t_1, \ldots, t_k \in (a, b)$, where (a, b) is from (A3), we set $\boldsymbol{t} = (t_1, \ldots, t_k)$ and

$$\tilde{J}_\varepsilon(\boldsymbol{\lambda}, \boldsymbol{t}) = J_\varepsilon(\boldsymbol{\lambda}, (t_1, 0, \ldots, 0), (t_2, 0, \ldots, 0), \ldots, (t_k, 0, \ldots, 0)).$$

Lemma 3.1. *If $(\boldsymbol{\lambda}, \boldsymbol{t})$ is a critical point of $\tilde{J}_\varepsilon$, then $(\boldsymbol{\lambda}, \boldsymbol{\xi})$ is a critical point of J_ε, where $\xi_i = (t_i, 0, \ldots, 0)$.*

Proof. This is an immediate consequence of Lemma 2.4. □

Let us now fix $k = 4$ and set

$$a_1 = a_3 = 1, \quad a_2 = a_4 = -1.$$

So we are looking for solutions to problem (1.1) with 2 positive and two negative spikes which are aligned along the x_1-direction with alternating signs. From Lemma 3.1, we need to find a critical point of the function $\tilde{J}_\varepsilon(\boldsymbol{\lambda}, \boldsymbol{t})$. The expansion obtained in Proposition 2.3 implies that our problem reduces to the study of critical points of a functional which is a small $\mathcal{C}^1$-perturbation of

$$\tilde{\Psi}(\boldsymbol{\Lambda}, \boldsymbol{t}) = \frac{1}{2}\sum_{i=1}^{4} \Lambda_i^2 h(t_i, t_i) - \sum_{i<j} (-1)^{i+j}\Lambda_i\Lambda_j g(t_i, t_j) - \log(\Lambda_1 \cdot \Lambda_2 \cdot \Lambda_3 \cdot \Lambda_4),$$

where $\boldsymbol{\Lambda} = (\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) \in (0, +\infty)^4$, $\boldsymbol{t} = (t_1, t_2, t_3, t_4) \in (a, b)^4$ and the functions g and h are the restrictions of G and H to the x_1-axis defined in the introduction. We recall that the function $\tilde{\Psi}$ is well defined in the set

$$\mathcal{M} := \Big\{ (\boldsymbol{\Lambda}, \boldsymbol{t}) \,\Big|\, \Lambda_i > 0,\ t_i \in (a, b)\ \forall i = 1, 2, 3, 4\ \ \&\ \ t_1 < t_2 < t_3 < t_4 \Big\}.$$

Observe that by assumption (1.3) the function $g(\cdot, s) = g(s, \cdot)$ is decreasing along the interval (s, b) and increasing along (a, s). Therefore

$$g(t_1, t_4) \le g(t_1, t_3) \le g(t_1, t_2), \qquad g(t_1, t_4) \le g(t_2, t_4) \le g(t_3, t_4) \quad \forall(\boldsymbol{\Lambda}, \boldsymbol{t}) \in \mathcal{M}. \tag{3.11}$$

Analogously,

$$g(t_2, t_4),\ g(t_1, t_3) \le g(t_2, t_3) \quad \forall(\boldsymbol{\Lambda}, \boldsymbol{t}) \in \mathcal{M}. \tag{3.12}$$

In this section we apply a max-min argument to characterize a topologically nontrivial critical value of the function $\tilde{\Psi}$ in the set $\mathcal{M}$. More precisely we will construct sets $\mathcal{D}$, K, $K_0 \subset \mathcal{M}$ satisfying the following properties:

(P1) $\mathcal{D}$ is an open set, K_0 and K are compact sets, K is connected and

$$K_0 \subset K \subset \mathcal{D} \subset \overline{\mathcal{D}} \subset \mathcal{M};$$

(P2) If we define the complete metric space $\mathcal{F}$ by

$$\mathcal{F} = \Big\{ \eta : K \to \mathcal{D} \,\Big|\, \eta \text{ continuous},\ \eta(\boldsymbol{\Lambda}, \boldsymbol{t}) = (\boldsymbol{\Lambda}, \boldsymbol{t})\ \forall(\boldsymbol{\Lambda}, \boldsymbol{t}) \in K_0 \Big\},$$

then

$$\tilde{\Psi}^* := \sup_{\eta\in\mathcal{F}} \min_{(\boldsymbol{\Lambda},\boldsymbol{t})\in K} \tilde{\Psi}(\eta(\boldsymbol{\Lambda}, \boldsymbol{t})) < \min_{(\boldsymbol{\Lambda},\boldsymbol{t})\in K_0} \tilde{\Psi}(\boldsymbol{\Lambda}, \boldsymbol{t}). \tag{3.13}$$

(P3) For every $(\boldsymbol{\Lambda}, \boldsymbol{t}) \in \partial\mathcal{D}$ such that $\tilde{\Psi}(\boldsymbol{\Lambda}, \boldsymbol{t}) = \Psi^*$, we have that $\partial\mathcal{D}$ is smooth at $(\boldsymbol{\Lambda}, \boldsymbol{t})$ and there exists a vector $\tau_{\boldsymbol{\Lambda},\boldsymbol{t}}$ tangent to $\partial\mathcal{D}$ at $(\boldsymbol{\Lambda}, \boldsymbol{t})$ so that $\tau_{\boldsymbol{\Lambda},\boldsymbol{t}} \cdot \nabla\tilde{\Psi}(\boldsymbol{\Lambda}, \boldsymbol{t}) \neq 0$.

Under these assumptions a critical point $(\boldsymbol{\Lambda}, \boldsymbol{t}) \in \mathcal{D}$ of $\tilde{\Psi}$ with $\tilde{\Psi}(\boldsymbol{\Lambda}, \boldsymbol{t}) = \tilde{\Psi}^*$ exists, as a standard deformation argument involving the gradient flow of $\tilde{\Psi}$ shows. Moreover, since properties (P2)-(P3) continue to hold also for a function which is $\mathcal{C}^1$-close to $\tilde{\Psi}$, then such a critical point will *survive* small $\mathcal{C}^1$-perturbations.

3.1. **Definition of $\mathcal{D}$.** We define

$$\mathcal{D}=\left\{(\boldsymbol{\Lambda},\boldsymbol{t})\in\mathcal{M}\;\middle|\;\Phi(\boldsymbol{\Lambda},\boldsymbol{t}):=\frac{1}{2}\sum_{i=1}^{4}\Lambda_i^2h(t_i,t_i)+\sum_{i<j}\Lambda_i\Lambda_jg(t_i,t_j)-\log(\Lambda_1\Lambda_2\Lambda_3\Lambda_4)<M\right\}$$

where $M>0$ is a sufficiently large number to be specified later. It is easy to check that the function Φ satisfies

$$\Phi(\boldsymbol{\Lambda},\boldsymbol{t})\to+\infty \text{ as } (\boldsymbol{\Lambda},\boldsymbol{t})\to\partial\mathcal{M}. \tag{3.14}$$

Indeed, for any $\Lambda>0$ and $t\in(a,b)$ we have

$$\frac{\Lambda^2}{2}h(t,t)-\log\Lambda\geq\frac{\Lambda^2}{4}h(t,t)+|\log\Lambda|+\Big(\frac{\Lambda^2}{4}H_0-2\log^+\Lambda\Big)$$

where $\log^+x=\max\{\log x,0\}$ denotes the positive part of the logarithm, and $H_0>0$ is the minimum value of the Robin's function in Ω (see (A.90)). Taking into account that the function $\frac{H_0}{4}x^2-2\log x$ minimizes for $x=2H_0^{-1/2}$, we deduce

$$\frac{\Lambda^2}{2}h(t,t)-\log\Lambda\geq\frac{\Lambda^2}{4}h(t,t)+|\log\Lambda|-2\log^+\frac{2}{\sqrt{H_0}}\quad\forall\Lambda>0,\ t\in(a,b). \tag{3.15}$$

Hence for any $(\boldsymbol{\Lambda},\boldsymbol{t})\in\mathcal{M}$ we get

$$\Phi(\boldsymbol{\Lambda},\boldsymbol{t})\geq\frac{1}{4}\sum_{i=1}^{4}\Lambda_i^2h(t_i,t_i)+\sum_{i=1}^{4}|\log\Lambda_i|+\sum_{i<j}\Lambda_i\Lambda_jg(t_i,t_j)-8\log^+\frac{2}{\sqrt{H_0}}. \tag{3.16}$$

(3.14) follows by using the properties of h and g (see Appendix A). In particular (3.14) implies that $\mathcal{D}$ is compactly contained in $\mathcal{M}$.

3.2. **Definition of K, K_0, and proof of (P1).** In this subsection we define the sets K, K_0 for which properties (P1)-(P2) hold. We consider the configurations $(\boldsymbol{\Lambda},\boldsymbol{t})$ such that $\Lambda_2=\Lambda_3$, i.e. configurations of the form

$$(\boldsymbol{\Lambda}(\boldsymbol{\mu}),\boldsymbol{t})=\left(\frac{\mu_1}{\sqrt{\mu}},\sqrt{\mu},\sqrt{\mu},\frac{\mu_4}{\sqrt{\mu}},t_1,t_2,t_3,t_4\right), \tag{3.17}$$

where $\boldsymbol{t}=(t_1,t_2,t_3,t_4)\in(a,b)^4$, and $\boldsymbol{\mu}=(\mu_1,\mu,\mu_4)\in(0,+\infty)^3$. Next we consider the open set

$$\left\{(\boldsymbol{\mu},\boldsymbol{t})\in(0,+\infty)^3\times(a,b)^4\;\middle|\;(\boldsymbol{\Lambda}(\boldsymbol{\mu}),\boldsymbol{t})\in\mathcal{M},\ \Phi(\boldsymbol{\Lambda}(\boldsymbol{\mu}),\boldsymbol{t})<\frac{M}{2}\right\}. \tag{3.18}$$

Since we do not know whether (3.18) is connected or not, so we will define U as a conveniently chosen connected component. Let $t_0\in(a,b)$ be fixed and choose $r_0>0$ sufficiently small such that

$$[t_0-4r_0,t_0+4r_0]\subset(a,b) \tag{3.19}$$

and

$$\frac{1}{2}h(t,t)+\frac{1}{2}h(s,s)-g(t,s)\leq0\ \ \forall t,s\in[t_0-4r_0,t_0+4r_0],\ t\neq s. \tag{3.20}$$

Setting $\boldsymbol{\mu}_0=(1,1,1)$, $\boldsymbol{t}_0=(t_0,t_0+r_0,t_0+2r_0,t_0+3r_0)$, then $(\boldsymbol{\Lambda}(\boldsymbol{\mu}_0),\boldsymbol{t}_0)\in\mathcal{M}$ and, consequently, $(\boldsymbol{\mu}_0,\boldsymbol{t}_0)$ belongs to (3.18) provided that M is sufficiently large. Now we are ready to define U, K and K_0:

$$U:=\text{ the connected component of (3.18) containing } (\boldsymbol{\mu}_0,\boldsymbol{t}_0),$$

$$K = \left\{ (\mathbf{\Lambda}(\boldsymbol{\mu}), \boldsymbol{t}) \in \mathcal{M} : \ (\boldsymbol{\mu}, \boldsymbol{t}) \in \overline{U} \right\},$$
$$K_0 = \left\{ (\mathbf{\Lambda}(\boldsymbol{\mu}), \boldsymbol{t}) \in \mathcal{M} : \ (\boldsymbol{\mu}, \boldsymbol{t}) \in \partial U \right\}.$$

Let us observe that, according to (3.14), the following inclusion holds:

$$K_0 \subset \left\{ (\mathbf{\Lambda}, \boldsymbol{t}) \in K \,\middle|\, \Phi(\mathbf{\Lambda}, \boldsymbol{t}) = \frac{M}{2} \right\}. \tag{3.21}$$

K is clearly isomorphic to $\overline{U}$ by the obvious isomorphism, and $K_0 \approx \partial U$. In particular, K and K_0 are compact sets and K is connected. Moreover we have $K_0 \subset K \subset \mathcal{D}$.

Since $\Lambda_2 = \Lambda_3$ by the definition of K, using (3.11) we obtain

$$-\sum_{i<j} (-i)^{i+j} \Lambda_i \Lambda_j g(t_i, t_j) \geq \Lambda_2 \Lambda_3 g(t_2, t_3) + \Lambda_1 \Lambda_4 g(t_1, t_4) \quad \forall (\mathbf{\Lambda}, \boldsymbol{t}) \in K. \tag{3.22}$$

Roughly speaking, the configurations in K have the crucial property that the negative interaction terms associated to the couples of points with the same sign are dominated by the positive interplay between the couples of points having opposite signs.

3.3. **An upper and a lower estimate for $\tilde{\mathbf{\Psi}}^*$.** Let $\eta \in \mathcal{F}$, so $\eta : K \to \mathcal{D}$ is a continuous function such that $\eta(\mathbf{\Lambda}, \boldsymbol{t}) = (\mathbf{\Lambda}, \boldsymbol{t})$ for any $(\mathbf{\Lambda}, \boldsymbol{t}) \in K_0$. Then we can compose the following maps

$$(0, +\infty)^3 \times (a, b)^4 \supset \overline{U} \longleftrightarrow K \xrightarrow{\eta} \eta(K) \subset \mathcal{D} \xrightarrow{\mathcal{H}} (0, +\infty)^3 \times (a, b)^4$$

where $\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_7) : \mathcal{D} \to (0, +\infty)^3 \times (a, b)^4$ is defined by

$$\mathcal{H}_1(\mathbf{\Lambda}, \boldsymbol{t}) = \Lambda_1 \Lambda_2, \quad \mathcal{H}_2(\mathbf{\Lambda}, \boldsymbol{t}) = \Lambda_2 \Lambda_3, \quad \mathcal{H}_3(\mathbf{\Lambda}, \boldsymbol{t}) = \Lambda_3 \Lambda_4,$$
$$\mathcal{H}_4(\mathbf{\Lambda}, \boldsymbol{t}) = t_1, \ \ \mathcal{H}_5(\mathbf{\Lambda}, \boldsymbol{t}) = t_2, \ \ \mathcal{H}_6(\mathbf{\Lambda}, \boldsymbol{t}) = t_3, \ \ \mathcal{H}_7(\mathbf{\Lambda}, \boldsymbol{t}) = t_4.$$

We set

$$T : \overline{U} \to (0, +\infty)^3 \times (a, b)^4$$

the resulting composition. Clearly T is a continuous map. We claim that $T = id$ on ∂U. Indeed, if $(\boldsymbol{\mu}, \boldsymbol{t}) \in \partial U$, then by construction $(\mathbf{\Lambda}(\boldsymbol{\mu}), \boldsymbol{t}) \in K_0$; consequently $\eta(\mathbf{\Lambda}(\boldsymbol{\mu}), \boldsymbol{t}) = (\mathbf{\Lambda}(\boldsymbol{\mu}), \boldsymbol{t})$, by which, using the definitions (3.17),

$$\mathcal{H}_1(\mathbf{\Lambda}(\boldsymbol{\mu}), \boldsymbol{t}) = \frac{\mu_1}{\sqrt{\mu}} \sqrt{\mu} = \mu_1,$$
$$\mathcal{H}_2(\mathbf{\Lambda}(\boldsymbol{\mu}), \boldsymbol{t}) = \sqrt{\mu}\sqrt{\mu} = \mu$$
$$\mathcal{H}_3(\mathbf{\Lambda}(\boldsymbol{\mu}), \boldsymbol{t}) = \sqrt{\mu} \frac{\mu_4}{\sqrt{\mu}} = \mu_4.$$

This proves that $T = id$ on ∂U. The theory of the topological degree assures that

$$\deg(T, U, (\boldsymbol{\mu}_0, \boldsymbol{t}_0)) = \deg(id, U, (\boldsymbol{\mu}_0, \boldsymbol{t}_0)) = 1.$$

Then there exists $(\boldsymbol{\mu}^\eta, \boldsymbol{s}^\eta) \in U$ such that $T(\boldsymbol{\mu}^\eta, \boldsymbol{s}^\eta) = (\boldsymbol{\mu}_0, \boldsymbol{t}_0)$, i.e., if we set $(\mathbf{\Lambda}^\eta, \boldsymbol{t}^\eta) := \eta(\mathbf{\Lambda}(\boldsymbol{\mu}^\eta), \boldsymbol{s}^\eta) \in \eta(K)$,

$$\Lambda_1^\eta \Lambda_2^\eta = \Lambda_2^\eta \Lambda_3^\eta = \Lambda_3^\eta \Lambda_4^\eta = 1. \tag{3.23}$$
$$\boldsymbol{t}^\eta = \boldsymbol{t}_0. \tag{3.24}$$

Using (3.20), and taking into account that $\Lambda_1^\eta = \Lambda_3^\eta$, $\Lambda_2^\eta = \Lambda_4^\eta$ by (3.23), we obtain

$$\frac{1}{2} (\Lambda_1^\eta)^2 h(t_1^0, t_1^0) + \frac{1}{2} (\Lambda_3^\eta)^2 h(t_3^0, t_3^0) - \Lambda_1^\eta \Lambda_3^\eta g(t_1^0, t_3^0) \leq 0, \tag{3.25}$$

$$\frac{1}{2}(\Lambda_2^\eta)^2 h(t_2^0,t_2^0)+\frac{1}{2}(\Lambda_4^\eta)^2 h(t_4^0,t_4^0)-\Lambda_2^\eta\Lambda_4^\eta g(t_2^0,t_4^0)\le 0. \tag{3.26}$$

Furthermore by (3.23) we also deduce

$$\Lambda_1^\eta\Lambda_4^\eta=\frac{1}{\Lambda_2^\eta}\frac{1}{\Lambda_3^\eta}=\frac{1}{\Lambda_2^\eta\Lambda_3^\eta}=1,\quad \Lambda_1^\eta\Lambda_2^\eta\Lambda_3^\eta\Lambda_4^\eta=(\Lambda_1^\eta\Lambda_2^\eta)(\Lambda_3^\eta\Lambda_4^\eta)=1. \tag{3.27}$$

Combining (3.24)-(3.25)-(3.26)-(3.27) with the definition of $\tilde\Psi$ we get

$$\tilde\Psi(\boldsymbol{\Lambda}^\eta,\boldsymbol{t}^\eta)\le g(t_1^0,t_2^0)+g(t_2^0,t_3^0)+g(t_3^0,t_4^0)+g(t_1^0,t_4^0).$$

Then we can estimate

$$\min_{(\boldsymbol{\Lambda},\boldsymbol{t})\in K}\tilde\Psi(\eta(\boldsymbol{\Lambda},\boldsymbol{t}))\le\tilde\Psi(\boldsymbol{\Lambda}^\eta,\boldsymbol{t}^\eta)\le g(t_1^0,t_2^0)+g(t_2^0,t_3^0)+g(t_3^0,t_4^0)+g(t_1^0,t_4^0).$$

By taking the supremum for all the maps $\eta\in\mathcal{F}$, we conclude

$$\tilde\Psi^*=\sup_{\eta\in\mathcal{F}}\min_{(\boldsymbol{\Lambda},\boldsymbol{t})\in K}\tilde\Psi(\eta(\boldsymbol{\Lambda},\boldsymbol{t}))\le g(t_1^0,t_2^0)+g(t_2^0,t_3^0)+g(t_3^0,t_4^0)+g(t_1^0,t_4^0). \tag{3.28}$$

On the other hand, by taking $\eta=id$ and using (3.15) and (3.22),

$$\tilde\Psi^*\ge\min_{(\boldsymbol{\Lambda},\boldsymbol{t})\in K}\tilde\Psi(\boldsymbol{\Lambda},\boldsymbol{t})\ge-8\log^+\frac{2}{\sqrt{H_0}}. \tag{3.29}$$

3.4. **Proof of (P2).** Let us first recall that the upper estimate for $\tilde\Psi^*$ obtained in (3.28) holds for any M sufficiently large. Then, by using (3.21), the max-min inequality (P2) will follow once we have proved that

$$\min_{(\boldsymbol{\Lambda},\boldsymbol{t})\in K,\,\Phi(\boldsymbol{\Lambda},\boldsymbol{t})=\frac{M}{2}}\tilde\Psi(\boldsymbol{\Lambda},\boldsymbol{t})\to+\infty\qquad\text{as } M\to+\infty. \tag{3.30}$$

To this aim, it will be convenient to provide a lower bound for the functional $\tilde\Psi$ over K. Combining (3.15) and (3.22) we get

$$\tilde\Psi(\boldsymbol{\Lambda},\boldsymbol{t})\ge\sum_{i=1}^4\frac{\Lambda_i^2}{4}h(t_i,t_i)+\sum_{i=1}^4|\log\Lambda_i|+\Lambda_2\Lambda_3 g(t_2,t_3)+\Lambda_1\Lambda_4 g(t_1,t_4)-8\log^+\frac{2}{\sqrt{H_0}} \tag{3.31}$$

for any $(\boldsymbol{\Lambda},\boldsymbol{t})\in K$.

Now we are going to prove (3.30). Indeed, let $(\boldsymbol{\Lambda}_n,\boldsymbol{t}_n)=(\Lambda_1^n,\Lambda_2^n,\Lambda_3^n,\Lambda_4^n,t_1^n,t_2^n,t_3^n,t_4^n)\in K$ be such that

$$\Phi(\boldsymbol{\Lambda}_n,\boldsymbol{t}_n)\to+\infty. \tag{3.32}$$

The definition of Φ implies that, up to a subsequence, the following four cases cover all the possibilities for which (3.32) may occur.

(1) *there exists $\hat\imath$ such that $\Lambda_{\hat\imath}^n\to 0$.*
(2) *there exists $\hat\imath$ such that $\Lambda_{\hat\imath}^n\to+\infty$.*
(3) *$t_1^n\to a$ or $t_4^n\to b$.*
(4) *for every i the numbers Λ_i^n are bounded from above and below by positive constants and there exist $\hat\imath<\hat\jmath$ such that $t_{\hat\jmath}^n-t_{\hat\imath}^n\to 0$.*

If case (1), (2) or (3) holds, then by (3.31), recalling (A.90), we get $\tilde\Psi(\boldsymbol{\Lambda}_n, \boldsymbol{t}_n) \to +\infty$, as required.

Assume that case (4) occurs. The definition of $\tilde\Psi$ combined with (3.22) implies

$$\tilde\Psi(\boldsymbol{\Lambda}_n, \boldsymbol{t}_n) \geq c\, g(t_2^n, t_3^n) - C$$

for suitable positive constants c, C. Therefore, if $\hat\imath \leq 2$ and $\hat\jmath \geq 3$, we get $t_3^n - t_2^n \to 0$, hence $\tilde\Psi(\boldsymbol{\Lambda}_n, \boldsymbol{t}_n) \to +\infty$.

It remains to consider the case when, up to a subsequence

$$t_3^n - t_2^n \geq a, \quad t_2^n - t_1^n \to 0,$$

or

$$t_3^n - t_2^n \geq a, \quad t_4^n - t_3^n \to 0$$

for some $a > 0$. Then we deduce $t_j^n - t_i^n \geq a$ for every $i \leq 2 < 3 \leq j$. Since the Green's function g is smooth on the compact sets disjoint from the diagonal, by the definition of $\tilde\Psi$ we get

$$\tilde\Psi(\boldsymbol{\Lambda}_n, \boldsymbol{t}_n) \geq c' g(t_1^n, t_2^n) + c' g(t_3^n, t_4^n) - C'$$

for some c', $C' > 0$ and then we conclude

$$\tilde\Psi(\boldsymbol{\Lambda}_n, \boldsymbol{t}_n) \to +\infty.$$

3.5. **Proof of (P3).** We shall prove that (P3) holds provided that M is sufficiently large. First we recall that the upper and the lower estimates for Ψ^* obtained in (3.28) and (3.29) holds for any M sufficiently large. Then we proceed by contradiction: assume that there exist $(\boldsymbol{\Lambda}_n, \boldsymbol{t}_n) = (\Lambda_1^n, \Lambda_2^n, \Lambda_3^n, \Lambda_4^n, t_1^n, t_2^n, t_3^n, t_4^n) \in \mathcal{M}$ and a vector $(\beta_1^n, \beta_2^n) \neq (0,0)$ such that:

$$\Phi(\boldsymbol{\Lambda}_n, \boldsymbol{t}_n) = n,$$

$$\tilde\Psi(\boldsymbol{\Lambda}_n, \boldsymbol{t}_n) = O(1),$$

$$\beta_1^n \nabla\tilde\Psi(\boldsymbol{\Lambda}_n, \boldsymbol{t}_n) + \beta_2^n \nabla\Phi(\boldsymbol{\Lambda}_n, \boldsymbol{t}_n) = 0.$$

The last expression means read as $\nabla\tilde\Psi(\boldsymbol{\Lambda}_n, \boldsymbol{t}_n)$ and $\nabla\Phi(\boldsymbol{\Lambda}_n, \boldsymbol{t}_n)$ are linearly dependent. Observe that, according to the Lagrange Theorem, this contradicts the nondegeneracy of $\nabla\tilde\Psi$ on the tangent space at the level Ψ^*.

Without loss of generality we may assume

$$(\beta_1^n)^2 + (\beta_2^n)^2 = 1 \text{ and } \beta_1^n + \beta_2^n \geq 0. \tag{3.33}$$

Considering $\Phi(\boldsymbol{\Lambda}_n, \boldsymbol{t}_n) + \tilde\Psi(\boldsymbol{\Lambda}_n, \boldsymbol{t}_n)$ and $\Phi(\boldsymbol{\Lambda}_n, \boldsymbol{t}_n) - \tilde\Psi(\boldsymbol{\Lambda}_n, \boldsymbol{t}_n)$ we obtain, respectively,

$$\sum_{i=1}^{4} (\Lambda_i^n)^2 h(t_i^n, t_i^n) + 2 \sum_{i<j,\, (-1)^{i+j}=-1} \Lambda_i^n \Lambda_j^n g(t_i^n, t_j^n) - 2\log(\Lambda_1^n \Lambda_2^n \Lambda_3^n \Lambda_4^n) = n + O(1) \tag{3.34}$$

and

$$2\Lambda_1^n \Lambda_3^n g(t_1^n, t_3^n) + 2\Lambda_2^n \Lambda_4^n g(t_2^n, t_4^n) = n + O(1). \tag{3.35}$$

The identities $\beta_1^n \frac{\partial\tilde\Psi}{\partial t_i}(\boldsymbol{\Lambda}_n, \boldsymbol{t}_n) + \beta_2^n \frac{\partial\Phi}{\partial t_i}(\boldsymbol{\Lambda}_n, \boldsymbol{t}_n) = 0$ imply

$$(\beta_1^n+\beta_2^n)(\Lambda_1^n)^2\frac{\partial h}{\partial t}(t_i^n,t_i^n) - \sum_{\substack{j=1\\ j\neq i}}^4((-1)^{i+j}\beta_1^n-\beta_2^n)\Lambda_i^n\Lambda_j^n\frac{\partial g}{\partial t}(t_i^n,t_j^n) = 0 \quad \forall i=1,2,3,4. \tag{3.36}$$

Moreover, from $\beta_1^n \frac{\partial\tilde\Psi}{\partial \Lambda_i}(\boldsymbol{\Lambda}_n, \boldsymbol{t}_n) + \beta_2^n \frac{\partial\Phi}{\partial \Lambda_i}(\boldsymbol{\Lambda}_n, \boldsymbol{t}_n) = 0$ we obtain the following four identities:

$$(\beta_1^n+\beta_2^n)(\Lambda_i^n)^2h(t_i^n,t_i^n) - \Lambda_i^n\sum_{j,j\neq i}((-1)^{i+j}\beta_1^n-\beta_2^n)\Lambda_j^n g(t_i^n,t_j^n) = \beta_1^n+\beta_2^n \quad \forall i=1,2,3,4, \tag{3.37}$$

by which, considering the sum in $i=1,2,3,4$,

$$(\beta_1^n+\beta_2^n)\sum_{i=1}^4(\Lambda_i^n)^2h(t_i^n,t_i^n) - 2\sum_{i<j}((-1)^{i+j}\beta_1^n-\beta_2^n)\Lambda_i^n\Lambda_j^n g(t_i^n,t_j^n) = 4(\beta_1^n+\beta_2^n) \tag{3.38}$$

which is equivalent to

$$\beta_1^n\big(\tilde\Psi(\boldsymbol{\Lambda}_n,\boldsymbol{t}_n)+\log(\Lambda_1^n\Lambda_2^n\Lambda_3^n\Lambda_4^n)\big)+\beta_2^n\big(n+\log(\Lambda_1^n\Lambda_2^n\Lambda_3^n\Lambda_4^n)\big) = 2(\beta_1^n+\beta_2^n). \tag{3.39}$$

Observe that by (3.34) we have $\log(\Lambda_1^n\Lambda_2^n\Lambda_3^n\Lambda_4^n) \geq -\frac{n}{2}+O(1)$, while, by (3.16), $(\Lambda_i^n)^2 \leq \frac{4}{H_0}n+O(1)$ and hence $\log(\Lambda_1^n\Lambda_2^n\Lambda_3^n\Lambda_4^n) \leq 2\log n + O(1)$. Then we easily obtain

$$n+\log(\Lambda_1^n\Lambda_2^n\Lambda_3^n\Lambda_4^n)\to+\infty \quad \text{ and } \quad \frac{\log(\Lambda_1^n\Lambda_2^n\Lambda_3^n\Lambda_4^n)}{n+\log(\Lambda_1^n\Lambda_2^n\Lambda_3^n\Lambda_4^n)} \leq o(1).$$

Multiplying (3.39) by β_1^n we get

$$\beta_1^n\beta_2^n = 2\beta_1^n\frac{\beta_1^n+\beta_2^n}{n+\log(\Lambda_1^n\Lambda_2^n\Lambda_3^n\Lambda_4^n)} - (\beta_1^n)^2\frac{O(1)+\log(\Lambda_1^n\Lambda_2^n\Lambda_3^n\Lambda_4^n)}{n+\log(\Lambda_1^n\Lambda_2^n\Lambda_3^n\Lambda_4^n)} \geq o(1).$$

Combining this with (3.33) we have

$$\beta_1^n\geq o(1),\quad \beta_2^n\geq o(1),\quad 2\geq\beta_1^n+\beta_2^n\geq 1+o(1). \tag{3.40}$$

Using (3.40), we can divide the identities (3.37) by $\beta_1^n+\beta_2^n$. Then we obtain:

$$(\Lambda_1^n)^2h(t_1^n,t_1^n)+\Lambda_1^n\Lambda_2^n g(t_1^n,t_2^n)-\frac{\beta_1^n-\beta_2^n}{\beta_1^n+\beta_2^n}\Lambda_1^n\Lambda_3^n g(t_1^n,t_3^n)+\Lambda_1^n\Lambda_4^n g(t_1^n,t_4^n)=1, \tag{3.41}$$

$$(\Lambda_2^n)^2h(t_2^n,t_2^n)+\Lambda_2^n\Lambda_1^n g(t_1^n,t_2^n)+\Lambda_2^n\Lambda_3^n g(t_2^n,t_3^n)-\frac{\beta_1^n-\beta_2^n}{\beta_1^n+\beta_2^n}\Lambda_2^n\Lambda_4^n g(t_2^n,t_4^n)=1, \tag{3.42}$$

$$(\Lambda_3^n)^2h(t_3^n,t_3^n)-\frac{\beta_1^n-\beta_2^n}{\beta_1^n+\beta_2^n}\Lambda_3^n\Lambda_1^n g(t_1^n,t_3^n)+\Lambda_3^n\Lambda_2^n g(t_2^n,t_3^n)+\Lambda_3^n\Lambda_4^n g(t_3^n,t_4^n)=1, \tag{3.43}$$

$$(\Lambda_4^n)^2h(t_4^n,t_4^n)+\Lambda_1^n\Lambda_4^n g(t_1^n,t_4^n)-\frac{\beta_1^n-\beta_2^n}{\beta_1^n+\beta_2^n}\Lambda_2^n\Lambda_4^n g(t_2^n,t_4^n)+\Lambda_3^n\Lambda_4^n g(t_3^n,t_4^n)=1. \tag{3.44}$$

Up to a subsequence, we may assume

$$t_i^n\to\bar t_i\in[a,b]\quad\forall i=1,2,3,4.$$

In what follows at many steps of the arguments we will pass to a subsequence, without further notice. We will often use the symbol c or C for denoting different positive constants independent on n. The value of c, C is allowed to vary from line to line (and also in the same formula). Motivated by (3.38), we distinguish five cases which will all lead to a contradiction.

Case 1. Avoiding blowing up of parameters I *Suppose the following holds:*

$$(\beta_1^n-\beta_2^n)\Lambda_1^n\Lambda_3^n g(t_1^n,t_3^n)\to+\infty,\quad (\beta_1^n-\beta_2^n)\Lambda_2^n\Lambda_4^n g(t_2^n,t_4^n)\to+\infty. \tag{3.45}$$

Then, in particular $\beta_1^n>\beta_2^n$ and, dividing (3.41) by $\frac{\beta_1^n-\beta_2^n}{\beta_1^n+\beta_2^n}\Lambda_1^n\Lambda_3^n g(t_1^n,t_3^n)$, we get

$$\frac{\Lambda_2^n}{\Lambda_3^n}\cdot\frac{\beta_1^n+\beta_2^n}{\beta_1^n-\beta_2^n}\le\frac{\Lambda_2^n}{\Lambda_3^n}\cdot\frac{\beta_1^n+\beta_2^n}{\beta_1^n-\beta_2^n}\frac{g(t_1^n,t_2^n)}{g(t_1^n,t_3^n)}\le 1+o(1). \tag{3.46}$$

where the first inequality follows by (3.11). Analogously, dividing (3.44) by $\frac{\beta_1^n-\beta_2^n}{\beta_1^n+\beta_2^n}\Lambda_2^n\Lambda_4^n g(t_2^n,t_4^n)$, and using again (3.11), we have

$$\frac{\Lambda_3^n}{\Lambda_2^n}\cdot\frac{\beta_1^n+\beta_2^n}{\beta_1^n-\beta_2^n}\le\frac{\Lambda_3^n}{\Lambda_2^n}\cdot\frac{\beta_1^n+\beta_2^n}{\beta_1^n-\beta_2^n}\frac{g(t_3^n,t_4^n)}{g(t_2^n,t_4^n)}\le 1+o(1). \tag{3.47}$$

(3.46) and (3.47) give

$$\frac{\beta_1^n+\beta_2^n}{\beta_1^n-\beta_2^n}\le 1+o(1)$$

which implies, using (3.40),

$$\beta_2^n=o(1),\quad \beta_1^n=1+o(1). \tag{3.48}$$

Inserting this into (3.46)-(3.47) we achieve

$$\Lambda_2^n=\Lambda_3^n(1+o(1)), \tag{3.49}$$

and

$$g(t_1^n,t_2^n)=g(t_1^n,t_3^n)(1+o(1)),\quad g(t_3^n,t_4^n)=g(t_2^n,t_4^n)(1+o(1)). \tag{3.50}$$

Using (3.48)-(3.50) and (3.45), the equations (3.41)-(3.44) lead to:

$$(\Lambda_1^n)^2h(t_1^n,t_1^n)+\Lambda_1^n\Lambda_4^n g(t_1^n,t_4^n)=o\big(\Lambda_1^n\Lambda_3^n g(t_1^n,t_3^n)\big), \tag{3.51}$$

$$(\Lambda_2^n)^2h(t_2^n,t_2^n)+\Lambda_2^n\Lambda_1^n g(t_1^n,t_2^n)+\Lambda_2^n\Lambda_3^n g(t_2^n,t_3^n)=(1+o(1))\Lambda_2^n\Lambda_4^n g(t_2^n,t_4^n), \tag{3.52}$$

$$(\Lambda_3^n)^2h(t_3^n,t_3^n)+\Lambda_2^n\Lambda_3^n g(t_2^n,t_3^n)+\Lambda_3^n\Lambda_4^n g(t_3^n,t_4^n)=(1+o(1))\Lambda_1^n\Lambda_3^n g(t_1^n,t_3^n), \tag{3.53}$$

$$(\Lambda_4^n)^2h(t_4^n,t_4^n)+\Lambda_1^n\Lambda_4^n g(t_1^n,t_4^n)=o\big(\Lambda_2^n\Lambda_4^n g(t_2^n,t_4^n)\big). \tag{3.54}$$

Combining (3.52)-(3.53) with (3.49)-(3.50) we obtain

$$\begin{aligned}\Lambda_2^n\Lambda_1^n g(t_1^n,t_2^n)&\le(1+o(1))\Lambda_2^n\Lambda_4^n g(t_2^n,t_4^n)=(1+o(1))\Lambda_3^n\Lambda_4^n g(t_3^n,t_4^n)\\&\le(1+o(1))\Lambda_1^n\Lambda_3^n g(t_1^n,t_3^n)=(1+o(1))\Lambda_2^n\Lambda_1^n g(t_1^n,t_2^n).\end{aligned}$$

Then all the above inequalities are actually equalities, by which (3.52)-(3.53) can be rewritten as

$$\begin{aligned}(\Lambda_2^n)^2h(t_2^n,t_2^n)+\Lambda_2^n\Lambda_3^n g(t_2^n,t_3^n)&=o\big(\Lambda_2^n\Lambda_4^n g(t_2^n,t_4^n)\big),\\(\Lambda_3^n)^2h(t_3^n,t_3^n)+\Lambda_2^n\Lambda_3^n g(t_2^n,t_3^n)&=o\big(\Lambda_1^n\Lambda_3^n g(t_1^n,t_3^n)\big).\end{aligned}$$

Now (3.12) applies and gives together with (3.49)

$$o(\Lambda_4^n)=\Lambda_3^n=(1+o(1))\Lambda_2^n=o(\Lambda_1^n).$$

Substituting in (3.51) and (3.54) yields

$$h(t_1^n,t_1^n),\ g(t_1^n,t_4^n)=o(g(t_1^n,t_3^n)),\quad h(t_4^n,t_4^n),\ g(t_1^n,t_4^n)=o(g(t_2^n,t_4^n)). \tag{3.55}$$

We will derive a contradiction from (3.50) and (3.55). Indeed, by $h(t_1^n,t_1^n)=o(g(t_1^n,t_3^n))$ we deduce $g(t_1^n,t_3^n)\to+\infty$, hence $|t_1^n-t_3^n|\to 0$. Analogously by $h(t_4^n,t_4^n)=o(g(t_2^n,t_4^n))$ we get $|t_2^n-t_4^n|\to 0$. Therefore we are in the following situation

$$t_1^n,\,t_2^n,\,t_3^n,\,t_4^n\to\bar t\in[a,b]\quad\forall i=1,2,3,4.$$

Now, if $\bar t=a$, then Lemma A.1 yields

$$h(t_1^n,t_4^n)=\frac{1+o(1)}{\sigma_N(N-2)(t_4^n+t_1^n-2a)^{N-2}}\le\frac{1+o(1)}{\sigma_N(N-2)(2t_1-2a)^{N-2}}=(1+o(1))h(t_1^n,t_1^n)$$

and therefore, using (3.55),

$$\frac{|t_1^n-t_3^n|^{N-2}}{|t_1^n-t_4^n|^{N-2}}=\frac{g(t_1^n,t_4^n)+h(t_1^n,t_4^n)}{g(t_1^n,t_3^n)+h(t_1^n,t_3^n)}\le\frac{g(t_1^n,t_4^n)+h(t_1^n,t_4^n)}{g(t_1^n,t_3^n)}=o(1)$$

and then $t_3^n-t_1^n=o(t_4^n-t_1^n)$. On the other hand, using again Lemma A.1,

$$h(t_1^n,t_4^n)=\frac{1+o(1)}{\sigma_N(N-2)(t_4^n+t_1^n-2a)^{N-2}}\le\frac{1+o(1)}{\sigma_N(N-2)(t_4-a)^{N-2}}=2^{N-2}(1+o(1))h(t_4^n,t_4^n).$$

Now (3.55) leads to

$$\frac{|t_2^n-t_4^n|^{N-2}}{|t_1^n-t_4^n|^{N-2}}=\frac{g(t_1^n,t_4^n)+h(t_1^n,t_4^n)}{h(t_2^n,t_4^n)+h(t_2^n,t_4^n)}\le\frac{g(t_1^n,t_4^n)+h(t_1^n,t_4^n)}{g(t_2^n,t_4^n)}=o(1),$$

hence $t_4^n-t_2^n=o(t_4^n-t_1^n)$. Combining this with $t_3^n-t_1^n=o(t_4^n-t_1^n)$ we obtain a contradiction. An analogous argument applies to the case $\bar t=b$. Finally assume $\bar t\in(a,b)$. Then $h(t_i^n,t_j^n)=O(1)$ for every i, j, therefore (3.55) yields

$$\frac{|t_i^n-t_j^n|^{N-2}}{|t_1^n-t_4^n|^{N-2}}=\frac{g(t_1^n,t_4^n)+O(1)}{g(t_i^n,t_j^n)+O(1)}=o(1)\qquad\text{for }(i,j)=(1,3),\,(2,4).$$

This gives $t_3^n-t_1^n=o(t_4^n-t_1^n)$ and $t_4^n-t_2^n=o(t_4^n-t_1^n)$ respectively, and the contradiction arises as above.

Case 2: Avoiding blowing up of parameters II. *Suppose the following holds:*

$$(\beta_1^n-\beta_2^n)\Lambda_1^n\Lambda_3^n g(t_1^n,t_3^n)\to+\infty,\quad(\beta_1^n-\beta_2^n)\Lambda_2^n\Lambda_4^n g(t_2^n,t_4^n)\le C.\tag{3.56}$$

The analogous holds by interchanging the roles of the couples of indexes $(1,3)$ *and* $(2,4)$.

Then in particular there holds $\beta_1^n>\beta_2^n$. Using (3.42), (3.44) and the second inequality in (3.56) we obtain

$$(\Lambda_2^n)^2h(t_2^n,t_2^n),\,\Lambda_1^n\Lambda_2^n g(t_1^n,t_2^n),\,\Lambda_2^n\Lambda_3^n g(t_2^n,t_3^n)\le C,\tag{3.57}$$

$$(\Lambda_4^n)^2h(t_4^n,t_4^n),\,\Lambda_1^n\Lambda_4^n g(t_1^n,t_4^n),\,\Lambda_3^n\Lambda_4^n g(t_3^n,t_4^n)\le C.\tag{3.58}$$

By inserting (3.57)-(3.58) into (3.41) and (3.43), we obtain

$$(\Lambda_1^n)^2h(t_1^n,t_1^n),\,(\Lambda_3^n)^2h(t_3^n,t_3^n)=\frac{\beta_1^n-\beta_2^n}{\beta_1^n+\beta_2^n}\Lambda_1^n\Lambda_3^n g(t_1^n,t_3^n)+O(1)\to+\infty.\tag{3.59}$$

We distinguish three cases. First assume that there exists $i_0\in\{1,2,3,4\}$ such that

$$t_i^n\to a\quad\forall 1\le i\le i_0,\qquad|t_i^n-a|\ge c\quad\forall i>i_0.\tag{3.60}$$

By adding (3.36) for $i = 1, \dots, i_0$ we obtain

$$(\beta_1^n + \beta_2^n) \sum_{i=1}^{i_0} (\Lambda_i^n)^2 \frac{\partial h}{\partial t}(t_i^n, t_i^n) - \sum_{i=1}^{i_0} \sum_{\substack{j=1 \\ j \neq i}}^{4} ((-1)^{i+j}\beta_1^n - \beta_2^n)\Lambda_i^n \Lambda_j^n \frac{\partial g}{\partial t}(t_i^n, t_j^n) = 0. \tag{3.61}$$

Now $|t_i^n - t_j^n| \geq c$ for $i \leq i_0$ and $j > i_0$ imply

$$\frac{\partial g}{\partial t}(t_i^n, t_j^n) = O(1) \quad \forall i \leq i_0, \forall j > i_0. \tag{3.62}$$

Considering the sum for $i, j \leq i_0$ we observe that by Lemma A.1

$$\frac{\partial g}{\partial t}(t_i^n, t_j^n) + \frac{\partial g}{\partial t}(t_j^n, t_i^n) = -\frac{\partial h}{\partial t}(t_i^n, t_j^n) - \frac{\partial h}{\partial t}(t_j^n, t_i^n) = \frac{2 + o(1)}{\sigma_N (t_i^n + t_j^n - 2a)^{N-1}} \quad \forall i, j \leq i_0,\, i \neq j.$$

Therefore, using again Lemma A.1, the identity of (3.61) becomes

$$\sum_{i=1}^{i_0} \frac{(\Lambda_i^n)^2(1 + o(1))}{(2t_i^n - 2a)^{N-1}} + 2 \sum_{\substack{i,j=1 \\ i<j}}^{i_0} \frac{(-1)^{i+j}\beta_1^n - \beta_2^n}{\beta_1^n + \beta_2^n} \cdot \frac{\Lambda_i^n \Lambda_j^n (1 + o(1))}{(t_j^n + t_i^n - 2a)^{N-1}} = \sum_{i \leq i_0 < j} O(\Lambda_i^n \Lambda_j^n). \tag{3.63}$$

In order to estimate the last sum, we will prove that

$$\Lambda_i^n \Lambda_j^n = o\left(\frac{(\Lambda_i^n)^2}{(2t_i^n - 2a)^{N-1}} \right) + O(1) \quad \forall i \leq i_0 < j. \tag{3.64}$$

Indeed, if $i \leq i_0 < j$ and $(i,j) \neq (1,3)$, then, either $j = 2$ or $j = 4$, and, as a consequence of (3.57)-(3.58), Λ_2^n, $\Lambda_4^n = O(1)$; therefore $\Lambda_i^n \Lambda_j^n \leq \frac{1}{2}(\Lambda_i^n)^2 + \frac{1}{2}(\Lambda_j^n)^2 \leq \frac{1}{2}(\Lambda_i^n)^2 + C$ and (3.64) holds true. On the other hand, using (3.59),

$$\Lambda_1^n \Lambda_3^n = (1 + o(1))(\Lambda_1^n)^2 \left(\frac{h(t_1^n, t_1^n)}{h(t_3^n, t_3^n)} \right)^{1/2} \leq (1 + o(1))(\Lambda_1^n)^2 \left(\frac{h(t_1^n, t_1^n)}{H_0} \right)^{1/2}$$

and (3.64) follows by using Lemma A.1.

Next, in order to estimate the second sum in (3.63), we claim that

$$\frac{\Lambda_i^n \Lambda_j^n}{(t_j^n + t_i^n - 2a)^{N-1}} = o\left(\frac{(\Lambda_i^n)^2}{(2t_i^n - 2a)^{N-1}} + \frac{(\Lambda_j^n)^2}{(2t_j^n - 2a)^{N-1}} \right) \quad \text{if } i, j \leq i_0, (-1)^{i+j} = -1. \tag{3.65}$$

Indeed, take, for instance, the couple $(i,j) = (1,2)$; the other cases are analogous. The claim is obvious if $\Lambda_2^n = o(\Lambda_1^n)$ or $\Lambda_1^n = o(\Lambda_2^n)$. Otherwise $c \leq \frac{\Lambda_2^n}{\Lambda_1^n} \leq C$ and then, using (3.57) and (3.59), $\frac{h(t_2^n, t_2^n)}{h(t_1^n, t_1^n)} = o(1)$, by which, applying Lemma A.1, $t_1^n - a = o(t_2^n - a)$. This in turn implies $t_1^n - a = o(t_1^n + t_2^n - 2a)$, and (3.65) follows.

Therefore, recalling that $\beta_1^n > \beta_2^n$, (3.63) becomes

$$\sum_{i=1}^{i_0} \frac{(\Lambda_i^n)^2(1 + o(1))}{(2t_i^n - 2a)^{N-1}} \leq C.$$

Taking into account that $\frac{(\Lambda_1^n)^2}{(2t_1^n - 2a)^{N-1}} \geq c \frac{(\Lambda_1^n)^2 h(t_1^n, t_1^n)}{2t_1^n - 2a} \to +\infty$ by Lemma A.1 and (3.59), the contradiction follows.

An analogous argument can be applied when there exists $i_0 \in \{1,2,3,4\}$ such that

$$t_i^n \to b \quad \forall i_0 \le i \le 4, \qquad |t_i^n - b| \ge c \quad \forall i < i_0. \tag{3.66}$$

So we may assume

$$t_i^n \to \bar t_i \in (a,b) \quad \forall i = 1,2,3,4. \tag{3.67}$$

According to the assumption (1.2) we have either $\frac{\partial h}{\partial t}(t_1^n, t_1^n) \le 0$ or $\frac{\partial h}{\partial t}(t_3^n, t_3^n) \ge 0$. Assume, for instance,

$$\frac{\partial h}{\partial t}(t_1^n, t_1^n) \le 0$$

(the case $\frac{\partial h}{\partial t}(t_3^n, t_3^n) \ge 0$ can be treated analogously). We set $\{1,2,3,4\} = I \cup J$ where

$$I = \{i \,:\, |t_i^n - t_1^n| = o(|t_1^n - t_3^n|)\}, \quad J = \{i\,;\, |t_i^n - t_1^n| \ge c(|t_1^n - t_3^n|)\}.$$

It is obvious that $I = \{1\}$ or $I = \{1,2\}$. Then, adding (3.36) for $i \in I$ we get

$$\sum_{i\in I} \sum_{\substack{j=1 \\ j\neq i}}^{4} ((-1)^{i+j}\beta_1^n - \beta_2^n)\Lambda_i^n \Lambda_j^n \frac{\partial g}{\partial t}(t_i^n, t_j^n) \le C(\Lambda_2^n)^2. \tag{3.68}$$

Observe that

$$\frac{\partial g}{\partial t}(t_1^n, t_2^n) + \frac{\partial g}{\partial t}(t_2^n, t_1^n) = -\frac{\partial h}{\partial t}(t_1^n, t_2^n) - \frac{\partial h}{\partial t}(t_2^n, t_1^n) = O(1)$$

and $\Lambda_2^n \le C$, $\Lambda_1^n\Lambda_2^n \le C$, by (3.57); therefore (3.68) becomes

$$\sum_{i\in I} \sum_{j\in J} ((-1)^{i+j}\beta_1^n - \beta_2^n)\Lambda_i^n \Lambda_j^n \frac{\partial g}{\partial t}(t_i^n, t_j^n) \le C. \tag{3.69}$$

According to the assumption (1.3) we have $\frac{\partial g}{\partial t}(t,s) > 0$ if $t < s$. Since all the sequences t_i^n lie in a compact subset of Ω, Lemma A.1 implies

$$c\,\frac{g(t_i^n, t_j^n)}{|t_i^n - t_j^n|} \le \frac{\partial g}{\partial t}(t_i^n, t_j^n) \le C\frac{g(t_i^n, t_j^n)}{|t_i^n - t_j^n|} \quad \forall i,j = 1,2,3,4,\ i<j. \tag{3.70}$$

On the other hand, if $i \in I$ and $j \in J$, then $i < j$ and $|t_i^n - t_j^n| \ge c|t_3^n - t_1^n|$ by the definition of I, J; therefore combining (3.69) and (3.70) we arrive at

$$(\beta_1^n - \beta_2^n)\Lambda_1^n\Lambda_3^n g(t_1^n, t_3^n) \le C \sum_{\substack{(i,j)\in I\times J \\ (i,j)\neq(1,3)}} |(-1)^{i+j}\beta_1^n - \beta_2^n|\Lambda_i^n\Lambda_j^n g(t_i^n, t_j^n) + C. \tag{3.71}$$

This contradicts (3.56)-(3.57)-(3.58).

Case 3: Avoiding the boundary. *Suppose the following holds:* $|\beta_1^n - \beta_2^n|\Lambda_1^n\Lambda_3^n g(t_1^n, t_3^n) = O(1)$, $|\beta_1^n - \beta_2^n|\Lambda_2^n\Lambda_4^n g(t_2^n, t_4^n) = O(1)$, $I_a := \{i = 1,2,3,4 \,|\, \bar t_i = a\} \neq \emptyset$ *and*

$$(\beta_1^n + \beta_2^n) \sum_{(i,j)\in I_a} (\Lambda_i^n)^2 h(t_i^n, t_i^n) + 2 \sum_{\substack{i,j\in I_a \\ i<j}} |(-1)^{i+j}\beta_1^n - \beta_2^n|\Lambda_i^n\Lambda_j^n g(t_i^n, t_j^n) \ge c. \tag{3.72}$$

Replacing I_a with I_b can be treated analogously.

First of all we observe that (3.35) implies

$$\beta_1^n - \beta_2^n = o(1). \tag{3.73}$$

Recalling that $(\beta_1^n)^2 + (\beta_2^n)^2 = 1$ it follows that
$$\beta_1^n + \beta_2^n = \sqrt{2} + o(1). \tag{3.74}$$
Using (3.41)-(3.44) we obtain
$$(\Lambda_i^n)^2 h(t_i^n, t_i^n) \leq C \quad \forall i = 1,2,3,4, \tag{3.75}$$
hence $\Lambda_i^n \leq C$ for all $i = 1,2,3,4$, and
$$\Lambda_1^n\Lambda_2^n g(t_1^n, t_2^n),\ \Lambda_1^n\Lambda_4^n g(t_1^n, t_4^n),\ \Lambda_2^n\Lambda_3^n g(t_2^n, t_3^n),\ \Lambda_3^n\Lambda_4^n g(t_3^n, t_4^n) \leq C. \tag{3.76}$$
Now we multiply (3.36) by $t_i^n - a$ and add for $i \in I_a$
$$\sum_{i\in I_a} (\Lambda_i^n)^2 \frac{\partial h}{\partial t}(t_i^n, t_i^n)(t_i^n - a) - \sum_{i \in I_a} \sum_{\substack{j=1\\ j\neq i}}^{4} \frac{(-1)^{i+j}\beta_1^n - \beta_2^n}{\beta_1^n + \beta_2^n} \Lambda_i^n \Lambda_j^n \frac{\partial g}{\partial t}(t_i^n, t_j^n)(t_i^n - a) = 0. \tag{3.77}$$
We estimate the terms in each sum in order to obtain a contradiction. Lemma A.1 implies
$$\frac{\partial h}{\partial t}(t_i^n, t_i^n) = -\frac{1+o(1)}{\sigma_N (2t_i^n - 2a)^{N-1}} = -(N-2)(1+o(1))\frac{h(t_i^n, t_i^n)}{2(t_i^n - a)}, \quad \forall i \in I_a.$$
By the definition of I_a, there holds $|t_i^n - t_j^n| \geq c$ for $i \in I_a$ and $j \notin I_a$. This implies
$$\frac{\partial g}{\partial t}(t_i^n, t_j^n) = O(1) \quad \forall i \in I_a, \forall j \notin I_a. \tag{3.78}$$
We split the second sum in (3.77) in two terms: those with $j \in I_a$ and those with $j \notin I_a$. We use again Lemma A.1 and, considering the sum for $i, j \in I_a$, we observe that
$$\begin{aligned}\frac{\partial g}{\partial t}(t_i^n, t_j^n)(t_i^n - a) + \frac{\partial g}{\partial t}(t_j^n, t_i^n)(t_j^n - a) &= -\frac{1}{\sigma_N |t_j^n - t_i^n|^{N-2}} + \frac{1+o(1)}{\sigma_N (t_j^n - 2a + t_i^n)^{N-2}} \\ &= -(N-2) g(t_i^n, t_j^n) + \frac{o(1)}{\sigma_N (t_j^n - 2a + t_i^n)^{N-2}} \qquad \forall i, j \in I_a,\ i \neq j.\end{aligned}$$
On the other hand, it is straightforward to prove that the function $\frac{\exp y}{|t-a|^{N-2}}$ is convex for $t \geq a$, $y \in \mathbb{R}$. Therefore
$$\begin{aligned}\frac{\Lambda_i^n \Lambda_j^n}{(t_j^n - 2a + t_i^n)^{N-2}} &= \frac{\exp\left(\frac{\log(\Lambda_i^n)^2}{2} + \frac{\log(\Lambda_j^n)^2}{2}\right)}{2^{N-2}\left(\frac{t_i^n + t_j^n}{2} - a\right)^{N-2}} \leq \frac{(\Lambda_i^n)^2}{2(2t_i^n - 2a)^{N-2}} + \frac{(\Lambda_j^n)^2}{2(2t_j^n - 2a)^{N-2}} \\ &\leq C((\Lambda_i^n)^2 h(t_i^n, t_i^n) + (\Lambda_j^n)^2 h(t_j^n, t_j^n)) \leq C \quad \forall i, j \in I_a,\ i \neq j.\end{aligned}$$
Therefore (3.77) becomes
$$(\beta_1^n + \beta_2^n) \sum_{i \in I_a} (\Lambda_i^n)^2 h(t_i^n, t_i^n) - 2 \sum_{\substack{(i,j) \in I_a\\ i<j}} ((-1)^{i+j}\beta_1^n - \beta_2^n)\Lambda_i^n \Lambda_j^n g(t_i^n, t_j^n) = o(1). \tag{3.79}$$
If $I_a = \{1\}$ or $I_a = \{1,2\}$, then the left hand sides of (3.72) and (3.79) coincide, in contradiction with the right hand sides. If $I_a = \{1,2,3,4\}$, then the contradiction arises by comparing (3.79) with (3.38) because of (3.74). So it remains to consider the case $I_a = \{1,2,3\}$. We sum the identities (3.37) for $i = 1,2,3$ and subtract (3.79) and we obtain
$$(\beta_1^n + \beta_2^n)\Lambda_1^n\Lambda_4^n g(t_1^n, t_4^n) - (\beta_1^n - \beta_2^n)\Lambda_2^n\Lambda_4^n g(t_2^n, t_4^n) + (\beta_1^n + \beta_2^n)\Lambda_3^n\Lambda_4^n g(t_3^n, t_4^n) = 3(\beta_1^n + \beta_2^n) + o(1).$$

However, combining this with (3.37) for $i = 4$ gives

$$(\beta_1^n + \beta_2^n)(\Lambda_4^n)^2 h(t_4^n, t_4^n) + 2(\beta_1^n + \beta_2^n) = o(1)$$

and the contradiction arises because of (3.74).

Case 4: Avoiding collisions. *Suppose the following holds:* $|\beta_1^n - \beta_2^n|\Lambda_1^n\Lambda_3^n g(t_1^n, t_3^n) = O(1)$, $|\beta_1^n - \beta_2^n|\Lambda_2^n\Lambda_4^n g(t_2^n, t_4^n) = O(1)$ *and there exists* $i_0 \neq j_0$ *such that* $\bar{t}_{i_0} = \bar{t}_{j_0} \in (a,b)$ *and*

$$|(-1)^{i_0+j_0}\beta_1^n - \beta_2^n|\Lambda_{i_0}^n\Lambda_{j_0}^n g(t_{i_0}^n, t_{j_0}^n) \geq c.$$

As in the previous case we immediately get (3.73)–(3.76). Hence, in particular, $\Lambda_i^n \leq C$ for any $i = 1,2,3,4$. Set $\bar{t} = \bar{t}_{i_0} = \bar{t}_{j_0} \in (a,b)$ and $I = \{i = 1,2,3,4 \,|\, \bar{t}_i = \bar{t}\}$. We split $I = I_1 \cup I_2$ where

$$I_1 = \left\{ i \in I \,\middle|\, \exists j \in I,\, j \neq i \text{ s.t. } \; |(-1)^{i+j}\beta_1^n - \beta_2^n| \frac{\Lambda_i^n \Lambda_j^n}{|t_i^n - t_j^n|^{N-1}} \to +\infty \right\},$$

and

$$I_2 = \left\{ i \in I \,\middle|\, \forall j \in I,\, j \neq i : \; |(-1)^{i+j}\beta_1^n - \beta_2^n| \frac{\Lambda_i^n \Lambda_j^n}{|t_i^n - t_j^n|^{N-1}} \leq C \right\}.$$

Since the sequences t_i^n lie in a compact subset of Ω for any $i \in I$, Lemma A.1 implies

$$\frac{\partial g}{\partial t}(t_i^n, t_j^n) = -\frac{t_i^n - t_j^n}{\sigma_N |t_i^n - t_j^n|^N} + O(1) \quad \forall i \in I,\, \forall j = 1,2,3,4,\; i \neq j.$$

Moreover, observe that $\frac{1}{|t_{i_0}^n - t_{j_0}^n|^{N-1}} \geq \sigma_N(N-2)\frac{g(t_{i_0}^n, t_{j_0}^n)}{|t_{i_0}^n - t_{j_0}^n|}$. Therefore, according to the assumptions, $i_0, j_0 \in I_1$. For any $i \in I_1$ we consider (3.36) and obtain

$$\sum_{j \in I_1,\, j \neq i} ((-1)^{i+j}\beta_1^n - \beta_2^n)\Lambda_i^n \Lambda_j^n \frac{t_i^n - t_j^n}{|t_i^n - t_j^n|^N} = O(1) \quad \forall i \in I_1. \tag{3.80}$$

Using (3.80) for i_0, we immediately get the existence of a third index $j \in I_1$, $j \neq i_0, j_0$. Therefore I_1 has actually at least three elements. Assume $I_1 = \{1,2,3,4\}$. We look at (3.80) for $i = 1$:

$$(\beta_1^n + \beta_2^n)\frac{\Lambda_1^n\Lambda_2^n}{|t_1^n - t_2^n|^{N-1}} + (\beta_1^n + \beta_2^n)\frac{\Lambda_1^n\Lambda_4^n}{|t_1^n - t_4^n|^{N-1}} = (\beta_1^n - \beta_2^n)\frac{\Lambda_1^n\Lambda_3^n}{|t_1^n - t_3^n|^{N-1}} + O(1) \to +\infty$$

which yields $\beta_1^n > \beta_2^n$. Dividing the identity by $(\beta_1^n - \beta_2^n)\frac{\Lambda_1^n\Lambda_3^n}{|t_1^n - t_3^n|^{N-1}}$, and using $|t_1^n - t_2^n| < |t_1^n - t_3^n|$, we get

$$\frac{\beta_1^n + \beta_2^n}{\beta_1^n - \beta_2^n} \leq \frac{\Lambda_3^n}{\Lambda_2^n}(1 + o(1)).$$

Next we consider (3.80) for $i = 4$ and proceed analogously, using now that $|t_3^n - t_4^n| < |t_2^n - t_4^n|$. This leads to: $\frac{\beta_1^n + \beta_2^n}{\beta_1^n - \beta_2^n} \leq \frac{\Lambda_2^n}{\Lambda_3^n}(1 + o(1))$, and so

$$\frac{\beta_1^n + \beta_2^n}{\beta_1^n - \beta_2^n} \leq 1 + o(1)$$

in contradiction with (3.73)-(3.74).

It remains to consider the case when I_1 has exactly three elements. If $I_1 = \{1,2,4\}$, then (3.80) for $i = 2$ gives

$$(\beta_1^n + \beta_2^n)\frac{\Lambda_1^n\Lambda_2^n}{|t_1^n - t_2^n|^{N-1}} = -(\beta_1^n - \beta_2^n)\frac{\Lambda_2^n\Lambda_4^n}{|t_1^n - t_4^n|^{N-1}} + O(1) \to +\infty,$$

which is absurd if $\beta_1^n \geq \beta_2^n$. On the other hand, by (3.80) for $i = 4$

$$(\beta_1^n + \beta_2^n)\frac{\Lambda_1^n\Lambda_4^n}{|t_1^n - t_4^n|^{N-1}} = (\beta_1^n - \beta_2^n)\frac{\Lambda_2^n\Lambda_4^n}{|t_2^n - t_4^n|^{N-1}} + O(1) \to +\infty$$

which gives the contradiction in the case $\beta_1^n < \beta_2^n$. An analogous argument applies to the case $I_1 = \{1,3,4\}$.

It remains to consider the cases $I_1 = \{1,2,3\}$ and $I_1 = \{2,3,4\}$. Assume, for instance, $I_1 = \{1,2,3\}$, the other case is similar. Then by (3.80) we obtain

$$\frac{\Lambda_1^n\Lambda_2^n}{|t_2^n - t_1^n|^{N-1}} = \frac{\beta_1^n - \beta_2^n}{\beta_1^n + \beta_2^n}\cdot\frac{\Lambda_1^n\Lambda_3^n}{|t_3^n - t_1^n|^{N-1}} + O(1) = \frac{\Lambda_2^n\Lambda_3^n}{|t_3^n - t_2^n|^{N-1}} + O(1) \to +\infty. \tag{3.81}$$

In particular we have $\beta_1^n > \beta_2^n$. Using (3.73)-(3.74), the first and the second equality in (3.81) give

$$\Lambda_2^n = o(\Lambda_3^n), \quad \Lambda_2^n = o(\Lambda_1^n),$$

respectively. Now we multiply the first identity in (3.81) by $t_2^n - t_1^n$ and the second by $t_3^n - t_2^n$ and, summing up, we obtain

$$(\beta_1^n + \beta_2^n)\frac{\Lambda_1^n\Lambda_2^n}{|t_2^n - t_1^n|^{N-2}} - (\beta_1^n - \beta_2^n)\frac{\Lambda_1^n\Lambda_3^n}{|t_3^n - t_1^n|^{N-2}} + (\beta_1^n + \beta_2^n)\frac{\Lambda_2^n\Lambda_3^n}{|t_3^n - t_2^n|^{N-2}} = o(1). \tag{3.82}$$

We may also assume

$$\Lambda_4^n \geq c. \tag{3.83}$$

Otherwise, if $\Lambda_4^n \to 0$, then (3.37) for $i = 4$ would give

$$(\Lambda_4^n)^2 h(t_4^n, t_4^n) + \Lambda_4^n(\Lambda_1^n g(t_1^n, t_4^n) + \Lambda_3^n g(t_3^n, t_4^n)) = 1 + \frac{\beta_1^n - \beta_2^n}{\beta_1^n + \beta_2^n}\Lambda_2^n\Lambda_4^n g(t_2^n, t_4^n) \geq 1$$

by which either $(\Lambda_4^n)^2 h(t_4^n, t_4^n) \geq \frac{1}{2}$ or $\Lambda_4^n(\Lambda_1^n g(t_1^n, t_4^n) + \Lambda_3^n g(t_3^n, t_4^n)) \geq \frac{1}{2}$. If $(\Lambda_4^n)^2 h(t_4^n, t_4^n) \geq \frac{1}{2}$, then $h(t_4^n, t_4^n) \to +\infty$, and, consequently, $t_4^n \to b$, so that we are again in the case 3. Otherwise, if $\Lambda_4^n(\Lambda_1^n g(t_1^n, t_4^n) + \Lambda_3^n g(t_3^n, t_4^n)) \geq \frac{1}{2}$, then $g(t_1^n, t_4^n) + g(t_3^n, t_4^n) \to +\infty$. So $t_4^n \to \bar{t}$ and then

$$\frac{\Lambda_1^n\Lambda_4^n}{|t_1^n - t_4^n|^{N-1}} + \frac{\Lambda_3^n\Lambda_4^n}{|t_3^n - t_4^n|^{N-1}} \geq \sigma_N(N-2)\Lambda_1^n\Lambda_4^n\frac{g(t_1^n, t_4^n)}{|t_1^n - t_4^n|} + \sigma_N(N-2)\Lambda_3^n\Lambda_4^n\frac{g(t_3^n, t_4^n)}{|t_3^n - t_4^n|} \to +\infty,$$

contradicting that $4 \notin I_1$.

Now we distinguish three cases. First assume

$$\Lambda_1^n,\ \Lambda_2^n,\ \Lambda_3^n \to 0. \tag{3.84}$$

Then (3.82) can be rewritten as

$$(\beta_1^n + \beta_2^n)\Lambda_1^n\Lambda_2^n g(t_1^n, t_2^n) - (\beta_1^n - \beta_2^n)\Lambda_1^n\Lambda_3^n g(t_1^n, t_3^n) + (\beta_1^n + \beta_2^n)\Lambda_2^n\Lambda_3^n g(t_2^n, t_3^n) = o(1).$$

We sum the identities (3.37) in $i = 1,2,3$ and, using the above estimate and (3.84), we obtain

$$(\beta_1^n + \beta_2^n)\Lambda_1^n\Lambda_4^n g(t_1^n, t_4^n) - (\beta_1^n - \beta_2^n)\Lambda_2^n\Lambda_4^n g(t_2^n, t_4^n) + (\beta_1^n + \beta_2^n)\Lambda_3^n\Lambda_4^n g(t_3^n, t_4^n) = 3(\beta_1^n + \beta_2^n) + o(1).$$

However, combining this with (3.37) for $i=4$ gives

$$(\beta_1^n+\beta_2^n)(\Lambda_4^n)^2 h(t_4^n,t_4^n)+2(\beta_1^n+\beta_2^n)=o(1)$$

and a contradiction arises because of (3.74).

Now assume that

$$\Lambda_1^n,\,\Lambda_2^n\to 0,\quad \Lambda_3^n\ge c. \tag{3.85}$$

Then $\Lambda_1^n\Lambda_2^n=o(\Lambda_2^n\Lambda_3^n)$. According to (3.81) we have $\frac{\Lambda_1^n\Lambda_2^n}{|t_2^n-t_1^n|^{N-1}}=(1+o(1))\frac{\Lambda_2^n\Lambda_3^n}{|t_3^n-t_2^n|^{N-1}}$, from which we deduce $t_2^n-t_1^n=o(t_3^n-t_2^n)$. Consequently $\frac{\Lambda_1^n\Lambda_2^n}{|t_2^n-t_1^n|^{N-2}}=o(\frac{\Lambda_2^n\Lambda_3^n}{|t_3^n-t_2^n|^{N-2}})$, which is equivalent to $\Lambda_1^n\Lambda_2^n g(t_1^n,t_2^n)=o(\Lambda_2^n\Lambda_3^n g(t_2^n,t_3^n))$. Now (3.76) implies $\Lambda_1^n\Lambda_2^n g(t_1^n,t_2^n)=o(1)$, hence (3.41) becomes

$$(\beta_1^n+\beta_2^n)\Lambda_1^n\Lambda_4^n g(t_1^n,t_4^n)=\beta_1^n+\beta_2^n+(\beta_1^n-\beta_2^n)\Lambda_1^n\Lambda_3^n g(t_1^n,t_3^n)+o(1)\ge\beta_1^n+\beta_2^n+o(1)$$

because $\beta_1^n>\beta_2^n$. Then $\Lambda_1^n\Lambda_4^n g(t_1^n,t_4^n)\ge c$, which implies $g(t_1^n,t_4^n)\to+\infty$ by (3.85). So, $t_4^n\to\bar t$ and then

$$\frac{\Lambda_1^n\Lambda_4^n}{|t_1^n-t_4^n|^{N-1}}\ge\sigma_N(N-2)\Lambda_1^n\Lambda_4^n\frac{g(t_1^n,t_4^n)}{|t_1^n-t_4^n|}\to+\infty,$$

in contradiction with $4\notin I_1$.

An analogous argument applies when

$$\Lambda_3^n,\,\Lambda_2^n\to 0,\quad \Lambda_1^n\ge c.$$

Finally, assume that

$$\Lambda_2^n\to 0,\quad \Lambda_1^n,\,\Lambda_3^n\ge c. \tag{3.86}$$

Then we obtain, using (3.35),

$$\Lambda_1^n\Lambda_3^n\le\frac{n}{g(t_1^n,t_3^n)}\le Cn|t_1^n-t_3^n|^{N-2}\le Cn(|t_1^n-t_2^n|^{N-2}+|t_2^n-t_3^n|^{N-2})\le Cn(\Lambda_1^n\Lambda_2^n+\Lambda_2^n\Lambda_3^n)$$

where the last inequality follows from (3.81). So, using (3.86), we deduce $c\le\Lambda_1^n\Lambda_3\le Cn\Lambda_2^n$, by which $\Lambda_2^n\ge\frac{c}{n}$. Combining this with (3.83) and (3.86) we obtain

$$\Lambda_1^n\cdot\Lambda_2^n\cdot\Lambda_3^n\cdot\Lambda_4^n\ge\frac{c}{n}.$$

Finally (3.35), (3.75) and (3.76) imply

$$\tilde\Psi^*=\tilde\Psi(\boldsymbol{\Lambda}_n,\boldsymbol{t}_n)=-\frac{n}{2}+O(1)-\log(\Lambda_1^n\Lambda_2^n\Lambda_3^n\Lambda_4^n)\le-\frac{n}{2}+O(1)+\log n\to-\infty$$

in contradiction with the lower estimate (3.29).

Case 5: Conclusion.

In order to not fall again in the cases 1-2, we assume:

$$|\beta_1^n-\beta_2^n|\Lambda_1^n\Lambda_3^n g(t_1^n,t_3^n)\le C,\quad |\beta_1^n-\beta_2^n|\Lambda_2^n\Lambda_4^n g(t_2^n,t_4^n)\le C.$$

So, as in the cases 3 and 4 we immediately get (3.73)–(3.76) and, in particular, $\Lambda_i^n\le C$ for any $i=1,2,3,4$. Moreover we may also assume

$$\Lambda_i^n\ge c\quad\forall i=1,2,3,4. \tag{3.87}$$

Indeed, assume for instance, that $\Lambda_1^n \to 0$. Then, by (3.37) for $i = 1$ we have that, either

$$(\Lambda_1^n)^2 h(t_1^n, t_1^n) \geq c, \tag{3.88}$$

or

$$\exists j = 2, 3, 4 \ \text{ such that } \ |(-1)^{1+j}\beta_1^n - \beta_2^n| \Lambda_1^n \Lambda_j^n g(t_1^n, t_j^n) \geq c. \tag{3.89}$$

If (3.88) holds, then $h(t_1^n, t_1^n) \to +\infty$, which implies $\bar{t}_1 = a$ or $\bar{t}_1 = b$ by (A.90), and we are back in the case 3. On the other hand, if (3.89) holds, then, $g(t_1^n, t_j^n) \to +\infty$ for some $j \neq 1$, which implies $\bar{t}_j = \bar{t}_1$, and we are either in the case 3 (if $\bar{t}_1 = a, b$) or in case 4 (if $\bar{t}_1 \in (a, b)$). Finally (3.35), (3.75), (3.76), (3.87) imply

$$\tilde{\Psi}^* = \tilde{\Psi}(\boldsymbol{\Lambda}_n, \boldsymbol{t}_n) = -\frac{n}{2} + O(1) \to -\infty$$

in contradiction with the lower estimate (3.29).

Appendix A. Some properties of the Green's function

Let Ω be a bounded domain with a $\mathcal{C}^2$-boundary. We denote by $G(x, y)$ the Green's function of $-\Delta$ on Ω under Dirichlet boundary conditions, and by $H(x, y)$ its regular part, as in the introduction. So H satisfies

$$\begin{cases} \Delta_y H(x,y) = 0 & y \in \Omega, \\ H(x,y) = \dfrac{1}{(N-2)\sigma_N |x-y|^{N-2}} & y \in \partial\Omega. \end{cases}$$

We recall that H is a smooth function in $\Omega \times \Omega$; moreover G and H are symmetric in x and y and $G, H > 0$ in $\Omega \times \Omega$.

The diagonal $H(x, x)$ is called the Robin's function of the domain Ω and satisfies

$$H(x,x) \to +\infty \qquad \text{as } \mathrm{d}(x) := \mathrm{dist}(x, \partial\Omega) \to 0. \tag{A.90}$$

Let H_0 be the minimum value of the Robin's function:

$$H_0 = \min_{\Omega} H(x,x) > 0.$$

Recall that the Robin's function of a convex bounded domain is strictly convex ([11]).

We need the following result concerning the behavior of the regular part $H(x, y)$ near the boundary. To this aim we fix $\delta > 0$ sufficiently small such that the projection onto $\partial\Omega$ is well defined in the region $\Omega_0 := \{x \in \Omega : \mathrm{d}(x) < \delta\}$; we denote this projection by $p : \Omega_0 \to \partial\Omega$. It is of class $\mathcal{C}^1$ because $\partial\Omega$ is of class $\mathcal{C}^2$. Moreover, for $x \in \Omega_0$, we write $\bar{x} = 2p(x) - x$ for the reflection of x at $\partial\Omega$ and $\nu_x = \frac{x-p(x)}{|x-p(x)|}$ for the inward unit normal at $p(x)$.

Lemma A.1. *Let Ω be a bounded domain with a $\mathcal{C}^2$-boundary. Then the following expansions hold uniformly for $x \in \Omega_0$ and $y \in \Omega$:*

$$H(x,y) = \frac{1}{(N-2)\sigma_N |\bar{x}-y|^{N-2}} + O\left(\frac{\mathrm{d}(x)}{|\bar{x}-y|^{N-2}}\right),$$

and

$$\frac{\partial H}{\partial \nu_x}(x,y) = \frac{1}{(N-2)\sigma_N} \frac{\partial}{\partial \nu_x}\left(\frac{1}{|\bar{x}-y|^{N-2}}\right) + O\left(\frac{1}{|\bar{x}-y|^{N-2}}\right).$$

Proof. During the proof we will often use the symbols c, C to denote different positive constants depending only on Ω. For any $x \in \Omega_0$ we introduce a diffeomorphism which straightens the boundary near $p(x)$. Let T_x be a rotation and translation of coordinates which maps $p(x)$ to 0 and the unit inward normal ν_x to the vector $\mathbf{e}_N := (0, \dots, 0, 1)$. Then $T_x(x) = (0, \dots, 0, \mathrm{d}(x))$, $T_x(\bar{x}) = (0, \dots, 0, -\mathrm{d}(x))$, and in some neighborhood of 0 the boundary $\partial(T_x\Omega)$ can be represented by

$$z_N = \rho_x(z'), \quad z' = (z_1, \dots, z_{N-1});$$

here ρ_x is a $\mathcal{C}^2$ function satisfying $\rho_x(0) = 0$ and $\nabla\rho_x(0) = 0$. Therefore we have

$$|z_N| \le C|z'|^2 \quad \text{on } \partial(T_x\Omega).$$

First we prove the following estimate for the boundary points:

$$\left| \frac{1}{|x-y|^{N-2}} - \frac{1}{|\bar{x}-y|^{N-2}} \right| \le C \frac{\mathrm{d}(x)}{|\bar{x}-y|^{N-2}} \quad \forall x \in \Omega_0,\ \forall y \in \partial\Omega. \tag{A.91}$$

In order to see this, we observe for $x \in \Omega_0$, $y \in \partial\Omega$, $z := T_x(y)$, that

$$\max\{\mathrm{d}(x), |z'|\} \le \min\{|x-y|, |\bar{x}-y|\}, \tag{A.92}$$

by which

$$\big| |x-y|^2 - |\bar{x}-y|^2 \big| = 4\mathrm{d}(x) z_N \le C\mathrm{d}(x)|z'|^2 \le C\mathrm{d}(x) \min\{|\bar{x}-y|^2, |x-y|^2\}. \tag{A.93}$$

The above inequality implies

$$c \le \frac{|\bar{x}-y|}{|x-y|} \le C \quad \forall x \in \Omega_0,\ \forall y \in \partial\Omega. \tag{A.94}$$

Taking into account that $|a^m - b^m| \le m|a-b|(a+b)^{m-1}$ for any $a, b \ge 0$ and $m \ge 1$, we have

$$\begin{aligned} \left| \frac{1}{|x-y|^{N-2}} - \frac{1}{|\bar{x}-y|^{N-2}} \right| &= \left| \frac{|\bar{x}-y|^{2(N-2)} - |x-y|^{2(N-2)}}{(|\bar{x}-y|^{N-2} + |x-y|^{N-2})|x-y|^{N-2}|\bar{x}-y|^{N-2}} \right| \\ &\le (N-2) \frac{(|\bar{x}-y|^2 + |x-y|^2)^{N-3}(|x-\bar{y}|^2 - |x-y|^2)}{(|\bar{x}-y|^{N-2} + |x-y|^{N-2})|x-y|^{N-2}|\bar{x}-y|^{N-2}} \end{aligned}$$

and (A.91) follows by using (A.93) and (A.94). So, for any $x \in \Omega_0$, the functions $H(x,y) - \frac{1}{\sigma_N(N-2)|\bar{x}-y|^{N-2}}$ and $\frac{1}{|\bar{x}-y|^{N-2}}$ are both harmonic in Ω in the variable y, and verify (A.91) on the boundary. Then the maximum principle applies and gives

$$\left| H(x,y) - \frac{1}{\sigma_N(N-2)|\bar{x}-y|^{N-2}} \right| \le C \frac{\mathrm{d}(x)}{|\bar{x}-y|^{N-2}} \quad \forall x \in \Omega_0,\ \forall y \in \Omega.$$

The first part of the thesis follows.

We go on with the normal derivative estimate. We claim the following estimate on the boundary:

$$\left| \frac{\partial H}{\partial \nu_x}(x,y) - \frac{(\bar{x}-y)\cdot\nu_x}{\sigma_N|\bar{x}-y|^N} \right| = \left| \frac{(y-x)\cdot\nu_x}{\sigma_N|x-y|^N} - \frac{(\bar{x}-y)\cdot\nu_x}{\sigma_N|\bar{x}-y|^N} \right| \le \frac{C}{|\bar{x}-y|^{N-2}} \quad \forall x \in \Omega_0,\ \forall y \in \partial\Omega. \tag{A.95}$$

Indeed, proceeding as for (A.91) we have

$$\left| \frac{1}{|x-y|^N} - \frac{1}{|\bar{x}-y|^N} \right| \le C \frac{\mathrm{d}(x)}{|\bar{x}-y|^N} \le \frac{C}{|\bar{x}-y|^{N-1}} \quad \forall x \in \Omega_0,\ \forall y \in \partial\Omega \tag{A.96}$$

where the second inequality holds since $\mathrm{d}(x) \le |\bar{x}-y|$ by (A.92). Moreover, for $x \in \Omega_0$, $y \in \partial\Omega$, $z := T_x(y)$,

$$\begin{aligned} |(y-x)\cdot\nu_x - (\bar{x}-y)\cdot\nu_x| &= |(z-\mathrm{d}(x)\mathbf{e}_N)\cdot\mathbf{e}_N - (-\mathrm{d}(x)\mathbf{e}_N - z)\cdot\mathbf{e}_N| \\ &= 2|z_N| \le C|z'|^2 \le C|\bar{x}-y|^2 \end{aligned} \tag{A.97}$$

where for the last inequality we have used (A.92). Thus we obtain for $x \in \Omega_0$ and $y \in \partial\Omega$:

$$\left|\frac{(y-x)\nu_x}{|x-y|^N} - \frac{(\bar{x}-y)\nu_x}{|\bar{x}-y|^N}\right| \le |(y-x)\nu_x|\left|\frac{1}{|x-y|^N} - \frac{1}{|\bar{x}-y|^N}\right| + \frac{|(y-x)\nu_x - (\bar{x}-y)\nu_x|}{|\bar{x}-y|^N}$$

and (A.95) follows from (A.94), (A.96), (A.97). Now, for $x \in \Omega_0$ fixed, the functions $\frac{\partial H}{\partial \nu_x}(x,y) - \frac{(\bar{x}-y)\cdot\nu_x}{\sigma_N|\bar{x}-y|^N}$ and $\frac{1}{|\bar{x}-y|^{N-2}}$ are harmonic in Ω with respect to the variable y, and verify (A.95) on the boundary. The maximum principle applies and gives

$$\left|\frac{\partial H}{\partial \nu_x}(x,y) - \frac{(\bar{x}-y)\cdot\nu_x}{\sigma_N|\bar{x}-y|^N}\right| \le \frac{C}{|\bar{x}-y|^{N-2}} \quad \forall x \in \Omega,\ \forall y \in \Omega_0.$$

In order to conclude observe that $\frac{\partial \bar{x}}{\partial \nu_x} = -\nu_x$, because $\frac{\partial p}{\partial \nu_x}(x) = 0$ for any $x \in \Omega_0$, so that

$$\frac{\partial}{\partial \nu_x}\left(\frac{1}{(N-2)|\bar{x}-y|^{N-2}}\right) = \frac{(\bar{x}-y)\cdot\nu_x}{|\bar{x}-y|^N} \quad \forall x \in \Omega_0,\ \forall y \in \Omega.$$

□

We conclude this section with the following lemma which is concerned with the behaviour of $G(\cdot,y)$ along half-lines through the domain starting from y. This implies (1.3) for convex domains.

Lemma A.2. *Let Ω be a convex and bounded domain with a smooth boundary. Then for any $x, y \in \Omega$, $x \ne y$, we have*

$$(x-y)\cdot\nabla_x G(x,y) < 0.$$

Proof. We use Lemma 3.1 in [15] which states that if Ω is a smooth and bounded domain in $\mathbb{R}^N$, then, for any $P \in \Omega$, $A, B \in \Omega$, $A \ne B$,

$$-\int_{\partial\Omega}(x-P)\cdot\nu_x \frac{\partial G(x,A)}{\partial \nu_x}\frac{\partial G(x,B)}{\partial \nu_x}ds = (2-N)G(A,B) + (P-A)\nabla_x G(A,B) + (P-B)\nabla_x G(B,A),$$

where ν_x is the unit inner normal at $x \in \partial\Omega$. Now assume that Ω is convex and take $P = B$. We deduce

$$(B-A)\nabla_x G(A,B) = -\int_{\partial\Omega}(x-B)\cdot\nu_x \frac{\partial G(x,A)}{\partial \nu_x}\frac{\partial G(x,B)}{\partial \nu_x}ds + (N-2)G(A,B)$$

which is strictly positive because $(x-B)\cdot\nu_x < 0$ for any $x \in \partial\Omega$ by the convexity of Ω, and because $\frac{\partial G(x,A)}{\partial \nu_x}$, $\frac{\partial G(x,B)}{\partial \nu_x} > 0$ on $\partial\Omega$. □

References

[1] T. Aubin: *Problémes isopérimétriques et espaces de Sobolev,* J. Differential Geometry **11** (1976), 573–598.

[2] A. Bahri, J. M. Coron: *On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain*, Comm. Pure Appl. Math. **41** (1988), 253–294.

[3] A. Bahri, Y. Li, O. Rey: *On a variational problem with lack of compactness: the topological effect of the critical points at infinity*, Calc. Var. Partial Differential Equations **3** (1995), 67–93.

[4] T. Bartsch: *Critical point theory on partially ordered Hilbert spaces*, J. Funct. Anal. **186** (2001), 117–152.

[5] T. Bartsch, T. D'Aprile, A. Pistoia: *On the existence of three-bubble nodal solutions for a slightly subcritical problem in a ball*, in preparation.

[6] T. Bartsch, A. Micheletti, A. Pistoia: *On the existence and the profile of nodal solutions of elliptic equations involving critical growth*, Calc. Var. Partial Differential Equations **26** (2006), 265–282.

[7] T. Bartsch, T. Weth: *A note on additional properties of sign changing solutions to superlinear elliptic equations*, Topol. Methods Nonlinear Anal. **22** (2003), 1–14.

[8] M. Ben Ayed, K. El Mehdi, F. Pacella: *Classification of low energy sign-changing solutions of an almost critical problem,* J. Funct. Anal. **50** (2007), 347–373.

[9] H. Brézis, L. A. Peletier: *Asymptotics for elliptic equations involving critical growth. Partial differential equations and the calculus of variations, vol. I*, Progr. Nonlinear Diff. Equ. Appl. Birkhäuser, Boston, MA **1** (1989), 149–192.

[10] L. Caffarelli, B. Gidas, J. Spruck: *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure Appl. Math. **42** (1989), 271–297.

[11] P. Cardaliaguet, R. Tahraoui: *On the strict concavity of the harmonic radius in dimension* $N \geq 3$, J. Math. Pures Appl. **81** (2002), 223–240.

[12] M. Del Pino, P. Felmer, M. Musso: *Multi-bubble solutions for slightly super-critical elliptic problems in domains with symmetries*, Bull. London Math. Soc. **35** (2003), 513–521.

[13] M. Del Pino, P. Felmer, M. Musso: *Multi-peak solutions for super-critical elliptic problems in domains with small holes*, J. Differential Equations **182** (2002), 511–540.

[14] M. Del Pino, P. Felmer, M. Musso: *Two-bubble solutions in the super-critical Bahri-Coron's problem*, Calc. Var. Partial Differential Equations **16** (2003), 113–145.

[15] M. Grossi, F. Takahashi: *Nonexistence of multi-bubble solutions to some elliptic equations on convex domains*, J. Funct. Anal. **259** (2010), 904–917.

[16] M. Flucher, J. Wei: *Semilinear Dirichlet problem with nearly critical exponent, asymptotic location of hot spots*, Manuscripta Math. **94** (1997), 337–346.

[17] Z. C. Han: *Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent*, Ann. Inst. H. Poincaré, Anal. Non Lin. **8** (1991), 159–174.

[18] J. Kazdan, F. Warner: *Remarks on some quasilinear elliptic equations,* Comm. Pure Appl. Math. **28** (1975), 567–597.

[19] M. Musso, A. Pistoia: *Tower of bubbles for almost critical problems in general domains,* J. Math. Pures Appl. **93** (2010), 1–40.

[20] A. Pistoia, O. Rey: *Multiplicity of solutions to the supercritical Bahri-Coron's problem in pierced domains,* Adv. Differential Equations **11** (2006), 647–666.

[21] S. I. Pohožaev: *On the eigenfunctions of the equation* $\Delta u + \lambda f(u) = 0$, Soviet Math. Dokl. **6** (1965), 1408–1411.

[22] O. Rey: *Blow-up points of solutions to elliptic equations with limiting nonlinearity*, Differential Integral Equations **4** (1991), 1155–1167.

[23] O. Rey: *The role of the Green's function in a nonlinear elliptic equation involving the critical Sobolev exponent*, J. Funct. Anal. **89** (1990), 1–52.

[24] O. Rey: *Proof of two conjectures of H. Brezis and L.A. Peletier*, Manuscripta Math. **65** (1989), 19–37.

[25] A. Pistoia, T. Weth: *Sign changing bubble tower solutions in a slightly subcritical semilinear Dirichlet problem*, Ann. Inst. H. Poincaré Anal. Non Linéaire **24** (2007), 325–340.

[26] G. Talenti: *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. **110** (1976), 353–372.

Thomas Bartsch, Mathematisches Institut, Justus-Liebig-Universität Giessen, Arndtstr. 2, 34392 Giessen, Germany.
E-mail address: `Thomas.Bartsch@math.uni-giessen.de`

Teresa D'Aprile, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica 1, 00133 Roma, Italy.
E-mail address: `daprile@mat.uniroma2.it`

Angela Pistoia, Dipartimento di Metodi e Modelli Matematici, Università di Roma "La Sapienza", via Antonio Scarpa 165, 00161 Roma, Italy.
E-mail address: `pistoia@dmmm.uniroma1.it`