Nonlinear Schrödinger equations near an infinite well potential

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Abstract

The paper deals with standing wave solutions of the dimensionless nonlinear Schrödinger equation

$$(NLS_\lambda) \qquad i\Phi_t(x,t) = -\Delta_x\Phi + V_\lambda(x)\Phi + g(x,|\Phi|)\Phi, \quad x \in \mathbb{R}^N,\ t \in \mathbb{R},$$

where the potential $V_\lambda : \mathbb{R}^N \to \mathbb{R}$ is close to an infinite well potential $V_\infty : \mathbb{R}^N \to \mathbb{R}$, i. e. $V_\infty = \infty$ on an exterior domain $\mathbb{R}^N \setminus \Omega$, $V_\infty|_\Omega \in L^\infty(\Omega)$, and $V_\lambda \to V_\infty$ as $\lambda \to \infty$ in a sense to be made precise. The nonlinearity may be of Gross-Pitaevskii type. A standing wave solution of (NLS_λ) with $\lambda = \infty$ vanishes on $\mathbb{R}^N \setminus \Omega$ and satisfies Dirichlet boundary conditions, hence it solves

$$(NLS_\infty) \qquad \begin{cases} i\Phi_t(x,t) = -\Delta_x\Phi + V_\lambda(x)\Phi + g(x,|\Phi|)\Phi, & x \in \Omega,\ t \in \mathbb{R} \\ \Phi(x,t) = 0 & x \in \partial\Omega,\ t \in \mathbb{R}. \end{cases}$$

We investigate when a standing wave solution Φ_∞ of the infinite well potential (NLS_∞) gives rise to nearby solutions Φ_λ of the finite well potential (NLS_λ) with $\lambda \gg 1$ large. Considering (NLS_∞) as a singular limit of (NLS_λ) we prove a kind of singular continuation type results.

Keywords: nonlinear Schrödinger equations, infinite well potential, steep potential well, singular limit, variational methods, topological methods, singular continuation

AMS subject classification: 35J20, 35J61, 35J91, 35Q55, 58E05

1 Introduction

Infinite well potentials like the infinite square well or the infinite spherical well are helpful as instructive models to describe confined particles in quantum mechanical systems. They are often used as a starting point for solving finite well problems. In this paper we investigate nonlinear Schrödinger equations, like the Gross-Pitaevskii equation, with a potential $V_\lambda : \mathbb{R}^N \to \mathbb{R}$ close to an infinite well potential $V_\infty : \mathbb{R}^N \to \mathbb{R}$. More precisely, $V_\infty = \infty$ on an exterior domain $\mathbb{R}^N \setminus \Omega$, and $V_\infty|_\Omega \in L^\infty(\Omega)$. As $\lambda \to \infty$ the potential depth of V_λ becomes infinite, i. e. $V_\lambda \to V_\infty$, in a sense to be made precise below. Our goal is to give rigorous proofs for the passage from the infinite well potential to the finite well potential.

We are interested in standing waves $\Phi(t,x) = e^{i\omega t}u(x)$ of the finite well nonlinear Schrödinger equation

$$(NLS_\lambda) \qquad i\Phi_t(x,t) = -\Delta_x\Phi(x,t) + V_\lambda(x)\Phi + g(x,|\Phi|)\Phi, \quad x \in \mathbb{R}^N,\ t \in \mathbb{R},$$

where $V_\lambda(x) \to V_\infty(x)$ as $\lambda \to \infty$. For $\lambda = \infty$ a solution should vanish in $\mathbb{R}^N \setminus \Omega$ and satisfy Dirichlet boundary conditions on Ω, hence it is a solution of the singular limit problem:

$$(NLS_\infty) \qquad \begin{cases} i\Phi_t(x,t) = -\Delta_x\Phi + V_\lambda(x)\Phi + g(x,|\Phi|)\Phi, & x \in \Omega,\ t \in \mathbb{R}, \\ \Phi(x,t) = 0 & x \in \partial\Omega,\ t \in \mathbb{R}. \end{cases}$$

The question we address in this paper is: suppose we know a standing wave solution Φ_∞ of (NLS_∞), does there exist a nearby solution Φ_λ of (NLS_λ) for λ large?

Standing wave solutions of (NLS_λ) correspond to solutions of the stationary nonlinear Schrödinger equation

$$(S_\lambda) \qquad \begin{cases} -\Delta u + V_\lambda(x)u = f(x,u) & \text{for } x \in \mathbb{R}^N; \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases}$$

where we sightly changed notation. For $\lambda = \infty$ we are similarly led to consider

$$(S_\infty) \qquad -\Delta u + V_\infty(x)u = f(x,u), \quad u \in H_0^1(\Omega),$$

as a singular limit of (S_λ) as $\lambda \to \infty$. The original question can now be reformulated as which solutions u_∞ of (S_∞) appear as limits of solutions u_λ of (S_λ). Solutions of (S_∞) can be obtained via variational or topological methods. We provide conditions on the convergence of $V_\lambda \to V_\infty$ and on f such that an isolated solution u_∞ of (S_∞) which can be found by variational or topological methods gives rise to a family of solutions u_λ of (S_λ). We include of course the generic case where u_∞ is a nondegenerate solution of (S_∞).

For the proofs we develop an abstract functional analytic approach in order to deal with the above type of singular limit problem. Our results may be thought of as being continuation results near a singular limit: For $\lambda < \infty$ we look for solutions of an equation $F_\lambda(u) = 0$ defined on the form domain of $-\Delta + V_\lambda$, a subspace of $H^1(\mathbb{R}^N)$, whereas the limit equation $F_\infty(u) = 0$ is only defined for u in the form domain of $-\Delta + V_\infty$ in Ω, which is a subspace of $H^1_0(\Omega)$. Some of the methods we develop can also be applied to more general nonlinear eigenvalue problems that are not necessarily of variational type.

The paper is organized as follows. In Section 2 we state our main results about (S_λ), and we discuss related results. Then in Section 3 we formulate the functional analytic setting which will be considered throughout the paper. Here we also state our main abstract results about solutions of nonlinear equations near a singular parameter limit. The abstract results as well as the results about (S_λ) will be proved in sections 4 – 6.

2 NLS near an infinite well potential

We begin with collecting our assumptions on the potentials V_λ. These are given in the form $V_\lambda = a_0 + \lambda a$, so the problem we consider is

$$(S_\lambda) \qquad \begin{cases} -\Delta u + (a_0(x) + \lambda a(x))u = f(x,u) & \text{for } x \in \mathbb{R}^N; \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases}$$

and the limit problem is

$$(S_\infty) \qquad -\Delta u + a_0(x)u = f(x,u), \quad u \in H^1_0(\Omega).$$

The distinguishing feature is that the potential $a \in L^\infty_{loc}(\mathbb{R}^N)$ satisfies $a \geq 0$ and $a^{-1}(0) = \overline{\Omega}$ with $\Omega \subset \mathbb{R}^N$ nonempty and open. Consequently, $V_\lambda(x) \to \infty$ as $\lambda \to \infty$ for $x \notin \overline{\Omega}$.

In order to describe the assumptions on a and a_0 we need some notation. For $x \in \mathbb{R}^N$ and $r > 0$ we set $B_r(x) := \{y \in \mathbb{R}^N : |y - x| < r\}$. We also set $K_r^c := \{x \in \mathbb{R}^n : |x|_\infty > r\}$. Let $\mu_N(-\Delta + V_\lambda, G)$ be the infimum of the spectrum of $-\Delta + V_\lambda$ on an open subset $G \subset \mathbb{R}^N$ with Neumann boundary conditions, i. e.

$$\mu_N(-\Delta + V_\lambda, G) = \inf_{\psi \in H^1(G)\setminus\{0\}} \frac{\int_G (|\nabla\psi|^2 + V_\lambda \psi^2)dx}{\|\psi\|^2_{L^2(G)}}.$$

Our basic hypotheses on the potential are:

(V_1) $a_0 \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ and $\operatorname{ess\,inf} a_0 > -\infty$.

(V_2) $a \in L^\infty_{\text{loc}}(\mathbb{R}^N)$, $a(x) \geq 0$ and $\Omega := \text{int }a^{-1}(0)$ is a non-empty open subset of $\mathbb{R}^N$ with Lipschitz boundary.

(V_3) There exists a sequence $R_j \to \infty$ such that

$$\lim_{\lambda\to\infty} \liminf_{j\to\infty} \mu_N(-\Delta + V_\lambda, K_{R_j}^c) = \infty.$$

The reader can find a discussion of condition (V_3), in particular various equivalent conditions, in [8]. Condition (V_3) holds, for instance, if a satisfies:

(V_4) There exist $M > 0$ and $r > 0$ such that

$$\mathrm{meas}(\{x \in B_r(y) : a(x) < M\}) \to 0 \quad \text{as } |y| \to \infty$$

where meas denotes the Lebesgue measure.

(V_3) implies that the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ is compact for $2 \le p < \infty$. Observe that (V_3) and (V_4) allow that Ω may be unbounded. For some results we require the stronger condition

(V_5) The form domain

$$E := \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} a_0^+ u^2 < \infty, \int_{\mathbb{R}^N} a u^2 < \infty \right\}$$

embeds compactly into $L^2(\mathbb{R}^N)$.

This holds, for instance, if $a_0(x) \to \infty$ or $a(x) \to \infty$ as $|x| \to \infty$, a condition usually satisfied for confining potentials. (V_5) also holds under the weaker condition

(V_6) For any $M > 0$ and any $r > 0$ there holds:

$$\mathrm{meas}\{x \in B_r(y) : a(x \le M\} \to 0 \quad \text{as } |y| \to \infty$$

A proof that (V_6) implies (V_5) can be found in [18]; see also [22].

Concerning the nonlinearity f we only require that

(f_1) f is a Carathéodory function, and there exist constants $C > 0$, $2 < p < 2^*$ such that

$$|f(x,t)| \le C(|t| + |t|^{p-1}) \quad \text{for } t \in \mathbb{R}, \text{ a. e. } x \in \mathbb{R}^N.$$

This includes the model nonlinearity $f(x,u) = W(x) \cdot |u|^{p-2}u$ with $2 < p < 2^* = 2N/(N-2)^+$ and $W \in L^\infty(\mathbb{R}^N)$, which appears in the Gross-Pitaevskii equation.

We define $E_\infty := E \cap H_0^1(\Omega)$ provided with the scalar product

$$\langle u, v\rangle := \int_\Omega (\nabla u \nabla v + (b + a_0)uv)\, dx$$

where $b := 1 - \operatorname{ess\,inf} a_0$. Setting $F(x,u) := \int_0^u f(x,t)\, dt$, it is well known that the functional $J_\infty : E_\infty \to \mathbb{R}$ defined by

$$\begin{aligned} J_\infty(u) &= \frac{1}{2}\int_\Omega (|\nabla u|^2 + a_0 u^2)\, dx - \int_\Omega F(x,u)\, dx \\ &= \frac{1}{2}\|u\|^2 - \int_\Omega \left(\frac{b}{2}u^2 + F(x,u)\right) dx \end{aligned}$$

is of class $\mathcal{C}^1$, and that critical points of J_∞ are solutions of (S_∞).

Recall that the critical groups of an isolated critical point u of a functional $J : E \to \mathbb{R}$ are defined as $C_k(J,u) := H_k(J^c, J^c \setminus \{u\})$ where $c := J(u)$. Here H_* is singular homology with coefficients in a commutative ring R with unit; typically $R = \mathbb{Z}$ or R is a field.

Now we can state our first result.

Theorem 2.1. *Assume $(V_1) - (V_3)$ and (f_1) hold. Let $u_\infty \in E_\infty$ be an isolated solution of (S_∞) with nontrivial critical groups $C_*(J_\infty, u_\infty)$. Then there exists $\Lambda \geq 1$ such that for each $\lambda \geq \Lambda$ there exists a solution $u_\lambda \in E$ of (S_λ) with $u_\lambda \to u_\infty$ in E as $\lambda \to \infty$.*

If $C_*(J_\infty, u_\infty) = 0$ then the solution u_∞ cannot be discovered using variational methods, and it can disappear under small perturbations. In our next result we strengthen the hypotheses by assuming that u_∞ has nontrivial index. Consider the functional

$$K_\infty : E_\infty \to \mathbb{R}, \quad K_\infty(u) = \int_\Omega \left(\frac{b}{2}u^2 + F(x,u)\right) dx,$$

and define its gradient $k_\infty = \nabla K_\infty : E_\infty \to E_\infty$ with respect to the above scalar product on E_∞. Then k_∞ is completely continuous because $p < 2^*$ in (f_1). The index of u_∞ is then defined by

$$\operatorname{ind}(k_\infty, u_\infty) := \deg(\operatorname{id} - k_\infty, B_\delta(u_\infty, E_\infty), 0).$$

Here $\deg$ denotes the Leray-Schauder degree, $\delta > 0$ is small so that u_∞ is the only solution of (S_∞) in the δ-ball $B_\delta(u_\infty, E_\infty)$ of u_∞ in E_∞.

Theorem 2.2. *Assume $(V_1), (V_2), (V_5)$ and (f_1) hold. Let $u_\infty \in E_\infty$ be an isolated solution of (S_∞) with nontrivial index* $\operatorname{ind}(k_\infty, u_\infty)$*. Then there exists a connected set*

$$\mathcal{S} \subset \{(\lambda, u) \in \mathbb{R} \times E : \ u \text{ solves } (S_\lambda)\} \subset \mathbb{R} \times E$$

such that $\mathcal{S}$ covers a parameter interval $[\Lambda, \infty)$ for some $\Lambda \geq 1$. Morevover, $u_n \to u_\infty$ for any sequence $(\lambda_n, u_n) \in E$ with $\lambda_n \to \infty$.

The assumption $\operatorname{ind}(k_\infty, u_\infty) \neq 0$ in Theorem 2.2 is stronger than the assumption $C_*(J_\infty, u_\infty) \neq 0$ in Theorem 2.1 because of the Poincaré-Hopf formula:

$$\operatorname{ind}(k_\infty, u_\infty) = \sum_{i=0}^{\infty} (-1)^i \operatorname{rank} C_i(J_\infty, u_\infty). \tag{2.1}$$

Surprisingly, the strong assumption (V_5) can be replaced by (V_3) if f satisfies

(f_1') f is differentiable in t, f and f_t are Carathéodory functions and there exist constants $C > 0$, $2 < p < 2^* = 2N/(N-2)^+$ such that

$$|f_t(x,t)| \le C(|t| + |t|^{p-2}) \quad \text{for } t \in \mathbb{R},\ \text{a. e. } x \in \mathbb{R}^N;$$

With this condition the functional J_λ is of class C^2.

Theorem 2.3. *Assume $(V_1) - (V_3)$ and (f_1') hold. Let $u_\infty \in E_\infty$ be an isolated solution of (S_∞) with nontrivial index $\operatorname{ind}(k_\infty, u_\infty)$. Then the conclusion of Theorem 2.2 holds.*

For our last result about (S_λ) we consider the case of a nondegenerate solution u_∞.

Theorem 2.4. *Assume $(V_1) - (V_3)$ and (f_1') hold. Let $u_\infty \in E_\infty$ be a nondegenerate solution of (S_∞). Then there exists $\Lambda \ge 1$ and a $\mathcal{C}^1$-function*

$$[\Lambda, \infty) \to E, \quad u \mapsto u_\lambda,$$

such that u_λ is a solution of (S_λ), and $u_\lambda \to u_\infty$ as $\lambda \to \infty$.

Remark 2.5. *It follows from results of Pankov [24] that the solutions which we obtain in Theorem 2.4 decay exponentially.*

Problem (S_λ) has found much interest in recent years after being first considered in [10, 8]. Most papers deal with potentials being positive and bounded away from 0, i. e. $\inf a_0 > 0$, exceptions being [9, 14]. The equation (S_λ) with asymptotically linear nonlinearity has been studied in [20, 21, 28, 29], with critical growth nonlinearity in [3, 4], with Neumann boundary conditions in exterior domains in [11]. In [9, 15, 27] multiplicity results have been obtained provided the bottom $\overline{\Omega}$ of the potential well consists of several connected components. Extensions to quasilinear problems can be found in [2], to the Schrödinger-Poisson system in [17].

In almost all earlier papers on the topic the authors made assumptions on a, a_0, f such that variational methods (e. g. the mountain pass theorem or some linking theorem) can be applied to show that (S_λ) has a solution u_λ. Then it is proved that u_λ converges as $\lambda \to \infty$ towards a solution u_∞ of the limit problem (S_∞). However, the limit u_∞ has not been prescribed in these papers as we do here. A notable exception, and the only one we are aware of, where the limit has been prescribed is [27, Theorem 1.2]. There the authors considered the one-dimensional problem

$$-u'' + (1 + \lambda a(x))u = |u|^{p-1}u, \quad u \in H^1(\mathbb{R}), \tag{2.2}$$

with the limit problem

$$\begin{cases} -u'' + u = |u|^{p-1}u, & x \in \Omega = (a_1, b_1) \cup (a_2, b_2), \\ \quad u(a_i) = u(b_i) = 0. \end{cases} \tag{2.3}$$

The solutions of (2.3) can be listed as $v_{i,j}$, $i, j \in \mathbb{Z}$, where $v_{\pm i,\pm j}$ are the unique solutions having $|i|$ zeroes in (a_1, b_1) and $|j|$ zeroes in (a_2, b_2). The authors find solutions u_λ of (2.2) such that $u_\lambda \to v_{i,j}$ as $\lambda \to \infty$. The proof is based on ODE methods and cannot be extended to dimensions $N \geq 2$. Observe that in the one-dimensional case the solutions $v_{i,j}$ are automatically non-degenerate, hence our Theorem 2.4 applies. Thus we improve and generalize [27, Theorem 1.2] considerably.

In contrast to all earlier papers we do not require global linking type hypotheses. Our results may be considered as a local version of these earlier results. As a consequence, we can deal with solutions of (S_∞) which are obtained not using a global linking structure. An example for this are almost critical problems like

$$-\Delta u = |u|^{2^*-2-\varepsilon}u, \quad u \in H_0^1(\Omega),$$

where in the limit for $\varepsilon \to 0$ the problem can be reduced via a Lyapunov-Schmidt reduction method to finding critical points of a finite-dimensional limit function; see [5, 6, 7, 23, 26]. For instance, in [6] the solutions have been obtained by finding a local minimum and a local mountain pass of the reduced functional. Since the reduced finite-dimensional functional is analytic, its critical points are isolated and have nontrivial index, but we do not know whether they are nondegenerate.

3 Critical points near a singular limit

Let E be a real Hilbert space with scalar product $\langle \cdot, \cdot \rangle$, and let $A : E \to E$ be a bounded self-adjoint linear operator. We require that $A \geq 0$ and that $E_\infty := \ker A \neq \{0\}$. E_∞ may be infinite-dimensional as in our application. Finally, let $K : E \to \mathbb{R}$ be a C^1-function, and set $k := \nabla K : E \to E$.

We are interested in finding critical points of the functional

$$J_\lambda : E \to \mathbb{R}, \quad J_\lambda(u) := \frac{1}{2}\|u\|^2 + \frac{\lambda}{2}\langle Au, u \rangle - K(u)$$

for λ large. Observe that $J_\lambda(u)$ is independent of λ for $u \in E_\infty$. Moreover, for $u \in E\backslash E_\infty$ we have $J_\lambda(u) \to \infty$ as $\lambda \to \infty$. We set $K_\infty = K|E_\infty$, $k_\infty := \nabla K_\infty : E_\infty \to E_\infty$, and consider

$$J_\infty : E_\infty \to \mathbb{R}, \quad J_\infty(u) = \frac{1}{2}\|u\|^2 - K_\infty(u),$$

as singular limit functional. Clearly, J_∞ is just the restriction of J_λ to E_∞.

Observe that $\langle Au, u\rangle > 0$ for $u \in E \setminus E_\infty$ and that

$$(3.1) \qquad u_n \rightharpoonup u \text{ in } E,\ \langle Au_n, u_n\rangle \to 0 \quad \Longrightarrow \quad Au_n \to 0,\ u \in E_\infty.$$

This can be seen by looking at the symmetric positive-semidefinite bilinear form $(u,v)_A := \langle Au, v\rangle$. The Schwarz inequality yields

$$\|Au\|^2 = (u, Au)_A \le \sqrt{(u,u)_A}\sqrt{(Au,Au)_A}.$$

Therefore $\langle Au, u\rangle = (u,u)_A = 0$ implies $Au = 0$. Similarly, $u_n \rightharpoonup u$, $\langle Au_n, u_n\rangle \to 0$ implies $Au_n \to 0 = Au$.

For $\lambda \ge 0$ and $u, v \in E$ we define

$$\langle u, v\rangle_\lambda := \langle u, v\rangle + \lambda\langle Au, v\rangle.$$

As a consequence of our hypotheses on A this is a scalar product on E, and it defines a norm $\|\cdot\|_\lambda$ on E which is equivalent to the given norm corresponding to $\lambda = 0$. Observe that the orthogonal complement of E_∞ with respect to $\langle\cdot,\cdot\rangle_\lambda$,

$$E_\infty^\perp = \{u \in E : \langle u, v\rangle_\lambda = 0 \text{ for all } v \in E_\infty\}$$

is independent of λ, hence the orthogonal projections $P : E \to E_\infty$ and $Q = \mathrm{id} - P : E \to E_\infty^\perp$ are independent of λ. We write $B_{r,\lambda}(0, E_\infty^\perp) := \{v \in E_\infty^\perp : \|v\|_\lambda \le r\}$ for the ball of radius $r > 0$ around $0 \in E_\infty^\perp$ with respect to the norm $\|\cdot\|_\lambda$. For $\delta > 0$ and $\lambda > 0$ and $u \in E_\infty$ we define

$$B_{\delta,\lambda}(u) := B_\delta(u, E_\infty) \times B_{\delta,\lambda}(0, E_\infty^\perp) \subset E.$$

Given a bounded linear map $L : E \to E$ we write

$$\|L\|_\lambda := \sup\{\|Lu\|_\lambda : u \in E,\ \|u\|_\lambda \le 1\}$$

for the operator norm of L with respect to $\|\cdot\|_\lambda$ on E.

For $\lambda > 0$ we define the nonlinear operators $k_\lambda = \nabla_\lambda K : E \to E$ and $\nabla_\lambda J_\lambda : E \to E$ by the equations

$$\langle k_\lambda(u), v\rangle_\lambda = \langle \nabla_\lambda K(u), v\rangle_\lambda = DK(u)[v] = \langle k(u), v\rangle,$$

and $\nabla_\lambda J_\lambda = \mathrm{id} - k_\lambda$. Observe that

$$\|k_\lambda(u)\|_\lambda = \sup_{\|v\|_\lambda \le 1} \langle k_\lambda(u), v\rangle_\lambda = \sup_{\|v\|_\lambda \le 1} \langle k(u), v\rangle \le \sup_{\|v\| \le 1} \langle k(u), v\rangle = \|k(u)\|\,. \tag{3.2}$$

If K is of class C^2 near u then the derivatives of $k = \nabla K$ and of $k_\lambda = \nabla_\lambda K$ satisfy

$$\langle Dk_\lambda(u)[v], w\rangle_\lambda = \langle Dk(u)[v], w\rangle \quad \text{for } u, v, w \in E. \tag{3.3}$$

We also deduce for $\lambda \ge 0$ and $u \in E$ that

$$\begin{aligned}\|Dk_\lambda(u)\|_\lambda^2 &= \sup_{\|v\|_\lambda \le 1} \langle Dk_\lambda(u)[v], Dk_\lambda(u)[v]\rangle_\lambda = \sup_{\|v\|_\lambda \le 1} \langle Dk(u)[v], Dk_\lambda(u)[v]\rangle \\ &\le \sup_{\|v\|_\lambda \le 1} \|Dk(u)\| \cdot \|v\| \cdot \|Dk_\lambda(u)\|_\lambda \cdot \|v\|_\lambda \le \|Dk(u)\| \cdot \|Dk_\lambda(u)\|_\lambda\end{aligned}$$

hence,

$$\|Dk_\lambda(u)\|_\lambda \le \|Dk(u)\|. \tag{3.4}$$

Similarly we obtain for $\lambda \ge 0$ and $u, v \in E$ that

$$\|Dk_\lambda(u) - Dk_\lambda(v)\|_\lambda \le \|Dk(u) - Dk(v)\|. \tag{3.5}$$

Now we collect some hypotheses on J_λ which we will impose in the various results.

- (J_1) J_∞ has an isolated critical point $u_\infty \in E_\infty$, and the critical groups of u_∞ as a critical point of J_∞ are nontrivial: $\mathcal{C}_*(J_\infty, u_\infty) \ne 0$.

We fix $\delta_0 > 0$ such that u_∞ is the only critical point of J_∞ in $B_{\delta_0}(u_\infty)$.

- (J_2) There exists $\lambda_0 > 0$ such that k is weakly sequentially continuous in $B_{\delta_0,\lambda_0}(u_\infty)$, i. e. if $u_n \in B_{\delta_0,\lambda_0}(u_\infty)$ and $u_n \rightharpoonup u$ then $k(u_n) \rightharpoonup k(u)$.
- (J_3) $B_{\delta_0,\lambda_n}(u_\infty) \ni u_n \rightharpoonup u,\ \lambda_n \to \infty \qquad \Longrightarrow \qquad k(u_n) \to k(u).$
- (J_4) There exists $\lambda_0 > 0$ such that k is completely continuous in $B_{\delta_0,\lambda_0}(u_\infty)$.

Condition (J_2) is rather harmless, also (J_3) holds under rather general assumptions on a, a_0, and f. Both are much weaker than requiring that k is completely continuous near u_∞ as in (J_4). (J_3) does imply that k_∞ is completely continuous in $B_{\delta_0}(u_\infty, E_\infty)$. Therefore J_∞ satisfies the Palais-Smale condition in $B_{\delta_0}(u_\infty, E_\infty)$, i. e. any Palais-Smale sequence $u_n \in B_{\delta_0}(u_\infty, E_\infty)$ for J_∞ has a convergent subsequence.

Theorem 3.1. *Suppose that* (J_1)*,* (J_2)*, and* (J_3) *hold. Then there exists* $\Lambda \geq 0$ *such that* J_λ *has a critical point* u_λ *for* $\lambda \in [\Lambda, \infty)$ *and such that* $u_\lambda \to u_\infty$ *as* $\lambda \to \infty$.

Our next result is based on degree theory. Recall that the index of u_∞ as fixed point of k_∞ is defined as:

$$\text{ind}(k_\infty, u_\infty) = \deg(\text{id} - k_\infty, B_\delta(u_\infty, E_\infty), u_\infty)$$

where $\deg$ denotes the Leray-Schauder degree, $0 < \delta \leq \delta_0$. This index is defined, for instance, if k_∞ is completely continuous in $B_{\delta_0}(u_\infty, E_\infty)$, hence if (J_2) or (J_3) holds, in particular if (J_4) holds.

Theorem 3.2. *Suppose* (J_1) *and* (J_4) *hold. Suppose moreover that the local fixed point index of* u_∞ *as a fixed point of* k_∞ *is nontrivial:*

$$\text{ind}(k_\infty, u_\infty) = \deg(\text{id} - k_\infty, U_\varepsilon(u_\infty, E_\infty), u_\infty) \neq 0.$$

Then there exists a connected set $\mathcal{S} \subset [\Lambda, \infty) \times E$ *covering the parameter interval* $[\Lambda, \infty)$ *for some* $\Lambda \geq 1$*, such that* $\nabla_\lambda J_\lambda(u) = 0$ *for every* $(\lambda, u) \in \mathcal{S}$*. Moreover, given a sequence* $(\lambda_n, u_n) \in \mathcal{S}$ *with* $\lambda_n \to \infty$ *there holds* $u_n \to u_\infty$.

Remark 3.3. a) *Recall that under the conditions of Theorem* 3.1 *the local index* $\text{ind}(K_\infty, u_\infty)$ *may be trivial. On the other hand, if the local index* $\text{ind}(K_\infty, u_\infty)$ *is nontrivial, then the critical groups* $\mathcal{C}_*(f, u_\infty)$ *are nontrivial. This follows from the Poincaré-Hopf formula* (2.1).
b) *Assumption* (J_4) *can be replaced by any assumption assuring that there is a degree theory for the maps* $\text{id} - k_\lambda$*. In the case of* (J_4) *one has the Leray-Schauder degree. If* k_λ *is, for instance, A-proper in the sense of [25], the generalized degree of Petryshin can be applied; see Theorem* 3.4 *below and its proof in Section* 5.

Surprisingly, the compactness condition can be considerable relaxed if K is C^2 near u_∞. We need the following condition on the differential $Dk(u_\infty)$.

(J_5) If $u_n \rightharpoonup u$ and $\|u_n\|_{\lambda_n}$ is bounded for some sequence $\lambda_n \to \infty$, then $Dk(u_\infty)[u_n] \to Dk(u_\infty)[u]$.

Theorem 3.4. *Suppose* K *is* C^2 *near* u_∞*,* (J_1)*,* (J_3)*, and* (J_5) *are satisfied. Then the conclusion of Theorem* 3.2 *holds true.*

Under the assumptions of Theorem 3.4 k_λ need not be compact, so we cannot work with the Leray-Schauder degree. Instead we will be able to use the degree for strict β_λ-set contractions where β_λ is the ball measure of noncompactness in E with respect to $\|\cdot\|_\lambda$; see [13] for definitions and the construction of the degree.

Finally we state a result in the nondegenerate setting.

Theorem 3.5. *Suppose that K is C^2 near u_∞, that u_∞ is a nondegenerate critical point of J_∞, and that (J_5) is satisfied. Then there exists $\Lambda \geq 0$ and a C^1-map $[\Lambda, \infty) \to E$, $\lambda \mapsto u_\lambda$, such that u_λ is the unique critical point of J_λ near u_∞ for $\lambda \in [\Lambda, \infty)$. Moreover, $u_\lambda \to u_\infty$ as $\lambda \to \infty$.*

4 Nontrivial critical groups

We first prove Theorem 3.1. Consider the isolated critical point $u_\infty \in E_\infty$ of J_∞ with nontrivial critical groups. Let (W, W_-) be a Gromoll-Meyer pair for u_∞ in $B_{\delta_0}(u_\infty, E_\infty)$. This means that:

(GM_1) $W \subset \operatorname{int} B_{\delta_0}(u_\infty, E_\infty)$ is a closed neighborhood of u_∞ in E_∞ containing no other critical point of J_∞.

(GM_2) There exist C^1-functions $g_i : U \cap E_\infty \to \mathbb{R}$, $i = 1, \ldots, l$, having 0 as regular value, such that $W = \bigcap_{i=1}^l g_i^0$ and $\partial W = W \cap \bigcup_{i=1}^l g_i^{-1}(0)$.

(GM_3) ∇J_∞ is transversal to each $g_i^{-1}(0)$; more precisely, for some $\alpha > 0$, some $j \in \{0, \ldots, n\}$:

$$\langle \nabla J_\infty(u), \nabla g_i(u) \rangle \leq -3\alpha < 0 \quad \text{for } u \in \partial W \cap g_i^{-1}(0),\ i = 1, \ldots, j,$$

and

$$\langle \nabla J_\infty(u), \nabla g_i(u) \rangle \geq 3\alpha > 0 \quad \text{for } u \in \partial W \cap g_i^{-1}(0),\ i = j+1, \ldots, l.$$

(GM_4) The exit set

$$\begin{aligned} W_- &= \left\{ u \in \partial W : \langle \nabla J_\infty(u), \nabla g_i(u) \rangle < 0 \text{ if } u \in g_i^{-1}(0), i = 1, ..., l \right\} \\ &= \bigcup_{i=1}^{j} g_i^{-1}(0) \end{aligned}$$

consists of those $u \in \partial W$ where $-\nabla J_\infty(u)$ points outside of W.

A construction of a Gromoll-Meyer pair can be found in [12, p. 49], where $l = 3$, $j = 1$. Using a pseudo-gradient vector field for J_∞ it is standard to show that

$$C_*(J_\infty, u_\infty) = H_*(J^c, J^c \setminus \{u_\infty\}) \cong H_*(W, W_-). \tag{4.1}$$

This uses that J_∞ satisfies the Palais-Smale condition in $B_{\delta_0}(u_\infty, E_\infty)$.

For $\delta > 0$ we set

$$W_{\delta,\lambda} := W \times B_{\delta,\lambda}(0, E_\infty^\perp).$$

Lemma 4.1. *If* (J_3) *holds then for every* $\varepsilon > 0$ *there exists* $\Lambda > 0$ *such that*

$$\sup_{u \in W_{\delta_0,\lambda}} \|k_\lambda(u) - k_\infty(Pu)\|_\lambda \le \varepsilon \qquad \text{for all } \lambda \ge \Lambda.$$

Proof. Arguing by contradiction, suppose there exist $\varepsilon > 0$, $\lambda_n \to \infty$, $u_n \in W_{\delta_0,\lambda_n}$ with

$$\|k_{\lambda_n}(u_n) - k_\infty(Pu_n)\|_{\lambda_n} \ge \varepsilon.$$

Then the sequence $(u_n)_n$ is bounded, and $\langle Au_n, u_n\rangle \to 0$, hence (3.1) applies and yields $u_n,\ Pu_n \rightharpoonup u \in E_\infty$ along a subsequence. Now (J_3) implies $k(u_n) \to k(u)$ and $k(Pu_n) \to k(u)$. Setting $v_n := k_{\lambda_n}(u_n) - k_\infty(Pu_n)$ we see that $\|v_n\|_{\lambda_n}$ is bounded uniformly in n as a consequence of (3.2). Applying (3.1) again shows that $v_n, Pv_n \rightharpoonup v \in E_\infty$ along a subsequence. This in turn implies

$$\begin{aligned}\varepsilon^2 &\le \|v_n\|_{\lambda_n}^2 = \langle k_{\lambda_n}(u_n), v_n\rangle_{\lambda_n} - \langle k_\infty(Pu_n), v_n\rangle \\ &= \langle k(u_n), v_n\rangle - \langle k(Pu_n), Pv_n\rangle \to \langle k(u), v\rangle - \langle k(u), v\rangle = 0\end{aligned}$$

which is absurd. □

Lemma 4.2. *For all* $0 < \delta \le \delta_0$ *there exists* $\Lambda_\delta > 0$ *such that for* $\lambda \ge \Lambda_\delta$ *and* $v \in B_{\delta,\lambda}(0, E_\infty^\perp)$*, there holds:*

$$\langle P\nabla_\lambda J_\lambda(u+v), \nabla g_i(u)\rangle \le -2\alpha \quad \text{for } u \in \partial W \cap g_i^{-1}(0),\ i = 1, \dots, j,$$

and

$$\langle P\nabla_\lambda J_\lambda(u+v), \nabla g_i(u)\rangle \ge 2\alpha \quad \text{for } u \in \partial W \cap g_i^{-1}(0),\ i = j+1, \dots, l,$$

and

$$\langle Q\nabla_\lambda J_\lambda(u+v), v\rangle_\lambda \ge \delta^2/2 \quad \text{for } u \in W,\ v \in B_{\delta,\lambda}(0, E_\infty^\perp).$$

Proof. We may assume that

$$m := \max_{i=1,\dots,l} \sup_{u \in g_i^{-1}(0)\cap W} \|\nabla g_i(u)\| < \infty.$$

According to Lemma 4.1, for $0 < \delta \le \delta_0$ there exists $\Lambda_\delta > 0$ such that

$$\|k_\lambda(u) - k_\infty(Pu)\|_\lambda \le \min\{\alpha/m, \delta/2\} \quad \text{if } \lambda \ge \Lambda_\delta,\ u \in W_{\delta_0,\lambda}.$$

Consequently, we obtain for $\lambda \ge \Lambda_\delta$, $i = 1, \dots, j$, $u \in W \cap g_i^{-1}(0)$ and $v \in B_{\delta_0,\lambda}(0, E_\infty^\perp)$ that

$$\begin{aligned}&\langle P\nabla_\lambda J_\lambda(u+v), \nabla g_i(u)\rangle \\ &\qquad \le \langle \nabla J_\infty(u), \nabla g_i(u)\rangle + \|P\nabla_\lambda J_\lambda(u+v) - \nabla J_\infty(u)\| \cdot \|\nabla g_i(u)\| \\ &\qquad = \langle \nabla J_\infty(u), \nabla g_i(u)\rangle + \|Pk_\lambda(u+v) - k_\infty(u)\| \cdot \|\nabla g_i(u)\| \\ &\qquad \le -3\alpha + \frac{\alpha}{m} m = -2\alpha.\end{aligned}$$

Similarly we obtain for $\lambda \geq \Lambda_\delta$, $i = j+1, \dots, l$, $u \in W \cap g_i^{-1}(0)$ and $v \in B_{\delta_0,\lambda}(0, E_\infty^\perp)$ that

$$\langle P\nabla_\lambda J_\lambda(u+v), \nabla g_i(u)\rangle \geq 2\alpha.$$

Finally, for $\lambda \geq \Lambda_\delta$, $u \in W$ and $v \in B_{\delta_0,\lambda}(0, E_\infty^\perp)$ there holds:

$$\begin{aligned}
\langle Q\nabla_\lambda J_\lambda(u+v), v\rangle_\lambda &= \|v\|_\lambda^2 - \langle k_\lambda(u+v) - k_\infty(u), v\rangle_\lambda \\
&\geq \|v\|_\lambda^2 - \|\langle k_\lambda(u+v) - k_\infty(u)\|_\lambda \|v\|_\lambda \\
&\geq \delta^2 - \frac{\delta}{2}\delta = \frac{\delta^2}{2}.
\end{aligned}$$

□

Lemma 4.2 implies that for $\delta > 0$ and $\lambda \geq \Lambda_\delta$, the set

$$\big(W_{\delta,\lambda}, W_- \times B_{\delta,\lambda}(0, E_\infty^\perp)\big) = \big(W \times B_{\delta,\lambda}(0, E_\infty^\perp), W_- \times B_{\delta,\lambda}(0, E_\infty^\perp)\big)$$

is a regular index pair for pseudo-gradient flows of J_λ in the sense of Conley index theory.

Lemma 4.3. *J_λ has a critical point $u_\lambda \in B_{\delta_0}(u_\infty, E_\infty) \times B_{\delta,\lambda}(0, E_\infty^\perp)$ if $0 < \delta \leq \delta_0$ and $\lambda \geq \Lambda_\delta$.*

Proof. If J_λ does not have a critical point in $W_{\delta,\lambda}$ then there exists a pseudo-gradient vector field V for J_λ in $W_{\delta,\lambda}$ such that the inequalities in Lemma 4.2 hold with V instead of $\nabla_\lambda J_\lambda$, α instead of 2α, and $\delta^2/4$ instead of $\delta^2/2$. Moreover,

$$\inf_{u \in W_{\delta,\lambda}} \|\nabla_\lambda J_\lambda(u)\|_\lambda > 0, \tag{4.2}$$

because if $u_n \in W_{\delta,\lambda}$ satisfies $\nabla_\lambda J_\lambda(u_n) = u_n - k_\lambda(u_n) \to 0$, then $u_n \rightharpoonup u \in B_{\delta_0}(u_\infty, E_\infty) \times B_{\delta,\lambda}(0, E_\infty^\perp)$ along a subsequence, hence $k_\lambda(u_n) \rightharpoonup k_\lambda(u)$ as a consequence of (J_2). This implies that $u \in B_{\delta_0}(u_\infty, E_\infty) \times B_{\delta,\lambda}(0, E_\infty^\perp) \subset B_{\delta_0}(u_\infty)$ is a critical point of J_λ. Observe that we do not prove strong convergence here, hence we do not prove, and neither need, the Palais-Smale condition in $W_{\delta,\lambda}$.

Now (4.2) implies that the flow associated to $-V$ provides a deformation of $W_{\delta,\lambda}$ to $W_- \times B_{\delta,\lambda}(0, E_\infty^\perp)$. This in turn implies

$$H_*\left(W_{\delta,\lambda}, W_- \times B_{\delta,\lambda}(0, E_\infty^\perp)\right) \cong 0$$

in contradiction with

$$H_*\left(W_{\delta,\lambda}, W_- \times B_{\delta,\lambda}(0, E_\infty^\perp)\right) \cong H_*(W, W_-) \cong C_*(J_\infty, u_\infty) \neq 0.$$

□

Proof of 3.1. The existence of a critical point $u_\lambda \in B_{\delta_0}(u_\infty, E_\infty) \times B_{\delta,\lambda}(0, E_\infty^\perp)$ of J_λ for $\lambda \geq \Lambda$ has been stated in Lemma 4.3. Clearly, $Qu_\lambda \to 0$ and $\nabla J_\infty(Pu_\lambda) \to 0$ as $\lambda \to \infty$. It follows that $u_\lambda \to u_\infty \in E_\infty$ because u_∞ is the only critical point of J_∞ in $B_{\delta_0}(u_\infty, E_\infty)$. □

Proof of 2.1. In order to apply Theorem 3.1 we set

$$E := \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} a_0 u^2 < \infty,\ \int_{\mathbb{R}^N} a u^2 < \infty \right\}$$

provided with the scalar product

$$\langle u, v \rangle := \int_{\mathbb{R}^N} (\nabla u, \nabla v + (b + a_0 + a) uv) dx$$

Here $b = 1 - \operatorname{ess\,inf} a_0$ is defined as in Section 2. The operator $A : E \to E$ is defined by the equation

$$\langle Au, v \rangle := \int_{\mathbb{R}^N} auv\, dx,$$

and the functional $K : E \to \mathbb{R}$ by

$$K(u) = -\int_{\mathbb{R}^N} \left(\frac{b}{2} u^2 + F(x, u) \right) dx.$$

A is a self-adjoint, positive semidefinite, and bounded linear operator. The kernel E_∞ of A consists of all $u \in E$ such that $u = 0$ a. e. in $\mathbb{R}^N \setminus \Omega$, hence $E_\infty = E \cap H_0^1(\Omega)$. This uses that the boundary of Ω is Lipschitz.

Solutions of (S_λ) are obtained as critical points of the $\mathcal{C}^1$-functional

$$\begin{aligned} J_\lambda(u) &= \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla u|^2 + (a_0 + \lambda a) u^2 \right) dx - \int_{\mathbb{R}^N} F(x, u)\, dx \\ &= \frac{1}{2} \left(\|u\|^2 + (\lambda - 1) \langle Au, u \rangle \right) - K(u). \end{aligned}$$

Observe that λ has to be replaced by $\lambda - 1$ because $\|\cdot\|_\lambda$ contains the summand $\langle Au, u \rangle$. Since $a = 0$ on Ω, we see that J_∞ is simply the restriction of J_λ to E_∞.

It remains to prove the conditions (J_2) and (J_3). In fact, (J_2) is an easy consequence of (f_1) because E imbeds into $L^p(\mathbb{R}^N)$ for $2 \leq p \leq 2^*$. In order to see (J_3), consider sequences $\lambda_n \to \infty$ and $B_{\delta_0, \lambda_n} \ni u_n \rightharpoonup u$. Then $u_n \to u$ strongly in $L^p(\mathbb{R}^N)$ for $2 < p < 2^*$ by [8, Lemma 4.2.]. And $u_n \to u$ strongly in $L^2(\mathbb{R}^N)$ follows from (V_3). This implies $k(u_n) \to k(u)$ in E because of the subcritical growth of f required in (f_1).

Theorem 2.1 is now an immediate consequence of Theorem 3.1. □

5 Nontrivial index

Proof of Theorem 3.2. From $\|u - k_\infty(Pu)\| \geq \|Pu - k_\infty(Pu)\|$ and using Lemma 4.1 we immediately deduce that there exists $\Lambda > 0$ such that for $\lambda \geq \Lambda$ and $0 < \delta \leq \delta_0$ small there holds

$$
\begin{aligned}
0 \neq \mathrm{ind}(k_\infty, u_\infty) &= \deg(\mathrm{id}_{E_\infty} - k_\infty, B_\delta(u_\infty, E_\infty), 0) \\
&= \deg(\mathrm{id}_E - k_\infty \circ P, B_{\delta,\lambda}(u_\infty), 0) \qquad (5.1)\\
&= \deg(\mathrm{id}_E - k_\lambda, B_{\delta,\lambda}(u_\infty), 0)\,.
\end{aligned}
$$

Since k is completely continuous in $B_{\delta_0,\lambda_0}(u_\infty)$ so is k_λ, hence the above degree is defined. Using (5.1), a standard continuation argument (see [1], for instance) shows that there exists a connected set $\mathcal{S} \subset [\Lambda, \infty) \times B_{\delta,\lambda}(u_\infty) \subset [\Lambda, \infty) \times E$ covering the parameter interval $[\Lambda, \infty)$, such that $\nabla_\lambda J_\lambda(u) = u - k_\lambda(u) = 0$ for every $(\lambda, u) \in \mathcal{S}$. Given a sequence $(\lambda_n, u_n) \in \mathcal{S}$ with $\lambda_n \to \infty$, using (5.1) and Lemma 4.1 once more, we deduce that $\|u_n - u_\infty\|_{\lambda_n} \to 0$. $\square$

Proof of Theorem 2.2. This follows from Theorem 3.2 as Theorem 2.1 follows from Theorem 3.1. We only need to observe that k is completely continuous as a consequence of (V_5) and (f_1), in particular (J_4) is satisfied. $\square$

For the proof of Theorem 3.4 we need the following lemma.

Lemma 5.1. *Suppose* (J_5) *is satisfied. Then*

$$\|Dk_\lambda(u_\infty) - Dk_\infty(u_\infty) \circ P\|_\lambda \to 0 \quad \text{as } \lambda \to \infty.$$

Proof. Arguing by contradiction, suppose that there exist sequences $\lambda_n \to \infty$, $u_n \in E$ with $\|u_n\|_{\lambda_n} = 1$, and

$$\|Dk_{\lambda_n}(u_\infty)[u_n] - Dk_\infty(u_\infty)[Pu_n]\|_\lambda \geq \varepsilon > 0.$$

Then $u_n \rightharpoonup u$ in E along a subsequence, and $u \in E_\infty$ by (3.1), hence also $Pu_n \rightharpoonup u$. Setting

$$v_n := Dk_{\lambda_n}(u_\infty)[u_n] - Dk_\infty(u_\infty)[Pu_n]$$

and using (3.4) we see that

$$\|v_n\|_{\lambda_n} \leq \|Dk_{\lambda_n}(u_\infty)[u_n]\|_{\lambda_n} + \|Dk(u_\infty)[Pu_n]\| \leq \|Dk(u_\infty)\| + \|Dk(u_\infty)\|$$

is bounded uniformly in n. We deduce, again by (3.1), that $v_n \rightharpoonup v$ in E along a subsequence, and that $v \in E_\infty$, hence also $Pv_n \rightharpoonup v$. Using condition (J_5) we obtain a contradiction:

$$
\begin{aligned}
\varepsilon^2 \leq \|v_n\|_{\lambda_n}^2 &= \langle Dk_{\lambda_n}(u_\infty)[u_n], v_n\rangle_{\lambda_n} - \langle Dk_\infty(u_\infty)[Pu_n], Pv_n\rangle \\
&= \langle Dk(u_\infty)[u_n], v_n\rangle - \langle Dk(u_\infty)[Pu_n], Pv_n\rangle \\
&\to \langle Dk(u_\infty)[u], v\rangle - \langle Dk(u_\infty)[u], v\rangle = 0.
\end{aligned}
$$

$\square$

Proof of Theorem 3.4. Let β_λ be the ball measure of non-compactness in E with respect to $\|\cdot\|_\lambda$, i. e. for a subset $A \subset E$

$$\beta_\lambda(A) = \inf\{r > 0 : A \text{ can be covered by finitely many } \|\cdot\|_\lambda\text{-balls of radius } r\}.$$

We claim that k_λ is a strict β_λ-set contraction in a neighborhood of u_∞ if λ is large. We refer to [13] for properties of this class of maps and the construction of a degree theory. It is sufficient to show that

$$\beta_\lambda(k_\lambda(A)) \le \frac{1}{2}\beta_\lambda(A) \quad \text{for } A \subset B_{\delta,\lambda}(u_\infty) \text{ if } \lambda \text{ is large and } \delta \text{ is small.} \tag{5.2}$$

For (5.2) it suffices to prove that $k_\lambda - k_\infty \circ P$ is $\|\cdot\|_\lambda$-Lipschitz continuous with Lipschitz constant $\frac{1}{2}$ because $k_\infty \circ P$ is completely continuous as a consequence of (J_3), and because the sum of a completely continuous map and a Lipschitz map with Lipschitz constant $\frac{1}{2}$ satisfies (5.2). Now the Lipschitz continuity of $k_\lambda - k_\infty \circ P$ follows easily from:

$$\begin{aligned}
&\|k_\lambda(u) - k_\infty(Pu) - (k_\lambda(v) - k_\infty(Pv))\|_\lambda \\
&\quad \le \|k_\lambda(u) - k_\lambda(v) - Dk_\lambda(u_\infty)[u-v])\|_\lambda \\
&\qquad + \|Dk_\lambda(u_\infty)[u-v] - Dk_\infty(u_\infty)[Pu - Pv]\|_\lambda \\
&\qquad + \|k_\infty(Pu) - k_\infty(Pv) - Dk_\infty(u_\infty)[Pu - Pv]\|_\lambda \\
&\quad \le \sup_{w \in B_{\delta,\lambda}(u_\infty)} \|Dk_\lambda(w)[u-v] - Dk_\lambda(u_\infty)[u-v]\|_\lambda \\
&\qquad + \|Dk_\lambda(u_\infty) - Dk_\infty(u_\infty) \circ P\|_\lambda \|u - v\|_\lambda \\
&\qquad + \sup_{w \in B_\delta(u_\infty, E_\infty)} \|Dk_\infty(w)[Pu - Pv] - Dk_\infty(u_\infty)[Pu - Pv]\|
\end{aligned}$$

Now $\sup_{w\in B_{\delta,\lambda}(u_\infty)} \|Dk_\lambda(w) - Dk_\lambda(u_\infty)\|_\lambda$ and $\sup_{w\in B_\delta(u_\infty,E_\infty)} \|Dk_\infty(w) - Dk_\infty(u_\infty)\|$ can be made arbitrarily small by making $\delta > 0$ small. And $\|Dk_\lambda(u_\infty) - Dk_\infty(u_\infty) \circ P\|_\lambda$ can be made arbitrarily small as $\lambda \to \infty$ as a consequence of Lemma 5.1.

Since k_λ is a strict β_λ-set contraction in a neighborhood of u_∞ for λ large, we may use the degree for strict β_λ-set contractions from [13] and argue as in the proof of Theorem 3.2 to conclude the proof of Theorem 3.4. $\square$

Proof of Theorem 2.3. Observe that (f_1') implies that

$$K : E \to \mathbb{R}, \quad K(u) = \int_{\mathbb{R}^N} \left(\frac{b}{2}u^2 + F(x,u)\right) dx,$$

is of class $\mathcal{C}^2$. It remains to prove (J_3) and (J_5). In fact, the proof of (J_3) proceeds as in the proof of Theorem 2.1. In order to see (J_5) consider a sequence $u_n \in E$ such that $u_n \rightharpoonup u$

and $\|u_n\|_{\lambda_n}$ is bounded for some sequence $\lambda_n \to \infty$, so that $u \in E_\infty = E \cap H_0^1(\Omega)$. Now assumption (V_3) yields a sequence $R_j \to \infty$ such that

$$\lim_{n\to\infty} \liminf_{j\to\infty} \frac{\|u_n - u\|_{\lambda_n}^2}{\int_{K_{R_j}^c} |u_n - u|^2} \to \infty,$$

which implies

$$\int_{K_{R_j}^c} |u_n - u|^2 \to 0 \quad \text{as } j, n \to \infty.$$

Since $u_n \to u$ in $L^2_{loc}(\mathbb{R}^N)$ we deduce that $u_n \to u$ in $L^2(\mathbb{R}^N)$. This implies that

$$\begin{aligned} |\langle Dk(u_\infty)[u_n] - Dk(u_\infty)[u], v\rangle| &= \left| \int_{\mathbb{R}^N} (b + f'(u_\infty))(u_n - u)v \, dx \right| \\ &\leq c\|u_n - u\|_{L^2(\Omega)}\|v\| \end{aligned}$$

hence $Dk(u_\infty)[u_n] \to Dk(u_\infty)[u]$ in E.

Now Theorem 2.3 follows from Theorem 3.4. □

6 The nondegenerate case

In this section we use the notation $f_\lambda = \mathrm{id}_E - k_\lambda : E \to E$. The proof of Theorem 3.5 is an immediate consequence of the following proposition.

Proposition 6.1. *For $\delta > 0$ small there exists $\Lambda_\delta \geq 1$ such that the map*

$$g_\lambda : B_{\delta,\lambda}(u_\infty) \to B_{\delta,\lambda}(u_\infty), \quad g_\lambda(u) := u - (\mathrm{id}_E - Dk_\infty(u_\infty) \circ P)^{-1} \circ f_\lambda(u),$$

is well defined and a contraction for $\lambda \geq \Lambda_\delta$.

Proof of Theorem 3.5. According to Proposition 6.1 there exists $\delta_0 > 0$ such that for $0 < \delta \leq \delta_0$ and $\lambda \geq \Lambda_\delta$, the Banach fixed point theorem yields a unique fixed point $u_\lambda \in B_{\delta,\lambda}(u_\infty)$ of g_λ, hence a zero of f_λ, i. e. a critical point of J_λ. The map

$$[\Lambda_{\delta_0}, \infty) \to E, \quad \lambda \mapsto u_\lambda,$$

is C^1 because f_λ is C^1 in λ. Finally, $\|u_\lambda - u_\infty\|_\lambda \to 0$ is also a consequence of Proposition 6.1. □

The proof of Proposition 6.1 is based on the following lemmata.

Lemma 6.2. *The bounded operator $L := \mathrm{id}_E - Dk_\infty(u_\infty) \circ P : E \to E$ is an isomorphism, and $\|L^{-1}\|_\lambda \le \alpha$ is bounded uniformly in λ.*

Proof. That L is an isomorphism follows easily from the assumption that u_∞ is a nondegenerate fixed point of k_∞, which means that $\mathrm{id}_{E_\infty} - Dk_\infty(u_\infty) : E_\infty \to E_\infty$ is an isomorphism. It is also clear that $\|L^{-1}\|_\lambda \le \max\{1, \|(\mathrm{id} - Dk_\infty(u_\infty))^{-1}\|\}$ because the norms on E_∞ do not depend on λ. □

Lemma 6.3. $\|f_\lambda(u_\infty)\|_\lambda \to 0$ *as* $\lambda \to \infty$.

Proof. Arguing by contradiction, suppose there exist $\varepsilon > 0$ and $\lambda_n \to \infty$ such that $v_n := f_{\lambda_n}(u_\infty) = u_\infty - k_{\lambda_n}(u_\infty) = k_\infty(u_\infty) - k_{\lambda_n}(u_\infty)$ satisfies $\|v_n\|_{\lambda_n} \ge \varepsilon$. Observe that $\|v_n\|_{\lambda_n}$ is bounded uniformly in n as a consequence of (3.2). Now (3.1) implies $v_n, Pv_n \rightharpoonup v \in E_\infty$ along a subsequence. This in turn implies:

$$\varepsilon^2 \le \|v_n\|_{\lambda_n}^2 = \langle k_\infty(u_\infty), v_n\rangle - \langle k_{\lambda_n}(u_\infty), v_n\rangle_{\lambda_n} = \langle k(u_\infty), Pv_n\rangle - \langle k(u_\infty), v_n\rangle \to 0$$

which is absurd. □

Proof of Proposition 6.1. By (3.5) there exists $\delta_1 > 0$ such that

$$\sup_{u \in B_{\delta_1,\lambda}(u_\infty)} \|Dk_\lambda(u) - Dk_\lambda(u_\infty)\|_\lambda \le \frac{1}{4\alpha} \quad \text{for all } \lambda \ge 0, \tag{6.1}$$

where $\alpha > 0$ is from Lemma 6.2. Now we fix $0 < \delta \le \delta_1$. Using Lemma 5.1 and Lemma 6.3 there exists Λ_δ such that

$$\|Dk_\lambda(u_\infty) - Dk_\infty(u_\infty) \circ P\|_\lambda \le \frac{1}{4\alpha} \quad \text{for } \lambda \ge \Lambda_\delta, \tag{6.2}$$

and

$$\|f_\lambda(u_\infty)\|_\lambda \le \frac{\delta}{2\alpha} \quad \text{for } \lambda \ge \Lambda_\delta. \tag{6.3}$$

Thus for $\lambda \ge \Lambda_\delta$ and $u, v \in B_{\delta,\lambda}(u_\infty)$ there holds

$$\begin{aligned} &\|k_\lambda(u) - k_\lambda(v) - Dk_\lambda(u_\infty)(u - v)\|_\lambda \\ &\quad \le \sup_{w \in B_{\delta,\lambda}(u_\infty)} \|Dk_\lambda(w) - Dk_\lambda(u_\infty)\|_\lambda \cdot \|u - v\|_\lambda \overset{(6.1)}{\le} \frac{1}{4\alpha}\|u - v\|_\lambda. \end{aligned} \tag{6.4}$$

Since $L = \mathrm{id}_E - Dk_\infty(u_\infty) \circ P$ we have $g_\lambda = L^{-1}(k_\lambda - Dk_\infty(u_\infty) \circ P)$. It follows that

$$\begin{aligned} \|g_\lambda(u) - g_\lambda(v)\|_\lambda &\le \alpha\|k_\lambda(u) - k_\lambda(v) - Dk_\infty(u_\infty)(P(u - v))\|_\lambda \\ &\le \alpha\|k_\lambda(u) - k_\lambda(v) - Dk_\lambda(u_\infty)(u - v)\|_\lambda \\ &\qquad + \alpha\|Dk_\lambda(u_\infty)(u - v) - Dk_\infty(u_\infty)(P(u - v))\|_\lambda \\ &\overset{(6.4)}{\le} \alpha\frac{1}{4\alpha}\|u - v\|_\lambda + \alpha\|Dk_\lambda(u_\infty) - Dk_\infty(u_\infty) \circ P\|_\lambda \cdot \|u - v\|_\lambda \\ &\overset{(6.2)}{\le} \frac{1}{4}\|u - v\|_\lambda + \frac{1}{4}\|u - v\|_\lambda = \frac{1}{2}\|u - v\|_\lambda. \end{aligned}$$

We also have

$$\|g_\lambda(u_\infty) - u_\infty\|_\lambda \le \|L^{-1}\|_\lambda \cdot \|f_\lambda(u_\infty)\|_\lambda \overset{(6.3)}{\le} \alpha\frac{\delta}{2\alpha} = \frac{\delta}{2}$$

hence, for $u \in B_{\delta,\lambda}(u_\infty)$ there holds:

$$\begin{aligned}\|g_\lambda(u) - u_\infty\|_\lambda &\le \|g_\lambda(u) - g_\lambda(u_\infty)\|_\lambda + \|g_\lambda(u_\infty) - u_\infty\|_\lambda \\ &\le \frac{1}{2}\|u - u_\infty\|_\lambda + \frac{\delta}{2} \le \frac{\delta}{2} + \frac{\delta}{2} = \delta.\end{aligned}$$

Therefore g_λ maps $B_{\delta,\lambda}(u_\infty)$ into itself. □

Proof of Theorem 2.4. As in the proof of Theorem 2.3 one sees that J_λ is of class $\mathcal{C}^2$ and that (J_5) holds. Therefore Theorem 2.4 follows from Theorem 3.5. □

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