

GROUND STATES OF A NONLINEAR CURL-CURL PROBLEM IN CYLINDRICALLY SYMMETRIC MEDIA

THOMAS BARTSCH, TOMÁŠ DOHNAL, MICHAEL PLUM, AND WOLFGANG REICHEL

ABSTRACT. We consider the nonlinear curl-curl problem $\nabla \times \nabla \times U + V(x)U = \Gamma(x)|U|^{p-1}U$ in \mathbb{R}^3 related to the nonlinear Maxwell equations for monochromatic fields. We search for solutions as minimizers (ground states) of the corresponding energy functional defined on subspaces (defocusing case) or natural constraints (focusing case) of $H(\text{curl}; \mathbb{R}^3)$. Under a cylindrical symmetry assumption on the functions V and Γ the variational problem can be posed in a symmetric subspace of $H(\text{curl}; \mathbb{R}^3)$. For a strongly defocusing case $\text{ess sup } \Gamma < 0$ with large negative values of Γ at infinity we obtain ground states by the direct minimization method. For the focusing case $\text{ess inf } \Gamma > 0$ the concentration compactness principle produces ground states under the assumption that zero lies outside the spectrum of the linear operator $\nabla \times \nabla \times + V(x)$. Examples of cylindrically symmetric functions V are provided for which this holds.

1. INTRODUCTION

For given functions $V \in L^\infty(\mathbb{R}^3)$, $\Gamma \in L^\infty_{loc}(\mathbb{R}^3) \setminus \{0\}$ we consider the nonlinear curl-curl problem

$$(1.1) \quad \nabla \times \nabla \times U + V(x)U = \Gamma(x)|U|^{p-1}U \quad \text{in } \mathbb{R}^3,$$

where $p > 1$, and look for real weak solutions

$$U \in X := H(\text{curl}; \mathbb{R}^3) \cap L^{p+1}_{|\Gamma|}(\mathbb{R}^3),$$

where $L^{p+1}_{|\Gamma|}(\mathbb{R}^3)$ denotes the space of L^{p+1} -functions with respect to the measure $|\Gamma| dx$ and $H(\text{curl}; \mathbb{R}^3)$ is the space of functions $U \in L^2(\mathbb{R}^3)$ for which $\text{curl } U$ is defined in the sense of distributions and $\text{curl } U \in L^2(\mathbb{R}^3)$; cf. Section 2 for more details on these spaces. The solutions $U \in X$ of (1.1) arise as critical points of the functional

$$J[U] = \int_{\mathbb{R}^3} \frac{1}{2} (|\nabla \times U|^2 + V(x)|U|^2) - \frac{\Gamma(x)}{p+1} |U|^{p+1} dx, \quad U \in X.$$

We find ground state solutions, i.e. minimizers of J within a certain subspace (defocusing case) or a natural constraint (focusing case) of X . Note that although we limit our attention to real solutions, the methods are in principle applicable in the complex case $U(x) \in \mathbb{C}^3$ as well.

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1.1. Variational aspects of the curl-curl problem. In the literature there are only few results on the nonlinear curl-curl problem. In [8] Benci, Fortunato opened the discussion about ground states for the problem

$$(1.2) \quad \nabla \times \nabla \times U = W'(|U|^2)U.$$

The problem was solved by Azzollini, Benci, D'Aprile, Fortunato in [4] using variational and symmetry-based methods. Using a different class of symmetries D'Aprile, Siciliano also obtained in [9] solutions of (1.2). Recently, Bartsch and Mederski [6] considered ground states as well as bound states of (1.2) on a bounded domain Ω with the boundary condition $\nu \times U = 0$ on $\partial\Omega$. In [16] Mederski considers (1.1) where, e.g., the right hand side is of the form $\Gamma(x)f(u)$ with $f(u) \sim |u|^{p-1}u$ if $|u| \gg 1$ and $f(u) \sim |u|^{q-1}u$ if $|u| \ll 1$ for $1 < p < 5 < q$ and where $\Gamma > 0$ is periodic and bounded, $V \leq 0$, $V \in L^{\frac{p+1}{p-1}}(\mathbb{R}^3) \cap L^{\frac{q+1}{q-1}}(\mathbb{R}^3)$.

Let us point out that on top of the common obstacle of J being unbounded from below in case $\Gamma > 0$, the variational formulation has the following additional difficulties:

- For $p > 1$ the space $H(\text{curl}; \mathbb{R}^3)$ does not embed into $L_{|\Gamma|}^{p+1}(\mathbb{R}^3)$ even when Γ is bounded. In the so-called focusing case $\Gamma > 0$ it is therefore hard to control the X -norm of any Palais-Smale sequence $(U_k)_{k \in \mathbb{N}}$, i.e., any sequence with $(J[U_k])_{k \in \mathbb{N}}$ bounded and $J'[U_k] \rightarrow 0$ as $k \rightarrow \infty$.
- Note that $\|\nabla U\|_2^2 = \|\nabla \times U\|_2^2 + \|\nabla \cdot U\|_2^2$. Hence restriction of J to the space $X_0 = \{U \in X : \nabla \cdot U = 0\}$ on one hand allows at least for $\Gamma \in L^\infty(\mathbb{R}^3)$ the embedding $X_0 \rightarrow L_{|\Gamma|}^{p+1}(\mathbb{R}^3)$ but on the other hand it generates an additional gradient term in the Euler-Lagrange equation.

Therefore, finding critical points of J directly in the whole space X is out of the scope of the current paper. Instead we will look for critical points on a suitable subspace by exploiting symmetries of (1.1). As proposed in [4], one such subspace is given by functions U of the form

$$(1.3) \quad U(x) = \frac{u(r, x_3)}{r} \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}, \quad r^2 = x_1^2 + x_2^2,$$

where $u : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ is a real valued, scalar function. Assuming cylindrical symmetry also for the potentials V and Γ , i.e., $V = V(r, x_3)$, $\Gamma = \Gamma(r, x_3)$, this ansatz leads to the equation

$$\left(-\partial_r^2 - \partial_{x_3}^2 - \frac{1}{r}\partial_r + \frac{1}{r^2} + V(r, x_3) \right) u = \Gamma(r, x_3)|u|^{p-1}u.$$

We also define the linear operator in the vector valued equation (1.1) as

$$(1.4) \quad \mathcal{L} := (\nabla \times \nabla \times) + V(x)$$

and study its spectrum $\sigma(\mathcal{L})$ when restricted to a suitable subspace of functions which exhibit symmetries like the functions given in (1.3). Under the above symmetry assumptions, we will study (1.1) in the following three scenarios:

- Fully radially symmetric case: $V = V(\rho)$, $\Gamma = \Gamma(\rho)$ with $\rho^2 = r^2 + x_3^2$.

- Strongly defocusing case: $\text{ess sup}_{\mathbb{R}^3} V < 0$ and $\Gamma(x) \leq -C(1+|x|)^\alpha$ for some constants $C > 0$ and $\alpha > \frac{3(p-1)}{2}$.
- Focusing case: $\text{ess inf}_{\mathbb{R}^3} \Gamma > 0$, $0 \notin \sigma(\mathcal{L})$. Examples of such potentials $V(r, x_3)$ are given. They are periodic in the x_3 -direction, satisfy $\lim_{r \rightarrow \infty} V(r, x_3) = V_\infty(x_3)$ and $\text{ess sup}_{\mathbb{R}} V_\infty(x_3) > 0 > \text{ess inf}_{\mathbb{R}} V$. Hence the potential has non-vanishing negative and, as $r \rightarrow \infty$, also non-vanishing positive part.

From a physical point of view the latter two scenarios may be criticized. Because Γ corresponds to the electric susceptibility of the considered medium, see Section 1.3, the strongly defocusing case implies unrealistically high defocusing nature of the material. And since $V(x)$ is proportional to $-n^2(x)$, where n is the refractive index, the condition of the non-vanishing positive part of V at infinity in the focusing case implies an imaginary refractive index. Hence it will be desirable to overcome these limitations in future work.

1.2. Main results. Now we state our main results. The first result is concerned with those solutions of (1.1) that are fully radially symmetric.

Theorem 1 (Fully radially symmetric case). *Let $p > 1$ and assume that $V, \Gamma \in L^\infty_{\text{loc}}(\mathbb{R}^3)$ and $0 \leq V\Gamma^{-1} \in L^{\frac{p}{p-1}}_{\text{loc}}(\mathbb{R}^3)$. Additionally suppose the full radial symmetry of V and Γ in \mathbb{R}^3 , i.e. $V(x) = \tilde{V}(|x|)$ and $\Gamma(x) = \tilde{\Gamma}(|x|)$ for almost all $x \in \mathbb{R}^3$ and $\tilde{V}, \tilde{\Gamma} \in L^\infty_{\text{loc}}([0, \infty))$. Under the full radial symmetry condition $U(x) = M^T U(Mx)$ for a.a. $x \in \mathbb{R}^3$ and all $M \in O(3)$, all distributional solutions $U \in L^p_{\text{loc}}(\mathbb{R}^3)$ of (1.1) satisfy $\nabla \times U = 0$ and have the form*

$$(1.5) \quad U(x) = s(|x|) \left(\frac{V(x)}{\Gamma(x)} \right)^{\frac{1}{p-1}} \frac{x}{|x|},$$

where $s : (0, \infty) \rightarrow \{-1, 1\}$ is an arbitrary measurable function. If additionally $(V\Gamma^{-1})^{\frac{2}{p-1}}, (V\Gamma^{-1})^{\frac{p+1}{p-1}}\Gamma \in L^1(\mathbb{R}^3)$ then $U \in X$ and hence it is a critical point of J .

Thus, the assumption of full radial symmetry does not lead to interesting solutions of (1.1). We therefore relax the fully radial symmetry and look for solutions having only cylindrical symmetry. For this purpose we use in Theorem 2 and Theorem 3 the space X_{G_1} which will be defined in Section 2 and may be thought of as the subspace of $X = H(\text{curl}; \mathbb{R}^3) \cap L^{p+1}_{|\Gamma|}(\mathbb{R}^3)$ consisting of vector fields of the form (1.3).

Theorem 2 (Strongly defocusing case). *Let $p > 1$ and assume that $V = V(r, x_3)$ and $\Gamma = \Gamma(r, x_3)$ have cylindrical symmetry and satisfy*

- (i) $\Gamma(x) \leq -C(1+|x|)^\alpha$ in \mathbb{R}^3 with $\alpha > \frac{3}{2}(p-1)$ and $C > 0$,
- (ii) $V \in L^\infty(\mathbb{R}^3)$ and $\text{ess sup}_{\mathbb{R}^3} V < 0$.

Then (1.1) has a ground state on X_{G_1} .

Theorem 3 (Focusing case). *Let $1 < p < 5$ and assume that $V = V(r, x_3)$ and $\Gamma = \Gamma(r, x_3)$ have cylindrical symmetry and satisfy*

- (i) $\text{ess inf}_{\mathbb{R}^3} \Gamma > 0$,
- (ii) $V, \Gamma \in L^\infty(\mathbb{R}^3)$ are 1-periodic in the x_3 -direction, i.e., $V(r, x_3) = V(r, x_3 + 1)$, $\Gamma(r, x_3) = \Gamma(r, x_3 + 1)$ for a.a. $r > 0, x_3 \in \mathbb{R}$,

(iii) $0 \notin \sigma(\mathcal{L})$.

Then (1.1) has a ground state on X_{G_1} , which is moreover a minimizer of J restricted to a natural constraint set (the so-called Nehari-Pankov manifold, cf. Section 5).

Examples of potentials $V(r, x_3)$ with $0 \notin \sigma(\mathcal{L})$ are given in Section 4. They have non-vanishing positive and negative parts.

1.3. Physical context of the problem. As we show below, equation (1.1) is a generalization of the Kerr nonlinear Maxwell's equations in three dimensions for monochromatic waves when higher harmonics are neglected. Solutions $U \in H^1(\mathbb{R}^3)$ then correspond to fully localized standing electromagnetic waves. The problem of localizing light in all three dimensions attracts strong interest in the physics community. This is partly due to the potential applications of such "light bullets" as information carriers in future optical logic and optical computing devices. Standing bullets, in particular, can be used in optical memory.

So far, to our knowledge, standing light bullets have not been observed in experiments: neither in homogeneous or periodic media nor in radial or cylindrical geometries, corresponding to the choice in this paper. Nevertheless, at least one theoretical prediction of such waves exists. In the Kerr nonlinear fiber Bragg grating (a cylindrical geometry with periodicity in the longitudinal direction) an asymptotic model for broad wavepackets and a small periodicity contrast supports localized waves, so called gap solitons [3]. The model is the system of one dimensional coupled mode equations and the gap solitons come in a family including standing solutions. Gap solitons have been experimentally observed with velocities as low as $0.23 \frac{c}{n}$, where c is the speed of light in vacuum and n the average refractive index of the fiber core [18], but not with velocity zero. On the other hand, moving localized pulses have been demonstrated in numerous other nonlinear geometries including standard optical fibers [19] and arrays of waveguides arranged in the plane [17]. In most physics articles theoretical predictions of light bullets are made based on the nonlinear Schrödinger equation (NLS). For instance in homogenous materials the NLS is known to have radially symmetric localized solutions, so called Townes solitons, in all dimensions [27]. In periodic media [22] and at interfaces of two periodic structures [10] standing ground state H^1 -solutions exist. The NLS is, however, only an asymptotic approximation of Maxwell's equations. Moreover, for inhomogeneous media in two and three dimensions the approximation has not been rigorously justified. This paper, in contrast, deals with the full three dimensional Maxwell problem.

The three dimensional Maxwell equations in the absence of charges and currents read

$$\begin{aligned} \nabla \times \mathcal{E} + \partial_t \mathcal{B} &= 0, & \nabla \cdot \mathcal{D} &= 0, \\ \nabla \times \mathcal{H} - \partial_t \mathcal{D} &= 0, & \nabla \cdot \mathcal{B} &= 0. \end{aligned}$$

Here $\mathcal{E}, \mathcal{H} : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ denote the electric and magnetic field, respectively, and $\mathcal{D}, \mathcal{B} : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ denote the displacement field and the magnetic induction, respectively. For the relation between the magnetic field \mathcal{H} and the magnetic induction \mathcal{B} we assume $\mathcal{B} = \mu_0 \mathcal{H}$ with μ_0 constant. By taking the curl of the first equation one finds

$$(1.6) \quad \nabla \times \nabla \times \mathcal{E} + \mu_0 \partial_t^2 \mathcal{D} = 0, \quad \nabla \cdot \mathcal{D} = 0.$$

For a Kerr-type nonlinear medium the material law between the electric field \mathcal{E} and the displacement field \mathcal{D} is given by

$$(1.7) \quad \mathcal{D} = \epsilon_0 (n^2(x)\mathcal{E} + \mathcal{P}_{\text{NL}}(x, \mathcal{E})) \quad \text{with} \quad \mathcal{P}_{\text{NL}}(x, \mathcal{E}) = \chi^{(3)}(x)(\mathcal{E} \cdot \mathcal{E})\mathcal{E},$$

where $n^2(x) = 1 + \chi^{(1)}(x)$ is the square of the linear refractive index and where \mathcal{P}_{NL} denotes the nonlinear part of the polarization. Note that in this section we use the notation $w \cdot z = w_1 z_1 + w_2 z_2 + w_3 z_3$ both for real and complex valued vectors $w, z \in \mathbb{C}^3$. The functions $\chi^{(1)}$ and $\chi^{(3)}$ denote the linear and cubic susceptibilities of the medium respectively. Although $\chi^{(3)}$ is generally a tensor, symmetries in the atomic structure of the material allow a reduction to a scalar, see [20, Sec. 2d]. The resulting second order equation for the electric field \mathcal{E} is then given by the quasilinear wave equation

$$(1.8) \quad \nabla \times \nabla \times \mathcal{E} + \frac{1}{c^2} \partial_t^2 (n(x)^2 \mathcal{E} + \chi^{(3)}(x)(\mathcal{E} \cdot \mathcal{E})\mathcal{E}) = 0, \quad (x, t) \in \mathbb{R}^4$$

together with $\nabla \cdot \mathcal{D} = 0$. Here $c = (\epsilon_0 \mu_0)^{-1/2}$ is the speed of light in vacuum. If \mathcal{E} solves (1.8), then \mathcal{D} is known from (1.7) and \mathcal{B} can be obtained from $\nabla \times \mathcal{E}$ by a time integration and thus also \mathcal{H} is known. Moreover, the fields \mathcal{D}, \mathcal{B} will be divergence free provided they are divergence free at some fixed time, e.g., $t = 0$.

The question of light bullets is that of the existence of solutions of Maxwell's equations in nonlinear dispersive media which are localized in space, i.e., which at all times t are decaying to 0 as $|x| \rightarrow \infty$.

In this paper we cannot give a complete answer to this question. Instead we will solve a related problem. Motivated by Fourier-expansion in time, one might look for a solution of (1.8) of the form $\mathcal{E}(x, t) = \sum_{k=0}^{\infty} (e^{-i(2k+1)\omega t} E_k(x) + \text{c.c.})$ with $E_k(x) \in \mathbb{C}^3$. If such solution existed under the additional assumption of localization, i.e., $\mathcal{E}(x, t)$ decaying to 0 as $|x| \rightarrow \infty$ for all t , then it would be a standing light bullet. Here we consider the simpler monochromatic ansatz

$$(1.9) \quad \mathcal{E}(x, t) = e^{-i\omega t} E(x) + \text{c.c.} \quad \text{with} \quad E(x) \in \mathbb{C}^3.$$

If we insert these monochromatic fields into the constitutive relation (1.7) and neglect the generation of higher harmonics, i.e., we cancel all terms with factors $e^{\pm 3i\omega t}$, then we obtain the new simplified constitutive relation $\mathcal{D} = \epsilon_0 (n^2(x)\mathcal{E} + \mathcal{P}_{\text{NL}}^{(a)}(x, \mathcal{E}))$ with

$$(1.10) \quad \mathcal{P}_{\text{NL}}^{(a)}(x, e^{-i\omega t} E + \text{c.c.}) = \chi^{(3)}(x) e^{-i\omega t} (2|E|^2 E + (E \cdot E) \bar{E}) + \text{c.c.}.$$

Note that here $E \cdot E = E_1^2 + E_2^2 + E_3^2 \in \mathbb{C}$ whereas $|E|^2 = |E_1|^2 + |E_2|^2 + |E_3|^2$ denotes the Hermitian inner product of E with itself. The second order elliptic equation for the E -field resulting from (1.6) is

$$(1.11) \quad \nabla \times \nabla \times E - \frac{\omega^2}{c^2} (n^2(x)E + \chi^{(3)}(x)(2|E|^2 E + (E \cdot E) \bar{E})) = 0, \quad x \in \mathbb{R}^3.$$

Note that the divergence conditions $\nabla \cdot \mathcal{D} = 0$ is automatically satisfied due to the monochromatic ansatz and the curl-curl structure of the equation.

Another model of the nonlinear polarization which effectively removes higher harmonics and results in equation (1.11) is given by time-averaging $\mathcal{E} \cdot \mathcal{E}$. In detail, for a T -periodic $\mathcal{E}(x, t) \cdot \mathcal{E}(x, t)$ one defines

$$(1.12) \quad \mathcal{P}_{\text{NL}}^{(\text{b})}(x, \mathcal{E}) = \chi^{(3)}(x) \frac{1}{T} \int_0^T \mathcal{E}(x, t) \cdot \mathcal{E}(x, t) dt \mathcal{E}(x, t),$$

see, e.g., [26, 28]. For \mathcal{E} as in (1.9), where $T = \pi/\omega$, we get the same for as in (1.10), i.e.

$$\mathcal{P}_{\text{NL}}^{(\text{a})}(x, e^{-i\omega t} E + \text{c.c.}) = \mathcal{P}_{\text{NL}}^{(\text{b})}(x, e^{-i\omega t} E + \text{c.c.}).$$

To sum up, we may say that a solution $E : \mathbb{R}^3 \rightarrow \mathbb{C}^3$ of (1.11) gives via (1.9) rise to a complete solution of the Maxwell system provided we consider the constitutive relation (1.10) or (1.12) instead of (1.7).

With the notation

$$V(x) := -\frac{\omega^2}{c^2} n^2(x), \quad \Gamma(x) := 3\frac{\omega^2}{c^2} \chi^{(3)}(x)$$

equation (1.11) reads

$$(1.13) \quad \nabla \times \nabla \times E + V(x)E = \frac{1}{3} \Gamma(x) (2|E|^2 E + (E \cdot E) \overline{E}) \text{ in } \mathbb{R}^3.$$

Restricting to real valued solutions $E \in H^1(\mathbb{R}^3)$, equation (1.13) is equivalent to (1.1) with $p = 3$.

1.4. Structure of the paper. The rest of the paper is structured as follows. In Section 2 Theorem 1 is first proved. Next, for the case of cylindrical symmetry of V and Γ a subspace of X is chosen in which minimization of J is possible. In Section 3 the strongly defocusing case (i.e. Theorem 2) is handled by the direct minimization method. Sections 4 and 5 treat the more delicate focusing case (i.e. Theorem 3). In Section 4 we study the spectrum of the linear operator in (1.1) and find examples of V for which zero lies outside the spectrum. This is a necessary condition for our minimization approach. Finally, in Section 5 J is minimized on the so called Nehari-Pankov manifold within the symmetric subspace using the concentration-compactness principle.

2. VARIATIONAL FORMULATION OF (1.1)

We begin with the definition of some spaces of vector valued functions $U : \mathbb{R}^3 \rightarrow \mathbb{R}^3$. For a measurable weight-function $\sigma : \mathbb{R}^3 \rightarrow (0, \infty)$ the corresponding weighted L^q -space for $1 \leq q < \infty$ is defined by

$$L_\sigma^q(\mathbb{R}^3) = \left\{ U : \mathbb{R}^3 \rightarrow \mathbb{R}^3 : \int_{\mathbb{R}^3} \sigma(x) |U|^q dx < \infty \right\}$$

with the norm

$$\|U\|_{\sigma, q} = \left(\int_{\mathbb{R}^3} \sigma(x) |U|^q dx \right)^{\frac{1}{q}}.$$

The space $H^1(\mathbb{R}^3)$ is defined by

$$H^1(\mathbb{R}^3) = \{U : \mathbb{R}^3 \rightarrow \mathbb{R}^3 : U^i, \frac{\partial U^i}{\partial x_j} \in L^2(\mathbb{R}^3) \text{ for } i, j = 1, 2, 3\},$$

with the norm

$$\|U\|_{H^1}^2 = \int_{\mathbb{R}^3} \sum_{i,j=1}^3 \left(\frac{\partial U^i}{\partial x_j} \right)^2 + |U|^2 dx.$$

The space $H(\text{curl}; \mathbb{R}^3)$ is defined by

$$H(\text{curl}; \mathbb{R}^3) = \{U : \mathbb{R}^3 \rightarrow \mathbb{R}^3 : U, \nabla \times U \in L^2(\mathbb{R}^3)\},$$

with the norm

$$\|U\|_{H(\text{curl})}^2 = \int_{\mathbb{R}^3} |\nabla \times U|^2 + |U|^2 dx$$

and where $\nabla \times U$ is understood in the distributional sense, i.e., it satisfies $\int_{\mathbb{R}^3} (\nabla \times U) \cdot \varphi dx = \int_{\mathbb{R}^3} U \cdot \text{curl } \varphi dx$ for all C^∞ -functions $\varphi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ with compact support.

Notice that for $U \in H^1(\mathbb{R}^3)$ we have the identity

$$\int_{\mathbb{R}^3} |\nabla \times U|^2 + (\nabla \cdot U)^2 dx = \int_{\mathbb{R}^3} \sum_{i,j=1}^3 \left(\frac{\partial U^i}{\partial x_j} \right)^2 dx$$

with $\nabla \cdot U$ denoting the distributional divergence of U . Therefore

$$(2.1) \quad H(\text{curl}; \mathbb{R}^3) \cap \{U : \nabla \cdot U = 0\} = H^1(\mathbb{R}^3) \cap \{U : \nabla \cdot U = 0\}.$$

This property will be used when we single out a suitable subspace of $H^1(\mathbb{R}^3)$, one in which we can solve (1.1). For this purpose we first study the symmetries of (1.1).

Lemma 4. *Assume that the locally bounded measurable functions $V, \Gamma : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ are radially symmetric.*

- (a) *If $U \in L_{loc}^p(\mathbb{R}^3; \mathbb{R}^3)$ is a distributional solution of (1.1) and $M \in O(3)$ then $\tilde{U}(x) := M^T U(Mx)$ also solves (1.1) in the sense of distributions.*
- (b) *Suppose $U : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ satisfies $U(x) = M^T U(Mx)$ for a.a. $x \in \mathbb{R}^3$ and all $M \in O(3)$. Then $U(x) = f(|x|) \frac{x}{|x|}$ for some $f : [0, \infty) \rightarrow \mathbb{R}$. If additionally $U \in L_{loc}^1(\mathbb{R}^3)$ then $\nabla \times U = 0$ in the sense of distributions.*

Proof. (a) Let $\varphi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a C^∞ -function with compact support, let $M \in O(3)$ and define

$$\psi(y) := M\varphi(M^T y), \quad y \in \mathbb{R}^3.$$

Then a direct computation yields

$$\nabla \times \nabla \times \psi(y) = M(\nabla \times \nabla \times \varphi)(M^T y)$$

and thus

$$(\mathcal{L}\psi)(y) = M(\mathcal{L}\varphi)(M^T y).$$

Therefore, if $U \in L^p_{loc}(\mathbb{R}^3; \mathbb{R}^3)$ satisfies

$$\int_{\mathbb{R}^3} U(y) \cdot (\mathcal{L}\psi)(y) - \Gamma(y)|U(y)|^{p-1}U(y) \cdot \psi(y) dy = 0 \text{ for all } \psi \in C_0^\infty(\mathbb{R}^3; \mathbb{R}^3)$$

then

$$\begin{aligned} & \int_{\mathbb{R}^3} \tilde{U}(x) \cdot (\mathcal{L}\varphi)(x) - \Gamma(x)|\tilde{U}(x)|^{p-1}\tilde{U}(x) \cdot \varphi(x) dx \\ &= \int_{\mathbb{R}^3} M^T U(y) \cdot (\mathcal{L}\varphi)(M^T y) - \Gamma(y)|U(y)|^{p-1}M^T U(y) \cdot \varphi(M^T y) dy \\ &= \int_{\mathbb{R}^3} M^T U(y) \cdot M^T \mathcal{L}\psi(y) - \Gamma(y)|U(y)|^{p-1}U(y) \cdot \psi(y) dy \\ &= \int_{\mathbb{R}^3} U(y) \cdot \mathcal{L}\psi(y) - \Gamma(y)|U(y)|^{p-1}U(y) \cdot \psi(y) dy \\ &= 0. \end{aligned}$$

(b) Let $x \in \mathbb{R}^3$ be such that $U(x) = M^T U(Mx)$ for all $M \in O(3)$. Then $U(x) = MU(x)$ for all those rotations M which leave x fixed, i.e., for all rotations around the axis $\mathbb{R}x$. Hence $U(x) \in \mathbb{R}x$ and we may write $U(x) = f(|x|)\frac{x}{|x|}$. Under the assumption $U \in L^1_{loc}(\mathbb{R}^3)$ we see that $f \in L^1(I)$ for any compact interval $I \subset (0, \infty)$. Therefore we may define the function $F(r) := \int_1^r f(s) ds$ for $r > 0$ which is absolutely continuous in \mathbb{R}^+ . Moreover, for any $R > 1$ using polar coordinates and Fubini's theorem we see that

$$\begin{aligned} \int_{B_R(0)} |F(|x|)| dx &= 4\pi \int_0^R \left| \int_1^r f(t) dt \right| r^2 dr \\ &\leq 4\pi \int_0^1 \int_r^1 |f(t)| dt r^2 dr + 4\pi \int_1^R \int_1^r |f(t)| dt r^2 dr \\ &\leq \frac{4\pi}{3} \int_0^1 |f(t)| t^3 dt + \frac{4\pi R^3}{3} \int_1^R |f(t)| dt \\ &\leq \frac{4\pi R^3}{3} \int_0^R |f(t)| t^2 dt \\ &= \frac{R^3}{3} \int_{B_R(0)} |U(x)| dx < \infty \end{aligned}$$

since $U \in L^1_{loc}(\mathbb{R}^3)$. Hence the function $F(|\cdot|)$ belongs to $L^1_{loc}(\mathbb{R}^3)$ and due to the absolute continuity of F it has the strong derivative $U(x)$ almost everywhere. Since both $F(|\cdot|)$ and U are $L^1_{loc}(\mathbb{R}^3)$, one can see that $U(x) = \nabla(F(|x|))$ in \mathbb{R}^3 in the weak sense. This implies $\nabla \times U = 0$ in the distributional sense. \square

Proof of Theorem 1 : Suppose $U \in L^p_{loc}(\mathbb{R}^3)$ is a distributional solution of (1.1). Lemma 4 shows that the requirement of full radial symmetry of V and Γ and the solution symmetry $U(x) = M^T U(Mx)$ for a.a. $x \in \mathbb{R}^3$ and all $M \in O(3)$ reduces (1.1) to the algebraic equation

$$V(x)U = \Gamma(x)|U|^{p-1}U \text{ in } \mathbb{R}^3.$$

Provided $0 \leq V\Gamma^{-1}$ the function U has the form (1.5). Now let us reversely assume that U has the form (1.5). Since $0 \leq V\Gamma^{-1} \in L^{\frac{p}{p-1}}_{loc}(\mathbb{R}^3)$ we see that $U \in L^p_{loc}(\mathbb{R}^3)$ and in particular $U \in L^1_{loc}(\mathbb{R}^3)$. Moreover, with an absolutely continuous function $F : (0, \infty) \rightarrow \mathbb{R}$ given by

$$F(t) = \int_1^t s(\tau) \left(\frac{\tilde{V}(\tau)}{\tilde{\Gamma}(\tau)} \right)^{1/(p-1)} d\tau, \quad t > 0$$

we have $U(x) = \nabla (F(|x|))$ in the distributional sense. As in Lemma 4 we find $F(|\cdot|) \in L^1_{loc}(\mathbb{R}^3)$ and, moreover, $\nabla \times U = 0$. Hence U solves (1.1). Finally, the assumption $(V\Gamma^{-1})^{\frac{2}{p-1}}, (V\Gamma^{-1})^{\frac{p+1}{p-1}}\Gamma \in L^1(\mathbb{R}^3)$ implies that any U defined by (1.5) belongs to the space X and thus is a critical point of J . \square

Although the $H(\text{curl}; \mathbb{R}^3)$ solutions in Theorem 1 are valid localized solutions of (1.1), in the rest of the paper we consider solutions that are not gradient fields.

Since the requirement of full radial symmetry does not lead to interesting solutions of (1.1), we look for solutions which are invariant only under a subgroup of $O(3)$ (this idea is due to Azzollini et. al. [4]). For this we define the following copy of $SO(2)$ as a subset of $O(3)$

$$G_0 := \left\{ \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} : \alpha \in \mathbb{R} \right\}.$$

Assume that the measurable weight $\sigma : \mathbb{R}^3 \rightarrow (0, \infty)$ satisfies $\sigma(Mx) = \sigma(x)$ for all $x \in \mathbb{R}^3$ and all $M \in G_0$. Then the group G_0 operates isometrically on $L^q_\sigma(\mathbb{R}^3)$, on $H(\text{curl}; \mathbb{R}^3)$ and on $H^1(\mathbb{R}^3)$ by the group action $U \mapsto M^T U(M\cdot)$. Due to this result we can now define the corresponding G_0 -fixed point subspaces of $L^q_\sigma(\mathbb{R}^3)$, $H(\text{curl}; \mathbb{R}^3)$ and $H^k(\mathbb{R}^3)$, $k \in \mathbb{N}$ by

$$\begin{aligned} L^q_{\sigma, G_0}(\mathbb{R}^3) &= \{U \in L^q_\sigma(\mathbb{R}^3) : U(x) = M^T U(Mx) \quad \forall x \in \mathbb{R}^3, \forall M \in G_0\}, \\ H_{G_0}(\text{curl}; \mathbb{R}^3) &= \{U \in H(\text{curl}; \mathbb{R}^3) : U(x) = M^T U(Mx) \quad \forall x \in \mathbb{R}^3, \forall M \in G_0\}, \\ H^k_{G_0}(\mathbb{R}^3) &= \{U \in H^k(\mathbb{R}^3) : U(x) = M^T U(Mx) \quad \forall x \in \mathbb{R}^3, \forall M \in G_0\}, \quad k \in \mathbb{N}, \\ X_{G_0} &= H_{G_0}(\text{curl}; \mathbb{R}^3) \cap L^{p+1}_{|\Gamma|}(\mathbb{R}^3). \end{aligned}$$

Observe that the functional J is invariant under the action of G_0 . Thus, by Palais' principle of symmetric criticality [21], [29], every critical point of $J|_{X_{G_0}}$ is also a critical point of J on X . Next we want to restrict the spaces $L^q_{\sigma, G_0}(\mathbb{R}^3)$, $H_{G_0}(\text{curl}; \mathbb{R}^3)$ and $H^k_{G_0}(\mathbb{R}^3)$ even further. In order to do so we need two lemmas – the first one being analogous to Lemma 4. We omit the proofs because they are contained in Lemma 1 and Proposition 1 in [4].

Lemma 5. *Suppose a measurable function $U : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ satisfies $U(x) = M^T U(Mx)$ for a.a. $x \in \mathbb{R}^3$ and all $M \in G_0$. Then there are unique measurable functions $Q, S, T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that*

$$U(x) = Q(x) + S(x) + T(x)$$

with

$$(2.2) \quad Q(x) = \frac{q(r, x_3)}{r} \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}, \quad S(x) = \frac{s(r, x_3)}{r} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}, \quad T(x) = \begin{pmatrix} 0 \\ 0 \\ t(r, x_3) \end{pmatrix}.$$

where $q, s, t : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ are measurable functions. If $U \in L^q_\sigma(\mathbb{R}^3)$ or $H^1(\mathbb{R}^3)$ then $Q, S, T \in L^q_\sigma(\mathbb{R}^3)$ or $H^1(\mathbb{R}^3)$, respectively.

Lemma 6. *Let Y be either the spaces $L^q_{\sigma, G_0}(\mathbb{R}^3)$, $H_{G_0}(\text{curl}; \mathbb{R}^3)$ or $H^k_{G_0}(\mathbb{R}^3)$, $k \in \mathbb{N}$ and define the map*

$$g_1 : \begin{cases} Y & \rightarrow Y, \\ U = Q + S + T & \mapsto Q - S - T. \end{cases}$$

The map g_1 is a linear isometry and satisfies $g_1 \circ g_1 = \text{Id}$. Hence $G_1 = \{\text{Id}, g_1\}$ is a group of order 2. Moreover, the functional $J|_{X_{G_0}}$ is invariant under the action of G_1 .

Remark. The proof of the isometry and invariance statement relies on the fact that pointwise $|U|^2 = |Q|^2 + |S|^2 + |T|^2$ and $|\nabla U|^2 = |\nabla Q|^2 + |\nabla S|^2 + |\nabla T|^2$, cf. [4]. However, for the curl only $|\nabla \times U|^2 = |\nabla \times Q|^2 + |\nabla \times (S + T)|^2$ holds. But this is sufficient for our claim.

This result allows to define the spaces

$$\begin{aligned} L^q_{\sigma, G_1}(\mathbb{R}^3) &= \{U \in L^q_{\sigma, G_0}(\mathbb{R}^3) : g_1 U = U\}, \\ H_{G_1}(\text{curl}; \mathbb{R}^3) &= \{U \in H_{G_0}(\text{curl}; \mathbb{R}^3) : g_1 U = U\}, \\ H^k_{G_1}(\mathbb{R}^3) &= \{U \in H^k_{G_0}(\mathbb{R}^3) : g_1 U = U\}, \quad k \in \mathbb{N}, \\ X_{G_1} &= H_{G_1}(\text{curl}; \mathbb{R}^3) \cap L^{p+1}_{|\Gamma|}(\mathbb{R}^3). \end{aligned}$$

All the spaces with the suffix G_1 may be thought of as the subspaces of $L^q_\sigma(\mathbb{R}^3)$, $H(\text{curl}; \mathbb{R}^3)$ and $H^k(\mathbb{R}^3)$ consisting of vector fields of the form (1.3). Again, Palais' principle of symmetric criticality ensures that every critical point of $J|_{X_{G_1}}$ is also a critical point of J on X . Finally, note that

$$(2.3) \quad H_{G_1}(\text{curl}; \mathbb{R}^3) = H^1_{G_1}(\mathbb{R}^3)$$

because the members of both spaces have vanishing divergence, cf. (2.1).

To summarize the results of this section recall that the energy functional related to (1.1) is

$$J[U] = \int_{\mathbb{R}^3} \frac{1}{2} (|\nabla \times U|^2 + V(x)|U|^2) - \frac{\Gamma(x)}{p+1} |U|^{p+1} dx,$$

which is well defined on $X = H(\text{curl}; \mathbb{R}^3) \cap L^{p+1}_{|\Gamma|}(\mathbb{R}^3)$. Due to Lemma 5, Palais' principle of symmetric criticality (cf. Palais [21], Willem [29]) and (2.3) can seek critical points of the functional J restricted to the subspace $X_{G_1} = H^1_{G_1}(\mathbb{R}^3) \cap L^{p+1}_{|\Gamma|}(\mathbb{R}^3)$ and these critical points will be solutions of (1.1). The elements of the subspace $H^1_{G_1}(\mathbb{R}^3)$ have the favorable property of vanishing divergence.

3. GROUND STATES IN THE DEFOCUSING CASE

We assume $p > 1$ and

$$(H\text{-defoc}) \quad \Gamma(x) \leq -C(1 + |x|)^\alpha \text{ in } \mathbb{R}^3 \text{ with } \alpha > \frac{3}{2}(p-1) \text{ and } C > 0.$$

We work in the following reflexive Banach space

$$X_{G_1} := H_{G_1}^1(\mathbb{R}^3) \cap L_{|\Gamma|}^{p+1}(\mathbb{R}^3)$$

where the norm on X_{G_1} is given by

$$\|U\|_X := \|U\|_{H^1} + \|U\|_{|\Gamma|, p+1}.$$

The basic tool for proving existence of ground states in the defocusing case is the following embedding result, which is due to Benci, Fortunato [7].

Lemma 7. *Assume $p > 1$ and (H-defoc). Then the space $L_{|\Gamma|}^{p+1}(\mathbb{R}^3)$ embeds continuously into $L^2(\mathbb{R}^3)$ and the space X_{G_1} embeds compactly into $L^2(\mathbb{R}^3)$.*

Proof. Let $\beta = \frac{p+1}{p-1}$ and $\beta' = \frac{p+1}{2}$. Then

$$\begin{aligned} \int_{\mathbb{R}^3} |U|^2 dx &= \int_{\mathbb{R}^3} |\Gamma(x)|^{-\frac{1}{\beta'}} |\Gamma(x)|^{\frac{1}{\beta'}} |U|^2 dx \\ &\leq \left(\int_{\mathbb{R}^3} |\Gamma(x)|^{-\frac{2}{p-1}} dx \right)^{\frac{p-1}{p+1}} \left(\int_{\mathbb{R}^3} |\Gamma(x)| |U|^{p+1} dx \right)^{\frac{2}{p+1}}, \end{aligned}$$

and the first integral is finite since by assumption (H-defoc) $|\Gamma(x)|^{-\frac{2}{p-1}} \leq C(1 + |x|)^{-\frac{2\alpha}{p-1}}$ and $-\frac{2\alpha}{p-1} < -3$. This proves the first part of the claim. For the second part, let us define the positive and continuous radially symmetric function $\rho : \mathbb{R}^3 \rightarrow (0, \infty)$ by setting $\rho(x) = 1$ for $|x| \leq 1$ and $\rho(x) = |x|^\gamma$ with $\gamma > 0$ so small that $\gamma\beta - \frac{2\alpha}{p-1} < 3$. Then $\rho(x) \rightarrow \infty$ as $|x| \rightarrow \infty$ and $\int_{\mathbb{R}^3} \rho(x)^\beta (1 + |x|)^{-\frac{2\alpha}{p-1}} dx < \infty$. We obtain

$$\begin{aligned} \|U\|_{\rho, 2}^2 &= \int_{\mathbb{R}^3} \rho(x) |\Gamma(x)|^{-\frac{1}{\beta'}} |\Gamma(x)|^{\frac{1}{\beta'}} |U|^2 dx \\ &\leq \left(\int_{\mathbb{R}^3} \rho(x)^\beta |\Gamma(x)|^{-\frac{2}{p-1}} dx \right)^{\frac{p-1}{p+1}} \left(\int_{\mathbb{R}^3} |\Gamma(x)| |U|^{p+1} dx \right)^{\frac{2}{p+1}} \\ &\leq C \|U\|_{|\Gamma|, p+1}^2. \end{aligned}$$

This shows that $L_{|\Gamma|}^{p+1}(\mathbb{R}^3)$ embeds continuously into $L_\rho^2(\mathbb{R}^3)$. Finally, by Theorem 3.1 of Benci, Fortunato [7] we have that $H^1(\mathbb{R}^3) \cap L_\rho^2(\mathbb{R}^3)$ embeds compactly into $L^2(\mathbb{R}^3)$. Both facts together imply the second claim of the lemma. \square

Lemma 8. *Assume $p > 1$, (H-defoc) and $V \in L^\infty(\mathbb{R}^3)$. Then the functional J is a weakly lower-semicontinuous, coercive C^1 -functional on X_{G_1} and hence has a minimizer.*

Proof. Since

$$J_1[U] = \int_{\mathbb{R}^3} \frac{1}{2} |\nabla \times U|^2 - \frac{\Gamma(x)}{p+1} |U|^{p+1} dx$$

is convex on X_{G_1} and

$$J_2[U] = \int_{\mathbb{R}^3} \frac{V(x)}{2} |U|^2 dx$$

is weakly continuous on X_{G_1} by Lemma 7 we find that the functional $J = J_1 + J_2$ is weakly lower-semicontinuous on X_{G_1} . Moreover, there exist constants $K_1, \dots, K_5 > 0$ such that the following estimates hold for $U \in X_{G_1}$:

$$\begin{aligned} J[U] &\geq \frac{1}{2} \|\nabla \times U\|_2^2 + \frac{1}{p+1} \|U\|_{|\Gamma|, p+1}^{p+1} - \frac{\|V\|_\infty}{2} \|U\|_2^2 \\ &\geq \frac{1}{2} \|\nabla \times U\|_2^2 + \frac{1}{p+1} \|U\|_{|\Gamma|, p+1}^{p+1} - K_1 \|U\|_{|\Gamma|, p+1}^2 \\ &\geq \frac{1}{2} \|\nabla \times U\|_2^2 + K_2 \|U\|_{|\Gamma|, p+1}^2 - K_3 \\ &\geq \frac{1}{2} \|\nabla \times U\|_2^2 + K_4 \|U\|_2^2 + \frac{K_2}{2} \|U\|_{|\Gamma|, p+1}^2 - K_3 \\ &\geq K_5 \|U\|_X^2 - K_3, \end{aligned}$$

which shows the coercivity of J . It is clear that the quadratic parts of the functional J are C^1 and it is standard (cf. Struwe [25]) to verify that the functional $\int_{\mathbb{R}^3} \Gamma(x) |U|^{p+1}$ has a Gâteaux derivative which depends continuously on $U \in X_{G_1}$. Hence J is a C^1 -functional on X_{G_1} and the minimizer of J is a weak solution of (1.1). \square

Proof of Theorem 2: We set $U_0(x) = sW(tx)$ for some vector-valued function $W \in C_0^\infty(\mathbb{R}^3)$ and take $s, t > 0$. Since $\text{ess sup}_{\mathbb{R}^3} V < 0$ we obtain

$$\begin{aligned} J[U_0] &= \int_{\mathbb{R}^3} \frac{1}{2} |\nabla \times U_0(x)|^2 - \frac{\Gamma(x)}{p+1} |U_0(x)|^{p+1} + \frac{V(x)}{2} |U_0(x)|^2 dx \\ &\leq t^{-3} s^2 \int_{\mathbb{R}^3} \frac{t^2}{2} |\nabla \times W(y)|^2 - \frac{s^{p-1} \Gamma(y/t)}{p+1} |W(y)|^{p+1} + \frac{\text{ess sup}_{\mathbb{R}^3} V}{2} |W(y)|^2 dy \\ &< 0 \end{aligned}$$

provided we first choose $t > 0$ so small that $\int_{\mathbb{R}^3} t^2 |\nabla \times W(y)|^2 + (\text{ess sup}_{\mathbb{R}^3} V) |W(y)|^2 dy < 0$ and then choose $s > 0$ sufficiently small. Thus the minimizer of J over X_{G_1} is non-trivial and therefore a ground state of (1.1) within X_{G_1} . \square

4. SPECTRUM OF THE LINEAR OPERATOR \mathcal{L}

In the focusing case we can only show the existence of ground states (cf. Section 5) when zero does not lie in the spectrum of the linear operator

$$\mathcal{L} := (\nabla \times \nabla \times) + V(r, x_3).$$

Of course an easy example is given by the class of potentials $V = V(r, x_3)$ with $\text{ess inf } V_{\mathbb{R}^3} > 0$. However, since $V(x)$ is proportional to $-n^2(x)$ with $n(x)$ being the refractive index, the physically interesting case consists of functions V which are negative (or at least have non-vanishing negative part). In this section we construct potentials V with non-vanishing negative part and where 0 lies in a spectral gap of the operator \mathcal{L} , cf. Lemma 15, Lemma 16, Lemma 17.

The construction of such examples needs various preparations. We consider \mathcal{L} as an operator defined on

$$D(\mathcal{L}) = H_{G_1}^2(\mathbb{R}^3) \subset L_{G_1}^2(\mathbb{R}^3)$$

and we will show in Lemma 12 that \mathcal{L} is a selfadjoint operator, whose spectrum has a particular additive structure whenever the potential is separable, i.e., $V(r, x_3) = W(r) + P(x_3)$, cf. Lemma 14. The key to these results is the following observation: if

$$U(x) = u(r, x_3) \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} \quad \text{with } r = \sqrt{x_1^2 + x_2^2}$$

then

$$(4.1) \quad \mathcal{L}U(x) = \left((Lu)(r, x_3) \right) \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}$$

with

$$(4.2) \quad L = -\frac{1}{r^3} \frac{\partial}{\partial r} \left(r^3 \frac{\partial}{\partial r} \right) - \frac{\partial^2}{\partial x_3^2} + V(r, x_3)$$

where the first two terms correspond to a five-dimensional Laplacian with cylindrical symmetry. Let us now start with the detailed analysis of the operators and their spectra.

Define the maps

$$\Psi_{\text{rad}} : \begin{cases} \mathbb{R}^4 & \rightarrow \mathbb{R}, \\ (y_1, \dots, y_4) & \mapsto \sqrt{y_1^2 + \dots + y_4^2} \end{cases} \quad \Psi : \begin{cases} \mathbb{R}^5 & \rightarrow \mathbb{R}^2, \\ (y_1, \dots, y_5) & \mapsto (\Psi_{\text{rad}}(y_1, \dots, y_4), y_5) \end{cases}$$

In the following we use the index *rad* for spaces of functions $u : (0, \infty) \rightarrow \mathbb{R}$ of the single radial variable r ; the index *cyl* refers to spaces of functions $u : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ of two variables r, x_3 . For the following Hilbert spaces we also denote in brackets the measure with

respect to which integration is performed.

$$\begin{aligned}
L_{\text{rad}}^2(r^3 dr) &= \{u : (0, \infty) \rightarrow \mathbb{R} : u \circ \Psi_{\text{rad}} \in L^2(\mathbb{R}^4)\} \\
&= \{u : (0, \infty) \rightarrow \mathbb{R} : u \in L_{r^3}^2(0, \infty)\}, \\
L_{\text{cyl}}^2(r^3 dr dx_3) &= \{u : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R} : u \circ \Psi \in L^2(\mathbb{R}^5)\} \\
&= \{u : (0, \infty) \rightarrow \mathbb{R} : u \in L_{r^3}^2((0, \infty) \times \mathbb{R})\}, \\
H_{\text{rad}}^1(r^3 dr) &= \{u : (0, \infty) \rightarrow \mathbb{R} : u \circ \Psi_{\text{rad}} \in H^1(\mathbb{R}^4)\} \\
&= \{u : (0, \infty) \rightarrow \mathbb{R} : u, u' \in L_{r^3}^2(0, \infty)\}, \\
H_{\text{cyl}}^1(r^3 dr dx_3) &= \{u : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R} : u \circ \Psi \in H^1(\mathbb{R}^5)\} \\
&= \left\{u : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R} : u, \frac{\partial u}{\partial r}, \frac{\partial u}{\partial x_3} \in L_{r^3}^2((0, \infty) \times \mathbb{R})\right\}, \\
H_{\text{rad}}^2(r^3 dr) &= \{u : (0, \infty) \rightarrow \mathbb{R} : u \circ \Psi_{\text{rad}} \in H^2(\mathbb{R}^4)\} \\
&= \left\{u : (0, \infty) \rightarrow \mathbb{R} : u, u', \frac{u'}{r}, u'' \in L_{r^3}^2(0, \infty)\right\}, \\
H_{\text{cyl}}^2(r^3 dr dx_3) &= \{u : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R} : u \circ \Psi \in H^2(\mathbb{R}^5)\} \\
&= \left\{u : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R} : u, \frac{\partial u}{\partial r}, \frac{\partial u}{\partial x_3}, \frac{1}{r} \frac{\partial u}{\partial r}, \frac{\partial^2 u}{\partial r^2}, \frac{\partial^2 u}{\partial x_3^2} \in L_{r^3}^2((0, \infty) \times \mathbb{R})\right\}.
\end{aligned}$$

These identities may be well known. For the sake of clarity we explain the last one for $H_{\text{cyl}}^2(r^3 dr dx_3)$ on the level of the derivatives of highest order: $u \circ \Psi$ has second order derivatives in $L^2(\mathbb{R}^5)$ if and only if for all $i, j \in \{1, 2, 3, 4\}$ we have

$$(4.3) \quad \left(\frac{\partial^2 u}{\partial r^2} - \frac{1}{r} \frac{\partial u}{\partial r} \right) \frac{y_i y_j}{r^2} + \frac{\partial u}{\partial r} \frac{\delta_{ij}}{r}, \frac{\partial^2 u}{\partial x_3^2}, \frac{\partial^2 u}{\partial r \partial x_3} \frac{y_i}{r} \in L^2(\mathbb{R}^5).$$

In view of the fact that $\int_{\mathbb{R}^3} \sum_{i,j=1}^3 \left(\frac{\partial^2 v(x)}{\partial x_i \partial x_j} \right)^2 dx = \int_{\mathbb{R}^3} (\Delta v(x))^2 dx$ for $v \in C_0^\infty(\mathbb{R}^3)$ we see that

$$u \circ \Psi, \Delta(u \circ \Psi) \in L^2(\mathbb{R}^5) \iff u \circ \Psi \in H^2(\mathbb{R}^5)$$

Hence, for (4.3) it is sufficient to have

$$(4.4) \quad u, \frac{1}{r} \frac{\partial u}{\partial r}, \frac{\partial^2 u}{\partial r^2}, \frac{\partial^2 u}{\partial x_3^2} \in L_{r^3}^2((0, \infty) \times \mathbb{R})$$

since the L^2 -norm of $\frac{\partial^2 u}{\partial r \partial x_3}$ may be estimated by the L^2 -norm of $\Delta(u \circ \Psi)$, i.e., by sums of $L_{r^3}^2((0, \infty) \times \mathbb{R})$ -norms of $\frac{1}{r} \frac{\partial u}{\partial r}, \frac{\partial^2 u}{\partial r^2}, \frac{\partial^2 u}{\partial x_3^2}$. Let us also show that (4.4) is necessary. If we square all the entries in (4.3) and add them up then we see that

$$\int_0^\infty \int_{-\infty}^\infty \left(\left(\frac{\partial^2 u}{\partial r^2} \right)^2 + \frac{3}{r^2} \left(\frac{\partial u}{\partial r} \right)^2 + \left(\frac{\partial^2 u}{\partial x_3^2} \right)^2 + \left(\frac{\partial^2 u}{\partial r \partial x_3} \right)^2 \right) r^3 dr dx_3 < \infty,$$

which implies that (4.4) is also necessary.

The spaces carry natural inner products. Instead of listing all of them we just write out the ones for $H_{\text{rad}}^2(r^3 dr)$ and $H_{\text{cyl}}^2(r^3 dr dx_3)$:

$$\begin{aligned}\langle u, v \rangle_{H_{\text{rad}}^2} &= \int_0^\infty \left(uv + u'v' + \frac{1}{r^2} u'v' + u''v'' \right) r^3 dr \\ \langle u, v \rangle_{H_{\text{cyl}}^2} &= \int_0^\infty \int_{-\infty}^\infty \left(uv + \frac{\partial u}{\partial r} \frac{\partial v}{\partial r} + \frac{\partial u}{\partial x_3} \frac{\partial v}{\partial x_3} + \frac{1}{r^2} \frac{\partial u}{\partial r} \frac{\partial v}{\partial r} + \frac{\partial^2 u}{\partial r^2} \frac{\partial^2 v}{\partial r^2} + \frac{\partial^2 u}{\partial x_3^2} \frac{\partial^2 v}{\partial x_3^2} \right) r^3 dr dx_3.\end{aligned}$$

The above identities of spaces have the following implication.

Lemma 9 (Hardy's inequality).

(i) *There exists a constant $C > 0$ such that for all $u \in H_{\text{cyl}}^1(r^3 dr dx_3)$*

$$\int_0^\infty \int_{-\infty}^\infty \frac{u^2}{r^2} r^3 dr dx_3 \leq \int_0^\infty \int_{-\infty}^\infty \left(\left(\frac{\partial u}{\partial r} \right)^2 + \left(\frac{\partial u}{\partial x_3} \right)^2 \right) r^3 dr dx_3.$$

(ii) *There exists a constant $C > 0$ such that for all $u \in H_{\text{cyl}}^2(r^3 dr dx_3)$*

$$\int_0^\infty \int_{-\infty}^\infty \frac{1}{r^2} \left(\frac{\partial u}{\partial x_3} \right)^2 r^3 dr dx_3 \leq \int_0^\infty \int_{-\infty}^\infty \left(\left(\frac{\partial^2 u}{\partial r^2} \right)^2 + \frac{1}{r^2} \left(\frac{\partial u}{\partial r} \right)^2 + \left(\frac{\partial^2 u}{\partial x_3^2} \right)^2 \right) r^3 dr dx_3.$$

Proof. We use the identities of spaces as explained above. Part (i) can be found as Theorem C in [5] set up in \mathbb{R}^5 where $r = \text{dist}(y, K)$ and $K = \{y \in \mathbb{R}^5 : y_1 = y_2 = y_3 = y_4 = 0\}$. Note that $H_0^1(\mathbb{R}^5 \setminus K) = H_0^1(\mathbb{R}^5)$, cf. Theorem 2.43 in [14], because as a subset of \mathbb{R}^5 the set K has zero 2-capacity, cf. Section 4.7.2 in [12]. Part (ii) is a consequence of (i) when applied to $\partial U / \partial y_5 \in H_0^1(\mathbb{R}^5)$. \square

Lemma 10. *The following identity holds between the group invariant spaces (denoted with suffix G_1) and the spaces of scalar functions with cylindrical symmetry (denoted with suffix cyl):*

$$\begin{aligned}L_{G_1}^2(\mathbb{R}^3) &= \left\{ u(r, x_3) \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} : u \in L_{\text{cyl}}^2(r^3 dr dx_3) \right\} \\ H_{G_1}^k(\mathbb{R}^3) &= \left\{ u(r, x_3) \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} : u \in H_{\text{cyl}}^k(r^3 dr dx_3) \right\}, \quad k = 1, 2.\end{aligned}$$

Proof. Let

$$U(x_1, x_2, x_3) = u(r, x_3) \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}$$

The identity of the L^2 -spaces is obvious since

$$\int_{\mathbb{R}^3} |U(x)|^2 dx = 2\pi \int_0^\infty \int_{-\infty}^\infty |u(r, x_3)|^2 r^3 dr dx_3.$$

Next we prove the identity of the H^1 -spaces. Clearly $U \in H^1(\mathbb{R}^3)$ if and only if $x_i u \in H^1(\mathbb{R}^3)$ for $i = 1, 2$, i.e., if and only if

$$u \delta_{ij} + \frac{x_i x_j}{r} \frac{\partial u}{\partial r}, \quad x_i \frac{\partial u}{\partial x_3}, \quad x_i u \in L^2(\mathbb{R}^3).$$

By squaring, summing from $i, j = 1, 2$ and rearranging terms this in turn is equivalent to

$$\int_0^\infty \int_{-\infty}^\infty \left(u^2 + \left(u + r \frac{\partial u}{\partial r} \right)^2 + r^2 \left(\frac{\partial u}{\partial x_3} \right)^2 + r^2 u^2 \right) r \, dr dx_3 < \infty.$$

The above is equivalent to $\frac{u}{r}, \frac{\partial u}{\partial r}, \frac{\partial u}{\partial x_3}, u \in L^2_{r^3}((0, \infty) \times \mathbb{R})$. Hardy's inequality of Lemma 9 tells us that the $L^2_{r^3}((0, \infty) \times \mathbb{R})$ -norm of the first term is bounded by the norm of the remaining terms, and hence $U \in H^1(\mathbb{R}^3)$ if and only if $u \in H^1_{\text{cyl}}(r^3 dr dx_3)$. Finally, let us prove the identity for the H^2 -spaces. The second derivatives of U lie in $H^2(\mathbb{R}^3)$ if and only if

$$\delta_{ij} \frac{x_k}{r} \frac{\partial u}{\partial r} + \delta_{ik} \frac{x_j}{r} \frac{\partial u}{\partial r} + \delta_{jk} \frac{x_i}{r} \frac{\partial u}{\partial r} - \frac{x_i x_j x_k}{r^3} \frac{\partial u}{\partial r} + \frac{x_i x_j x_k}{r^2} \frac{\partial^2 u}{\partial r^2}, \quad \delta_{ij} \frac{\partial u}{\partial x_3} + \frac{x_i x_j}{r} \frac{\partial^2 u}{\partial r \partial x_3} \in L^2(\mathbb{R}^3)$$

for $i, j, k = 1, 2$. By squaring, summing from $i, j, k = 1, 2$ and rearranging this becomes

$$\begin{aligned} & \int_0^\infty \int_{-\infty}^\infty \left(3 \left(\frac{\partial u}{\partial r} \right)^2 + \left(2 \frac{\partial u}{\partial r} + r \frac{\partial^2 u}{\partial r^2} \right)^2 \right) r \, dr dx_3 < \infty \\ & \int_0^\infty \int_{-\infty}^\infty \left(\left(\frac{\partial u}{\partial x_3} \right)^2 + \left(\frac{\partial u}{\partial x_3} + r \frac{\partial^2 u}{\partial r \partial x_3} \right)^2 \right) r \, dr dx_3 < \infty \\ & \int_0^\infty \int_{-\infty}^\infty \left(\frac{\partial^2 u}{\partial x_3^2} \right)^2 r^3 \, dr dx_3 < \infty \end{aligned}$$

Therefore a necessary and sufficient condition for $U \in H^2(\mathbb{R}^3)$ is given by

$$\int_0^\infty \int_{-\infty}^\infty \left(\frac{1}{r^2} \left(\frac{\partial u}{\partial r} \right)^2 + \left(\frac{\partial^2 u}{\partial r^2} \right)^2 + \frac{1}{r^2} \left(\frac{\partial u}{\partial x_3} \right)^2 + \left(\frac{\partial^2 u}{\partial r \partial x_3} \right)^2 + \left(\frac{\partial^2 u}{\partial x_3^2} \right)^2 \right) r^3 \, dr dx_3 < \infty$$

By the relation $\|D^2 u\|_{L^2(\mathbb{R}^3)} = \|\Delta u\|_{L^2(\mathbb{R}^3)}$ and by Hardy's inequality of Lemma 9 the above is equivalent to

$$\int_0^\infty \int_{-\infty}^\infty \left(\frac{1}{r^2} \left(\frac{\partial u}{\partial r} \right)^2 + \left(\frac{\partial^2 u}{\partial r^2} \right)^2 + \left(\frac{\partial^2 u}{\partial x_3^2} \right)^2 \right) r^3 \, dr dx_3 < \infty$$

which means that U has second derivatives in $L^2(\mathbb{R}^3)$ if and only if $\frac{1}{r} \frac{\partial u}{\partial r}, \frac{\partial^2 u}{\partial r^2}, \frac{\partial^2 u}{\partial x_3^2} \in L^2_{r^3}((0, \infty) \times \mathbb{R})$. In view of the definition of $H^2_{\text{cyl}}(r^3 dr dx_3)$ this establishes the claim. \square

Lemma 11. *Let $V \in L^\infty(\mathbb{R}^3)$ and suppose $V = V(r, x_3)$ has cylindrical symmetry. Then the operator $L : D(L) := H^2_{\text{cyl}}(r^3 dr dx_3) \subset L^2_{\text{cyl}}(r^3 dr dx_3) \rightarrow L^2_{\text{cyl}}(r^3 dr dx_3)$ given by (4.2) is selfadjoint.*

Proof. For $u \in D(L)$ we have

$$(Lu) \circ \Psi = -\Delta(u \circ \Psi) + (Vu) \circ \Psi,$$

i.e., L coincides with the five-dimensional Schrödinger operator $-\Delta + V$ in the space of functions with cylindrical symmetry. \square

Lemma 12. *Let $V \in L^\infty(\mathbb{R}^3)$ and suppose $V = V(r, x_3)$ has cylindrical symmetry. The operator $\mathcal{L} := (\nabla \times \nabla \times) + V(r, x_3)$ defined on $D(\mathcal{L}) = H_{G_1}^2(\mathbb{R}^3) \subset L_{G_1}^2(\mathbb{R}^3) \rightarrow L_{G_1}^2(\mathbb{R}^3)$ is selfadjoint and $\sigma(\mathcal{L}) = \sigma(L)$.*

Proof. First we check the symmetry of \mathcal{L} . Let $U, \tilde{U} \in D(\mathcal{L})$, i.e., by Lemma 10,

$$U(x) = u(r, x_3) \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}, \quad \tilde{U}(x) = \tilde{u}(r, x_3) \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}$$

for some $u, \tilde{u} \in D(L)$. Thus

$$\begin{aligned} \langle \mathcal{L}U, \tilde{U} \rangle_{L^2(\mathbb{R}^3)} &= \left\langle \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} (Lu)(r, x_3), \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} \tilde{u}(r, x_3) \right\rangle_{L^2(\mathbb{R}^3)} \\ (4.5) \quad &= 2\pi \langle Lu, \tilde{u} \rangle_{L_{\text{cyl}}^2} \\ &= 2\pi \langle u, L\tilde{u} \rangle_{L_{\text{cyl}}^2} \text{ since } L \text{ is selfadjoint} \\ &= \langle U, \mathcal{L}\tilde{U} \rangle_{L^2(\mathbb{R}^3)}. \end{aligned}$$

To show that \mathcal{L} is selfadjoint it suffices to show that for some $\mu \in \mathbb{R}$ the operator

$$\mathcal{L} - \mu \text{Id} : D(\mathcal{L}) \rightarrow L_{G_1}^2(\mathbb{R}^3)$$

is onto, cf. [11], Theorem 4.2. We choose any μ in the resolvent set of L , e.g. $\mu = -\|V\|_\infty - 1$. Let $F \in L_{G_1}^2(\mathbb{R}^3)$, i.e., there exists $f \in L_{\text{cyl}}^2(r^3 dr dx_3)$ with

$$F(x) = f(r, x_3) \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}.$$

Since μ lies in the resolvent set of L we can find $u \in D(L) = H_{\text{cyl}}^2(r^3 dr dx_3)$ such that $Lu - \mu u = f$. Defining

$$U(x) = u(r, x_3) \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}$$

and using (4.1) we get $\mathcal{L}U - \mu U = F$. This finishes the proof of the selfadjointness of \mathcal{L} and also of the identity of the resolvent sets of \mathcal{L} and of L and hence $\sigma(\mathcal{L}) = \sigma(L)$ follows. \square

We assume now that our cylindrical waveguide geometry is 1-periodic along the x_3 -direction, i.e., $V(r, x_3) = V(r, x_3 + 1)$ for a.a. $r > 0$ and $x_3 \in \mathbb{R}$. Besides the cylindrical symmetry and the periodicity in x_3 -direction, the existence of ground states in the focusing case relies on the assumption that $0 \notin \sigma(\mathcal{L})$. In the following we will construct an example of a potential

$V = V(r, x_3)$ which is 1-periodic w.r.t. x_3 , with $0 \notin \sigma(\mathcal{L})$ and with $\sigma(\mathcal{L}) \cap (-\infty, 0) \neq \emptyset$. Recall that the physical significance of the sign of V has been explained at the beginning of this section.

Let us assume that the linear potential V is separable, i.e.

$$(4.6) \quad V(r, x_3) = W(r) + P(x_3).$$

with $W \in L^\infty(0, \infty)$, $P \in L^\infty(\mathbb{R})$ and $P(x_3 + 1) = P(x_3)$ for all $x_3 \in \mathbb{R}$ (later we will assume that P is piecewise continuous in order to have (4.7)). The splitting of the potential implies a splitting of the operator L as follows (so far we consider this only on a formal level): if $u(r, x_3) = v(r)w(x_3)$ then

$$(Lu)(r, x_3) = w(x_3)(L_r v)(r) + v(r)(L_p w)(x_3)$$

where L_r, L_p are given by the following differential expressions

$$L_r = -\frac{1}{r^3} \frac{\partial}{\partial r} \left(r^3 \frac{\partial}{\partial r} \right) + W(r), \quad L_p = -\frac{\partial^2}{\partial x_3^2} + P(x_3).$$

We can give these differential expressions the meaning of proper selfadjoint operators by specifying their domains of definition properly as in the following lemma.

Lemma 13. *Let $W \in L^\infty(0, \infty)$, $P \in L^\infty(\mathbb{R})$. The operator L_p defined on $D(L_p) = H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is selfadjoint. The operator L_r defined on $D(L_r) = H_{\text{rad}}^2(r^3 dr) \subset L_{\text{rad}}^2(r^3 dr) \rightarrow L_{\text{rad}}^2(r^3 dr)$ is selfadjoint.*

Proof. The statement for L_p is clear. The statement for L_r follows from the observation that for $v \in H_{\text{rad}}^2(r^3 dr)$

$$(L_r v) \circ \Psi_{\text{rad}} = -\Delta(v \circ \Psi_{\text{rad}}) + (Wv) \circ \Psi_{\text{rad}},$$

i.e., L_r coincides with the four dimensional Schrödinger operator $-\Delta + W(r)$ with radial symmetry. \square

Now we are in a position to state that for $V(r, x_3) = W(r) + P(x_3)$ the spectrum of \mathcal{L} can be computed by separation of variables.

Lemma 14. *Let $W \in L^\infty(0, \infty)$, $P \in L^\infty(\mathbb{R})$ and $V(r, x_3) = W(r) + P(x_3)$. Then $\sigma(\mathcal{L}) = \sigma(L) = \sigma(L_r) + \sigma(L_p)$.*

Proof. Let us define the subspace

$$D_0 = \left\{ \sum_{k=1}^N v_k(r) w_k(x_3) : N \in \mathbb{N}, v_k \in D(L_r), w_k \in D(L_p) \text{ for } k = 1, \dots, N. \right\}$$

and the operator $L_r + L_p$ on D_0 by

$$(L_r + L_p) \left(\sum_{k=1}^N v_k w_k \right) := \sum_{k=1}^N ((L_r v_k) w_k + v_k (L_p w_k)).$$

The subspace D_0 is dense in $L^2_{\text{cyl}}(r^3 dr dx_3)$ because $D(L_r) \subset L^2_{\text{rad}}(r^3 dr)$ and $D(L_p) \subset L^2(\mathbb{R})$ are dense. Since $L_r + L_p$ is symmetric, it is therefore closable. Let us recall the definition of the closure of an operator and its domain:

$$D(\overline{L_r + L_p}) = \{u \in L^2_{\text{cyl}}(r^3 dr dx_3) : \exists (u_n)_{n \in \mathbb{N}} \text{ in } D_0, z \in L^2_{\text{cyl}}(r^3 dr dx_3) \text{ s.t.} \\ u_n \rightarrow u \text{ in } L^2_{\text{cyl}}(r^3 dr dx_3) \text{ and } (L_r + L_p)u_n \rightarrow z \text{ in } L^2_{\text{cyl}}(r^3 dr dx_3)\}.$$

For $u \in D(\overline{L_r + L_p})$ one defines $\overline{L_r + L_p}(u) := z$. By Theorem VIII.33 and its Corollary from [24], the selfadjointness of L_r , L_p is passed on and yields selfadjointness of $\overline{L_r + L_p}$. Note also that $L_r + L_p = L|_{D_0}$ and hence $\overline{L_r + L_p} = \overline{L|_{D_0}}$. Since by Lemma 11 the operator L defined on $H^2_{\text{cyl}}(r^3 dr dx_3)$ is a selfadjoint extension of $L|_{D_0}$ and hence also of the operator $\overline{L|_{D_0}} = \overline{L_r + L_p}$ which is already selfadjoint, we find that $\overline{L_r + L_p} = L$. Again by Theorem VIII.33 and its Corollary from [24] we find the claim $\sigma(L) = \sigma(\overline{L_r + L_p}) = \sigma(L_r) + \sigma(L_p) = \sigma(L_r) + \sigma(L_p)$, where the last equality holds since the two spectra $\sigma(L_r), \sigma(L_p)$ are closed and bounded from below. \square

Next to the periodicity and boundedness of P let us now sharpen the assumption by requiring additionally that P is piecewise continuous. Then the spectrum of L_p is purely continuous and consists of the union of countably many intervals

$$(4.7) \quad \sigma(L_p) = \bigcup_{k=1}^{\infty} [\nu_{2k-1}, \nu_{2k}] \text{ with } \nu_{2k-1} < \nu_{2k} \leq \nu_{2k+1},$$

see Theorem XIII.90 in [23]. We assume that the first gap is open, i.e.,

$$\nu_2 < \nu_3.$$

Next we describe the radial part of the spectrum of L under some special assumptions on the potential W . We start with some properties of Bessel functions.

Lemma 15. *Let J_1 denote the order one Bessel function which is regular at 0 and K_1 the order one modified Bessel function which decreases exponentially at infinity. Let $0 < j_1 < j_2 < \dots$ be the positive zeroes of J_1 and $0 < j'_1 < j'_2 < \dots$ be the positive zeroes of J'_1 . Let*

$$\eta_* = \sqrt{(j_1)^2 - (j'_1)^2}, \quad \eta^* := \sqrt{(j'_2)^2 - (j_1)^2}.$$

Then $\eta_ < \eta^*$ and for every $\eta \in [\eta_*, \eta^*]$ there exists a unique value $\xi = \xi(\eta) \in (j'_1, j_1)$ with the properties*

$$(4.8) \quad \frac{J_1(\xi)}{\xi J'_1(\xi)} = \frac{K_1(\eta)}{\eta K'_1(\eta)} \quad \text{and} \quad (j_1)^2 < \xi^2 + \eta^2 < (j'_2)^2.$$

Proof. Define $\tilde{g} : (0, j'_2) \setminus \{j'_1\} \rightarrow \mathbb{R}$ and $\tilde{h} : (0, \infty) \rightarrow \mathbb{R}$ by

$$\tilde{g}(\xi) := \frac{J_1(\xi)}{\xi J'_1(\xi)}, \quad \tilde{h}(\eta) := \frac{K_1(\eta)}{\eta K'_1(\eta)}.$$

Let us mention that the properties of \tilde{g}, \tilde{h} used in this proof are proved in Lemma 23 in the Appendix. Since $\tilde{g}(\xi) \rightarrow 1$ as $\xi \rightarrow 0+$ and $\tilde{h}(\eta) \rightarrow -1$ as $\eta \rightarrow 0+$, the two functions can be

extended continuously to 0. Moreover, on $[0, j'_1)$ the function \tilde{g} is strictly increasing from 1 to $+\infty$, on (j'_1, j'_2) it increases strictly from $-\infty$ to $+\infty$ with a zero at j_1 . The function \tilde{h} is negative and strictly increases on $[0, \infty)$ from -1 to 0. Suppose a value $\eta > 0$ is given. By the strict monotonicity of \tilde{g} we can find a unique solution $\xi = \xi(\eta)$ of $\tilde{g}(\xi) = \tilde{h}(\eta)$ within the interval (j'_1, j_1) . Now we want to ensure that the pair $(\xi(\eta), \eta)$ satisfies the constraint in (4.8). Since $\xi(\eta) \in (j'_1, j_1)$ the constraint $(j_1)^2 < \xi^2 + \eta^2 < (j'_2)^2$ is certainly satisfied if we impose the following restriction on η :

$$\eta^2 \in ((j_1)^2 - (j'_1)^2, (j'_2)^2 - (j_1)^2) = (\eta_*^2, \eta^{*2}).$$

Note that $\eta^* > \eta_*$ is equivalent to $2(j_1)^2 < (j'_1)^2 + (j'_2)^2$. This inequality can be checked using the numerical values in Table 9.5 of [2]. Up to an error in omitted digits the values are $j_1 = 3.83171$, $j'_1 = 1.84118$, and $j'_2 = 5.33144$ so that $2(j_1)^2 < (j'_1)^2 + (j'_2)^2$ holds. \square

Lemma 16. *Assume $\nu_2 < \nu_3$ and choose values μ_0, W_∞ such that*

$$(4.9) \quad -\nu_3 < \mu_0 < -\nu_2 < -\nu_1 < W_\infty.$$

Let $\eta \in [\eta_, \eta^*]$ and let $\xi(\eta)$ be as in Lemma 15. If we define*

$$\delta := \frac{\eta}{\sqrt{W_\infty - \mu_0}} \quad \text{and} \quad W_0 := \mu_0 - \left(\frac{\xi(\eta)}{\delta} \right)^2$$

as well as

$$(4.10) \quad W(r) = \begin{cases} W_0, & 0 \leq r < \delta, \\ W_\infty, & r \geq \delta, \end{cases}$$

then μ_0 is an eigenvalue of L_r . There are no other eigenvalues below the essential spectrum $[W_\infty, \infty)$ and hence

$$\sigma(L_r) = \{\mu_0\} \cup [W_\infty, \infty).$$

Proof. Due to the form of W as in (4.10) we have $\sigma(L_r) \subset [W_0, \infty)$ and $\sigma_{\text{ess}}(L_r) = [W_\infty, \infty)$. Now consider the eigenvalue equation

$$(4.11) \quad -u''(r) - \frac{3}{r}u'(r) = (-W(r) + \mu)u.$$

Let us check that neither W_0 nor W_∞ are eigenvalues of L_r . First suppose $\mu = W_0$ is an eigenvalue. Note that $-W(r) + W_0 \leq 0$ on $[0, \infty)$ and < 0 on (δ, ∞) . Multiplication of (4.11) with a corresponding eigenfunction u and integration with respect to the measure $r^3 dr$ on $(0, \infty)$ yields a positive left hand side and a negative right hand side. Hence $\mu = W_0$ is not an eigenvalue. Now suppose $\mu = W_\infty$ is an eigenvalue and u is a corresponding eigenfunction. Then $-W(r) + W_0 = 0$ for $r \geq \delta$ so that $u(r) = \text{const} \cdot r^{-2}$ on $[\delta, \infty)$. But no matter how r^{-2} extends to $[0, \delta)$ the function u does not belong to $L^2_{\text{rad}}(r^3 dr)$ because $\int_\delta^\infty r^{-4} \cdot r^3 dr = \infty$. So $\mu = W_\infty$ is also not an eigenvalue of L_r .

Since $\min \sigma(L_r) \geq W_0$ and since neither W_0 nor W_∞ are eigenvalues of L_r , we are looking for solutions of (4.11) with $W_0 < \mu < W_\infty$. As $W(r)$ only takes the values W_0, W_∞ , equation

(4.11) is transformed via $u(r) = r^{-1}v(\sqrt{\mu - W_0}r)$, $s := \sqrt{\mu - W_0}r$ into

$$s^2 v'' + s v' + (s^2 - 1)v = 0 \text{ for } 0 \leq s \leq \delta \sqrt{\mu - W_0}$$

and via $u(r) = r^{-1}w(\sqrt{W_\infty - \mu}r)$, $s := \sqrt{W_\infty - \mu}r$ into

$$s^2 w'' + s w' - (s^2 + 1)w = 0 \text{ for } \delta \sqrt{W_\infty - \mu} \leq s < \infty.$$

Thus, $v(s) = \alpha J_1(s)$ is a multiple of the order one Bessel function which is regular at 0 and $w(s) = \beta K_1(s)$ is a multiple of the order one modified Bessel function which is exponentially decaying at infinity. Altogether we obtain

$$u(r) = \begin{cases} \alpha r^{-1} J_1(\sqrt{\mu - W_0}r) & \text{for } 0 \leq r \leq \delta, \\ \beta r^{-1} K_1(\sqrt{W_\infty - \mu}r) & \text{for } \delta \leq r < \infty. \end{cases}$$

We need to choose α, β, μ in order to obtain a C^1 -function at $r = \delta$. This leads to the equation

$$g(\mu) := \frac{J_1(\sqrt{\mu - W_0}\delta)}{\sqrt{\mu - W_0}\delta J_1'(\sqrt{\mu - W_0}\delta)} = \frac{K_1(\sqrt{W_\infty - \mu}\delta)}{\sqrt{W_\infty - \mu}\delta K_1'(\sqrt{W_\infty - \mu}\delta)} =: h(\mu)$$

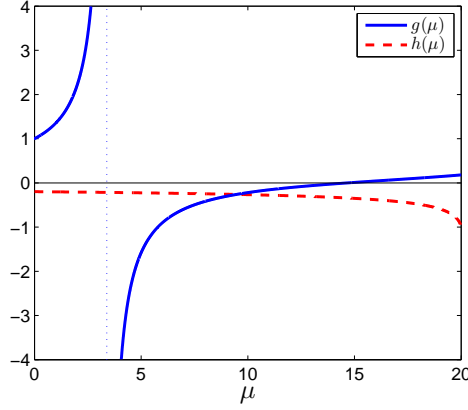
and our choice of W_0 and δ such that $\sqrt{\mu_0 - W_0}\delta = \xi$ and $\sqrt{W_\infty - \mu_0}\delta = \eta(\xi)$ guarantees the C^1 -matching at $\mu = \mu_0$. We have therefore verified that μ_0 is indeed an eigenvalue of L_r . It remains to show that there is no other eigenvalue.

We analyze the two sides of the equation $g(\mu) = h(\mu)$ independently. We have already mentioned in the proof of the preceding lemma that $\frac{K_1(x)}{xK_1'(x)}$ is a negative and increasing function of x with $\lim_{x \rightarrow 0+} \frac{K_1(x)}{xK_1'(x)} = -1$. Likewise the function $\frac{J_1(x)}{xJ_1'(x)}$ satisfies $\lim_{x \rightarrow 0+} \frac{J_1(x)}{xJ_1'(x)} = 1$, has its zeroes at j_1, j_2, \dots and its poles at j_1', j_2', \dots and is increasing between 0 and j_1' and between two consecutive poles. The proof of these statements is given in the Appendix. Thus, as μ runs through $[W_0, W_\infty]$, the function $g(\mu)$ starts from the value 1 and increases up to its first pole at $W_0 + \left(\frac{j_1'}{\delta}\right)^2$. On the interval $(W_0 + \left(\frac{j_1'}{\delta}\right)^2, W_0 + \left(\frac{j_2'}{\delta}\right)^2]$ it increases from $-\infty$ to 0 and on $[W_0 + \left(\frac{j_2'}{\delta}\right)^2, W_\infty]$ it increases from 0 to a positive value. The function $h(\mu)$ stays negative and strictly decreases to the value -1 as μ ranges through the interval $[W_0, W_\infty]$, cf. Figure 4 for a plot of an example of the two functions. Therefore, on $[W_0, W_\infty]$ the two functions g, h intersect exactly once provided W_∞ lies between the first zero and the second pole of g , i.e., provided

$$(4.12) \quad \left(\frac{j_1'}{\delta}\right)^2 < W_\infty - W_0 < \left(\frac{j_2'}{\delta}\right)^2.$$

(4.12) is equivalent to $j_1^2 < \xi^2 + \eta^2 < (j_2')^2$. The latter is guaranteed by our choice of ξ, η and from Lemma 15. \square

As we have seen from Lemma 16, the piecewise constant function W given in (4.10) together with the choice of μ_0, W_∞ , the restriction on η and the definition of δ and W_0 ensures the existence of exactly one eigenvalue of L_r below the essential spectrum. Therefore these

FIGURE 1. Functions g and h for $\delta = 1, W_0 = 0, W_\infty = 20$.

conditions can be considered as assumptions that ensure guiding of a single linear mode in the cylindrical waveguide.

Lemma 17. *Assume that P is piecewise continuous, 1-periodic such that $\sigma(L_p)$ has a bounded first open gap. Assume that W is as in (4.10) and μ_0, W_∞, δ and W_0 are chosen as in Lemma 16. Then 0 is not in the spectrum of L .*

Proof. Because of Lemma 14, Lemma 16 and (4.7) we see that

$$\sigma(L) = \bigcup_{k=1}^{\infty} [\mu_0 + \nu_{2k-1}, \mu_0 + \nu_{2k}] \cup [\nu_1 + W_\infty, \infty).$$

Thus, the inequalities $-\nu_3 < \mu_0 < -\nu_2 < -\nu_1 < W_\infty$ from Lemma 16 guarantee that 0 lies in the resolvent of L . \square

Remark. Let us verify that the constructed potential V takes positive values on sets of positive measure for $r > \delta$. Since $V(x) = -\frac{\omega^2}{c^2}n^2(x)$, this implies the unphysical situation of an imaginary refractive index. It is an open problem to construct a negative $V = V(r, x_3)$, which satisfies the condition $0 \notin \sigma(\mathcal{L})$. First note that

$$\begin{aligned} \nu_1 = \min \sigma(L_p) &= \inf_{\psi \in H^1(\mathbb{R}) \setminus \{0\}} \frac{\int_{\mathbb{R}} \psi'^2 + P(x)\psi^2 dx}{\int_{\mathbb{R}} \psi^2 dx} \\ &< \inf_{\psi \in H^1(\mathbb{R}) \setminus \{0\}} \frac{\int_{\mathbb{R}} \psi'^2 + (\text{ess sup } P)\psi^2 dx}{\int_{\mathbb{R}} \psi^2 dx} \\ &= \text{ess sup } P, \end{aligned}$$

where the strict inequality comes from the fact that P is a periodic potential which generates an operator with a true first open gap and hence must be non-constant. Together with $W_\infty > -\nu_1$, cf. (4.9) we obtain

$$\text{ess sup } V = W_\infty + \text{ess sup } P > 0.$$

5. GROUND STATES IN THE FOCUSING CASE

We assume

$$(H\text{-foc}) \quad 1 < p < 5, \quad V, \Gamma \in L^\infty(\mathbb{R}^3), \quad \operatorname{ess\,inf}_{\mathbb{R}^3} \Gamma > 0, \quad \text{and} \quad 0 \notin \sigma(\mathcal{L}).$$

For the existence of a Palais-Smale sequence our only assumption is (H-foc). In the previous section we found examples of cylindrical potentials V of the form $V(r, x_3) = W(r) + P(x_3)$ with periodicity of P which produce $0 \notin \sigma(\mathcal{L})$. Nevertheless, also other potentials V may satisfy this condition. Note that for the potential V from the previous section the resulting operator \mathcal{L} has negative spectrum, positive spectrum and 0 is in the resolvent set. The analysis of this chapter will, however, work also in the case where $0 < \min \sigma(\mathcal{L})$.

For the final step of the proof, i.e. the non-triviality of the limit of a suitable Palais-Smale sequence and the existence of a nontrivial ground state, we use the assumption of cylindrical symmetry and periodicity in x_3 of both V and Γ .

Since \mathcal{L} is a selfadjoint operator on

$$D(\mathcal{L}) = H_{G_1}^2(\mathbb{R}^3) \subset L_{G_1}^2(\mathbb{R}^3)$$

the spectral theorem yields the existence of a spectral resolution $(P_\lambda)_{\lambda \in \mathbb{R}}$ and hence of projections

$$P^+ = \int_0^\infty 1 \, dP_\lambda, \quad P^- = \int_{-\infty}^0 1 \, dP_\lambda$$

and of the spectral decomposition $H_{G_1}^1(\mathbb{R}^3) = H^+ \oplus H^-$ with $H^\pm = P^\pm(H_{G_1}^1(\mathbb{R}^3))$. We define $U^\pm = P^\pm U$ for all $U \in H_{G_1}^1(\mathbb{R}^3)$. Notice that \mathcal{L} defines the bilinear form $b(U, V) := \int_{\mathbb{R}^3} (\nabla \times U) \cdot (\nabla \times V) + V(x)U \cdot V \, dx$ which is positive/negative definite on the spaces H^\pm by construction. Therefore, we may define the scalar product

$$\langle U, V \rangle := b(U^+, V^+) - b(U^-, V^-) \quad \text{on } H_{G_1}^1(\mathbb{R}^3) \times H_{G_1}^1(\mathbb{R}^3)$$

which has the property that H^+ is orthogonal to H^- , i.e., that P^\pm are orthogonal projections. We denote the norm on $H_{G_1}^1(\mathbb{R}^3)$ corresponding to $\langle \cdot, \cdot \rangle$ by $||| \cdot |||$. It is equivalent to the H^1 -norm $\| \cdot \|_{H^1}$ and has the property

$$|||U||| = \pm \int_{\mathbb{R}^3} |\nabla \times U|^2 + V(x)|U|^2 \, dx \quad \text{for all } U \in H^\pm$$

such that

$$|||U|||^2 = |||U^+|||^2 + |||U^-|||^2 \quad \text{for all } U \in H_{G_1}^1(\mathbb{R}^3).$$

If the potential V has the unphysical property that $\operatorname{ess\,inf} V > 0$ (imaginary refractive index in all of \mathbb{R}^3) then $H^- = \{0\}$. But in general $H^- \neq \{0\}$ and therefore the functional J has linking geometry. In any case J is unbounded from below on $H_{G_1}^1(\mathbb{R}^3)$ so that a direct minimization is impossible. Therefore we choose to minimize over the Nehari-Pankov manifold

$$N := \{U \in H_{G_1}^1(\mathbb{R}^3) \setminus \{0\} : J'[U]\Phi = 0 \, \forall \Phi \in [U] \oplus H^-\}.$$

This approach is analogous to that in [22]. Later, in Lemma 21, we will see that the constraint set N does not produce Lagrange multipliers. Note also that for $U \in N$

$$J[U] = \frac{p-1}{2(p+1)} \int_{\mathbb{R}^3} \Gamma(x) |U|^{p+1} dx = \frac{p-1}{2(p+1)} \int_{\mathbb{R}^3} |\nabla \times U|^2 + V(x) |U|^2 dx.$$

Remarks: (a) Another common approach to obtain a critical point of J is minimization under the constraint $\|U\|_{L^2} = 1$. This however produces a Lagrange multiplier κ , which is generally nonzero, so that the minimizer does not solve (1.1) but the equation $\mathcal{L}U + \kappa U = \Gamma(r, x_3) |U|^{p-1} U$.

(b) If $H^- = \{0\}$ then another common approach consists in minimizing $K(U) = \int_{\mathbb{R}^3} |\nabla \times U|^2 + V(x) |U|^2 dx$ under the constraint $\|U\|_{\Gamma, p+1} = 1$. This produces a Lagrange multiplier which however can be scaled out. If H^- is not trivial then one may still find a critical point of K under this constraint.

Lemma 18. *Under the assumption (H-foc) there exist values $\epsilon_1, \epsilon_2, C > 0$ such that*

$$\|U\|_{H^1} \geq \epsilon_1, \quad J[U] \geq \epsilon_2, \quad \|U\|_{H^1} \leq C J[U]^{\frac{p}{p+1}}$$

for all $U \in N$.

Proof. For $U \in N$ we have

$$\begin{aligned} |||U^+|||^2 &= \int_{\mathbb{R}^3} |\nabla \times U^+|^2 + V(x) |U^+|^2 dx \\ &= \langle U, U^+ \rangle \\ &= J'[U] U^+ + \int_{\mathbb{R}^3} \Gamma(x) |U|^{p-1} U \cdot U^+ dx \\ &= \underbrace{J'[U] (U - U^-)}_{=0} + \int_{\mathbb{R}^3} \Gamma(x) |U|^{p-1} U \cdot U^+ dx, \end{aligned}$$

where the first term in the last equation vanishes due to the definition of the manifold N . Hence, by using Hölder's and Sobolev's inequality and again the definition of N we obtain

$$\begin{aligned} (5.1) \quad |||U^+|||^2 &\leq \left(\int_{\mathbb{R}^3} (\Gamma(x) |U|^p)^{\frac{p+1}{p}} dx \right)^{\frac{p}{p+1}} \left(\int_{\mathbb{R}^3} |U^+|^{p+1} dx \right)^{\frac{1}{p+1}} \\ &\leq C_1 \|\Gamma\|_{\infty}^{\frac{1}{p+1}} \left(\int_{\mathbb{R}^3} \Gamma(x) |U|^{p+1} dx \right)^{\frac{p}{p+1}} \|U^+\|_{H^1} \\ &\leq C_2 \left(\frac{2(p+1)}{p-1} J[U] \right)^{\frac{p}{p+1}} |||U^+|||. \end{aligned}$$

We may repeat the above argument with U^- and find

$$\begin{aligned} |||U^-|||^2 &= - \int_{\mathbb{R}^3} |\nabla \times U^-|^2 + V(x)|U^-|^2 dx \\ &= \langle U, U^- \rangle \\ &= - \underbrace{J'[U]U^-}_{=0} - \int_{\mathbb{R}^3} \Gamma(x)|U|^{p-1}U \cdot U^- dx, \end{aligned}$$

and from there the same Hölder- and Sobolev-estimates as before lead to

$$|||U^-||| \leq C_2 \left(\frac{2(p+1)}{p-1} J[U] \right)^{\frac{p}{p+1}}.$$

Together with (5.1) this establishes the third of the three claims.

To see the first of the three claims, we use (5.1) and the corresponding estimate for U^- to get

$$|||U^+|||^2 \leq C_3 \|U\|_{L^{p+1}}^p |||U^+|||, \quad |||U^-|||^2 \leq C_3 \|U\|_{L^{p+1}}^p |||U^-|||$$

from which we obtain $|||U||| \leq \sqrt{2}C_3 \|U\|_{L^{p+1}}^p$. Due to the Sobolev inequality and since $|||\cdot|||$ and $\|\cdot\|_{H^1}$ are equivalent we also have $\|U\|_{L^{p+1}}^p \leq C_4 |||U|||^p$. Hence $|||U||| \leq \sqrt{2}C_3C_4 |||U|||^p$ and thus $\|U\|_{H^1} \leq C_5 \|U\|^p$. Since $U \neq 0$ by the definition of N we obtain the first of the three estimates. Finally, the second estimate follows from the first and the third. \square

Lemma 19. *Assume (H-foc). The map*

$$G : \begin{cases} H_{G_1}^1(\mathbb{R}^3) \setminus H^- & \rightarrow H_{G_1}^1(\mathbb{R}^3), \\ U & \mapsto \langle \nabla J[U], U^+ \rangle \frac{U^+}{|||U^+|||^2} + P^- \nabla J[U]. \end{cases}$$

is a C^1 -map. If $U \in N$ and $X_U := [U] + H^-$ then the map

$$\partial_{X_U} G(U) : X_U \rightarrow X_U$$

is negative definite uniformly with respect to bounded subsets of N , i.e., if $N_0 \subset N$ is bounded, then there exists a value $\delta > 0$ such that

$$\langle \partial_{X_U} G(U)v, v \rangle \leq -\delta |||v|||^2 \text{ for all } v \in X_U \text{ and all } U \in N_0.$$

In particular $\partial_{X_U} G(U) : X_U \rightarrow X_U$ has a bounded inverse.

Proof. For $U \in N$ and $v \in H_{G_1}^1(\mathbb{R}^3)$ we have

$$\begin{aligned} G'(U)v &= \langle \nabla^2 J[U]v, U^+ \rangle \frac{U^+}{|||U^+|||^2} + \langle \nabla J[U], v^+ \rangle \frac{U^+}{|||U^+|||^2} + \langle \nabla J[U], U^+ \rangle \frac{v^+}{|||U^+|||^2} \\ &\quad - 2\langle U^+, v \rangle \langle \nabla J[U], U^+ \rangle \frac{U^+}{|||U^+|||^4} + (\nabla^2 J[U]v)^-. \end{aligned}$$

If we take $v \in X_U = [U] + H^- = [U^+] + H^-$, i.e., $v = tU + w$ with $t \in \mathbb{R}$, $w \in H^-$ then the above formula simplifies to

$$\partial_{X_U} G(U)v = \langle \nabla^2 J[U]v, U^+ \rangle \frac{U^+}{\|U^+\|^2} + (\nabla^2 J[U]v)^- = P_{X_U} \nabla^2 J[U]v,$$

where $P_{X_U} : H \rightarrow X_U$ denotes the orthogonal projection with respect to $\langle \cdot, \cdot \rangle$. Therefore

$$\begin{aligned} \langle \partial_{X_U} G(U)v, v \rangle &= \langle \nabla^2 J[U]v, v \rangle \\ &= t^2 \int_{\mathbb{R}^3} |\nabla \times U|^2 + V(x)|U|^2 - p\Gamma(x)|U|^{p+1} dx \\ &\quad + \int_{\mathbb{R}^3} |\nabla \times w|^2 + V(x)|w|^2 - p\Gamma(x)|U|^{p-1}|w|^2 dx \\ &\quad + 2t \int_{\mathbb{R}^3} \nabla \times U \cdot \nabla \times w + V(x)U \cdot w - p\Gamma(x)|U|^{p-1}U \cdot w dx \end{aligned}$$

and by using $U \in N$ we obtain

$$\langle \partial_{X_U} G(U)v, v \rangle = - \|w\|^2 - \int_{\mathbb{R}^3} \Gamma(x)|U|^{p-1} (t^2(p-1)|U|^2 + p|w|^2 + 2t(p-1)U \cdot w) dx.$$

Now we use the identity

$$\begin{aligned} t^2(p-1)|U|^2 + p|w|^2 + 2t(p-1)U \cdot w &= t^2 \frac{p-1}{p} |U|^2 + \left| \sqrt{p}w + \frac{p-1}{\sqrt{p}}tU \right|^2 \\ &\geq t^2 \frac{p-1}{p} |U|^2 \end{aligned}$$

to deduce

$$\langle \partial_{X_U} G(U)v, v \rangle \leq - \|w\|^2 - t^2 \frac{p-1}{p} \int_{\mathbb{R}^3} \Gamma(x)|U|^{p+1} dx.$$

Next we obtain from Lemma 18 that $J[U] = (\frac{1}{2} - \frac{1}{p+1}) \int_{\mathbb{R}^3} \Gamma(x)|U|^{p+1} dx \geq \epsilon_2$ and hence we find

$$\langle \partial_{X_U} G(U)v, v \rangle \leq - \|w\|^2 - 2t^2 \frac{p+1}{p} \epsilon_2 \leq -C \|w + tU\|^2$$

by using the boundedness of N_0 . This finishes the proof. \square

Lemma 20. *Assume (H-foc). The set N is a C^1 -manifold such that*

$$N = G^{-1}\{0\} \text{ and } T_U N = \text{Ker } G'(U)$$

for every $U \in N$.

Proof. As before, for $U \in N$ let $X_U = [U] + H^-$ and denote by $P_{X_U} : H_{G_1}^1(\mathbb{R}^3) \rightarrow X_U$ the orthogonal projection w.r.t. $\langle \cdot, \cdot \rangle$. Recall that the map $G : H_{G_1}^1(\mathbb{R}^3) \setminus H^- \rightarrow H_{G_1}^1(\mathbb{R}^3)$ satisfies

$$(5.2) \quad G(X_U \setminus H^-) \subset X_U \text{ for all } U \in H_{G_1}^1(\mathbb{R}^3) \setminus H^-.$$

Now fix $U \in N = G^{-1}(\{0\})$. There exists an open neighbourhood \mathcal{O} of U in $H_{G_1}^1(\mathbb{R}^3) \setminus H^-$ such that

$$(5.3) \quad X_V \cap X_U^\perp = \{0\} \text{ for all } V \in \mathcal{O}.$$

For $V \in \mathcal{O}$ we have by (5.2), (5.3) the equivalence

$$G(V) = 0 \Leftrightarrow (P_{X_U} \circ G)(V) = 0.$$

This shows that $N \cap \mathcal{O} = (P_{X_U} \circ G|_{\mathcal{O}})^{-1}(\{0\})$. But the map $P_{X_U} \circ G|_{\mathcal{O}}: \mathcal{O} \rightarrow X_U$ is a submersion, i.e., its derivative is surjective at every point of \mathcal{O} , because

$$(P_{X_U} \circ G)'(V) = P_{X_U} \circ G'(V) : H_{G_1}^1(\mathbb{R}^3) \xrightarrow{G'(V)} X_V \xrightarrow{P_{X_U}} X_U$$

and the first map $G'(V) : H_{G_1}^1(\mathbb{R}^3) \rightarrow X_V$ is surjective by Lemma 19 and the second map $P_{X_U} : X_V \rightarrow X_U$ is an isomorphism by (5.3). Therefore, the submersion theorem of [1], Theorem 3.5.4, applies and states that $N \cap \mathcal{O}$ is a submanifold of $H_{G_1}^1(\mathbb{R}^3) \setminus H^-$ and $T_U N = \text{Ker } G'(U)$. \square

Notice that any nontrivial solution $U \in H_G^1(\mathbb{R}^3)$ of (1.1) belongs to the Nehari-Pankov manifold N . As a consequence, one can show that the constraint N produces a zero Lagrange multiplier. The following is a much stronger statement.

Lemma 21. *Assume (H-foc). Let N_0 be a bounded subset of N . Then there exists a constant $C_0 > 0$ such that the following holds: if $U \in N_0$ and $\nabla J[U] = \tau + \sigma$ where $\tau \in T_U N$ is the tangential component of $\nabla J[U]$ and $\sigma \perp \tau$ is the transversal component of $\nabla J[U]$ then*

$$\|\nabla J[U]\|_{H^1} \leq C_0 \|\tau\|_{H^1}.$$

Proof. By Lemma 19 the map $\partial_{X_U} G(U) : X_U \rightarrow X_U$ has a bounded inverse and hence a closed range. Moreover, $\partial_{X_U} G(U)|_{X_U} = P_{X_U} \nabla^2 J[U]$ is symmetric as a composition of a second derivative and an orthogonal projection. Therefore

$$\text{Rg } \partial_{X_U} G(U)|_{X_U} = (\text{Ker } \partial_{X_U} G(U)|_{X_U})^\perp = (T_U N)^\perp.$$

If we consider

$$\nabla J[U] = \tau + \sigma \text{ with } \tau \in T_U N \text{ and } \sigma \in (T_U N)^\perp$$

then there exists $h \in X_U$ such that

$$(5.4) \quad \nabla J[U] = \tau + \partial_{X_U} G(U)h.$$

Hence, using $h \in X_U = [U] + H^-$ and thus $\langle \nabla J[U], h \rangle = 0$ we get from Lemma 19

$$\langle \partial_{X_U} G(U)h, h \rangle = \langle -\tau, h \rangle \leq -\delta \|h\|_{H^1}^2.$$

Using the Cauchy-Schwarz inequality and the equivalence of the norms $\|\cdot\|_{H^1}$ and $\|\cdot\|$ we get

$$\|h\|_{H^1} \leq C_0 \|\tau\|_{H^1}.$$

By (5.4) and the boundedness of $G'(U)$ on bounded subsets of N this implies the claim. \square

By the previous lemma the tangential component of the gradient of J at a point in N controls the entire gradient. As a consequence there exists special minimizing sequences of

$J|_N$, where the tangential part of the gradient converges to zero (a consequence of Ekeland's variational principle) and hence the entire gradient converges to 0.

Lemma 22. *Assume (H-foc). There exists a bounded Palais-Smale sequence $(U_k)_{k \in \mathbb{N}}$ in N such that*

$$J[U_k] \rightarrow c := \inf_N J, \quad J'[U_k] \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Proof. As a consequence of Ekeland's variational principle, cf. Struwe [25], there exists a minimizing sequence $(U_k)_{k \in \mathbb{N}}$ of $J|_N$ such that $(J|_N)'(U_k) \rightarrow 0$, hence $J'(U_k) \rightarrow 0$ by Lemma 21. \square

Proof of Theorem 3: Let $(U_k)_{k \in \mathbb{N}}$ be the Palais-Smale sequence from Lemma 22. If any subsequence of $(U_k)_{k \in \mathbb{N}}$ converges in $L^{p+1}(\mathbb{R}^3)$ to zero, then by the definition of the Nehari-Pankov manifold this sequence also converges to zero in $H^1(\mathbb{R}^3)$. This is impossible by Lemma 18. Therefore, the concentration compactness principle (cf. Lions [15] or Lemma 1.21 in Willem [29] suitably adapted to the vectorial case) implies that for every radius $R > 0$

$$\liminf_{k \in \mathbb{N}} \sup_{y \in \mathbb{R}^3} \int_{B_R(y)} |U_k|^2 dx > 0.$$

Fix a value $R > 0$. Then there exists a subsequence again denoted $(U_k)_{k \in \mathbb{N}}$, centers $y_k \in \mathbb{R}^3$ and $\eta > 0$ such that

$$(5.5) \quad \int_{B_R(y_k)} |U_k|^2 dx \geq \eta \text{ for all } k \in \mathbb{N}.$$

By possibly adding the value 1 to R we may assume that $y_k^3 \in \mathbb{Z}$ and (5.5) still holds. Now we claim that $\rho_k^2 := (y_k^1)^2 + (y_k^2)^2$ is bounded. Assume the contrary. Due to the symmetries in $H_{G_1}^1(\mathbb{R}^3)$ we have that

$$\int_{B_R(y_k)} |U_k|^2 dx = \int_{B_R(\tilde{y}_k)} |U_k|^2 dx$$

whenever the point \tilde{y}_k is such that

$$(P) \quad y_k^3 = \tilde{y}_k^3 \text{ and } (y_k^1)^2 + (y_k^2)^2 = (\tilde{y}_k^1)^2 + (\tilde{y}_k^2)^2.$$

Notice that the number of disjoint balls $B_R(\tilde{y}_k)$ with centers \tilde{y}_k satisfying (P) tends to infinity if $\rho_k \rightarrow \infty$ as $k \rightarrow \infty$. But this is impossible since the L^2 -norm of $(U_k)_{k \in \mathbb{N}}$ is bounded. Thus, if we define

$$\rho := R + \sup_{k \in \mathbb{N}} \rho_k$$

then

$$\int_{B_\rho(0,0,y_k^3)} |U_k|^2 dx \geq \eta \text{ for all } k \in \mathbb{N}.$$

Set

$$\bar{U}_k(x_1, x_2, x_3) := U_k(x_1, x_2, x_3 + y_k^3).$$

Then, due to the periodicity of V, Γ in the x_3 -variable we have

$$\bar{U}_k \in N, \quad J[\bar{U}_k] = J[U_k] \rightarrow c, \quad \|J'[\bar{U}_k]\|_* = \|J'[U_k]\|_* \rightarrow 0 \text{ as } k \rightarrow \infty$$

and

$$\int_{B_\rho(0)} |\bar{U}_k|^2 dx \geq \eta \text{ for all } k \in \mathbb{N}.$$

Now we may take a weakly converging subsequence (again denoted by $(\bar{U}_k)_{k \in \mathbb{N}}$) with $\bar{U}_k \rightharpoonup \bar{U} \neq 0$ in $H_{G_1}^1(\mathbb{R}^3)$ for some $\bar{U} \in H_{G_1}^1(\mathbb{R}^3)$. Moreover, for every $\varphi \in H_{G_1}^1(\mathbb{R}^3) \cap C_0^\infty(\mathbb{R}^3)$ we have $J'[\bar{U}]\varphi = \lim_{k \rightarrow \infty} J'[\bar{U}_k]\varphi = 0$ due to weak convergence and the compact embedding $H_{G_1}^1(\mathbb{R}^3) \rightarrow L^q(K)$, $K = \text{supp } \varphi$ and $q \in [1, 6)$. Hence \bar{U} is a critical point of J so that $\bar{U} \in N$. Hence

$$\begin{aligned} J[\bar{U}] &= \left(\frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbb{R}^3} \Gamma(x) |\bar{U}|^{p+1} dx \\ &\leq \liminf_{k \in \mathbb{N}} \left(\frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbb{R}^3} \Gamma(x) |\bar{U}_k|^{p+1} dx \\ &= \liminf_{k \in \mathbb{N}} J[\bar{U}_k] \\ &= c \end{aligned}$$

and therefore “=” holds and \bar{U} is a minimizer of J restricted to N . \square

APPENDIX

Lemma 23. *Let J_1 denote the order one Bessel function which is regular at 0. Then the function*

$$\alpha(x) := \frac{J_1(x)}{xJ_1'(x)}$$

satisfies $\lim_{x \rightarrow 0+} \alpha(x) = 1$ and is strictly increasing on the intervals $(0, j_1')$, (j_k', j_{k+1}') for $k = 1, 2, 3, \dots$. Let K_1 denote the order one modified Bessel function which decreases exponentially at infinity. Then the function

$$\beta(x) := \frac{K_1(x)}{xK_1'(x)}$$

satisfies $\lim_{x \rightarrow 0+} \beta(x) = -1$ and is negative and strictly increasing on $(0, \infty)$.

Proof. Since J_1 is analytic and since $J_1(0) = 0$, $J_1'(0) = 1/2$ the relation $\alpha(x) \rightarrow 1$ as $x \rightarrow 0$ follows immediately. Differentiating $\alpha(x)$, we need to show

$$\frac{x(J_1'(x))^2 - J_1(x)(J_1'(x) + xJ_1''(x))}{x^2(J_1'(x))^2} > 0$$

and by using the differential equation for J_1 this amounts to

$$x \left((J_1'(x))^2 + \left(1 - \frac{1}{x^2} \right) J_1(x)^2 \right) > 0,$$

i.e.,

$$x^2(J_1'(x))^2 > (1 - x^2)J_1(x)^2 \text{ for } x \in (0, \infty).$$

If we multiply the differential equation

$$x^2 J_1''(x) + xJ_1'(x) = (1 - x^2)J_1(x)$$

by $J_1'(x)$ and integrate from 0 to x we obtain

$$\int_0^x s^2 \frac{d}{ds} (J_1'(s))^2 + 2s(J_1'(s))^2 ds = \int_0^x (1-s^2) \frac{d}{ds} (J_1(s)^2) ds.$$

Integration by parts and using $J_1(0) = 0$ leads to

$$x^2(J_1'(x))^2 = (1-x^2)J_1(x)^2 + \int_0^x 2sJ_1(s)^2 ds > (1-x^2)J_1(x)^2 \text{ for } x \in (0, \infty)$$

and hence the result is proved.

Now we turn to the statement for β . First we recall that $xK_0(x) \rightarrow 0$ (cf. [13], 8.447), $K_1(x) \rightarrow \infty$ as $x \rightarrow 0$ (cf. [13], 8.451(6.)) and that $xK_1'(x) + K_1(x) = -xK_0(x)$ (cf. [13], 8.486(12.)). This implies $\beta(x) \rightarrow -1$ as $x \rightarrow 0$. For the monotonicity of $\beta(x)$ it suffices by differentiation to prove that

$$x(K_1'(x))^2 - K_1(x)K_1'(x) - xK_1(x)K_1''(x) > 0$$

and using the differential equation this amounts to showing that

$$x^2(K_1'(x))^2 > (1+x^2)K_1(x)^2 \text{ for all } x \in (0, \infty).$$

If we multiply the differential equation

$$x^2K_1''(x) + xK_1'(x) = (1+x^2)K_1(x)$$

by $K_1'(x)$ and integrate from x to ∞ we obtain

$$\int_x^\infty s^2 \frac{d}{ds} (K_1'(s))^2 + 2s(K_1'(s))^2 ds = \int_x^\infty (1+s^2) \frac{d}{ds} (K_1(s)^2) ds.$$

Integration by parts and using the exponential decay of K_1, K_1', K_1'' at infinity leads to

$$-x^2(K_1'(x))^2 = -(1+x^2)K_1(x)^2 - \int_x^\infty 2sK_1(s)^2 ds < -(1+x^2)K_1(x)^2 \text{ for } x \in (0, \infty).$$

This proves the result. □

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TH. BARTSCH

INSTITUT FÜR MATHEMATIK, UNIVERSITY OF GIESSEN

D-35392 GIESSEN, GERMANY

E-mail address: `thomas.bartsch@math.uni-giessen.de`

T. DOHNAL

FACHBEREICH MATHEMATIK, UNIVERSITY OF DORTMUND

D-44221 DORTMUND, GERMANY

E-mail address: `tomas.dohnal@math.tu-dortmund.de`

M. PLUM

INSTITUT FÜR ANALYSIS, KARLSRUHE INSTITUTE OF TECHNOLOGY (KIT)

D-76128 KARLSRUHE, GERMANY

E-mail address: `michael.plum@kit.edu`

W. REICHEL

INSTITUT FÜR ANALYSIS, KARLSRUHE INSTITUTE OF TECHNOLOGY (KIT),

D-76128 KARLSRUHE, GERMANY

E-mail address: `wolfgang.reichel@kit.edu`