NORMALIZED SOLUTIONS FOR A SYSTEM OF COUPLED CUBIC SCHRÖDINGER EQUATIONS ON $\mathbb{R}^3$

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Abstract. We consider the system of coupled elliptic equations

$$\begin{cases} -\Delta u - \lambda_1 u = \mu_1 u^3 + \beta u v^2 \\ -\Delta v - \lambda_2 v = \mu_2 v^3 + \beta u^2 v \end{cases} \quad \text{in } \mathbb{R}^3,$$

and study the existence of positive solutions satisfying the additional condition

$$\int_{\mathbb{R}^3} u^2 = a_1^2 \quad \text{and} \quad \int_{\mathbb{R}^3} v^2 = a_2^2.$$

Assuming that a_1, a_2, μ_1, μ_2 are positive fixed quantities, we prove existence results for different ranges of the coupling parameter $\beta > 0$. The extension to systems with an arbitrary number of components is discussed, as well as the orbital stability of the corresponding standing waves for the related Schrödinger systems.

1. Introduction

This paper concerns the existence of solutions $(\lambda_1, \lambda_2, u, v) \in \mathbb{R}^2 \times H^1(\mathbb{R}^3, \mathbb{R}^2)$ to the system of elliptic equations

$$\begin{cases} -\Delta u - \lambda_1 u = \mu_1 u^3 + \beta u v^2 \\ -\Delta v - \lambda_2 v = \mu_2 v^3 + \beta u^2 v \end{cases} \quad \text{in } \mathbb{R}^3, \tag{1.1}$$

satisfying the additional condition

$$\int_{\mathbb{R}^3} u^2 = a_1^2 \quad \text{and} \quad \int_{\mathbb{R}^3} v^2 = a_2^2. \tag{1.2}$$

One refers to this type of solutions as to *normalized solutions*, since (1.2) imposes a normalization on the L^2-masses of u and v. This fact implies that λ_1 and λ_2 cannot be determined a priori, but are part of the unknown.

The problem under investigation comes from the research of solitary waves for the system of coupled Schrödinger equations

$$\begin{cases} -\iota \partial_t \Phi_1 = \Delta \Phi_1 + \mu_1 |\Phi_1|^2 \Phi_1 + \beta |\Phi_2|^2 \Phi_1 \\ -\iota \partial_t \Phi_2 = \Delta \Phi_2 + \mu_2 |\Phi_2|^2 \Phi_2 + \beta |\Phi_1|^2 \Phi_2 \end{cases} \quad \text{in } \mathbb{R} \times \mathbb{R}^3, \tag{1.3}$$

having applications in nonlinear optics and in the Hartree-Fock approximation for Bose-Einstein condensates with multiple states; see [13, 29].

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It is well known that three quantities are conserved in time along trajectories of (1.3): the *energy*

$$J_{\mathbb{C}}(\Phi_1,\Phi_2)=\int_{\mathbb{R}^3}\left(\frac{1}{2}|\nabla\Phi_1|^2-\frac{\mu_1}{4}|\Phi_1|^4\right)\\ +\int_{\mathbb{R}^3}\left(\frac{1}{2}|\nabla\Phi_2|^2-\frac{\mu_2}{4}|\Phi_2|^4\right)-\frac{\beta}{2}\int_{\mathbb{R}^3}|\Phi_1|^2|\Phi_2|^2,$$

and the *masses*

$$\int_{\mathbb{R}^3}|\Phi_1|^2 \quad\text{and}\quad \int_{\mathbb{R}^3}|\Phi_2|^2.$$

A solitary wave of (1.3) is a solution having the form

$$\Phi_1(t,x)=e^{-i\lambda_1 t}u(x) \quad\text{and}\quad \Phi_2(t,x)=e^{-i\lambda_2 t}v(x)$$

for some $\lambda_1,\lambda_2\in\mathbb{R}$, where (u,v) solves (1.1). Two different approaches are possible: one can either regard the frequencies λ_1,λ_2 as fixed, or include them in the unknown and prescribe the masses. In this latter case, which seems to be particularly interesting from the physical point of view, λ_1 and λ_2 appear as Lagrange multipliers with respect to the mass constraint.

The problem with fixed λ_i has been widely investigated in the last ten years, and, at least for systems with 2 components and existence of positive solutions (i. e. $u,v>0$ in $\mathbb{R}^3$), the situation is quite well understood. A complete review of the available results in this context goes beyond the aim of this paper; we refer the interested reader to [1, 2, 3, 6, 7, 11, 21, 22, 25, 27, 30, 36, 37, 38, 39, 41, 42] and to the references therein.

In striking contrast, very few papers deal with the existence of normalized solutions. Up to our knowledge, the only known results are the ones in [5, 32, 34, 40]. In [34], the authors consider (1.1) in bounded domains of $\mathbb{R}^N$, or the problem with trapping potentials in the whole space $\mathbb{R}^N$ (the presence of a trapping potential makes the two problems essentially equivalent), with $N\leq 3$. In both cases, they proved existence of positive solutions with small masses a_1 and a_2, and the orbital stability of the associated solitary waves, see Theorem 1.3 therein. It is remarkable that they can work essentially without assumptions on μ_1,μ_2 and β. The requirement that the masses have to be small gives their result a bifurcation flavor. In [32,40] the authors consider the defocusing setting $\mu_1,\mu_2<0$ in regime of competition $\beta<0$ in bounded domains. In the defocusing competitive case $\mu_1,\mu_2,\beta<0$ existence is an easy consequence of standard Lusternik-Schnirelmann theory because the functional is bounded from below. Supposing that all the components have the same mass, they prove existence of infinitely many solutions and occurrence of phase-separation as $\beta\to-\infty$. Concerning [5], we postpone a discussion of the results therein in the following paragraphs.

In the present paper we address a situation which is substantially different compared to those considered in the papers [32,34,40]. We study system (1.1) in $\mathbb{R}^3$ in the focusing setting $\mu_1,\mu_2>0$, so that the functional is unbounded from below on the constraint. We prove the existence of positive normalized solutions for different ranges of the coupling parameter $\beta>0$, without any assumption on the masses a_1,a_2. Our approach is variational: we find solutions of (1.1)-(1.2) as critical points of the energy functional

$$J(u,v)=\int_{\mathbb{R}^3}\left(\frac{1}{2}|\nabla u|^2-\frac{\mu_1}{4}u^4\right)+\int_{\mathbb{R}^3}\left(\frac{1}{2}|\nabla v|^2-\frac{\mu_2}{4}v^4\right)-\frac{\beta}{2}\int_{\mathbb{R}^3}u^2v^2, \tag{1.4}$$

on the constraint $T_{a_1}\times T_{a_2}$, where for $a\in\mathbb{R}$ we define

$$T_a:=\left\{u\in H^1(\mathbb{R}^3):\int_{\mathbb{R}^3}u^2=a^2\right\}. \tag{1.5}$$

The main results are the following:

Theorem 1.1. *Let $a_1,a_2,\mu_1,\mu_2>0$ be fixed. There exists $\beta_1>0$ depending on a_i and μ_i such that if $0<\beta<\beta_1$, then (1.1)-(1.2) has a solution $(\tilde\lambda_1,\tilde\lambda_2,\tilde u,\tilde v)$ such that $\tilde\lambda_1,\tilde\lambda_2<0$, and $\tilde u$ and $\tilde v$ are both positive and radial.*

For our next result we introduce a Pohozaev-type constraint as follows:

$$V:=\{(u,v)\in T_{a_1}\times T_{a_2}:G(u,v)=0\}, \tag{1.6}$$

where

$$G(u,v)=\int_{\mathbb{R}^3}\left(|\nabla u|^2+|\nabla v|^2\right)-\frac{3}{4}\int_{\mathbb{R}^3}\left(\mu_1u^4+2\beta u^2v^2+\mu_2v^4\right).$$

We shall see that V contains all solutions of (1.1)-(1.2).

Theorem 1.2. *Let $a_1,a_2,\mu_1,\mu_2>0$ be fixed. There exists $\beta_2>0$ depending on a_i and μ_i such that, if $\beta>\beta_2$, then (1.1)-(1.2) has a solution $(\bar\lambda_1,\bar\lambda_2,\bar u,\bar v)$ such that $\bar\lambda_1,\bar\lambda_2<0$, and $\bar u$ and $\bar v$ are both positive and radial. Moreover, $(\bar\lambda_1,\bar\lambda_2,\bar u,\bar v)$ is a ground state solution in the sense that*

$$\begin{aligned}J(\bar u,\bar v)&=\inf\{J(u,v):(u,v)\in V\}\\&=\inf\{J(u,v):(u,v)\text{ is a solution of (1.1)-(1.2) for some }\lambda_1,\lambda_2\}\end{aligned}$$

holds.

Remark 1.3. a) The values of β_1 and β_2 can be explicitly estimated; see (3.3) and Remark 4.5 below. In particular, we point out that they are not obtained by means of any limit process, so that one should not think that β_1 is very small and β_2 is very large. For instance, if $\mu_1=\mu_2$ and $a_1\le a_2$, then the proof of Theorem 1.1 works for

$$\beta_1=\mu_1\left(\sqrt{1+\frac{a_1^2}{a_2^2}}-1\right).$$

Nevertheless, it remains an open problem to obtain sharp bounds.

b) The variational characterizations of the solutions obtained in Theorems 1.1 and 1.2 are different. The solution from Theorem 1.1 has Morse index 2 as critical point of J constrained to $T_{a_1}\times T_{a_2}$. On the other hand, the solution from Theorem 1.2 is a mountain pass solution of J on the constraint. It can also be obtained as a minimizer of J on the Pohozaev-type submanifold of the constraint.

c) Our results can be extended with minor changes to systems with general exponents of type

$$\begin{cases}-\Delta u_1-\lambda_1u_1=\mu_1|u_1|^{2p_1-2}u_1+\beta|u_1|^{r-2}|u_2|^ru_1\\-\Delta u_2-\lambda_2u_2=\mu_2|u_2|^{2p_2-2}u_2+\beta|u_1|^r|u_2|^{r-2}u_2\end{cases}\quad\text{in }\mathbb{R}^N \tag{1.7}$$

(or the k components analogue) with $N \le 4$, provided we restrict ourselves to a L^2-*supercritical* and *Sobolev subcritical* setting:

$$2 + \frac{4}{N} < 2p_i, 2r < \frac{2N}{N-2}.$$

Moreover, the proofs do not use the evenness of the functional. Thus one may replace the terms u^4, v^4 in (1.4) by general nonlinearities $f(u), g(v)$ which are not odd. Similarly the coupling term u^2v^2 in the functional may be replaced by a nonsymmetric one. We decided not to include this kind of generality since it would make the statement of our results and the proofs very technical.

d) Also in the case of fixed frequencies for system (1.1) there exist values $0 < \beta_1' < \beta_2'$ such that the problem has a positive solution whenever $\beta < \beta_1'$ or $\beta > \beta_2'$ [1,37], see also [27]. In this setting, it is known that if $\lambda_1 \ge \lambda_2$, $\mu_1 \ge \mu_2$, and one of the inequalities is strict, then $\beta_1' < \beta_2'$, and for $\beta \in [\beta_1', \beta_2']$ the problem has no positive solution [6, 37]. On the other hand, the non-existence range (in terms of β) can disappear. This is the case, for instance, if $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 1$. Then (1.1) has positive solutions for all $\beta > 0$. Since in the context of normalized solutions the values λ_i are not prescribed, it is an interesting open problem whether there are conditions on a_1, a_2, μ_1, μ_2 such that positive solutions of (1.1)-(1.2) exist for all $\beta > 0$.

e) Despite the similarity between our results and those in [1, 37], the proofs differ substantially. First, while in [1, 37] the approach is based on the research of critical points constrained on Nehari-type sets associated to the problem, here no Nehari manifold is available, since λ_1 and λ_2 are part of the unknown; as a consequence, we shall directly investigate the geometry of the functional on the product of the L^2-spheres $T_{a_1} \times T_{a_2}$ in order to apply a suitable minimax theorem. We also point out that in [1, 37], as well as in all the contributions related to the problem with fixed frequencies, one of the main difficulties is represented by the fact that one search for solutions having both $u \not\equiv 0$ and $v \not\equiv 0$. Here this problem is still present, and actually it assumes a more subtle form, in the following sense: let us suppose that we can find a Palais-Smale sequence for J on $T_{a_1} \times T_{a_2}$, and suppose that this sequence is weakly convergent in H^1 to a limit (u, v). Due to the lack of compactness of the embedding $H^1(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3)$, a delicate step consists in showing that $(u, v) \in T_{a_1} \times T_{a_2}$, so that it satisfies (1.2). Notice that the lack of compactness persists also if we restrict ourselves to a radial setting. As a consequence, we emphasize that it is not sufficient to rule out the possibility that in the weak limit $u \equiv 0$ or $v \equiv 0$. We have also to prevent the loss of part of the mass of one of the components in the passage to the limit.

Both theorems rest upon a suitable minimax argument, where an important role is played by the *ground state levels* $\ell(a_1, \mu_1)$ and $\ell(a_2, \mu_2)$ associated to the scalar problems

$$\begin{cases} -\Delta w - \lambda w = \mu w^3 & \text{in } \mathbb{R}^3 \\ \int_{\mathbb{R}^3} w^2 = a^2 \end{cases}$$

with $a = a_1$ and $\mu = \mu_1$, or with $a = a_2$ and $\mu = \mu_2$, respectively. We refer to Section 2 for the precise definition of $\ell(a, \mu)$. In this perspective, it is interesting to emphasize the different relations between the critical values of Theorems 1.1 and 1.2 with $\ell(a_1, \mu_1)$ and $\ell(a_2, \mu_2)$.

Proposition 1.4. *With the notation of Theorems 1.1 and 1.2, we have*

$$J(\bar u,\bar v) < \min\{\ell(a_1,\mu_1),\ell(a_2,\mu_2)\} \le \max\{\ell(a_1,\mu_1),\ell(a_2,\mu_2)\} < J(\tilde u,\tilde v).$$

In [5] the authors consider systems of the type of (1.7) looking also for solutions satisfying (1.2). The results obtained in [5] have no intersection with the one of the present paper because there $2 < p_1 < 2+4/N < p_2 < 6$. A common feature is that one looks for constrained critical points in a situation where the functional is unbounded from below on the constraint. Already in the scalar case it is known that, when the underlying equations are set on all the space, looking to critical points which are not global minima of the associated functional may present new difficulties (with respect to the minimizing problem), see [8, 17]. In particular a standard approach following the Compactness Concentration Principle of P.L. Lions [23,24] is hardly applicable. We also mention [4,18,26] for multiplicity results in that direction, and [33] for normalized solutions in bounded domains.

In the second part of the paper we partially generalize the previous results to the $k \ge 2$ components system

$$\begin{cases} -\Delta u_i - \lambda_i u_i = \sum_{j=1}^k \beta_{ij} u_j^2 u_i & \text{in } \mathbb{R}^3 \\ u_i \in H^1(\mathbb{R}^3) \end{cases} \qquad i=1,\dots,k, \tag{1.8}$$

with the normalization condition

$$\int_{\mathbb{R}^3} u_i^2 = a_i^2 \qquad i=1,\dots,k. \tag{1.9}$$

We always suppose that $\beta_{ij} = \beta_{ji}$ for every $i \ne j$. Notice that problem (1.1)-(1.2) falls in this setting with $k=2$, $u=u_1$, $v=u_2$, $\beta_{ii}=\mu_i$ and $\beta_{12}=\beta$.

From a variational point of view, thanks to the fact that $\beta_{ij}=\beta_{ji}$ solutions of (1.8)-(1.9) are critical points of

$$J(u_1,\dots,u_k) := \int_{\mathbb{R}^3} \left(\frac{1}{2}\sum_{i=1}^k |\nabla u_i|^2 - \frac{1}{4}\sum_{i,j=1}^k \beta_{ij} u_i^2 u_j^2 \right)$$

on the constraint $T_{a_1}\times\dots\times T_{a_k}$, where T_a has been defined in (1.5). Notice that the definition of the functional J depends on k and the matrix β_{ij}), but we will not stress such dependence to keep the notation as simple as possible.

The first result which we present is the extension of Theorem 1.2 to any $k \ge 3$.

Theorem 1.5. *Let $k \ge 2$, and let $a_i, \beta_{ii}, \beta_{ij} > 0$ be positive constant, such that the following inequality holds:*

$$\frac{\left(\sum_{i=1}^k a_i^2\right)^3}{\left(\sum_{i,j=1}^k \beta_{ij} a_i^2 a_j^2\right)^2} < \min_{\substack{\mathcal{I}\subset\{1,\dots,k\} \\ |\mathcal{I}|\le k-1}} \frac{1}{\left[\max_{i\in\mathcal{I}}\{\beta_{jj}a_j\} + \frac{k-2}{k-1}\max_{\substack{i\ne j \\ i,j\in\mathcal{I}}}\left\{\beta_{ij}a_i^{1/2}a_j^{1/2}\right\}\right]^2}, \tag{1.10}$$

where $|\mathcal{I}|$ denotes the cardinality of the set $\mathcal{I}$. Then (1.8)-(1.9) has a solution $(\bar\lambda_1,\dots,\bar\lambda_k,\bar u_1,\dots,\bar u_k)$ such that $\bar\lambda_i < 0$, and $\bar u_i$ is positive and radial for every i. Moreover,

$$J(\bar u_1,\dots,\bar u_k) = \inf\left\{J(u_1,\dots,u_k) : (u_1,\dots,u_k) \textit{ is a solution of } (1.8)\text{-}(1.9)\right\},$$

that is $(\bar\lambda_1,\dots,\bar\lambda_k,\bar u_1,\dots,\bar u_k)$ *is a ground state solution.*

Some remarks are in order.

Remark 1.6. a) The set of parameters fulfilling condition (1.10) is not empty. For instance, if $a_i = a$ for every i, $\beta_{ii} > 0$ are fixed and $\beta_{ij} = \beta$ for every $i \neq j$, then (1.10) is satisfied provided β is sufficiently large. More in general, if $\beta_{ii} > 0$, $\beta_{ij} = \beta$ for every $i \neq j$, and

$$\frac{\left(\sum_i a_i^2\right)^3 \left(\frac{k-2}{k-1}\right)^2 \max_{i\neq j}\{a_i a_j\}}{\left(\sum_{i\neq j} a_i^2 a_j^2\right)^2} < 1,$$

then (1.10) is satisfied provided β is sufficiently large.

b) At a first glance (1.10) seems unclear if compared with the simple condition $\beta > \beta_2$ appearing in Theorem 1.2. On the contrary, for β_{ii} and a_i fixed and $k = 2$, it is easy to check that (1.10) is fulfilled provided β_{12} is larger than a positive threshold β_2' (which can be explicitly computed). We observed that the value β_2 in Theorem 1.2 can be estimated, see Remark 4.5. Actually, using (1.10) we expect to have a better estimate (in the sense that $\beta_2' \le \beta_2$); the price to pay is that the derivation of (1.10) requires a lot of extra work.

c) A condition somehow similar to (1.10) appears also for the problem with fixed frequencies λ_i, see Theorem 2.1 in [25].

Regarding the extension of Theorem 1.1 to systems with an arbitrary number of components, we have a much weaker result.

Proposition 1.7. *Let* $a_i, \beta_{ii} > 0$ *be fixed positive constant. There exists* $\beta_0 > 0$ *such that if* $|\beta_{ij}| < \beta_0$ *for every* $i \neq j$*, then system* (1.8)-(1.9) *has a solution* $(\tilde\lambda_1,\dots,\tilde\lambda_k,\tilde u_1,\dots,\tilde u_k)$ *such that* $\lambda_i < 0$*, and* u_i *is positive and radial for every* i*.*

The proof is based on a simple application of the implicit function theorem, and is omitted for the sake of brevity. Notice that using a perturbative argument we can allow some (or all) the couplings β_{ij} to take negative values. On the other hand, being β_0 obtained by a limit argument, it cannot be estimated from below and it could be very small; in this sense Proposition 1.7 is weaker than Theorem 1.1, where an explicit estimate for β_1 is available.

Let us now turn to the question of the orbital stability of the solitary waves of

$$-\iota\partial_t\Phi_j = \Delta\Phi_j + \beta_{jj}|\Phi_j|^2\Phi_j + \sum_{k\neq j}\beta_{kj}|\Phi_k|^2\Phi_j \qquad \text{in } \mathbb{R}\times\mathbb{R}^3,\ j=1,\dots,k, \tag{1.11}$$

associated to the solutions found in Theorem 1.5 (or Theorem 1.2 if $k = 2$). In this framework, we can adapt the classical Berestycki-Cazenave argument [9] (see also [10, 20] for more detailed proofs) and prove the following:

Theorem 1.8. *Let* $k \ge 2$*, and* $(\bar\lambda_1,\dots,\bar\lambda_k,\bar u_1,\dots,\bar u_k)$ *be the solution obtained in Theorem 1.5 (or in Theorem 1.2 if* $k = 2$*). Then the associated solitary wave is orbitally unstable.*

Regarding the stability of the solutions found in Theorem 1.1 and Proposition 1.7, a Berestycki-Cazenave-type argument does not seem to be applicable, since these solutions are characterized by a different minimax construction with respect to those in Theorems 1.2 and 1.5. Therefore, the stability remains open in these cases.

The orbital stability of solutions to weakly coupled Schrödinger equations associated to power-type systems like (1.7) has been studied in several papers (we refer to [12, 28, 31, 35] and to the references therein), but the available results mainly regard the L^2-subcritical setting setting $2p < 1 + 4/N$, and the problem with fixed frequencies. In particular, we point out that Theorem 1.8 does not follow by previous contributions.

2. Preliminaries

In the first part of the section, we collect some facts concerning the cubic NLS equation, which will be used later. Let us consider the scalar problem

$$\begin{cases} -\Delta w + w = w^3 & \text{in } \mathbb{R}^3 \\ w > 0 & \text{in } \mathbb{R}^3 \\ w(0) = \max w \quad \text{and} \quad w \in H^1(\mathbb{R}^3). \end{cases} \tag{2.1}$$

It is well known that (2.1) has a unique solution, denoted by w_0 and that this solution is radial. In what follows we set

$$C_0 := \int_{\mathbb{R}^3} w_0^2 \quad \text{and} \quad C_1 := \int_{\mathbb{R}^3} w_0^4. \tag{2.2}$$

For $a, \mu \in \mathbb{R}$ fixed, let us search for $(\lambda, w) \in \mathbb{R} \times H^1(\mathbb{R}^3)$, with $\lambda < 0$ in $\mathbb{R}^3$, solving

$$\begin{cases} -\Delta w - \lambda w = \mu w^3 & \text{in } \mathbb{R}^3 \\ w(0) = \max w \quad \text{and} \quad \int_{\mathbb{R}^3} w^2 = a^2. \end{cases} \tag{2.3}$$

Solutions w of (2.3) can be found as critical points of $I_\mu : H^1(\mathbb{R}^3) \mapsto \mathbb{R}$, defined by

$$I_\mu(w) = \int_{\mathbb{R}^3} \left(\frac{1}{2}|\nabla w|^2 - \frac{\mu}{4} w^4 \right), \tag{2.4}$$

constrained on the L^2-sphere T_a, and λ appears as Lagrange multipliers. It is well known that they can be obtained by the solutions of (2.1) by scaling.

Let us introduce the set

$$\mathcal{P}(a, \mu) := \left\{ w \in T_a : \int_{\mathbb{R}^3} |\nabla w|^2 = \frac{3\mu}{4} \int_{\mathbb{R}^3} w^4 \right\}. \tag{2.5}$$

The role of $\mathcal{P}(a, \mu)$ is clarified by the following result.

Lemma 2.1. *If w is a solution of (2.3), then $w \in \mathcal{P}(a, \mu)$. In addition the positive solution w of (2.3) minimizes I_μ on $\mathcal{P}(a, \mu)$.*

Proof. The proof of the first part is a simple consequence of the Pohozaev identity. We refer to Lemma 2.7 in [17] for more details. For the last part we refer to Lemma 2.10 in [17]. □

Proposition 2.2. *Problem (2.3) has a unique positive solution $(\lambda_{a,\mu}, w_{a,\mu})$ defined by*

$$\lambda_{a,\mu} := -\frac{C_0^2}{\mu^2 a^4} \quad \textit{and} \quad w_{a,\mu}(x) := \frac{C_0}{\mu^{3/2} a^2} w_0 \left(\frac{C_0}{\mu a^2} x \right).$$

The function $w_{a,\mu}$ satisfies

$$\int_{\mathbb{R}^3} |\nabla w_{a,\mu}|^2 = \frac{3C_0C_1}{4\mu^2 a^2} \tag{2.6}$$

$$\int_{\mathbb{R}^3} w_{a,\mu}^4 = \frac{C_0C_1}{\mu^3 a^2}. \tag{2.7}$$

$$\ell(a,\mu) := I_\mu(w_{a,\mu}) = \frac{C_0C_1}{8\mu^2 a^2}. \tag{2.8}$$

The value $\ell(a,\mu)$ is called least energy level *of problem* (2.3).

Proof. It is not difficult to directly check that $w_{a,\mu}$ defined in the proposition is a solution of (2.3) for $\lambda = \lambda_{a,\mu} < 0$. By [19], it is the only positive solution. To obtain (2.6) and (2.7), we can use the explicit expression of $w_{a,\mu}$: by a change of variables

$$\int_{\mathbb{R}^3} |\nabla w_{a,\mu}|^2 = \frac{C_0}{\mu^2 a^2} \int_{\mathbb{R}^3} |\nabla w_0|^2 = \frac{3C_0}{4\mu^2 a^2} \int_{\mathbb{R}^3} w_0^4,$$

where the last equality follows by Lemma 2.1 with $a^2 = C_0$ and $\mu = 1$. This gives (2.6). In a similar way, one can also prove (2.7) and (2.8). □

Working with systems with several components, it will be useful to have a characterization of the best constant in a Gagliardo-Nirenberg inequality in terms of C_0 and C_1. To obtain it, we observe at first that if $w_a := w_{a,C_0/a^2}$, then w_a is the unique positive solution of

$$\begin{cases} -\Delta w + w = \frac{C_0}{a^2} w^3 & \text{in } \mathbb{R}^3 \\ w(0) = \max w \quad \text{and} \quad \int_{\mathbb{R}^3} w^2 = a^2, \end{cases}$$

and hence is a minimizer of $I_{a,C_0/a^2}$ on $\mathcal{P}(a, C_0/a^2)$. Our next result shows that this level can also be characterized as an infimum of a Rayleigh-type quotient, defined by

$$\mathcal{R}_a(w) := \frac{8\left(\int_{\mathbb{R}^3} |\nabla w|^2\right)^3}{27\left(\frac{C_0}{a^2}\int_{\mathbb{R}^3} w^4\right)^2}.$$

Lemma 2.3. *There holds*

$$\inf_{\mathcal{P}(a,C_0/a^2)} I_{a,C_0/a^2} = \inf_{T_a} \mathcal{R}_a.$$

Proof. We refer to the proof of the forthcoming Lemma 5.4, where the corresponding equality is proved for systems, and which then includes the present result as a particular case. □

Let us recall the following Gagliardo-Nirenberg inequality: there exists a universal constant $S > 0$ such that

$$\int_{\mathbb{R}^3} w^4 \le S \left(\int_{\mathbb{R}^3} w^2\right)^{1/2} \left(\int_{\mathbb{R}^3} |\nabla w|^2\right)^{3/2} \qquad \text{for all } w \in H^1(\mathbb{R}^3). \tag{2.9}$$

In particular, the optimal value of S can be found as

$$\frac{1}{S^2} = \inf_{w \in H^1(\mathbb{R}^3)\setminus\{0\}} \frac{\left(\int_{\mathbb{R}^3} w^2\right)\cdot\left(\int_{\mathbb{R}^3} |\nabla w|^2\right)^3}{\left(\int_{\mathbb{R}^3} w^4\right)^2} = \inf_{w\in T_a} \frac{a^2\left(\int_{\mathbb{R}^3} |\nabla w|^2\right)^3}{\left(\int_{\mathbb{R}^3} w^4\right)^2}, \tag{2.10}$$

where the last equality comes from the fact that the ratio on the right hand side is invariant with respect to multiplication of w with a positive number.

Lemma 2.4. *In the previous notation, we have*

$$S^2 = \frac{64}{27C_0C_1},$$

where C_0 and C_1 have been defined in (2.2).

Proof. Multiplying and dividing the last term in (2.10) by $8a^2/(27C_0^2)$, we deduce that

$$\frac{1}{S^2} = \frac{27C_0^2}{8a^2} \inf_{w \in T_a} \mathcal{R}_a(w).$$

Hence, by Proposition 2.2 and Lemma 2.3, we infer that

$$\frac{1}{S^2} = \frac{27C_0^2}{8a^2} I_{C_0/a^2}(w_{a,C_0/a^2}) = \frac{27C_0C_1}{64}. \qquad \square$$

3. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1, which is based upon a two-dimensional linking argument.

In order to avoid compactness issues, we work in a radial setting. This means that we search for solutions of (1.1)-(1.2) as critical points of J constrained on $S_{a_1} \times S_{a_2}$, where for any $a \in \mathbb{R}$ the set S_a is defined by

$$S_a := \left\{ w \in H^1_{\text{rad}}(\mathbb{R}^3) : \int_{\mathbb{R}^3} w^2 = a^2 \right\}, \tag{3.1}$$

and $H^1_{\text{rad}}(\mathbb{R}^3)$ denotes the subset of $H^1(\mathbb{R}^3)$ containing all the functions which are radial with respect to the origin. Recall that $H^1_{\text{rad}}(\mathbb{R}^3) \hookrightarrow L^4(\mathbb{R}^3)$ with compact embedding, and the fact that critical points of J constrained on $S_{a_1} \times S_{a_2}$ (thus in a radial setting) are true critical points of J constrained in the full product $T_{a_1} \times T_{a_2}$ is a consequence of the Palais' principle of symmetric criticality.

In order to describe the minimax structure, it is convenient to introduce some notation. We define, for $s \in \mathbb{R}$ and $w \in H^1(\mathbb{R}^3)$, the radial dilation

$$(s \star w)(x) := e^{\frac{3s}{2}} w(e^s x). \tag{3.2}$$

It is straightforward to check that if $w \in S_a$, then $(s \star w) \in S_a$ for every $s \in \mathbb{R}$.

Lemma 3.1. *For every $\mu > 0$ and $w \in H^1(\mathbb{R}^3)$, there holds:*

$$\begin{aligned} I_\mu(s \star w) &= \frac{e^{2s}}{2} \int_{\mathbb{R}^3} |\nabla w|^2 - \frac{e^{3s}}{4} \mu \int_{\mathbb{R}^3} w^4 \\ \frac{\partial}{\partial s} I_\mu(s \star w) &= e^{2s} \left(\int_{\mathbb{R}^3} |\nabla w|^2 - \frac{3e^s}{4} \mu \int_{\mathbb{R}^3} w^4 \right). \end{aligned}$$

In particular, if $w = w_{a,\mu}$, then

$$\frac{\partial}{\partial s} I_\mu(s \star w_{a,\mu}) \text{ is } \begin{cases} > 0 & \text{if } s < 0 \\ = 0 & \text{if } s = 0 \\ < 0 & \text{if } s > 0. \end{cases}$$

For the reader's convenience, we recall that I_μ denotes the functional for the scalar equation, see (2.4), and $w_{a,\mu}$ has been defined in Proposition 2.2.

Proof. For the first part, it is sufficient to use the definition of $s \star w$ and a change of variables in the integrals. For the second part, we observe that

$$\frac{\partial}{\partial s} I_\mu(s \star w_{a,\mu}) \text{ is } \begin{cases} > 0 & \text{if } s < \bar{s} \\ = 0 & \text{if } s = \bar{s} \\ < 0 & \text{if } s > \bar{s}, \end{cases}$$

where $\bar{s} \in \mathbb{R}$ is uniquely defined by

$$e^{\bar{s}} = \frac{4 \int_{\mathbb{R}^3} |\nabla w_{a,\mu}|^2}{3\mu \int_{\mathbb{R}^3} w_{a,\mu}^4}.$$

Recalling that $w_{a,\mu} \in \mathcal{P}(a,\mu)$, see Lemma 2.1, we deduce that $e^{\bar{s}} = 1$, i.e. $\bar{s} = 0$. □

For $a_1, a_2, \mu_1, \mu_2 > 0$ let $\beta_1 = \beta_1(a_1, a_2, \mu_1, \mu_2) > 0$ be defined by the equation:

$$\max\left\{\frac{1}{a_1^2\mu_1^2}, \frac{1}{a_2^2\mu_2^2}\right\} = \frac{1}{a_1^2(\mu_1+\beta_1)^2} + \frac{1}{a_2^2(\mu_2+\beta_1)^2} \tag{3.3}$$

Lemma 3.2. *For $0 < \beta < \beta_1$ there holds:*

$$\inf\{J(u,v) : (u,v) \in \mathcal{P}(a_1, \mu_1+\beta) \times \mathcal{P}(a_2, \mu_2+\beta)\} > \max\{\ell(a_1,\mu_1), \ell(a_2,\mu_2)\}$$

where $\ell(a_i, \mu_i)$ is defined by (2.8).

Proof. Using Young's inequality and recalling the definition of I_μ (see (2.4)), we obtain for $(u,v) \in \mathcal{P}(a_1, \mu_1+\beta) \times \mathcal{P}(a_2, \mu_2+\beta)$:

$$\begin{aligned} J(u,v) &= I_{\mu_1}(u) + I_{\mu_2}(v) - \frac{\beta}{2}\int_{\mathbb{R}^3} u^2 v^2 \\ &\geq I_{\mu_1}(u) + I_{\mu_2}(v) - \frac{\beta}{4}\int_{\mathbb{R}^3} u^4 - \frac{\beta}{4}\int_{\mathbb{R}^3} v^4 \\ &= I_{\mu_1+\beta}(u) + I_{\mu_2+\beta}(v) \geq \inf_{u \in \mathcal{P}(a_1,\mu_1+\beta)} I_{\mu_1+\beta}(u) + \inf_{v \in \mathcal{P}(a_2,\mu_2+\beta)} I_{\mu_2+\beta}(v) \\ &= \ell(a_1, \mu_1+\beta) + \ell(a_2, \mu_2+\beta) \end{aligned}$$

Therefore, the claim is satisfied provided

$$\max\{\ell(a_1,\mu_1), \ell(a_2,\mu_2)\} < \ell(a_1,\mu_1+\beta) + \ell(a_2,\mu_2+\beta),$$

that is (by Proposition 2.2)

$$\max\left\{\frac{C_0C_1}{8a_1^2\mu_1^2}, \frac{C_0C_1}{8a_2^2\mu_2^2}\right\} < \frac{C_0C_1}{8a_1^2(\mu_1+\beta)^2} + \frac{C_0C_1}{8a_2^2(\mu_2+\beta)^2}. \tag{3.4}$$

Clearly, this holds for $0 < \beta < \beta_1$. □

Now we fix $0 < \beta < \beta_1 = \beta_1(a_1, a_2, \mu_1, \mu_2)$ and choose $\varepsilon > 0$ such that

$$\begin{aligned} &\inf\{J(u,v) : (u,v) \in \mathcal{P}(a_1,\mu_1+\beta) \times \mathcal{P}(a_2,\mu_2+\beta)\} \\ &\qquad > \max\{\ell(a_1,\mu_1), \ell(a_2,\mu_2)\} + \varepsilon. \end{aligned} \tag{3.5}$$

We introduce

$$w_1 := w_{a_1,\mu_1+\beta} \quad \text{and} \quad w_2 := w_{a_2,\mu_2+\beta}, \tag{3.6}$$

and for $i = 1, 2$,

$$\varphi_i(s) := I_{\mu_i}(s \star w_i) \quad \text{and} \quad \psi_i(s) := \frac{\partial}{\partial s} I_{\mu_i+\beta}(s \star w_i). \tag{3.7}$$

Lemma 3.3. *For $i = 1, 2$ there exists $\rho_i < 0$ and $R_i > 0$, depending on ε and on β, such that*

(i) $0 < \varphi_i(\rho_i) < \varepsilon$ *and* $\varphi_i(R_i) \le 0$;
(ii) $\psi_i(s) > 0$ *for any* $s < 0$ *and* $\psi_i(s) < 0$ *for every* $s > 0$. *In particular,* $\psi_i(\rho_i) > 0$ *and* $\psi_i(R_i) < 0$.

Proof. By Lemma 3.1, we deduce that $\varphi_i(s) \to 0^+$ as $s \to -\infty$, and $\varphi_i(s) \to -\infty$ as $s \to +\infty$. Thus there exist ρ_i and R_i satisfying (i). Condition (ii) follows directly from Lemma 3.1. $\square$

Let $Q := [\rho_1, R_1] \times [\rho_2, R_2]$, and let

$$\gamma_0(t_1, t_2) := (t_1 \star w_1, t_2 \star w_2) \in S_{a_1} \times S_{a_2} \qquad \forall (t_1, t_2) \in \overline{Q}.$$

We introduce the minimax class

$$\Gamma := \left\{ \gamma \in \mathcal{C}\left(\overline{Q}, S_{a_1} \times S_{a_2}\right) : \gamma = \gamma_0 \text{ on } \partial Q \right\}.$$

The minimax structure of the problem is enlightened by (3.5) and the following two lemmas.

Lemma 3.4. *There holds*

$$\sup_{\partial Q} J(\gamma_0) \le \max\{\ell(a_1, \mu_1), \ell(a_2, \mu_2)\} + \varepsilon.$$

Proof. Notice that

$$J(u, v) = I_{\mu_1}(u) + I_{\mu_2}(v) - \frac{\beta}{2} \int_{\mathbb{R}^3} u^2 v^2 \le I_{\mu_1}(u) + I_{\mu_2}(v)$$

for every $(u, v) \in S_{a_1} \times S_{a_2}$, since $\beta > 0$. Therefore, by Lemma 3.3 we infer that

$$\begin{aligned} J(t_1 \star w_1, \rho_2 \star w_2) &\le I_{\mu_1}(t_1 \star w_1) + I_{\mu_2}(\rho_2 \star w_2) \le I_{\mu_1}(t_1 \star w_1) + \varepsilon \\ &\le \sup_{s \in \mathbb{R}} I_{\mu_1}(s \star w_1) + \varepsilon. \end{aligned}$$

In order to estimate the last term, by Proposition 2.2 it is easy to check that

$$w_{a_i, \mu_i} = (\bar{s}_i \star w_i) \quad \text{for} \quad e^{\bar{s}_i} := \frac{4 \int_{\mathbb{R}^3} |\nabla w_i|^2}{3 \int_{\mathbb{R}^3} \mu_i w_i^4} = \frac{\mu_i + \beta}{\mu_i}.$$

As a consequence, observing also that $s_1 \star (s_2 \star w) = (s_1 + s_2) \star w$ for every $s_1, s_2 \in \mathbb{R}$ and $w \in H^1(\mathbb{R}^3)$, we have

$$\sup_{s \in \mathbb{R}} I_{\mu_1}(s \star w_1) = \sup_{s \in \mathbb{R}} I_{\mu_1}(s \star w_{a_1, \mu_1}). \tag{3.8}$$

As a consequence of Lemma 3.1 the supremum on the right hand side is achieved for $s = 0$, and hence

$$J(t_1 \star w_1, \rho_2 \star w_2) \le \ell(a_1, \mu_1) + \varepsilon \qquad \forall t_1 \in [\rho_1, R_1], \tag{3.9}$$

and in a similar way one can show that

$$J(\rho_1 \star w_1, t_2 \star w_2) \le \ell(a_2, \mu_2) + \varepsilon \qquad \forall t_2 \in [\rho_2, R_2]. \tag{3.10}$$

The value of $J(\gamma_0)$ on the remaining sides of ∂Q is smaller: indeed by Lemma 3.3 and (3.8)

$$\begin{aligned} J(t_1 \star w_1, R_2 \star w_2) &\le I_{\mu_1}(t_1 \star w_1) + I_{\mu_2}(R_2 \star w_2) \\ &\le \sup_{s \in \mathbb{R}} I_{\mu_1}(s \star w_1) = \ell(a_1, \mu_1) \end{aligned} \tag{3.11}$$

for every $t_1 \in [\rho_1, R_1]$, and analogously

$$J(R_1 \star w_1, t_2 \star w_2) \le \ell(a_2, \mu_2) \qquad \forall t_2 \in [\rho_2, R_2]. \tag{3.12}$$

Collecting together (3.9)-(3.12), the thesis follows. □

Now we show that the class Γ "links" with $\mathcal{P}(a_1, \mu_1 + \beta) \times \mathcal{P}(a_2, \mu_2 + \beta)$.

Lemma 3.5. *For every $\gamma \in \Gamma$, there exists $(t_{1,\gamma}, t_{2,\gamma}) \in Q$ such that $\gamma(t_{1,\gamma}, t_{2,\gamma}) \in \mathcal{P}(a_1, \mu_1 + \beta) \times \mathcal{P}(a_2, \mu_2 + \beta)$.*

Proof. For $\gamma \in \Gamma$, we use the notation $\gamma(t_1, t_2) = (\gamma_1(t_1, t_2), \gamma_2(t_1, t_2)) \in S_{a_1} \times S_{a_2}$. Let us consider the map $F_\gamma : Q \to \mathbb{R}^2$ defined by

$$F_\gamma(t_1, t_2) := \left(\frac{\partial}{\partial s} I_{\mu_1+\beta}(s \star \gamma_1(t_1, t_2)) \Big|_{s=0}, \frac{\partial}{\partial s} I_{\mu_2+\beta}(s \star \gamma_2(t_1, t_2)) \Big|_{s=0} \right).$$

From

$$\begin{aligned} &\frac{\partial}{\partial s} I_{\mu_i+\beta}(s \star \gamma_i(t_1, t_2)) \Big|_{s=0} \\ &\qquad = \frac{\partial}{\partial s} \left(\frac{e^{2s}}{2} \int_{\mathbb{R}^3} |\nabla \gamma_i(t_1, t_2)|^2 - \frac{e^{3s}}{4} (\mu_i + \beta) \int_{\mathbb{R}^3} \gamma_i^4(t_1, t_2) \right) \Big|_{s=0} \\ hspace1cm &= \int_{\mathbb{R}^3} |\nabla \gamma_i(t_1, t_2)|^2 - \frac{3}{4} (\mu_i + \beta) \int_{\mathbb{R}^3} \gamma_i^4(t_1, t_2) \end{aligned}$$

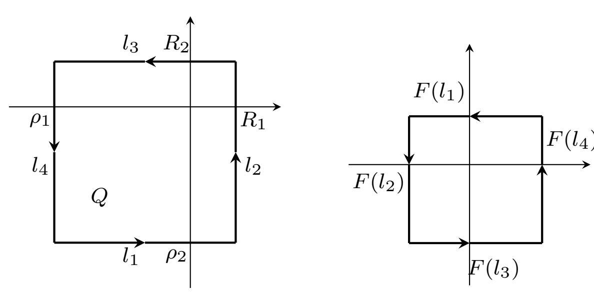
we deduce that

$$F_\gamma(t_1, t_2) = (0, 0) \quad \text{if and only if} \quad \gamma(t_1, t_2) \in \mathcal{P}(a_1, \mu_1 + \beta) \times \mathcal{P}(a_2, \mu_2 + \beta).$$

In order to show that $F_\gamma(t_1, t_2) = (0, 0)$ has a solution in Q for every $\gamma \in \Gamma$, we can check that the oriented path $F_\gamma(\partial^+ Q)$ has winding number equal to 1 with respect to the origin of $\mathbb{R}^2$, so that standard degree theory gives the desired result. In doing this, we observe at first that $F_\gamma(\partial^+ Q) = F_{\gamma_0}(\partial^+ Q)$ depends only on the choice of γ_0, and not on γ. Then we compute

$$\begin{aligned} F_{\gamma_0}(t_1, t_2) = \Bigg(& e^{2t_1} \left(\int_{\mathbb{R}^3} |\nabla w_1|^2 - \frac{3e^{t_1}}{4} (\mu_1 + \beta) \int_{\mathbb{R}^3} w_1^4 \right), \\ & e^{2t_2} \left(\int_{\mathbb{R}^3} |\nabla w_2|^2 - \frac{3e^{t_2}}{4} (\mu_1 + \beta) \int_{\mathbb{R}^3} w_2^4 \right) \Bigg) = (\psi_1(t_1), \psi_2(t_2)), \end{aligned}$$

where we recall that the definition of ψ_i has been given in (3.7). Therefore, the restriction of F_{γ_0} on ∂Q is completely described by Lemma 3.3-(ii), see the picture below:



In particular, we have that the topological degree

$$\deg(F_\gamma, Q, (0,0)) = \iota(F_{\gamma_0}(\partial^+ Q), (0,0)) = 1,$$

where $\iota(\sigma, P)$ denotes the winding number of the curve σ with respect to the point P. Hence there exists $(t_{1,\gamma}, t_{2,\gamma}) \in Q$ such that $F_\gamma(t_{1,\gamma}, t_{2,\gamma}) = (0,0)$, which, as observed, is the desired result. □

Lemmas 3.4 and 3.5 permit to apply the minimax principle (Theorem 3.2 in [14]) to J on Γ. In this way, we could obtain a Palais-Smale sequence for the constrained functional J on $S_{a_1} \times S_{a_2}$, but the boundedness of the Palais-Smale sequence would be unknown. In order to find a bounded Palais-Smale sequence, we shall adapt the trick introduced by one of the authors in [17] in the present setting.

Lemma 3.6. *There exists a Palais-Smale sequence (u_n, v_n) for J on $S_{a_1} \times S_{a_2}$ at the level*

$$c := \inf_{\gamma \in \Gamma} \max_{(t_1,t_2) \in Q} J(\gamma(t_1,t_2)) > \max\{\ell(a_1,\mu_1), \ell(a_2,\mu_2)\},$$

satisfying the additional condition

$$\int_{\mathbb{R}^3} \left(|\nabla u_n|^2 + |\nabla v_n|^2\right) - \frac{3}{4}\left(\int_{\mathbb{R}^3} \mu_1 u_n^4 + \mu_2 v_n^4 + 2\beta u_n^2 v_n^2\right) = o(1), \tag{3.13}$$

where $o(1) \to 0$ as $n \to \infty$. Furthermore, $u_n^-, v_n^- \to 0$ a.e. in $\mathbb{R}^3$ as $n \to \infty$.

Proof. We consider the augmented functional $\tilde{J} : \mathbb{R} \times S_{a_1} \times S_{a_2} \to \mathbb{R}$ defined by $\tilde{J}(s,u,v) := J(s \star u, s \star v)$. Let also

$$\tilde{\gamma}_0(t_1,t_2) := (0, \gamma_0(t_1,t_2)) = (0, t_1 \star w_1, t_2 \star w_2),$$

and

$$\tilde{\Gamma} := \{\tilde{\gamma} \in \mathcal{C}(Q, \mathbb{R} \times S_{a_1} \times S_{a_2}) : \tilde{\gamma} = \tilde{\gamma}_0 \text{ on } \partial Q\}.$$

We wish to apply the minimax principle Theorem 3.2 in [14] to the functional $\tilde{J}$ with the minimax class $\tilde{\Gamma}$, in order to find a Palais-Smale sequence for $\tilde{J}$ at level

$$\tilde{c} := \inf_{\tilde{\gamma} \in \tilde{\Gamma}} \sup_{(t_1,t_2) \in Q} \tilde{J}(\tilde{\gamma}(t_1,t_2)).$$

Notice that, since $\tilde{J}(\tilde{\gamma}_0) = J(\gamma_0)$ on ∂Q, by Lemmas 3.4 and 3.5, the assumptions of the minimax principle will be satisfied if we show that $\tilde{c} = c$. This equality is a simple consequence of the definition: firstly, since $\Gamma \subset \tilde{\Gamma}$, we have $\tilde{c} \leq c$. Secondly, using the notation

$$\tilde{\gamma}(t_1,t_2) = (s(t_1,t_2), \gamma_1(t_1,t_2), \gamma_2(t_1,t_2)),$$

for any $\tilde{\gamma} \in \tilde{\Gamma}$ and $(t_1,t_2) \in Q$ it results that

$$\tilde{J}(\tilde{\gamma}(t_1,t_2)) = J(s(t_1,t_2) \star \gamma_1(t_1,t_2), s(t_1,t_2) \star \gamma_2(t_1,t_2)),$$

and $(s(\cdot)\star\gamma_1(\cdot), s(\cdot)\star\gamma_2(\cdot)) \in \Gamma$. Thus $\tilde{c} = c$, and the minimax principle is applicable.

Notice that, using the notation of Theorem 3.2 in [14], we can choose the minimizing sequence $\tilde{\gamma}_n = (s_n, \gamma_{1,n}, \gamma_{2,n})$ for $\tilde{c}$ satisfying the additional conditions $s_n \equiv 0$, $\gamma_{1,n}(t_1,t_2) \geq 0$ a.e. in $\mathbb{R}^N$ for every $(t_1,t_2) \in Q$, $\gamma_{2,n}(t_1,t_2) \geq 0$ a.e. in $\mathbb{R}^N$ for every $(t_1,t_2) \in Q$. Indeed, the first condition comes from the fact that

$$\begin{aligned}\tilde{J}(\tilde{\gamma}(t_1,t_2)) &= J(s(t_1,t_2) \star \gamma_1(t_1,t_2), s(t_1,t_2) \star \gamma_2(t_1,t_2)) \\ &= \tilde{J}(0, s(t_1,t_2) \star \gamma_1(t_1,t_2), s(t_1,t_2) \star \gamma_2(t_1,t_2)).\end{aligned}$$

The remaining ones are a consequence of the fact that $\tilde J(s,u,v) = \tilde J(s,|u|,|v|)$.

In conclusion, Theorem 3.2 in [14] implies that there exists a Palais-Smale sequence $(\tilde s_n, \tilde u_n, \tilde v_n)$ for $\tilde J$ on $\mathbb{R} \times S_{a_1} \times S_{a_2}$ at level $\tilde c$, and such that

$$\lim_{n\to\infty} |\tilde s_n| + \mathrm{dist}_{H^1}\left((\tilde u_n, \tilde v_n), \tilde\gamma_n(Q)\right) = 0. \tag{3.14}$$

To obtain a Palais-Smale sequence for J at level c satisfying (3.13), it is possible to argue as in [17, Lemma 2.4] with minor changes. The fact that $u_n^-, v_n^- \to 0$ a.e. in $\mathbb{R}^N$ as $n \to \infty$ comes from (3.14). Finally, the lower estimate for c comes from Lemma 3.4. □

To complete the proof of Theorem 1.1, we aim at showing that (u_n, v_n) is strongly convergent in $H^1(\mathbb{R}^3, \mathbb{R}^2)$ to a limit (u,v). Once this has been achieved the claim follows because

$$dJ|_{S_{a_1}\times S_{a_2}}(u_n,v_n) \to 0 \quad \text{and} \quad (u_n,v_n) \in S_{a_1}\times S_{a_2}$$

for all n. A first step in this direction is given by the following statement.

Lemma 3.7. *The sequence $\{(u_n,v_n)\}$ is bounded in $H^1(\mathbb{R}^3,\mathbb{R}^2)$. Furthermore, there exists $\bar C > 0$ such that*

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 + |\nabla v_n|^2 \geq \bar C \qquad \text{for all } n.$$

Proof. Using (3.13), we have

$$J(u_n,v_n) = \frac{1}{6}\left(\int_{\mathbb{R}^3} |\nabla u_n|^2 + |\nabla v_n|^2\right) - o(1),$$

where $o(1) \to 0$ as $n \to \infty$. Therefore, the desired results follow from the fact that $J(u_n,v_n) \to c > 0$. □

By the previous lemma, up to a subsequence $(u_n,v_n) \to (\tilde u, \tilde v)$ weakly in $H^1(\mathbb{R}^3)$, strongly in $L^4(\mathbb{R}^3)$ (by compactness of the embedding $H^1_{\mathrm{rad}}(\mathbb{R}^3) \hookrightarrow L^4(\mathbb{R}^3)$), and a. e. in $\mathbb{R}^3$; in particular, both $\tilde u$ and $\tilde v$ are nonnegative in $\mathbb{R}^3$; we explicitly remark that we cannot deduce strong convergence in $L^2(\mathbb{R}^3)$, so that we cannot conclude that $(\tilde u, \tilde v) \in S_{a_1} \times S_{a_2}$. Observe that as a consequence of $dJ|_{S_{a_1}\times S_{a_2}}(u_n,v_n) \to 0$ there exist two sequences of real numbers $(\lambda_{1,n})$ and $(\lambda_{2,n})$ such that

$$\begin{aligned}\int_{\mathbb{R}^3} \left(\nabla u_n \cdot \nabla\varphi + \nabla v_n \cdot \nabla \psi - \mu_1 u_n^3 \varphi - \mu_2 v_n^3 \psi - \beta u_n v_n (u_n \psi + v_n \varphi)\right) \\ - \int_{\mathbb{R}^3} (\lambda_{1,n} u_n \varphi + \lambda_{2,n} \psi) = o(1) \|(\varphi,\psi)\|_{H^1}\end{aligned} \tag{3.15}$$

for every $(\varphi,\psi) \in H^1(\mathbb{R}^3,\mathbb{R}^2)$, with $o(1) \to 0$ as $n \to \infty$. For more details we refer to Lemma 2.2 of [5].

Lemma 3.8. *Both $(\lambda_{1,n})$ and $(\lambda_{2,n})$ are bounded sequences, and at least one of them is converging, up to a subsequence, to a strictly negative value.*

Proof. The value of the $(\lambda_{i,n})$ can be found using $(u_n, 0)$ and $(0, v_n)$ as test functions in (3.15):

$$\lambda_{1,n} a_1^2 = \int_{\mathbb{R}^3} \left(|\nabla u_n|^2 - \mu_1 u_n^4 - \beta u_n^2 v_n^2\right) - o(1)$$
$$\lambda_{2,n} a_2^2 = \int_{\mathbb{R}^3} \left(|\nabla v_n|^2 - \mu_2 v_n^4 - \beta u_n^2 v_n^2\right) - o(1),$$

with $o(1) \to 0$ as $n \to \infty$. Hence the boundedness of $(\lambda_{i,n})$ follows by the boundedness of (u_n, v_n) in H^1 and in L^4. Moreover, by (3.13) and Lemma 3.7

$$\begin{aligned}\lambda_{1,n} a_1^2 + \lambda_{2,n} a_2^2 &= \int_{\mathbb{R}^3} \left(|\nabla u_n|^2 + |\nabla v_n|^2 - \mu_1 u_n^4 - \mu_2 v_n^4 - 2\beta u_n^2 v_n^2\right) - o(1) \\ &= -\frac{1}{3} \int_{\mathbb{R}^3} \left(|\nabla u_n|^2 + |\nabla v_n|^2\right) + o(1) \le -\frac{\bar{C}}{6}\end{aligned}$$

for every n sufficiently large, so that at least one sequence of $(\lambda_{i,n})$ is negative and bounded away from 0. □

From now on, we consider converging subsequences $\lambda_{1,n} \to \lambda_1 \in \mathbb{R}$ and $\lambda_{2,n} \to \lambda_2 \in \mathbb{R}$. The sign of the limit values plays an essential role in our argument, as clarified by the next statement.

Lemma 3.9. *If $\lambda_1 < 0$ (resp. $\lambda_2 < 0$) then $u_n \to \bar{u}$ (resp. $v_n \to \bar{v}$) strongly in $H^1(\mathbb{R}^3)$.*

Proof. Let us suppose that $\lambda_1 < 0$. By weak convergence in $H^1(\mathbb{R}^3)$, strong convergence in $L^4(\mathbb{R}^3)$, and using (3.15), we have

$$\begin{aligned}o(1) &= (dJ(u_n, v_n) - dJ(\tilde{u}, \tilde{v}))\,[(u_n - \tilde{u}, 0)] - \lambda_1 \int_{\mathbb{R}^3} (u_n - \tilde{u})^2 \\ &= \int_{\mathbb{R}^3} |\nabla(u_n - \tilde{u})|^2 - \lambda_1 (u_n - \tilde{u})^2 + o(1),\end{aligned}$$

with $o(1) \to 0$ as $n \to \infty$. Since $\lambda_1 < 0$, this is equivalent to the strong convergence in H^1. The proof in the case $\lambda_2 < 0$ is similar. □

Remark 3.10. It is important to observe that Lemmas 3.7-3.9 do not depend on the value of β. This implies that we can use them in the proof of Theorem 1.2.

Conclusion of the proof of Theorem 1.1. By (3.15), the convergence of $(\lambda_{1,n})$ and $(\lambda_{2,n})$, and the weak convergence $(u_n, v_n) \rightharpoonup (\tilde{u}, \tilde{v})$, we have that $(\tilde{u}, \tilde{v})$ is a solution of (1.1). It remains to prove that it satisfies (1.2). Without loss of generality, by Lemma 3.8 we can suppose that $\lambda_1 < 0$, and hence (see Lemma 3.9) $u_n \to \tilde{u}$ strongly in $H^1(\mathbb{R}^3)$. If $\lambda_2 < 0$, we can infer in the same way that $v_n \to \tilde{v}$ strongly in $H^1(\mathbb{R}^3)$, which completes the proof. Now we argue by contradiction and assume that $\lambda_2 \ge 0$ and $v_n \not\to \tilde{v}$ strongly in $H^1(\mathbb{R}^3)$. Notice that, by regularity, any weak solution of (1.1) is smooth. Since both $\tilde{u}, \tilde{v} \ge 0$ in $\mathbb{R}^N$, we have that

$$-\Delta \tilde{v} = \lambda_2 \tilde{v} + \mu_2 \tilde{v}^3 + \beta \tilde{u}^2 \tilde{v} \ge 0 \qquad \text{in } \mathbb{R}^3,$$

and hence we can apply Lemma A.2 in [16], deducing that $\tilde{v} \equiv 0$. In particular, this implies that $\tilde{u}$ solves

$$\begin{cases} -\Delta \tilde{u} - \lambda_1 \tilde{u} = \mu_1 \tilde{u}^3 & \text{in } \mathbb{R}^3 \\ \tilde{u} > 0 & \text{in } \mathbb{R}^3 \\ \int_{\mathbb{R}^3} \tilde{u}^2 = a_1, \end{cases} \tag{3.16}$$

so that $\tilde u \in \mathcal{P}(a_1,\mu_1)$ and $I_{\mu_1}(\tilde u) = \ell(a_1,\mu_1)$ (recall (2.5) and Proposition 2.2). But then, using (3.13) and $\tilde u \in \mathcal{P}(a_1,\mu_1)$, we obtain

$$\begin{aligned} c = \lim_{n\to\infty} J(u_n,v_n) &= \lim_{n\to\infty} \frac{1}{8}\int_{\mathbb{R}^3} \left(\mu_1 u_n^4 + 2\beta u_n^2 v_n^2 + \mu_2 v_n^4\right) \\ &= \frac{\mu_1}{8}\int_{\mathbb{R}^3} \tilde u^4 = I_{\mu_1}(\tilde u) = \ell(a_1,\mu_1), \end{aligned} \tag{3.17}$$

in contradiction with Lemma 3.6. □

Remark 3.11. In the conclusion of the proof of Theorem 1.1 we used the uniqueness, up to translation, of the positive solution to (3.16) to deduce that, being $\tilde u$ a positive solution of (3.16), its level $I_{\mu_1}(\tilde u_1)$ is equal to $\ell(a_1,\mu_1)$. Such a uniqueness result is known for systems as (1.1) only if β is very small (see [15]). This is what prevents us to extend Theorem 1.1 to systems with several components without requiring the coupling parameters to be very small. In particular, we observe that the minimax construction can be extended to systems with an arbitrary number of components with some extra work.

4. Proof of Theorem 1.2

This section is divided into two parts. In the first one, we show the existence of a positive solution $(\bar u,\bar v)$, in the second one we characterize it as a ground state, in the sense that

$$\begin{aligned} J(\bar u,\bar v) &= \inf\{J(u,v) : (u,v) \in V\} \\ &= \inf\left\{J(u,v) : (u,v) \text{ is a solution of (1.1)-(1.2) for some } \lambda_1,\lambda_2\right\}. \end{aligned}$$

The proof of Theorem 1.2 is based upon a mountain pass argument, and, compared with the proof of Theorem 1.1, it is closer to the proof of the existence of normalized solutions for the single equation. We shall often consider, for $(u,v) \in S_{a_1}\times S_{a_2}$, the function

$$J(s\star(u,v)) = \frac{e^{2s}}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + |\nabla v|^2\right) - \frac{e^{3s}}{4}\int_{\mathbb{R}^3}\left(\mu_1 u^4 + 2\beta u^2 v^2 + \mu_2 v^4\right),$$

where $s\star(u,v) = (s\star u, s\star v)$ for short, and $s\star u$ is defined in (3.2). Recall that if $(u,v)\in S_{a_1}\times S_{a_2}$, then also $s\star(u,v)\in S_{a_1}\times S_{a_2}$. As an immediate consequence of the definition, the following holds:

Lemma 4.1. *Let $(u,v)\in S_{a_1}\times S_{a_2}$. Then*

$$\lim_{s\to-\infty}\int_{\mathbb{R}^3} |\nabla(s\star u)|^2 + |\nabla(s\star v)|^2 = 0, \quad \lim_{s\to+\infty}\int_{\mathbb{R}^3} |\nabla(s\star u)|^2 + |\nabla(s\star v)|^2 = +\infty,$$

and

$$\lim_{s\to-\infty} J(s\star(u,v)) = 0^+, \quad \lim_{s\to-\infty} J(s\star(u,v)) = -\infty.$$

The next lemma enlighten the mountain pass structure of the problem.

Lemma 4.2. *There exists $K>0$ sufficiently small such that for the sets*

$$A := \left\{(u,v)\in S_{a_1}\times S_{a_2} : \int_{\mathbb{R}^3} |\nabla u|^2 + |\nabla v|^2 \le K\right\}$$

and

$$B := \left\{(u,v)\in S_{a_1}\times S_{a_2} : \int_{\mathbb{R}^3} |\nabla u|^2 + |\nabla v|^2 = 2K\right\}$$

there holds

$$J(u) > 0 \text{ on } A \quad \text{and} \quad \sup_A J < \inf_B J.$$

Proof. By the Gagliardo-Nirenberg inequality (2.9)

$$\int_{\mathbb{R}^3} \left(\mu_1 u^4 + 2\beta u^2 v^2 + \mu_2 v^4\right) \le C \int_{\mathbb{R}^3} \left(u^4 + v^4\right) \le C \left(\int_{\mathbb{R}^3} |\nabla u|^2 + |\nabla v|^2 \right)^{3/2}$$

for every $(u,v) \in S_{a_1} \times S_{a_2}$, where $C > 0$ depends on $\mu_1, \mu_2, \beta, a_1, a_2 > 0$ but not on the particular choice of (u,v). Now, if $(u_1, v_1) \in B$ and $(u_2, v_2) \in A$ (with K to be determined), we have

$$\begin{aligned} J(u_1,v_1) - J(u_2,v_2) &\ge \frac{1}{2} \left(\int_{\mathbb{R}^3} |\nabla u_1|^2 + |\nabla v_1|^2 - \int_{\mathbb{R}^3} |\nabla u_2|^2 + |\nabla v_2|^2 \right) \\ &\quad - \frac{1}{4} \int_{\mathbb{R}^3} \left(\mu_1 u_1^4 + 2\beta u_1^2 v_1^2 + \mu_2 v_1^4\right) \\ &\ge \frac{K}{2} - \frac{C}{4} (2K)^{3/2} \ge \frac{K}{4} \end{aligned}$$

provided $K > 0$ is sufficiently small. Furthermore, making K smaller if necessary, we have also

$$J(u_2, v_2) \ge \frac{1}{2} \left(\int_{\mathbb{R}^3} |\nabla u_2|^2 + |\nabla v_2|^2 \right) - \frac{C}{4} \left(\int_{\mathbb{R}^3} |\nabla u_2|^2 + |\nabla v_2|^2 \right)^{3/2} > 0 \tag{4.1}$$

for every $(u_2, v_2) \in A$. □

In order to introduce a suitable minimax class, we recall that w_{a_1,μ_1} (resp. w_{a_2,μ_2}) is the unique positive solution of (2.3) with $a = a_1$ and $\mu = \mu_1$ (resp. $a = a_2$ and $\mu = \mu_2$). Now we define

$$C := \left\{ (u,v) \in S_{a_1} \times S_{a_2} : \int_{\mathbb{R}^3} |\nabla u|^2 + |\nabla v|^2 \ge 3K \text{ and } J(u,v) \le 0 \right\}. \tag{4.2}$$

It is clear by Lemma 4.1 that there exist $s_1 < 0$ and $s_2 > 0$ such that

$$s_1 \star (w_{a_1,\mu_1}, w_{a_2,\mu_2}) =: (\bar{u}_1, \bar{v}_1) \in A \quad \text{and} \quad s_2 \star (w_{a_1,\mu_1}, w_{a_2,\mu_2}) =: (\bar{u}_2, \bar{v}_2) \in C.$$

Finally we define

$$\Gamma := \left\{ \gamma \in \mathcal{C}([0,1], S_{a_1} \times S_{a_2}) : \gamma(0) = (\bar{u}_1, \bar{v}_1) \text{ and } \gamma(1) = (\bar{u}_2, \bar{v}_2) \right\}. \tag{4.3}$$

By Lemma 4.2 and by the continuity of the L^2-norm of the gradient in the topology of H^1, it follows that the mountain pass lemma is applicable for J on the minimax class Γ. Arguing as in Lemma 3.6, we deduce the following:

Lemma 4.3. *There exists a Palais-Smale sequence (u_n, v_n) for J on $S_{a_1} \times S_{a_2}$ at the level*

$$d := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)),$$

satisfying the additional condition (3.13)*:*

$$\int_{\mathbb{R}^3} \left(|\nabla u_n|^2 + |\nabla v_n|^2\right) - \frac{3}{4} \left(\int_{\mathbb{R}^3} \mu_1 u_n^4 + \mu_2 v_n^4 + 2\beta u_n^2 v_n^2 \right) = o(1),$$

with $o(1) \to 0$ as $n \to \infty$. Furthermore, $u_n^-, v_n^- \to 0$ a.e. in $\mathbb{R}^3$ as $n \to \infty$.

As in the previous section, the last part of the proof consists in showing that $(u_n, v_n) \to (\bar u, \bar v)$ in $H^1(\mathbb{R}^3, \mathbb{R}^2)$, and $(\bar u, \bar v)$ is a solution of (1.1)-(1.2). This can be done similarly to the case $\beta > 0$ small, recalling also Remark 3.10. Firstly, thanks to (3.13), up to a subsequence $(u_n, v_n) \to (\bar u, \bar v)$ weakly in $H^1(\mathbb{R}^3, \mathbb{R}^2)$, strongly in $L^4(\mathbb{R}^3, \mathbb{R}^2)$, a. e. in $\mathbb{R}^3$. By weak convergence and by Lemma 3.8, (u, v) is a solution of (1.1) for some $\lambda_1, \lambda_2 \in \mathbb{R}$. Moreover, we can also suppose that one of these parameters, say λ_1, is strictly negative. Thus, Lemma 3.9 implies that $u_n \to \bar u$ strongly in $H^1(\mathbb{R}^3)$. If by contradiction $v_n \not\to \bar v$ strongly in $H^1(\mathbb{R}^3)$, then $\lambda_2 \ge 0$, and by Lemma A.2 in [16] we deduce that $\bar v \equiv 0$. As in (3.17), this implies that $d = \ell(a_1, \mu_1)$ (defined in Proposition 2.2), and it remains to show that this gives a contradiction, which is the object of the following lemma.

Lemma 4.4. *There exists $\beta_2 > 0$ sufficiently large such that*

$$\sup_{s \in \mathbb{R}} J(s \star (w_{a_1,\mu_1}, w_{a_2,\mu_2})) < \min\{\ell(a_1, \mu_1), \ell(a_2, \mu_2)\} \qquad \text{for all } \beta > \beta_2.$$

Proof. By Lemma 3.1, for any $\varepsilon > 0$ there exists $s_\varepsilon \ll -1$ such that

$$I_{\mu_1}(s \star w_{a_1,\mu_1}) + I_{\mu_2}(s \star w_{a_2,\mu_2}) < \varepsilon \qquad \text{for all } s < s_\varepsilon.$$

For such values of s we have $J(s \star (w_{a_1,\mu_1}, w_{a_2,\mu_2})) < \varepsilon$, because $\beta > 0$. If $s \ge s_\varepsilon$, then the interaction term can be bounded from below:

$$\int_{\mathbb{R}^3} (s \star w_{a_1,\mu_1})^2 (s \star w_{a_2,\mu_2})^2 = e^{3s} \underbrace{\int_{\mathbb{R}^3} w_{a_1,\mu_1}^2 w_{a_2,\mu_2}^2}_{=: C_2 = C_2(a_1,a_2,\mu_1,\mu_2) > 0} \ge C_2 e^{3 s_\varepsilon}.$$

As a consequence, recalling that $\sup_s I_{\mu_i}(s \star w_{a_i,\mu_i}) = I_{\mu_i}(w_{a_i,\mu_i}) = \ell(a_i, \mu_i)$ (see again Lemma 3.4), we have

$$\begin{aligned} J(s \star (w_{a_1,\mu_1}, w_{a_2,\mu_2})) &\le I_{\mu_1}(s \star w_{a_1,\mu_1}) + I_{\mu_2}(s \star w_{a_2,\mu_2}) - \frac{C_2}{2} e^{3 s_\varepsilon} \beta \\ &\le \ell(a_1, \mu_1) + \ell(a_2, \mu_2) - \frac{C_2}{2} e^{3 s_\varepsilon} \beta, \end{aligned}$$

and the last term is strictly smaller than $\min\{\ell(a_1, \mu_1), \ell(a_2, \mu_2)\}$ provided β is sufficiently large. □

Remark 4.5. From the previous proof one can obtain an explicit estimate of β_2 in Theorem 1.2, in the following way. Choose as ε any value smaller than $\min\{\ell(a_1, \mu_1), \ell(a_2, \mu_2)\}$. Then one can explicitly estimate s_ε using Lemma 3.1 (the smaller is ε, the larger is $|s_\varepsilon|$). Once that ε is fixed and s_ε is estimated, it remains to solve the inequality

$$\ell(a_1, \mu_1) + \ell(a_2, \mu_2) - \frac{C_2(a_1, a_2, \mu_1, \mu_2)}{2} e^{3 s_\varepsilon} \beta < \min\{\ell(a_1, \mu_1), \ell(a_2, \mu_2)\}$$

with respect to β. An optimization in $0 < \varepsilon < \min\{\ell(a_1, \mu_1), \ell(a_2, \mu_2)\}$ reveals that β_2 can be chosen as

$$\begin{aligned} \beta_2 = {} & [\ell(a_1, \mu_1) + \ell(a_2, \mu_2) - \min\{\ell(a_1, \mu_1), \ell(a_2, \mu_2)\}] \\ & \cdot \left. \frac{2 e^{-3 s_\varepsilon}}{C_2(a_1, a_2, \mu_1, \mu_2)} \right|_{\varepsilon = \min\{\ell(a_1,\mu_1), \ell(a_2,\mu_2)\}}. \end{aligned}$$

Existence of a positive solution at level d. In our contradiction argument, we are supposing that $v_n \not\to \bar{v}$ strongly in $H^1(\mathbb{R}^3)$, and hence we have observed that $\bar{v} \equiv 0$ and $d = \ell(a_1, \mu_1)$. Let us consider the path

$$\gamma(t) := (((1-t)s_1 + ts_2) \star (w_{a_1,\mu_1}, w_{a_2,\mu_2}))\,.$$

Clearly, $\gamma \in \Gamma$, so that by Lemma 4.4

$$d \le \sup_{t\in[0,1]} J(\gamma(t)) \le \sup_{s\in\mathbb{R}} J(s \star (w_{a_1,\mu_1}, w_{a_2,\mu_2})) < \ell(a_1, \mu_1),$$

a contradiction. □

Let us now turn to the variational characterization for $(\bar{u}, \bar{v})$. In what follows we aim at proving that

$$\begin{aligned} J(\bar{u}, \bar{v}) &= \inf\{J(u,v) : (u,v) \in V\} \\ &= \inf\{J(u,v) : (u,v) \text{ is a solution of (1.1)-(1.2) for some } \lambda_1, \lambda_2\} \end{aligned}$$

Let us recall the definitions of A, see Lemma 4.2, and of C, see (4.2). We set

$$A^+ := \{(u,v) \in A : u, v \ge 0 \text{ a.e. in } \mathbb{R}^3\}$$

and

$$C^+ := \{(u,v) \in C : u, v \ge 0 \text{ a.e. in } \mathbb{R}^3\}.$$

For any $(u_1, v_1) \in A^+$ and $(u_2, v_2) \in C^+$, let

$$\Gamma(u_1, v_1, u_2, v_2) := \{\gamma \in \mathcal{C}([0,1], S_{a_1} \times S_{a_2}) : \gamma(0) = (u_1, v_1) \text{ and } \gamma(1) = (u_2, v_2)\}\,.$$

Lemma 4.6. *The sets A^+ and C^+ are connected by arcs, so that*

$$d = \inf_{\gamma\in\Gamma(u_1,v_1,u_2,v_2)} \max_{t\in[0,1]} J(\gamma(t)) \tag{4.4}$$

for every $(u_1, v_1) \in A^+$ and $(u_2, v_2) \in C^+$.

Proof. The proof is similar to the one of Lemma 2.8 in [17]. Equality (4.4) follows easily, once we show that A^+ and C^+ are connected by arcs (recall the definition of Γ, see (4.3), and also that $\bar{u}_1, \bar{v}_1, \bar{u}_2, \bar{v}_2 \ge 0$ in $\mathbb{R}^N$).

Let $(u_1, v_1), (u_2, v_2) \in S_{a_1} \times S_{a_2}$ be nonnegative functions such that

$$\int_{\mathbb{R}^3} |\nabla u_1| + |\nabla v_1|^2 = \int_{\mathbb{R}^3} |\nabla u_2| + |\nabla v_2|^2 = \alpha^2 \tag{4.5}$$

for some $\alpha > 0$. We define, for $s \in \mathbb{R}$ and $t \in [0, \pi/2]$,

$$h(s,t)(x) := (\cos t (s \star u_1)(x) + \sin t (s \star u_2)(x), \cos t (s \star v_1)(x) + \sin t (s \star v_2)(x))\,.$$

Although h depends on (u_1, v_1) and (u_2, v_2), we will not stress such dependence in order to simplify the notation. Setting $h = (h_1, h_2)$, we have that $h_1(s,t), h_2(s,t) \ge 0$ a.e. in $\mathbb{R}^N$, and by direct computations it is not difficult to check that

$$\int_{\mathbb{R}^3} h_1^2(s,t) = a_1^2 + \sin(2t) \int_{\mathbb{R}^3} u_1 u_2$$

$$\int_{\mathbb{R}^3} h_2^2(s,t) = a_2^2 + \sin(2t) \int_{\mathbb{R}^3} v_1 v_2$$

$$\int_{\mathbb{R}^3} |\nabla h_1(s,t)| + |\nabla h_2(s,t)|^2 = e^{2s} \left(\alpha^2 + \sin(2t) \int_{\mathbb{R}^3} \nabla u_1 \cdot \nabla u_2 + \nabla v_1 \cdot \nabla v_2 \right)$$

for all $(s,t) \in \mathbb{R} \times [0, \pi/2]$. From these expressions, and recalling (4.5) and the fact that $u_1, v_1, u_2, v_2 \ge 0$ a. e. in $\mathbb{R}^N$, it is possible to deduce that there exists $C > 0$ (depending on (u_1, v_1) and (u_2, v_2)) such that

$$Ce^{2s} \le \int_{\mathbb{R}^3} |\nabla h_1(s,t)|^2 + |\nabla h_2(s,t)|^2 \le 2\alpha^2 e^{2s}$$
$$a_1^2 \le \int_{\mathbb{R}^3} h_1^2(s,t) \le 2a_1^2 \quad \text{and} \quad a_2^2 \le \int_{\mathbb{R}^3} h_2^2(s,t) \le 2a_2^2$$

Thus, we can define for $(s,t) \in \mathbb{R} \times [0, \pi/2]$ the function

$$\hat{h}(s,t)(x) := \left(a_1 \frac{h_1(s,t)}{\|h_1(s,t)\|_{L^2}}, a_2 \frac{h_2(s,t)}{\|h_2(s,t)\|_{L^2}} \right).$$

Notice that $\hat{h}(s,t) \in S_{a_1} \times S_{a_2}$ for every (s,t). It results

$$\frac{\min\{a_1^2, a_2^2\} C e^{2s}}{2 \max\{a_1^2, a_2^2\}} \le \int_{\mathbb{R}^3} |\nabla \hat{h}_1(s,t)|^2 + |\nabla \hat{h}_2(s,t)|^2 \le \frac{2\alpha^2 e^{2s} \max\{a_1^2, a_2^2\}}{\min\{a_1^2, a_2^2\}}. \tag{4.6}$$

Furthermore, using again (4.5) (and replacing if necessary C with a smaller quantity), it is possible to check that

$$\int_{\mathbb{R}^3} \hat{h}_1^4(s,t) \ge Ce^{3s} \quad \text{and} \quad \int_{\mathbb{R}^3} \hat{h}_2^4(s,t) \ge Ce^{3s} \tag{4.7}$$

for all $(s,t) \in \mathbb{R} \times [0, \pi/2]$.

These estimates permits to prove the desired result. Let $(u_1, v_1), (u_2, v_2) \in A^+$, and let $\hat{h}$ de defined as in the previous discussion. By (4.6) there exists $s_0 > 0$ such that

$$\int_{\mathbb{R}^3} |\nabla \hat{h}_1(-s_0,t)|^2 + |\nabla \hat{h}_2(-s_0,t)|^2 \le K \qquad \text{for all } t \in \left[0, \frac{\pi}{2}\right],$$

where K has been defined in Lemma 4.2. For this choice of s_0, let

$$\sigma_1(r) := \begin{cases} -r \star (u_1, v_1) = \hat{h}(-r, 0) & 0 \le r \le s_0 \\ \hat{h}(-s_0, r - s_0) & s_0 < r \le s_0 + \frac{\pi}{2} \\ \left(r - 2s_0 - \frac{\pi}{2}\right) \star (u_2, v_2) = \hat{h}\left(r - 2s_0 - \frac{\pi}{2}, \frac{\pi}{2}\right) & s_0 + \frac{\pi}{2} < r \le 2s_0 + \frac{\pi}{2}. \end{cases}$$

It is not difficult to check that σ is a continuous path connecting (u_1, v_1) and (u_2, v_2) and lying in A^+. To conclude that A^+ is connected by arcs, it remains to analyse the cases when condition (4.5) is not satisfied. Suppose for instance

$$\int_{\mathbb{R}^3} |\nabla u_1|^2 + |\nabla v_1|^2 > \int_{\mathbb{R}^3} |\nabla u_2|^2 + |\nabla v_2|^2.$$

Then, by Lemma 4.1, there exists $s_1 < 0$ such that

$$\int_{\mathbb{R}^3} |\nabla (s_1 \star u_1)|^2 + |\nabla (s_1 \star v_1)|^2 = \int_{\mathbb{R}^3} |\nabla u_2|^2 + |\nabla v_2|^2.$$

Therefore, to connect (u_1, v_1) and (u_2, v_2) by a path in A^+ we can at first connect (u_1, v_1) with $s_1 \star (u_1, v_1)$, and then connect this point with (u_2, v_2).

To prove that also C^+ is connected by arcs, let us fix $(u_1, v_1), (u_2, v_2) \in C^+$, and suppose that (4.5) holds (as before, we can always reduce to this case). By (4.6) and (4.7), there exists $s_0 > 0$ such that

$$\int_{\mathbb{R}^3} |\nabla \hat{h}_1(s_0,t)|^2 + |\nabla \hat{h}_2(s_0,t)|^2 \ge 3K \quad \text{and} \quad J(\hat{h}(s_0,t)) \le 0$$

for all $t \in [0, \pi/2]$. For this choice of s_0, we set

$$\sigma_2(r) := \begin{cases} r \star (u_1, v_1) = \hat{h}(r, 0) & 0 \le r \le s_0 \\ \hat{h}(s_0, r - s_0) & s_0 < r \le s_0 + \frac{\pi}{2} \\ \left(2s_0 + \frac{\pi}{2} - r\right) \star (u_2, v_2) = \hat{h}\left(2s_0 + \frac{\pi}{2} - r, \frac{\pi}{2}\right) & s_0 + \frac{\pi}{2} < r \le 2s_0 + \frac{\pi}{2}, \end{cases}$$

which is the desired continuous path connecting (u_1, v_1) and (u_2, v_2) in C^+. □

As we shall see, the previous lemma will be the key in proving the variational characterization of $(\bar{u}, \bar{v})$. Let us recall the set

$$V := \{(u, v) \in T_{a_1} \times T_{a_2} : G(u, v) = 0\},$$

from (1.6), and its radial subset

$$V_{\mathrm{rad}} := \{(u, v) \in S_{a_1} \times S_{a_2} : G(u, v) = 0\}, \tag{4.8}$$

where

$$G(u, v) = \int_{\mathbb{R}^3} \left(|\nabla u|^2 + |\nabla v|^2\right) - \frac{3}{4} \int_{\mathbb{R}^3} \left(\mu_1 u^4 + 2\beta u^2 v^2 + \mu_2 v^4\right).$$

Lemma 4.7. *If (u, v) is a solution of* (1.1)-(1.2) *for some $\lambda_1, \lambda_2 \in \mathbb{R}$, then $(u, v) \in V$.*

Proof. The Pohozaev identity for system (1.1) reads

$$\frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + |\nabla v|^2 = \int_{\mathbb{R}^3} \frac{3}{2} \left(\lambda_1 u^2 + \lambda_2 v^2\right) + \frac{3}{4} \left(\mu_1 u^4 + 2\beta u^2 v^2 + \mu_2 v^4\right). \tag{4.9}$$

On the other hand, testing (1.1) with (u, v), we find

$$\lambda_1 \int_{\mathbb{R}^3} u^2 = \int_{\mathbb{R}^3} |\nabla u|^2 - \int_{\mathbb{R}^3} \left(\mu_1 u^4 + \beta u^2 v^2\right)$$
$$\lambda_2 \int_{\mathbb{R}^3} v^2 = \int_{\mathbb{R}^3} |\nabla v|^2 - \int_{\mathbb{R}^3} \left(\beta u^2 v^2 + \mu_2 v^4\right)$$

which substituted into (4.9) give the desired result. □

For $(u, v) \in T_{a_1} \times T_{a_2}$, let us set

$$\Psi_{(u,v)}(s) := J(s \star (u, v)),$$

where as before $s \star (u, v) = (s \star u, s \star v)$ for short, and $s \star u$ is defined in (3.2).

It is not difficult to check that V, V_{rad} are not empty. Actually, directly from the definition, one has much more.

Lemma 4.8. *For every $(u, v) \in T_{a_1} \times T_{a_2}$, there exists a unique $s_{(u,v)} \in \mathbb{R}$ such that $(s_{(u,v)} \star (u, v)) \in V$. Moreover, $s_{(u,v)}$ is the unique critical point of $\Psi_{(u,v)}$, which is a strict maximum.*

Lemma 4.9. *There holds*

$$\inf_V J = \inf_{V_{\mathrm{rad}}} J.$$

Proof. In order to prove the lemma we assume by contradiction that there exists $(u, v) \in V$ such that

$$0 < J(u, v) < \inf_{V_{\mathrm{rad}}} J. \tag{4.10}$$

For $u \in H^1(\mathbb{R}^3)$ let u^* denotes its Schwarz spherical rearrangement. By the properties of Schwarz symmetrization we have that $J(u^*, v^*) \le J(u, v)$ and $G(u^*, v^*) \le G(u, v) = 0$. Thus there exists $s_0 \le 0$ such that $G(s_0 \star (u^*, v^*)) = 0$. We claim that

$$J(s_0 \star (u^*, v^*)) \le e^{2s_0} J(u^*, v^*).$$

Indeed using that $G(s_0 \star (u^*, v^*)) = G(u, v) = 0$ we have

$$\begin{aligned} J(s_0 \star (u^*, v^*)) &= \frac{e^{2s_0}}{6} \int_{\mathbb{R}^3} |\nabla u^*|^2 + |\nabla v^*|^2 \\ &\le \frac{e^{2s_0}}{6} \int_{\mathbb{R}^3} |\nabla u|^2 + |\nabla v|^2 = e^{2s_0} J(u, v). \end{aligned} \tag{4.11}$$

Thus

$$0 < J(u, v) < \inf_{V_{\mathrm{rad}}} J \le J(s_0 \star (u^*, v^*)) \le e^{2s_0} J(u, v)$$

which contradicts $s_0 \le 0$. □

We are ready to complete the proof of Theorem 1.2.

Conclusion of the proof of Theorem 1.2. Recalling that any solution of (1.1)-(1.2) stays in V, if we have

$$J(\bar{u}, \bar{v}) = d \le \inf\{J(u, v) : (u, v) \in V_{\mathrm{rad}}\} \tag{4.12}$$

the thesis follows in view of Lemma 4.9. In order to prove (4.12) we choose an arbitrary $(u, v) \in V_{\mathrm{rad}}$ and show that $J(u, v) \ge d$. At first, since $(u, v) \in V_{\mathrm{rad}}$ implies $(|u|, |v|) \in V_{\mathrm{rad}}$ and $J(u, v) = J(|u|, |v|)$, it is not restrictive to suppose that $u, v \ge 0$ a.e. in $\mathbb{R}^3$. Let us consider the function $\Psi_{(u,v)}$. By Lemma 4.1 there exists $s_0 \gg 1$ such that $(-s_0) \star (u, v) \in A^+$ and $s_0 \star (u, v) \in C^+$. Therefore, the continuous path

$$\gamma(t) := ((2t - 1)s_0) \star (u, v) \qquad t \in [0, 1]$$

connects A^+ with C^+, and by Lemmas 4.6 and 4.8 we infer that

$$d \le \max_{t \in [0,1]} J(\gamma(t)) = J(u, v).$$

Since this holds for all the elements in V_{rad}, equality (4.12) follows. □

5. Systems with many components

In this section we prove Theorem 1.5. The problem under investigation is (1.8)-(1.9): we search for solutions to

$$\begin{cases} -\Delta u_i - \lambda_i u_i = \sum_{j=1}^k \beta_{ij} u_j^2 u_i & \text{in } \mathbb{R}^3 \\ u_i \in H^1(\mathbb{R}^3) \end{cases} \qquad i = 1, \dots, k,$$

satisfying

$$\int_{\mathbb{R}^3} u_i^2 = a_i^2 \qquad i = 1, \dots, k.$$

Dealing with multi-components systems, we adopt the notation $\mathbf{u} := (u_1, \dots, u_k)$. The first part of the proof is similar to the one of Theorem 1.2, therefore, we only sketch it.

For $\mathbf{u} \in S_{a_1} \times \dots \times S_{a_k}$ (recall definition (3.1)) and $s \in \mathbb{R}$, we consider

$$J(s \star \mathbf{u}) = \frac{e^{2s}}{2} \int_{\mathbb{R}^3} \sum_i |\nabla u_i|^2 - \frac{e^{3s}}{4} \int_{\mathbb{R}^3} \sum_{i,j} \beta_{ij} u_i^2 u_j^2$$

and

$$G(\mathbf{u}) = \int_{\mathbb{R}^3} \sum_{i=1}^k |\nabla u_i|^2 - \frac{3}{4} \int_{\mathbb{R}^3} \sum_{i,j=1}^k \beta_{ij} u_i^2 u_j^2. \tag{5.1}$$

It is not difficult to extend Lemma 4.2 for $k > 2$, proving the following:

Lemma 5.1. *There exists $K > 0$ small enough such that*

$$0 < \sup_A J < \inf_B J \quad \text{and} \quad G(u), J(u) > 0 \quad \forall u \in A,$$

where

$$A := \left\{ \mathbf{u} \in S_{a_1} \times S_{a_k} : \int_{\mathbb{R}^3} \sum_{i=1}^k |\nabla u_i|^2 \le K \right\}$$

and

$$B := \left\{ \mathbf{u} \in S_{a_1} \times S_{a_k} : \int_{\mathbb{R}^3} \sum_{i=1}^k |\nabla u_i|^2 = 2K \right\}.$$

We also introduce the set

$$C := \left\{ \mathbf{u} \in S_{a_1} \times \cdots \times S_{a_k} : \int_{\mathbb{R}^3} \sum_{i=1}^k |\nabla u_i|^2 \ge 3K \text{ and } J(\mathbf{u}) \le 0 \right\}, \tag{5.2}$$

and we recall the definition of $w_{a,\mu}$, given in Proposition 2.2. It is clear that there exist $s_1 < 0$ and $s_2 > 0$ such that

$$s_1 \star (w_{a_1,\beta_{11}}, \dots, w_{a_k,\beta_{kk}}) =: \hat{\mathbf{u}} \in A$$

and

$$s_2 \star (w_{a_1,\beta_{11}}, \dots, w_{a_k,\beta_{kk}}) =: \tilde{\mathbf{u}} \in C.$$

Setting

$$\Gamma := \{\gamma \in \mathcal{C}([0,1], S_{a_1} \times \cdots \times S_{a_k}) : \gamma(0) = \hat{\mathbf{u}},\ \gamma(1) = \tilde{\mathbf{u}}\},$$

by Lemma 5.1, it is possible to argue as in Lemma 4.3, showing that there exists a Palais-Smale sequence $(\mathbf{u}_n)$ for J at level

$$d := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)),$$

satisfying the additional condition

$$G(\mathbf{u}) = \int_{\mathbb{R}^3} \sum_{i=1}^k |\nabla u_i|^2 - \frac{3}{4} \int_{\mathbb{R}^3} \sum_{i,j=1}^k \beta_{ij} u_i^2 u_j^2 = o(1), \tag{5.3}$$

with $o(1) \to 0$ as $n \to \infty$. Moreover $u_{i,n}^- \to 0$ a.e. in $\mathbb{R}^3$ as $n \to \infty$, for any i. Notice that the value d depends on all the masses a_i and on all the couplings β_{ij}.

It remains to show that $\mathbf{u}_n \to \bar{\mathbf{u}}$ strongly in H^1, and the limit is a solution of (1.8)-(1.9). In order to do this, we argue as for the 2-components system: thanks to (5.3), up to a subsequence $\mathbf{u}_n \to \bar{\mathbf{u}}$ weakly in $H^1(\mathbb{R}^3, \mathbb{R}^k)$, strongly in $L^4(\mathbb{R}^3, \mathbb{R}^k)$, a.e. in $\mathbb{R}^3$. As before we arrive at the conclusion that $\bar{\mathbf{u}}$ is a solution of (1.8) for some $\lambda_1, \dots, \lambda_k \in \mathbb{R}$. We can also suppose that one of these parameters, say λ_1, is strictly negative. Thus, Lemma 3.9 implies that $u_{1,n} \to \bar{u}_1$ strongly in $H^1(\mathbb{R}^3)$. If by contradiction $u_{j,n} \not\to \bar{u}_j$ strongly in $H^1(\mathbb{R}^3)$ for some j, then $\lambda_j \ge 0$, and by Lemma A.2 in [16] we deduce that $\bar{u}_j \equiv 0$. To complete the proof, we aim at showing that $\bar{u}_i \not\equiv 0$ for every i, and to do this it is necessary to substantially modify the argument used for Theorem 1.1.

We divide the set of indexes $\{1, \dots, k\}$ in two subsets:

$$\mathcal{I}_1 := \{i \in \{1, \dots, k\} : \lambda_i < 0\} \quad \text{and} \quad \mathcal{I}_2 := \{i \in \{1, \dots, k\} : \lambda_i \geq 0\}.$$

Notice that $1 \in \mathcal{I}_1$, so that the cardinality of $\mathcal{I}_2$ is at most $k-1$, and that the absurd assumption can be written as $\mathcal{I}_2 \neq \emptyset$. Up to a relabelling, we can suppose for the sake of simplicity that

$$\mathcal{I}_1 := \{1, \dots, m\} \quad \text{and} \quad \mathcal{I}_2 := \{m+1, \dots, k\}$$

for some $1 \leq m < k$. By strong convergence (and by the maximum principle)

$$\begin{cases} -\Delta \bar{u}_i - \lambda_i \bar{u}_i = \sum_{j \in \mathcal{I}_1} \beta_{ij} \bar{u}_i \bar{u}_j^2 & \text{in } \mathbb{R}^3 \\ \bar{u}_i > 0 & \text{in } \mathbb{R}^3 \qquad \forall i \in \mathcal{I}_1, \\ \int_{\mathbb{R}^3} \bar{u}_i^2 = a_i^2, \end{cases}$$

while $\bar{u}_i \equiv 0$ for every $i \in \mathcal{I}_2$. As in Lemma 4.7, this implies that $(\bar{u}_1, \dots, \bar{u}_m) \in V_{\text{rad}}^{\mathcal{I}_1}$, where

$$V_{\text{rad}}^{\mathcal{I}_1} := \left\{ \mathbf{u} \in S_{a_1} \times \dots \times S_{a_m} : \int_{\mathbb{R}^3} \sum_{i=1}^m |\nabla u_i|^2 = \frac{3}{4} \int_{\mathbb{R}^3} \sum_{i,j=1}^m \beta_{ij} u_i^2 u_j^2 \right\}.$$

Therefore

$$J(\bar{\mathbf{u}}) = J(\bar{u}_1, \dots, \bar{u}_m, 0, \dots, 0) \geq \inf_{V_{\text{rad}}^{\mathcal{I}_1}} J. \tag{5.4}$$

Notice that in the last term we used J to denote the functional associated to a m components system, while in the previous terms J is used for the functional associated to the full k components system. This should not be a source of misunderstanding.

The value $J(\mathbf{u})$ can also be characterized in a different way: by (5.3), strong L^4-convergence, and recalling that $(\bar{u}_1, \dots, \bar{u}_m) \in V_{\text{rad}}^{\mathcal{I}_1}$, we have also

$$\begin{aligned} d = \lim_{n \to \infty} J(\mathbf{u}_n) &= \lim_{n \to \infty} \frac{1}{8} \int_{\mathbb{R}^3} \sum_{i,j=1}^k \beta_{ij} u_{i,n}^2 u_{j,n}^2 \\ &= \frac{1}{8} \int_{\mathbb{R}^3} \sum_{i,j=1}^m \beta_{ij} \bar{u}_i^2 \bar{u}_j^2 = J(\bar{u}_1, \dots \bar{u}_m, 0, \dots, 0) = J(\bar{\mathbf{u}}). \end{aligned} \tag{5.5}$$

A comparison between (5.4) and (5.5) reveals that

$$d \geq \inf_{V_{\text{rad}}^{\mathcal{I}_1}} J. \tag{5.6}$$

To find a contradiction, we shall provide an estimate from above on d, an estimate from below on $\inf_{V_{\text{rad}}^{\mathcal{I}_1}} J$, and show that these are not compatible with (5.6).

Upper estimate on d. First of all, we state the extension of Lemma 4.8.

Lemma 5.2. *For every $\mathbf{u} \in S_{a_1} \times \dots \times S_{a_k}$, there exists a unique $s_{\mathbf{u}} \in \mathbb{R}$ such that $(s_{\mathbf{u}} \star (u, v)) \in V_{\text{rad}}$. Moreover, $s_{\mathbf{u}}$ is the unique critical point of $\Psi_{\mathbf{u}}(s) := J(s \star \mathbf{u})$, which is a strict maximum.*

We shall now prove two variational characterizations for d.

Lemma 5.3. *It results that*
$$d = \inf_{V_{\rm rad}} J,$$
where
$$V_{\rm rad} := \left\{ \mathbf{u} \in S_{a_1} \times \cdots \times S_{a_k} : \int_{\mathbb{R}^3} \sum_{i=1}^k |\nabla u_i|^2 = \frac{3}{4} \int_{\mathbb{R}^3} \sum_{i,j=1}^k \beta_{ij} u_i^2 u_j^2 \right\}.$$

Proof. This can be done arguing as in the proof of Theorem 1.2. Firstly, we introduce the sets
$$A^+ := \left\{ \mathbf{u} \in A : u_i \geq 0 \text{ a.e. in } \mathbb{R}^3 \text{ for every } i \right\}$$
and
$$C^+ := \left\{ \mathbf{u} \in C : u_i \geq 0 \text{ a.e. in } \mathbb{R}^3 \text{ for every } i \right\},$$
where A and C have been defined in Lemma 5.1 and (5.2), respectively. Slightly modifying the proof of Lemma 4.6, one can check that A^+ and C^+ are connected by arcs, so that for any $\mathbf{u} \in A^+$ and $\mathbf{u}' \in C^+$ it results that
$$d = \inf_{\gamma \in \Gamma(\mathbf{u},\mathbf{u}')} \max_{t \in [0,1]} J(\gamma(t)),$$
where
$$\Gamma(\mathbf{u},\mathbf{u}') := \left\{ \gamma \in \mathcal{C}([0,1], S_{a_1} \times \cdots \times S_{a_k}) : \gamma(0) = \mathbf{u},\ \gamma(1) = \mathbf{u}' \right\}.$$
As in the conclusion of the proof of Theorem 1.2, from this it follows that
$$d \leq \inf_{V_{\rm rad}} J. \tag{5.7}$$
We have to check that also the reverse inequality holds. To this aim, we claim that
$$\text{for any path } \gamma \text{ from } A^+ \text{ to } C^+ \text{ there exists } t \in (0,1) \text{ such that } \gamma(t) \in V_{\rm rad}. \tag{5.8}$$
Once that the claim is proved, it is possible to observe that for any such γ
$$\inf_{V_{\rm rad}} J \leq J(\gamma(t)) \leq \max_{t \in [0,1]} J(\gamma(t)),$$
and taking the infimum over all the admissible γ from A^+ to C^+, we deduce that
$$\inf_{V_{\rm rad}} J \leq d,$$
which together with (5.7) completes the proof. Thus, it remains only to verify claim (5.8). Notice that
$$\mathbf{u} \in V_{\rm rad} \quad \Longleftrightarrow \quad G(\mathbf{u}) = 0,$$
where G has been defined in (5.1). There we showed that $G(\mathbf{u}) > 0$ for every $\mathbf{u} \in A$. Moreover, $J(\mathbf{u}) \leq 0$ for every $\mathbf{u} \in C^+$, which directly implies
$$G(\mathbf{u}) \leq -\frac{1}{4} \int_{\mathbb{R}^3} \sum_{i,j} \beta_{ij} u_i^2 u_j^2 < 0 \qquad \text{for all } \mathbf{u} \in C.$$
Thus, by continuity, for any $\mathbf{u} \in A^+$, any $\mathbf{u}' \in C^+$, and any $\gamma \in \Gamma(\mathbf{u},\mathbf{u}')$, there exists $t \in (0,1)$ such that $G(\gamma(t)) = 0$, which proves the claim. ∎

We introduce a Rayleigh-type quotient as
$$\mathcal{R}(\mathbf{u}) := \frac{8 \left(\int_{\mathbb{R}^3} \sum_{i=1}^k |\nabla u_i|^2 \right)^3}{27 \left(\int_{\mathbb{R}^3} \sum_{i,j=1}^k \beta_{ij} u_i^2 u_j^2 \right)^2}.$$

Lemma 5.4. *There holds that*

$$d = \inf_{V_{\rm rad}} J = \inf_{S_{a_1} \times \cdots \times S_{a_k}} \mathcal{R}.$$

Proof. If $\mathbf{u} \in V_{\rm rad}$, then

$$\frac{4 \int_{\mathbb{R}^3} \sum_{i=1}^k |\nabla u_i|^2}{3 \int_{\mathbb{R}^3} \sum_{i,j} \beta_{ij} u_i^2 u_j^2} = 1 \quad \text{and} \quad J(u_1, \dots, u_k) = \frac{1}{6} \int_{\mathbb{R}^3} \sum_{i=1}^k |\nabla u_i|^2.$$

Therefore

$$J(\mathbf{u}) = \frac{1}{6} \int_{\mathbb{R}^3} \sum_{i=1}^k |\nabla u_i|^2 \cdot \left(\frac{4 \int_{\mathbb{R}^3} \sum_{i=1}^k |\nabla u_i|^2}{3 \int_{\mathbb{R}^3} \sum_{i,j} \beta_{ij} u_i^2 u_j^2} \right)^2 = \mathcal{R}(\mathbf{u}),$$

which proves that $\inf_{V_{\rm rad}} J \geq \inf_{S_{a_1} \times \cdots \times S_{a_k}} \mathcal{R}$. On the other hand, it is not difficult to check that

$$\mathcal{R}(s \star \mathbf{u}) = \mathcal{R}(\mathbf{u}) \qquad \text{for all } s \in \mathbb{R}, \ \mathbf{u} \in S_{a_1} \times \cdots \times S_{a_k}.$$

By Lemma 5.2, we conclude that

$$\mathcal{R}(\mathbf{u}) = \mathcal{R}(s_{\mathbf{u}} \star \mathbf{u}) = J(s_{\mathbf{u}} \star \mathbf{u}) \geq \inf_{V_{\rm rad}} J$$

for every $\mathbf{u} \in S_{a_1} \times \cdots \times S_{a_k}$. □

The previous characterization makes it possible to derive an upper bound on d.

Lemma 5.5. *With C_0 and C_1 defined in* (2.2)*, there holds*

$$d \leq \frac{C_0 C_1 \left(\sum_i a_i^2\right)^3}{8 \left(\sum_{i,j} \beta_{ij} a_i^2 a_j^2\right)^2}$$

.

Proof. By Lemma 5.4, we have

$$d \leq \mathcal{R}\left(w_{a_1, C_0/a_1^2}, \dots, w_{a_k, C_0/a_k^2}\right),$$

where we recall that $w_{a,\mu}$ has been defined in Proposition 2.2. Using the explicit expression of $w_{a_i, C_0/a_i^2}$, we can compute

$$\int_{\mathbb{R}^3} w_{a_i, C_0/a_i^2}^2 w_{a_j, C_0/a_j^2}^2 = \frac{a_i^2 a_j^2}{C_0^2} \int_{\mathbb{R}^3} w_0^4 = \frac{a_i^2 a_j^2 C_1}{C_0^2}.$$

Recalling also (2.6) and (2.7), we deduce that

$$\mathcal{R}\left(w_{a_1, C_0/a_1^2}, \dots, w_{a_k, C_0/a_k^2}\right) = \frac{\left(\displaystyle\sum_i \frac{C_1 a_i^2}{C_0}\right)^3}{8 \left(\displaystyle\sum_{i,j} \beta_{ij} \frac{a_i^2 a_j^2 C_1}{C_0^2}\right)^2} = \frac{C_0 C_1 \left(\sum_i a_i^2\right)^3}{8 \left(\sum_{i,j} \beta_{ij} a_i^2 a_j^2\right)^2},$$

and the lemma follows. □

Lower estimate for $\inf_{V_{\rm rad}^{\mathcal{I}_1}} J$. Recall that we supposed, for the sake of simplicity, that $\mathcal{I}_1 = \{1, \dots, m\}$ for some $1 \le m < k$. Let us introduce

$$\mathcal{R}_{\mathcal{I}_1}(u_1, \dots, u_m) := \frac{8 \left(\int_{\mathbb{R}^3} \sum_{i=1}^m |\nabla u_i|^2 \right)^3}{27 \left(\int_{\mathbb{R}^3} \sum_{i,j=1}^m \beta_{ij} u_i^2 u_j^2 \right)^2}.$$

Exactly as in Lemma 5.4, one can prove that

$$\inf_{V_{\rm rad}^{\mathcal{I}_1}} J = \inf_{S_{a_1} \times \cdots \times S_{a_m}} \mathcal{R}_{\mathcal{I}_1}. \tag{5.9}$$

Lemma 5.6.

$$\inf_{V_{\rm rad}^{\mathcal{I}_1}} J \ge \frac{C_0 C_1}{8 \left[\max\limits_{1 \le j \le m} \{\beta_{jj} a_j\} + \frac{m-1}{m} \max\limits_{1 \le i \ne j \le m} \{\beta_{ij} a_i^{1/2} a_j^{1/2}\} \right]^2}.$$

Proof. We recall that

$$\sum_{1 \le i \ne j \le m} x_i x_j \le \frac{m-1}{m} \left(\sum_{i=1}^m x_i \right)^2 \qquad \text{for all } m \in \mathbb{N},\ x_1, \dots, x_m > 0.$$

Thus, by the Young's and the Gagliardo-Nirenberg's inequalities we have

$$\begin{aligned}
\sum_{i,j=1}^m \int_{\mathbb{R}^3} \beta_{ij} u_i^2 u_j^2 &\le \sum_{i,j=1}^m \frac{\beta_{ij}}{2} \left(\int_{\mathbb{R}^3} u_i^4 \right)^{\frac12} \left(\int_{\mathbb{R}^3} u_j^4 \right)^{\frac12} \\
&\le S \sum_{i,j=1}^m \beta_{ij} \sqrt{a_i a_j} \left(\int_{\mathbb{R}^3} |\nabla u_i|^2 \right)^{\frac34} \left(\int_{\mathbb{R}^3} |\nabla u_j|^2 \right)^{\frac34} \\
&\le S \left[\max_{1 \le j \le m} \{\beta_{jj} a_j\} \sum_{i=1}^m \left(\int_{\mathbb{R}^3} |\nabla u_i|^2 \right)^{\frac32} \right. \\
&\qquad \left. + \max_{1 \le i \ne j \le m} \{\beta_{ij} \sqrt{a_i a_j}\} \sum_{i \ne j} \left(\int_{\mathbb{R}^3} |\nabla u_i|^2 \right)^{\frac34} \left(\int_{\mathbb{R}^3} |\nabla u_j|^2 \right)^{\frac34} \right] \\
&\le S \left[\max_{1 \le j \le m} \{\beta_{jj} a_j\} + \frac{m-1}{m} \max_{1 \le i \ne j \le m} \{\beta_{ij} \sqrt{a_i a_j}\} \right] \left(\sum_{i=1}^m \int_{\mathbb{R}^3} |\nabla u_i|^2 \right)^{\frac32}
\end{aligned}$$

for every $(u_1, \dots, u_m) \in S_{a_1} \times \cdots \times S_{a_m}$. Thanks to the characterization of S in terms of C_0 and C_1, Lemma 2.4, and the definition of $\mathcal{R}_{\mathcal{I}_1}$, we deduce that

$$\inf_{S_{a_1} \times \cdots \times S_{a_m}} \mathcal{R}_{\mathcal{I}_1} \ge \frac{C_0 C_1}{8 \left[\max\limits_{1 \le j \le m} \{\beta_{jj} a_j\} + \frac{m-1}{m} \max\limits_{1 \le i \ne j \le m} \{\beta_{ij} a_i^{1/2} a_j^{1/2}\} \right]^2},$$

and the desired result follows (recall also (5.9)). □

Before proceeding with the conclusion of the proof, we observe that, as for systems with 2 components, the ground state radial level coincides with the ground state level in the all space, in the following sense.

Lemma 5.7. *There holds that*

$$\inf_V J = \inf_{V_{\mathrm{rad}}} J$$

where

$$V := \{\mathbf{u} \in T_{a_1} \times \cdots \times T_{a_k} : G(\mathbf{u}) = 0\}\,, \tag{5.10}$$

Proof. The proof relies on Lemma 5.2, which also holds when $\mathbf{u} \in T_{a_1} \times \cdots \times T_{a_k}$, and follows the line of the proof of Lemma 4.9. □

Conclusion of the proof of Theorem 1.5. We want to show that, under assumption (1.10), inequality (5.6) cannot be satisfied. If

$$\frac{C_0 C_1 \left(\sum_{i=1}^k a_i^2\right)^3}{8\left(\displaystyle\sum_{i,j=1}^k \beta_{ij} a_i^2 a_j^2\right)^2} < \frac{C_0 C_1}{8\left[\displaystyle\max_{1 \le j \le m}\{\beta_{jj} a_j\} + \frac{m-1}{m} \max_{1 \le i \ne j \le m}\{\beta_{ij} a_i^{1/2} a_j^{1/2}\}\right]^2} \tag{5.11}$$

then by Lemmas 5.5, 5.6 and 5.7 the theorem follows. A condition which implies the validity of (5.11), and hence of the theorem, is assumption (1.10). □

Remark 5.8. We emphasize the main difference between the concluding arguments in Theorem 1.2 and 1.5. In the former case to obtain a contradiction one has to compare the value d with two fixed quantities $\ell(a_1, \mu_1)$ and $\ell(a_2, \mu_2)$, which do not depend on β; on one side, argueing by contradiction one has $d = \ell(a_i, \mu_i)$ for some i; on the other hand, we have seen that it is sufficient to take β very large to have $d < \min\{\ell(a_1, \mu_1), \ell(a_2, \mu_2)\}$, which gives a contradiction. For systems with many components the situation is much more involved: the crucial equality for Theorem 1.5 is (5.6), which involves two quantities *both depending* on the coupling parameters. It would be tenting to think that the natural assumption in Theorem 1.5 is $\beta_{ij} \ge \bar\beta$ for every $i \ne j$. But if we make some β_{ij} too large, than both the sides in (5.6) becomes very small, and without any condition on the other parameters (β_{ij} and a_i) it seems hard to obtain a contradiction. Notice also that we do not have any control on the set $\mathcal{I}_1$, which makes the problem even more involved and imposes an assumption involving all the possible choices of $\mathcal{I}_1$.

For all these reasons we think that condition (1.10), which seems strange at a first glance, is not so unnatural.

6. Orbital stability

This section is devoted to the proof of Theorem 1.8, and we focus on a general k components system. Let $(\bar\lambda_1, \dots, \bar\lambda_k, \bar u_1, \dots, \bar u_k)$ be the solution of (1.8) found in Theorem 1.5. The crucial fact is that, by Lemma 5.7, $J(\bar{\mathbf{u}}) = \inf_V J$, where we recall that $V := \{\mathbf{u} \in T_{a_1} \times \cdots \times T_{a_k} : G(\mathbf{u}) = 0\}$, with

$$G(\mathbf{u}) = \int_{\mathbb{R}^3} \sum_{i=1}^k |\nabla u_i|^2 - \frac{3}{4}\int_{\mathbb{R}^3} \sum_{i,j=1}^k \beta_{ij} u_i^2 u_j^2.$$

The dynamics of (1.11) takes place in $H^1(\mathbb{R}^3, \mathbb{C}^k)$. By using similar arguments as in the proof of Lemmas 4.9 and 5.7, with (u^*, v^*) replaced by $(|u|, |v|)$, one can show that

$$\inf_{V_{\mathbb{C}}} J = \inf_V J$$

where

$$V_{\mathbb{C}} := \left\{ \mathbf{u} \in T^{\mathbb{C}}_{a_1} \times \cdots \times T^{\mathbb{C}}_{a_k} : G(\mathbf{u}) = 0 \right\},$$

and

$$T^{\mathbb{C}}_a := \left\{ u \in H^1(\mathbb{R}^3, \mathbb{C}^k) : \int_{\mathbb{R}^3} |u|^2 = a^2 \right\}.$$

Let us introduce the function

$$g_{\mathbf{u}}(t) := \frac{t^2}{2} \int_{\mathbb{R}^3} \sum_{i=1}^k |\nabla u_i|^2 - \frac{t^3}{4} \int_{\mathbb{R}^3} \sum_{i,j=1}^k \beta_{ij} |u_i|^2 |u_j|^2,$$

defined for $t > 0$. Notice that $g_{\mathbf{u}}(t) = J(\log t \star \mathbf{u})$. It is clear that for any $\mathbf{u} \in H^1(\mathbb{R}^3, \mathbb{C}^k)$ there exists a unique critical point $t_{\mathbf{u}} > 0$ for $g_{\mathbf{u}}$, which is a strict maximum, and that $\log t_{\mathbf{u}} \star \mathbf{u} \in V_{\mathbb{C}}$. Moreover, the function $g_{\mathbf{u}}$ is concave in $(t_{\mathbf{u}}, +\infty)$.

Lemma 6.1. *Let $d := \inf\{J(\mathbf{u}) : \mathbf{u} \in V_{\mathbb{C}}\}$. Then*

$$G(\mathbf{u}) < 0 \quad \Longrightarrow \quad G(\mathbf{u}) \le J(\mathbf{u}) - d.$$

Proof. By a direct computation $G(\mathbf{u}) = g'_{\mathbf{u}}(1)$. Thus, the condition $G(\mathbf{u}) < 0$ implies that $t_{\mathbf{u}} < 1$, and $g_{\mathbf{u}}$ is concave in $(t_{\mathbf{u}}, +\infty)$. As a consequence,

$$g_{\mathbf{u}}(1) \ge g_{\mathbf{u}}(t_{\mathbf{u}}) + (1 - t_{\mathbf{u}}) g'_{\mathbf{u}}(1) \ge g_{\mathbf{u}}(t_{\mathbf{u}}) + G(\mathbf{u}) \ge d + G(\mathbf{u}),$$

and since $g_{\mathbf{u}}(1) = J(\mathbf{u})$ the thesis follows. □

Conclusion of the proof of Theorem 1.5. Let $\mathbf{u}_s := s \star \bar{\mathbf{u}}$. Since $\bar{\mathbf{u}} \in V_{\mathbb{C}}$, it follows that $G(\mathbf{u}_s) < 0$ for every $s > 0$. Let $\Phi^s = (\Phi^s_1, \dots, \Phi^s_k)$ be the solution of system (1.11) with initial datum $\mathbf{u}_s$, defined on the maximal interval $(T_{\min}, T_{\max})$. By continuity, provided $|t|$ is sufficiently small we have $G(\Phi^s(t)) < 0$. Therefore, by Lemma 6.1 and recalling that the energy is conserved along trajectories of (1.11), we have

$$G(\Phi^s(t)) \le J(\Phi^s(t)) - d = J(\mathbf{u}_s) - d =: -\delta < 0$$

for any such t, and by continuity again we infer that $G(\Phi^s(t)) \le -\delta$ for every $t \in (T_{\min}, T_{\max})$. To obtain a contradiction, we recall that the virial identity (see Proposition 6.5.1 in [10] for the identity associated to the scalar equation; dealing with a gradient-type system, the computations are very similar) establishes that

$$f''_s(t) = 8G(\Phi^s(t)) \le -8\delta < 0 \quad \text{for} \quad f_s(t) := \int_{\mathbb{R}^3} |x|^2 \sum_{i=1}^k |\Phi^s_i(t,x)|^2 \, dx$$

and as a consequence

$$0 \le f_s(t) \le -\delta t^2 + O(t) \qquad \text{for all } t \in (-T_{\min}, T_{\max}).$$

Since the right hand side becomes negative for $|t|$ sufficiently large, it is necessary that both $T_{\min}$ and $T_{\max}$ are bounded. This proves that, for a sequence of initial data arbitrarily close to $\bar{\mathbf{u}}$, we have blow-up in finite time, implying orbital instability. □

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