A NATURAL CONSTRAINT APPROACH TO NORMALIZED SOLUTIONS OF NONLINEAR SCHRÖDINGER EQUATIONS AND SYSTEMS

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Abstract. The paper deals with the existence of normalized solutions to the system

$$\begin{cases} -\Delta u - \lambda_1 u = \mu_1 u^3 + \beta u v^2 & \text{in } \mathbb{R}^3 \\ -\Delta v - \lambda_2 v = \mu_2 v^3 + \beta u^2 v & \text{in } \mathbb{R}^3 \\ \int_{\mathbb{R}^3} u^2 = a_1^2 \quad \text{and} \quad \int_{\mathbb{R}^3} v^2 = a_2^2 \end{cases}$$

for any $\mu_1, \mu_2, a_1, a_2 > 0$ and $\beta < 0$ prescribed. We present a new approach that is based on the introduction of a natural constraint associated to the problem. We also show that, as $\beta \to -\infty$, phase separation occurs for the solutions that we find.

Our method can be adapted to scalar nonlinear Schrödinger equations with normalization constraint, and leads to alternative and simplified proofs to some results already available in the literature.

1. Introduction

Various physical phenomena, such as the occurrence of phase-separation in Bose-Einstein condensates with multiple states, or the propagation of mutually incoherent wave packets in nonlinear optics, are modeled by the system of coupled nonlinear Schrödinger equations

$$\begin{cases} -\iota \partial_t \Phi_1 = \Delta \Phi_1 + \mu_1 |\Phi_1|^2 \Phi_1 + \beta |\Phi_2|^2 \Phi_1 \\ -\iota \partial_t \Phi_2 = \Delta \Phi_2 + \mu_2 |\Phi_2|^2 \Phi_2 + \beta |\Phi_1|^2 \Phi_2 \end{cases} \qquad t, x \in \mathbb{R} \times \mathbb{R}^N, \tag{1.1}$$

see e.g. [1, 16, 18, 27, 48]. In the models, Φ_i is the wave function of the i-th component, the dimension of the ambient space is $N \leq 3$, and the real parameters μ_i and β represent the intra-spaces and inter-species scattering length, describing respectively the interaction between particles of the same component or of different components. In particular, the positive sign of μ_i (and of β) stays for attractive interaction, while the negative sign stays for repulsive interaction.

A fundamental step in the comprehension of the dynamics of the system consists in studying the possible existence and properties of solitary waves, solutions to (1.1) of type $\Phi_i(t, x) = e^{-i\lambda_i t} u_i(x)$, with $\lambda_i \in \mathbb{R}$ and $u_i : \mathbb{R}^N \to \mathbb{R}$. This ansatz leads to the following elliptic system for the densities u_1 and u_2:

$$\begin{cases} -\Delta u_1 - \lambda_1 u_1 = \mu_1 u_1^3 + \beta u_2^2 u_1 \\ -\Delta u_2 - \lambda_2 u_2 = \mu_2 u_2^3 + \beta u_1^2 u_2 \end{cases} \quad \text{in } \mathbb{R}^N. \tag{1.2a}$$

This paper concerns the existence of *normalized solutions* to (1.2a) in dimension $N = 3$, i.e. the existence of real numbers $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ and of functions $(u_1, u_2) \in$

$H^1(\mathbb{R}^3, \mathbb{R}^2)$ satisfying (1.2a) together with the normalization condition

$$\int_{\mathbb{R}^3} u_1^2 = a_1^2 \quad \text{and} \quad \int_{\mathbb{R}^3} u_2^2 = a_2^2, \tag{1.2b}$$

for a-priori given a_1, $a_2 > 0$, μ_1, μ_2, $\beta \in \mathbb{R}$. In what follows we refer to a solution of (1.2a)-(1.2b) simply as to a solution to (1.2). We emphasize that, prescribing the *masses* a_i from the beginning, the *frequencies* λ_i are included in the unknown. A somehow dual approach consists in fixing the frequencies λ_i from the beginning, and leave the masses free.

Normalized solutions are particularly interesting from a physical point of view, since the mass $\|\Phi_i(t,\cdot)\|_{L^2} = \|u_i\|_{L^2}$ has often a clear physical meaning. In the aforementioned contexts, it represents the number of particles of each component in Bose-Einstein condensates, or the power supply in the nonlinear optics framework. But despite this physical relevance, most of the papers deal with the problem with fixed frequencies, see e.g. [2,10,14,22,25,26,28,35,36,37,38,47,49] and the references therein, while problem (1.2) is far from being well understood.

In order to clarify the difficulties that one has to face when searching for normalized solutions, in what follows we introduce some notation and review the few known results regarding (1.2).

Let $\mu_i =: \beta_{ii}$, $\beta =: \beta_{12} = \beta_{21}$, and for any $a > 0$ let us consider

$$S_a := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} u^2 = a^2 \right\}. \tag{1.3}$$

Solutions to (1.2) are critical points of the *energy functional*

$$J(u_1, u_2) = \int_{\mathbb{R}^3} \frac{1}{2} \sum_{i=1}^{2} |\nabla u_i|^2 - \frac{1}{4} \sum_{i,j=1}^{2} \beta_{ij} u_i^2 u_j^2, \tag{1.4}$$

on the constraint $S_{a_1} \times S_{a_2}$ with (λ_1, λ_2) Lagrange multipliers. We are interested in *positive* normalized solutions, i.e. normalized solutions with $u_1, u_2 > 0$ in $\mathbb{R}^N$. Concerning the terminology, we often identify a solution $(\lambda_1, \lambda_2, u_1, u_2)$ of (1.2) with its last components (u_1, u_2), with some abuse of notation. This is justified by the fact that we obtain (u_1, u_2) as critical points of the above constrained functional and (λ_1, λ_2) are determined as Lagrange multipliers.

Some papers concern the existence of positive normalized solution when $\mathbb{R}^N$ is replaced by a bounded domain Ω, or when a trapping potential is included in the equation; we refer to [33], where essentially no assumption is imposed on μ_1, μ_2, β, but where the masses a_1 and a_2 are supposed to be small, and to [30,46], which regard the *defocusing-repulsing* case $\mu_1, \mu_2, \beta < 0$ with equal masses $a_1 = a_2 = 1$. Notice that, if $\mu_1, \mu_2, \beta < 0$ and Ω is bounded, the existence of a single normalized solution can be proved quite easily by minimization arguments, and indeed in [30,46] the authors are mainly interested in multiplicity results and occurrence of phase-separation.

Let us consider now the *focusing* case $\mu_1, \mu_2 > 0$ in the whole space $\mathbb{R}^N$. When (1.2) is considered in dimension $N = 1$, the constrained functional is bounded from below, and for arbitrary $a_i, \mu_i, \beta > 0$ a positive normalized solution can be found minimizing $J|_{S_{a_1} \times S_{a_2}}$ and using concentration-compactness arguments. This approach, used in [29] (see also [12, Section 5]), fails if $N = 2, 3$, since $J|_{S_{a_1} \times S_{a_2}}$ is unbounded both from above and from below. Thus, in higher dimensions one is induced to apply minimax methods, as successfully done in [9]. In the paper [9] we considered the *attractive* case $\beta > 0$ in $\mathbb{R}^3$ (the 2-dimensional case is particularly delicate, see the forthcoming Remark 1.8). We proved that, for arbitrary masses a_i and parameters μ_i, there exist $\bar\beta_2 > \bar\beta_1 > 0$ (depending on the data) such that for both $0 < \beta < \bar\beta_1$ and $\beta > \bar\beta_2$ system (1.2) has a positive radial solution; in case $\beta > \bar\beta_2$ this solution is of mountain pass type, while for $0 < \beta < \bar\beta_1$ the solution is obtained with a 2-dimensional linking. This is somehow reminiscent to what happens for the unconstrained problem with fixed frequencies [2, 36]. But despite the similarity between the results in [9] and those in [2,36], the proofs differ substantially: the approach in [2, 36] is indeed based on the research of critical points for the *action functional*

$$\mathcal{A}(u_1, u_2) := J(u_1, u_2) - \sum_{i=1}^{2} \frac{\lambda_i}{2} \int_{\mathbb{R}^3} u_i^2$$

constrained on Nehari-type sets associated to the problem, while apparently no Nehari manifold is available in the framework of normalized solutions because λ_1 and λ_2 are part of the unknown, and (u_1, u_2) cannot be used as variation for $J|_{S_{a_1}\times S_{a_2}}$ in (u_1, u_2). Further difficulties in dealing with the normalization constraint are that the existence of *bounded* Palais-Smale sequences requires new arguments (the classical method used to prove the boundedness of any Palais-Smale sequence for unconstrained Sobolev-subcritical problem does not work), that Lagrange multipliers have to be controlled, and that weak limits of Palais-Smale sequences do not necessarily lie on $S_{a_1} \times S_{a_2}$. For all these reasons, the proofs in [9] are quite delicate and cannot be directly extended to cover the case $\beta < 0$. The existence of normalized solutions for the focusing-repulsive case $\mu_i > 0$, $\beta < 0$ was then, up to now, completely open. This is the object of our first main result.

Theorem 1.1. *Let $N = 3$, and let $\mu_1, \mu_2, a_1, a_2 > 0$ and $\beta < 0$ be fixed. Then (1.2) has a solution $(\lambda_1, \lambda_2, \bar u_1, \bar u_2)$ with $\lambda_i < 0$, and $\bar u_i$ is positive in $\mathbb{R}^3$ and radially symmetric.*

For a_1, a_2, μ_1 and μ_2 fixed, we find then a family $\{(\lambda_{1,\beta}, \lambda_{2,\beta}, \bar u_{1,\beta}, \bar u_{2,\beta}) : \beta < 0\}$. Our next result shows that phase-separation occurs as $\beta \to -\infty$.

Theorem 1.2. *Let $N = 3$, and let $\mu_1, \mu_2, a_1, a_2 > 0$ be fixed. Then, as $\beta \to -\infty$, we have (up to a subsequence):*

- *(i)* $(\lambda_{1,\beta}, \lambda_{2,\beta}) \to (\lambda_1, \lambda_2)$, *with* $\lambda_1, \lambda_2 \le 0$;
- *(ii)* $(\bar u_{1,\beta}, \bar u_{2,\beta}) \to (\bar u_1, \bar u_2)$ *in* $\mathcal{C}^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ *and in* $H^1_{\mathrm{loc}}(\mathbb{R}^N)$;
- *(iii)* $\bar u_1$ *and* $\bar u_2$ *are nonnegative Lipschitz continuous functions having disjoint positivity sets, in the sense that* $\bar u_1 \bar u_2 \equiv 0$ *in* $\mathbb{R}^N$;
- *(iv) the difference* $\bar u_1 - \bar u_2$ *is a sign-changing radial solution of*

$$-\Delta w - \lambda_1 w^+ + \lambda_2 w^- = \mu_1 (w_1^+)^3 - \mu_2 (w_1^-)^3 \qquad \text{in } \mathbb{R}^N.$$

In order to prove Theorem 1.1, we devise a new approach, substantially different with respect to the one in [9], based upon the introduction of a further constraint. Let

$$G(u_1, u_2) = \sum_{i=1}^{2} \int_{\mathbb{R}^3} |\nabla u_i|^2 - \frac{3}{4} \sum_{i,j=1}^{2} \int_{\mathbb{R}^3} \beta_{ij} u_i^2 u_j^2, \tag{1.5}$$

and let

$$\mathcal{P} := \{(u_1, u_2) \in S_{a_1} \times S_{a_2} \mid G(u_1, u_2) = 0\}. \tag{1.6}$$

As proved in [9, Lemma 4.6], any solution of (1.2) stays in $\mathcal{P}$. The solution that was obtained in [9, Theorem 1.2] for $\beta > 0$ large by a mountain pass argument on $S_{a_1} \times S_{a_2}$ was characterized as minimizer of J on $\mathcal{P}$. In the present paper we show that one can actually apply min-max methods to J constrained to $\mathcal{P}$ in order to obtain solutions of (1.2).

Theorem 1.3. *The set $\mathcal{P}$ is a $\mathcal{C}^1$-manifold, and moreover:*

(i) *If there exists a Palais-Smale sequence $\{(\tilde u_{1,n}, \tilde u_{2,n})\}$ for J restricted to $\mathcal{P}$ at level $\ell \in \mathbb{R}$, then there exists a possibly different Palais-Smale sequence $\{(u_{1,n}, u_{2,n})\} \subset \mathcal{C}_c^\infty(\mathbb{R}^3)$ for J restricted to $S_{a_1} \times S_{a_2}$ at the same level $\ell \in \mathbb{R}$.*

(ii) *If (u_1, u_2) is a critical point of J restricted on $\mathcal{P}$, then (u_1, u_2) is a critical point of J restricted on $S_{a_1} \times S_{a_2}$, and hence a solution to (1.2).*

One often refers to property (ii) saying that $\mathcal{P}$ *is a natural constraint*. Clearly it has codimension 1 in $S_{a_1} \times S_{a_2}$. Roughly speaking, the manifold $\mathcal{P}$ plays, for problem (1.2), the role of the Nehari manifold for equations with fixed frequencies.

With the previous result in hands, we prove Theorem 1.1 finding a critical point of mountain pass type for the constrained functional $J|_{\mathcal{P}}$.

We point out that this natural constraint approach is very flexible and, suitably modified, permits also to recover the known existence and multiplicity results regarding normalized solutions for the nonlinear Schrödinger equation

$$\begin{cases} -\Delta u - \lambda u = f(u) & \text{in } \mathbb{R}^N \\ u > 0, u \in H^1(\mathbb{R}^N) \\ \int_{\mathbb{R}^N} u^2 = a^2, \end{cases} \tag{1.7}$$

under appropriate assumptions on f. Solutions to (1.7) are critical points of the functional

$$I(u) := \int_{\mathbb{R}^N} |\nabla u|^2 - F(u), \quad F(s) := \int_0^s f(\sigma)\, d\sigma, \tag{1.8}$$

on the sphere S_a. The case of the pure power nonlinearity $f(s) = |s|^{p-2}s$ can be treated using the results available for the problem with fixed $\lambda < 0$, properly scaling the equation; such an approach fails when f is inhomogeneous. For inhomogeneous f two different pictures are possible, depending on whether or not I can be globally minimized on S_a. For the power nonlinearity, the former case, called L^2*-subcritical*, takes place if $2 < p < 2 + 4/N$, and was firstly considered in [41, 42]. Afterwards it was also addressed with the aid of the concentration-compactness principle [23, 24]. If $2 + 4/N \le p < 2N/(N-2)$, then $I|_{S_a}$ cannot be minimized, and the problem is considerably more involved. The so called L^2*-critical* case $p = 2 + 4/N$ is particularly delicate, and will be discussed in Remark 1.8. The L^2-supercritical and Sobolev subcritical case $2 + 4/N < p < 2N/(N-2)$ was considered only in the two papers [7, 21]. In [21] it is proved the existence of a mountain pass positive normalized solution. In [7], putting in evidence the "fountain" type structure of $I|_{S_a}$, the authors proved the existence of infinitely many normalized solutions. The precise assumptions considered in [7, 21] on the nonlinearity f are the following:

$(f1)$ $f : \mathbb{R} \to \mathbb{R}$ is continuous and odd;

($f2$) $N \geq 2$, and there exists $\alpha, \beta \in \mathbb{R}$,

$$2 + \frac{4}{N} < \alpha \leq \beta < 2^* := \begin{cases} +\infty & \text{if } N = 1, 2 \\ \frac{2N}{N-2} & \end{cases},$$

such that

$$0 < \alpha F(s) \leq f(s)s \leq \beta F(s) \qquad \forall s \in \mathbb{R} \setminus \{0\};$$

In this paper we give an alternative simple proof of the existence and multiplicity results in [7, 21]. We emphasize that here we use the additional assumption ($f3$) below, which is not needed in [7, 21].

Theorem 1.4. *Let $N \geq 2$, $a > 0$, and let f satisfy ($f1$), ($f2$), and*

($f3$) *the map $\tilde{F}(s) := f(s)s - 2F(s)$ is of class $\mathcal{C}^1$, and*

$$\tilde{F}'(s)s > \left(2 + \frac{4}{N}\right) \tilde{F}(s).$$

Then (1.7) has infinitely many radial solutions $\{u_k : k \geq 1\}$ with increasing energy, and u_1 is positive in $\mathbb{R}^N$.

Our proof of Theorem 1.4 is based upon the search for critical points of I constrained on

$$\mathcal{M} := \{u \in S_a : G(u) = 0\}, \tag{1.9}$$

where

$$\begin{aligned} G(u) :&= \int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} \left(\frac{N}{2} f(u)u - NF(u)\right) \\ &= \int_{\mathbb{R}^N} |\nabla u|^2 - \frac{N}{2} \int_{\mathbb{R}^N} \tilde{F}(u). \end{aligned} \tag{1.10}$$

It turns out that $\mathcal{M}$ is a natural constraint, as expressed by the following statement.

Theorem 1.5. *Under ($f1$)-($f3$), the set $\mathcal{M}$ is a $\mathcal{C}^1$ manifold, and moreover:*

(i) *If there exists a Palais-Smale sequence $\{\tilde{u}_n\}$ for I restricted to $\mathcal{M}$ at level $\ell \in \mathbb{R}$, then there exists a possibly different Palais-Smale sequence $\{u_n\} \subset \mathcal{C}_c^\infty(\mathbb{R}^3)$ for I restricted to S_a at the same level $\ell \in \mathbb{R}$.*

(ii) *If u is a critical point of I restricted on $\mathcal{M}$, then u is a critical point of I restricted on S_a, and hence a solution to (1.7).*

We will see that the constrained functional I restricted to $\mathcal{M}$ is bounded from below, coercive, and satisfies the Palais-Smale condition. Therefore, Theorem 1.4 will be a simple consequence of the equivariant Lusternik-Schirelman theory.

Remark 1.6. Assumption ($f3$) is not needed in [7, 21] for proving the existence of solutions of (1.7) (actually it is required in [21] in order to treat the case $N = 1$). It is an interesting question whether ($f3$) can be omitted in Theorem 1.4 also with our approach. Then $\mathcal{M}$ will not be a manifold anymore but it still contains all solutions of (1.7). This suggests that Theorem 1.4 could be approached using the critical point theory on metric spaces from [15]. In any case, we observe that for a wide class of nonlinearities, such as those of type

$$f(s) = \sum_{i=1}^{m} \mu_i |s|^{p_i - 2} s \quad \text{with} \quad \mu_i > 0,$$

this assumption is already included in $(f2)$. Notice also that, even if $\tilde{F} \in \mathcal{C}^1$, the function f need not be differentiable in the origin.

We conclude the introduction mentioning further problems which we believe could be treated with our natural constraint approach, and discussing why we do not consider (1.2) in $\mathbb{R}^2$.

Remark 1.7. Even though we focused on system (1.2a), we can treat more general power type problems such as

$$\begin{cases} -\Delta u_1 - \lambda_1 u_1 = \mu_1 |u_1|^{p_1-2} u_1 + \beta |u_1|^{r_1-2} |u_2|^{r_2} u_1 & \text{in } \mathbb{R}^N \\ -\Delta u_2 - \lambda_2 u_2 = \mu_2 |u_2|^{p_2-2} u_2 + \beta |u_1|^{r_1} |u_2|^{r_2-2} u_2 & \text{in } \mathbb{R}^N \\ \int_{\mathbb{R}^N} u_1^2 = a_1^2 \quad \int_{\mathbb{R}^N} u_2^2 = a_2^2, \end{cases} \tag{1.11}$$

or even systems with right hand sides $\partial_1 F(u_1, u_2)$, $\partial_2 F(u_1, u_2)$, under appropriate assumptions on F. Systems with an arbitrary number of components can be considered as well, i.e. also in these contexts it is possible to introduce the set $\mathcal{P}$, and to prove that it is a natural constraint, in the sense specified by Theorem 1.3. Notice that (1.2) is a particular case of (1.11), and we mention that existence results under different assumptions on the data of the problem have been obtained in [8, 9, 20]. We believe that some of the results therein could be re-proved using $\mathcal{P}$ and adapting the method used here.

More generally, we believe that our approach can be adapted in many situations in which a Pohozaev-type identity without boundary terms is available. With regard to this, we mention that the three problems (1.2), (1.7) and (1.11) considered in this paper have been studied also in bounded domains instead that in the whole space, see [17, 32, 33] and the references therein. In such situations it is not clear how to define $\mathcal{P}$ or $\mathcal{M}$, since the Pohozaev identity involves boundary terms which are not necessarily well defined for $u \in H^1(\Omega)$.

Remark 1.8. The existence of normalized solutions in the L^2-critical case is a very delicate problem. Let us consider the stationary NLS equation

$$-\Delta u - \lambda u = |u|^{p-2} u \quad \text{in } \mathbb{R}^N, \quad \int_{\mathbb{R}^N} u^2 = a^2. \tag{1.12}$$

If either $2 < p < 2 + 4/N$ or $2 + 4/N < p < 2N/(N-2)$, for any $a > 0$ the problem has a unique positive radial solution, which can be obtained by scaling the unique positive radial solution of

$$-\Delta w + w = |w|^{p-2} w \quad \text{in } \mathbb{R}^N.$$

If on the other hand $p = 2 + 4/N$, which is for instance the case of the cubic NLS equation (i.e. $p = 4$) in $\mathbb{R}^2$, then there exists a uniquely determined $\bar{a} > 0$ (depending only on the dimension) such that (1.12) with $a = \bar{a}$ has infinitely many positive radial solutions (corresponding to different λ), while for $a \neq \bar{a}$ (1.12) has no positive solution at all. This makes the L^2-critical problem extremely peculiar to treat, and as far as we know there is no result concerning inhomogeneous f in this case. In the same spirit, even though we could introduce the set $\mathcal{P}$, we cannot treat system (1.2a) in $\mathbb{R}^2$ with our technique, which is tailor-made for the L^2-supercritical and Sobolev-subcritical context.

Organization of the paper. Theorem 1.3 is the object of Section 2. The result is then used in the proof of existence of solutions to (1.2), Theorem 1.1, which is the content of Section 3. Theorem 1.2, is treated in Subsection 3.4. Sections 4 and 5 are devoted to the proofs of Theorems 1.5 and 1.4 respectively.

Notation. For the sake of brevity, we often write $\mathbf{u}$ instead of (u_1, u_2) for vector valued functions in $H^1(\mathbb{R}^3, \mathbb{R}^2)$. We recall that $\beta_{ii} := \mu_i$ and $\beta_{12} = \beta_{21} := \beta$. If $\mathcal{N}$ is a $\mathcal{C}^1$-manifold, we denote by $T_P\mathcal{N}$ the tangent space to $\mathcal{N}$ in the point $P \in \mathcal{N}$. Throughout the paper C will always denote a positive constant, whose value is allowed to change also from line to line.

2. A natural constraint for elliptic systems

In this section we aim at proving that the set $\mathcal{P}$, introduced in (1.6), is a natural constraint in the sense specified by Theorem 1.3. Actually, we will prove the following slightly stronger statement.

Theorem 2.1. *The set $\mathcal{P} \subset S_{a_1} \times S_{a_2} \subset H^1(\mathbb{R}^3, \mathbb{R}^2)$ is a $\mathcal{C}^1$-submanifold, and moreover:*

- (i) *If $\{(u_{1,n}, u_{2,n})\} \subset \mathcal{C}_c^\infty(\mathbb{R}^3) \cap \mathcal{P}$ is a Palais-Smale sequence for J restricted to $\mathcal{P}$ at a certain level $\ell \in \mathbb{R}$, then $\{(u_{1,n}, u_{2,n})\}$ is a Palais-Smale sequence for J restricted to $S_{a_1} \times S_{a_2}$.*
- (ii) *If there exists a Palais-Smale sequence $\{(\tilde{u}_{1,n}, \tilde{u}_{2,n})\}$ for J restricted to $\mathcal{P}$ at level $\ell \in \mathbb{R}$, then there exists a possibly different Palais-Smale sequence $\{(u_{1,n}, u_{2,n})\} \subset \mathcal{C}_c^\infty(\mathbb{R}^3)$ for J restricted to $\mathcal{P}$ at the same level $\ell \in \mathbb{R}$. Moreover $\|u_{i,n} - \tilde{u}_{i,n}\|_{H^1} \to 0$ as $n \to \infty$ for $i = 1, 2$.*
- (iii) *If there exists a Palais-Smale sequence $\{(\tilde{u}_{1,n}, \tilde{u}_{2,n})\}$ for J restricted to $\mathcal{P}$ at level $\ell \in \mathbb{R}$, then there exists a possibly different Palais-Smale sequence $\{(u_{1,n}, u_{2,n})\} \subset \mathcal{C}_c^\infty(\mathbb{R}^3)$ for J restricted to $S_{a_1} \times S_{a_2}$ at the same level $\ell \in \mathbb{R}$. Moreover $\|u_{i,n} - \tilde{u}_{i,n}\|_{H^1} \to 0$ as $n \to \infty$ for $i = 1, 2$.*
- (iv) *Let (u_1, u_2) be a critical point of J restricted on $\mathcal{P}$. Then (u_1, u_2) is a critical point of J restricted on $S_{a_1} \times S_{a_2}$, and hence a solution to (1.2).*

The first step consists in showing that $\mathcal{P}$ is a manifold.

Lemma 2.2. *The set $\mathcal{P}$ is a $\mathcal{C}^1$-submanifold of codimension 1 in $S_{a_1} \times S_{a_2}$, hence a $\mathcal{C}^1$-submanifold of codimension 3 in $H^1(\mathbb{R}^3, \mathbb{R}^2)$.*

Proof. As subset of $H^1(\mathbb{R}^3, \mathbb{R}^2)$, the constraint $\mathcal{P}$ is defined by $G(u_1, u_2) = 0$, $G_1(u_1) = 0$, $G_2(u_2) = 0$, where

$$G_i(u_i) := a_i^2 - \int_{\mathbb{R}^3} u_i^2$$

for $i = 1, 2$, and G is defined in (1.5). Since the functions G and G_i are of class $\mathcal{C}^1$, we have only to check that

$$d(G_1, G_2, G) : H^1(\mathbb{R}^3, \mathbb{R}^2) \to \mathbb{R}^3 \quad \text{is surjective.} \tag{2.1}$$

If this is not true, $dG(u_1, u_2)$ has to be linearly dependent from $dG_1(u_1)$ and $dG_2(u_2)$, i.e. there exist $\nu_1, \nu_2 \in \mathbb{R}$ such that

$$2\sum_{i=1}^{2} \int_{\mathbb{R}^3} \nabla u_i \cdot \nabla \varphi_i - \frac{3}{2} \sum_{i,j=1}^{2} \int_{\mathbb{R}^3} \beta_{ij} u_i u_j (u_i \varphi_j + \varphi_i u_j) = 2\sum_{i=1}^{2} \nu_i \int_{\mathbb{R}^3} u_i \varphi_i$$

for every $(\varphi_1, \varphi_2) \in H^1(\mathbb{R}^3, \mathbb{R}^2)$. This means that (u_1, u_2) is a solution to

$$\begin{cases} -\Delta u_i - \nu_i u_i = \frac{3}{2} \sum_{i=1}^2 \beta_{ij} u_i u_j^2 & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} u_i^2 = a_i^2 \end{cases}$$

for $i = 1, 2$. But then, applying [9, Lemma 4.6], we conclude that

$$\sum_{i=1}^2 \int_{\mathbb{R}^3} |\nabla u_i|^2 - \frac{9}{8} \sum_{i,j=1}^2 \int_{\mathbb{R}^3} \beta_{ij} u_i^2 u_j^2 = 0.$$

Recalling that $G(u_1, u_2) = 0$, this implies that

$$\sum_{i=1}^2 \int_{\mathbb{R}^3} |\nabla u_i|^2 = 0, \quad \text{and hence} \quad (u_1, u_2) = (0, 0),$$

in contradiction with the fact that $(u_1, u_2) \in S_{a_1} \times S_{a_2}$. $\square$

We define for $s \in \mathbb{R}$ and $w \in H^1(\mathbb{R}^3)$ the function

$$(s \star w)(x) = e^{3s/2} w(e^s x). \tag{2.2}$$

One can easily check that $\|s \star w\|_{L^2(\mathbb{R}^3)} = \|w\|_{L^2(\mathbb{R}^3)}$ for every $s \in \mathbb{R}$. As a consequence, given $(u_1, u_2) \in S_{a_1} \times S_{a_2}$, it results that

$$s \star \mathbf{u} = s \star (u_1, u_2) := (s \star u_1, s \star u_2) \in S_{a_1} \times S_{a_2}$$

for every $s \in \mathbb{R}$. We consider the real valued function

$$\Psi_{\mathbf{u}}(s) := J(s \star \mathbf{u}).$$

By changing variables in the integrals, we obtain

$$\Psi_{\mathbf{u}}(s) = \frac{e^{2s}}{2} \int_{\mathbb{R}^3} \sum_{i=1}^2 |\nabla u_i|^2 - \frac{e^{3s}}{4} \int_{\mathbb{R}^3} \sum_{i,j=1}^2 \beta_{ij} u_i^2 u_j^2. \tag{2.3}$$

Let us introduce

$$\mathcal{E} := \left\{ (u_1, u_2) \in S_{a_1} \times S_{a_2} : \sum_{i,j=1}^2 \beta_{ij} \int_{\mathbb{R}^3} u_i^2 u_j^2 > 0 \right\}. \tag{2.4}$$

By the Hölder inequality, it follows straightforwardly that $\mathcal{E} = S_{a_1} \times S_{a_2}$ in case $-\sqrt{\mu_1 \mu_2} < \beta < +\infty$, while for $\beta \le -\sqrt{\mu_1 \mu_2}$ it results that $\mathcal{E} \subset S_{a_1} \times S_{a_2}$ with strict inclusion. Notice also that, thanks to the continuity of the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^4(\mathbb{R}^3)$, the set $\mathcal{E}$ is an open subset of $S_{a_1} \times S_{a_2}$ in the H^1 topology. The role of $\mathcal{E}$ is clarified by the following statement.

Lemma 2.3. *For any $\mathbf{u} = (u_1, u_2) \in S_{a_1} \times S_{a_2}$, a value $s \in \mathbb{R}$ is a critical point of $\Psi_{\mathbf{u}}$ if and only if $s \star \mathbf{u} \in \mathcal{P}$. It results that:*

(i) If $\mathbf{u} \in \mathcal{E}$, then there exists a unique critical point $s_{\mathbf{u}} \in \mathbb{R}$ for $\Psi_{\mathbf{u}}$, which is a strict maximum point, and is defined by

$$\exp(s_{\mathbf{u}}) = \frac{4 \int_{\mathbb{R}^3} \sum_i |\nabla u_i|^2}{3 \int_{\mathbb{R}^3} \sum_{i,j} \beta_{ij} u_i^2 u_j^2}. \tag{2.5}$$

In particular, if $\mathbf{u} \in \mathcal{P}$, then $s_{\mathbf{u}} = 0$.

(ii) If $\mathbf{u} = (u_1, u_2) \notin \mathcal{E}$, then $\Psi_{\mathbf{u}}$ has no critical points in $\mathbb{R}$.

The proof is a simple consequence of (2.3) and the definition of $\mathcal{P}$ and $\mathcal{E}$.

In the following statement we describe the structure of $T_{\mathbf{u}}(S_{a_1} \times S_{a_2})$ in points of $\mathcal{P}$.

Lemma 2.4. *For any* $\mathbf{u} \in \mathcal{P} \cap \mathcal{C}_c^\infty(\mathbb{R}^3, \mathbb{R}^2)$, *we have*

$$T_{\mathbf{u}}(S_{a_1} \times S_{a_2}) = T_{\mathbf{u}} S_{a_1} \times T_{\mathbf{u}} S_{a_2} = T_{\mathbf{u}} \mathcal{P} \oplus \mathbb{R} \left. \frac{d}{ds} \right|_{s=0} (s \star \mathbf{u}).$$

Proof. First observe that for $w \in S_a \cap \mathcal{C}_c^\infty(\mathbb{R}^3, \mathbb{R})$ the path $\gamma : \mathbb{R} \to S_a$, $s \mapsto s * w$, is of class C^1 with derivative given by $\gamma'(s)(x) = \frac{3}{2} e^{3s/2} w(e^s x) + e^{3s/2} \nabla w(e^s x) \cdot (e^s x)$. Consequently $\left.\frac{d}{ds}\right|_{s=0} (s \star \mathbf{u}) \in T_{\mathbf{u}}$ is well defined for $\mathbf{u} \in \mathcal{P} \cap \mathcal{C}_c^\infty(\mathbb{R}^3, \mathbb{R}^2)$. By Lemma 2.2, we know that $\mathcal{P}$ has codimension 1 with respect to $S_{a_1} \times S_{a_2}$, and hence it is sufficient to show that

$$\left. \frac{d}{ds} \right|_{s=0} (s \star \mathbf{u}) \notin T_{\mathbf{u}} \mathcal{P},$$

that is

$$dG(u_1, u_2) \left[\left. \frac{d}{ds} \right|_{s=0} (s \star \mathbf{u}) \right] \neq 0,$$

with G defined in (1.5). For any $w \in \mathcal{C}_c^\infty(\mathbb{R}^3)$, we can compute

$$\begin{aligned} dG(u_1, u_2) \left[\left. \frac{d}{ds} \right|_{s=0} (s \star \mathbf{u}) \right] &= 2 \sum_i \int_{\mathbb{R}^3} \left[\frac{3}{2} |\nabla u_i|^2 + \nabla u_i \cdot \nabla (\nabla u_i \cdot x) \right] \\ &- \frac{9}{2} \sum_{i,j} \int_{\mathbb{R}^3} \beta_{ij} u_i^2 u_j^2 - 3 \sum_{i,j} \int_{\mathbb{R}^3} \beta_{ij} u_i u_j^2 \nabla u_i \cdot x \end{aligned} \tag{2.6}$$

Observing that

$$\nabla (\nabla u_i \cdot x) = (\nabla^2 u_i)[x] + \nabla u_i$$

and using the divergence theorem, the first integral on the right hand side in (2.6) can be developed as

$$\begin{aligned} &\int_{\mathbb{R}^3} \left[\frac{3}{2} |\nabla u_i|^2 + \nabla u_i \cdot \nabla (\nabla u_i \cdot x) \right] \\ &\qquad = \int_{\mathbb{R}^3} \left[\frac{3}{2} |\nabla u_i|^2 + \frac{1}{2} \nabla (|\nabla u_i|^2) \cdot x + |\nabla u_i|^2 \right] = \int_{\mathbb{R}^3} |\nabla u_i|^2. \end{aligned}$$

Concerning the second and the third integral, again by the divergence theorem it results that

$$\begin{aligned} &\frac{9}{2} \sum_{i,j} \int_{\mathbb{R}^3} \beta_{ij} u_i^2 u_j^2 + 3 \sum_{i,j} \int_{\mathbb{R}^3} \beta_{ij} u_i u_j^2 \nabla u_i \cdot x \\ &\qquad = \frac{9}{2} \sum_{i,j} \int_{\mathbb{R}^3} \beta_{ij} u_i^2 u_j^2 + \frac{3}{4} \sum_{i,j} \int_{\mathbb{R}^3} \beta_{ij} \nabla (u_i^2 u_j^2) \cdot x = \frac{9}{4} \sum_{i,j} \int_{\mathbb{R}^3} \beta_{ij} u_i^2 u_j^2. \end{aligned}$$

Coming back to (2.6), and using the definition of $\mathcal{P}$, we finally conclude

$$\begin{aligned} &dG(u_1, u_2) \left[\left. \frac{d}{ds} (s \star (u_1, u_2)) \right|_{s=0} \right] \\ &\qquad = 2 \sum_i \int_{\mathbb{R}^3} |\nabla u_i|^2 - \frac{9}{4} \sum_{i,j} \int_{\mathbb{R}^3} \beta_{ij} u_i^2 u_j^2 = - \sum_i \int_{\mathbb{R}^3} |\nabla u_i|^2 \neq 0, \end{aligned}$$

which completes the proof. □

Remark 2.5. In general the variation
$$\frac{d}{ds}\bigg|_{s=0}(s\star\mathbf{u})$$
is not in $H^1(\mathbb{R}^3,\mathbb{R}^2)$; this is why we require $\mathbf{u}\in\mathcal{C}_c^\infty(\mathbb{R}^3,\mathbb{R}^2)$ in the lemma. Actually it would have been enough to suppose that $\mathbf{u}\in H^2(\mathbb{R}^3,\mathbb{R}^2)$ decays sufficiently fast so that the previous variation stays in $H^1(\mathbb{R}^3,\mathbb{R}^2)$.

In Lemma 2.4, we showed that the tangent space to $S_{a_1}\times S_{a_2}$ in a point $\mathbf{u}\in\mathcal{P}\cap\mathcal{C}_c^\infty(\mathbb{R}^3,\mathbb{R}^2)$ splits as direct sum of the tangent space to $\mathcal{P}$ plus a 1-dimensional subspace of type $\mathbb{R}(v_1,v_2)$ for a suitable variation (v_1,v_2). The crucial fact for Theorem 2.1 is that any point of $\mathcal{P}$ is critical for J, by definition, with respect to variations in $\mathbb{R}(v_1,v_2)$. This is why criticality on $\mathcal{P}$ implies criticality on $S_{a_1}\times S_{a_2}$, which is rigorously proved in the following lemma.

Lemma 2.6. *If $(u_1,u_2)\in\mathcal{C}_c^\infty(\mathbb{R}^3,\mathbb{R}^2)\cap\mathcal{P}$, then*
$$dJ(u_1,u_2)\left[\frac{d}{ds}\bigg|_{s=0}(s\star(u_1,u_2))\right]=0.$$

Proof. If $\mathbf{u}=(u_1,u_2)\in\mathcal{C}_c^\infty(\mathbb{R}^3,\mathbb{R}^2)\cap\mathcal{P}$, then by Lemma 2.3 we have $s_{\mathbf{u}}=0$, and
$$0=\Psi_{\mathbf{u}}'(0)=\frac{d}{ds}\bigg|_{s=0}J(s\star\mathbf{u})=\left[dJ(s\star\mathbf{u})\left[\frac{d}{ds}(s\star\mathbf{u})\right]\right]_{s=0}.$$
The thesis follows. □

We prove now a simple preliminary result which we shall use many times in the rest of the paper.

Lemma 2.7. *Let $\{u_n\}\subset H^1(\mathbb{R}^3)$, $\{s_n\}\subset\mathbb{R}$, and let us suppose that $u_n\to u$ strongly in $H^1(\mathbb{R}^3)$ and $s_n\to s\in\mathbb{R}$, as $n\to\infty$. Then $s_n\star u_n\to s\star u$ strongly in $H^1(\mathbb{R}^3)$ as $n\to\infty$.*

Proof. By definition $s_n\star u_n\to s\star u$ a. e. in $\mathbb{R}^3$, and
$$\|s_n\star u_n\|_{H^1}^2=e^{2s_n}\int_{\mathbb{R}^3}|\nabla u_n|^2+\int_{\mathbb{R}^3}u_n^2\le C$$
for every n. Hence, up to a subsequence $s_n\star u_n\rightharpoonup s\star u$ weakly in H^1, and moreover we have the convergence of the norms $\|s_n\star u_n\|_{H^1}\to\|s\star u\|_{H^1}$. This argument works for all the possible subsequences. □

We are finally ready for the:

Proof of Theorem 2.1. For the proof of (i), let $\{\mathbf{u}_n\}\subset\mathcal{C}_c^\infty(\mathbb{R}^3,\mathbb{R}^2)\cap\mathcal{P}$ be a Palais-Smale sequence for $J|_{\mathcal{P}}$. We denote by $T_{\mathbf{u}}^*(S_{a_1}\times S_{a_2})$ the dual space to $T_{\mathbf{u}}(S_{a_1}\times S_{a_2})$, and by $\|\cdot\|$ the $H^1(\mathbb{R}^3,\mathbb{R}^2)$ norm. By Lemma 2.4
$$\begin{aligned}\|dJ(\mathbf{u}_n)\|_{T_{\mathbf{u}}^*(S_{a_1}\times S_{a_2})}&=\sup\left\{\left|dJ(\mathbf{u}_n)[\varphi]\right|:\varphi\in T_{\mathbf{u}}(S_{a_1}\times S_{a_2}),\ \|\varphi\|\le1\right\}\\&=\sup\left\{|dJ(\mathbf{u}_n)[\phi]+dJ(\mathbf{u}_n)[\psi]|:\begin{array}{l}\varphi=\phi+\psi,\|\varphi\|\le1\\ \phi\in T_{\mathbf{u}}\mathcal{P},\ \psi\in\mathbb{R}\left(\frac{d}{ds}\big|_{s=0}(s\star\mathbf{u}_n)\right)\end{array}\right\}.\end{aligned}$$

Since Lemma 2.6 yields $dJ(\mathbf{u}_n)[\psi] = 0$, we deduce that

$$\begin{aligned}\|dJ(\mathbf{u}_n)\|_{(T_{\mathbf{u}}(S_{a_1}\times S_{a_2}))^*} &= \sup\{|dJ(\mathbf{u}_n)[\phi]| : \phi \in T_{\mathbf{u}}\mathcal{P},\ \|\phi\| \le 1\} \\ &= \|dJ(\mathbf{u}_n)\|_{(T_{\mathbf{u}}(\mathcal{P}))^*} \to 0\end{aligned}$$

as $n \to \infty$. Here we used the fact that $\{\mathbf{u}_n\}$ is a Palais-Smale sequence for J restricted to $\mathcal{P}$. This proves point (i).

Concerning (ii), we show first that $\mathcal{P} \cap \mathcal{C}_c^\infty(\mathbb{R}^3, \mathbb{R}^2)$ is dense in $\mathcal{P}$. Let $\mathbf{u} \in \mathcal{P}$. By density in H^1, there exists $\{\mathbf{u}_n\} \subset \mathcal{C}_c^\infty(\mathbb{R}^3, \mathbb{R}^2) \cap (S_{a_1} \times S_{a_2})$ such that $\mathbf{u}_n \to \mathbf{u}$ strongly in $H^1(\mathbb{R}^3, \mathbb{R}^2)$. The problem is that $\mathbf{u}_n \notin \mathcal{P}$ in general, but this can be easily settled in the following way: first, since $\mathbf{u} \in \mathcal{P} \subset \mathcal{E}$ with $\mathcal{E}$ from (2.4), and since $\mathcal{E}$ is open, $\mathbf{u}_n \in \mathcal{E}$ for sufficiently large n. Then we can consider the uniquely determined $s_n := s_{\mathbf{u}_n}$, defined by (2.5). By strong convergence, it is immediate that $s_n \to 0$ as $n \to +\infty$, so that Lemma 2.7 implies that $s_n \star \mathbf{u}_n \to \mathbf{u}$ strongly in $H^1(\mathbb{R}^3)$. Moreover, by definition $s_n \star \mathbf{u}_n \in \mathcal{P}$ for every n, and hence the proof of the density is complete.

Let now $\{\tilde{\mathbf{u}}_n\}$ be a Palais-Smale sequence for J on $\mathcal{P}$, and let $\varepsilon_m \to 0^+$ as $m \to \infty$. For every n and m, by density there exists $\mathbf{u}_{n,m} \in \mathcal{P} \cap \mathcal{C}_c^\infty(\mathbb{R}^3)$ such that $\|\mathbf{u}_{n,m} - \tilde{\mathbf{u}}_n\|_{H^1} < \varepsilon_m$, and it is clear that the diagonal sequence $\mathbf{u}_n := \mathbf{u}_{n,n}$ satisfies all the requirements in point (ii).

Points (iii) and (iv) follow now straightforwardly. □

3. Existence of normalized solutions for competing system

This section is devoted to the proof of Theorem 1.1. The proof is divided into three main steps: in the first part we study some useful properties of the unique radial ground state solution of the scalar Schrödinger equation. With these, we prove the existence of a Palais-Smale sequence for J constrained on $\mathcal{P}$, which, by Theorem 2.1, provides a Palais-Smale sequence for J on $S_{a_1} \times S_{a_2}$; in the last part of the proof, we discuss the convergence of the Palais-Smale sequence to a solution of (1.2).

3.1. **The ground state of the cubic Schrödinger equation.** In this section we consider general $a, \mu > 0$. Let us introduce

$$S_a^r := \{u \in S_a : u \text{ is radially symmetric with respect to } 0\}.$$

We denote by $w = w_{a,\mu}$ the unique function solving, for some $\nu < 0$, the problem

$$\begin{cases} -\Delta w - \nu w = \mu w^3 & \text{in } \mathbb{R}^3 \\ w > 0 & \text{in } \mathbb{R}^3 \\ w \in S_a^r \end{cases} \tag{3.1}$$

(we refer e.g. to [9, Proposition 2.2] for existence, uniqueness, and basic properties of w). From the variational point of view, w is characterized as a mountain pass critical point of the functional

$$I(u) = I_\mu(u) := \int_{\mathbb{R}^3} \frac{1}{2}|\nabla u|^2 - \frac{\mu}{4} u^4 \tag{3.2}$$

on S_a^r, and is the *least energy solution* of the problem (i.e. the solution having minimal energy among all the nontrivial solutions). The energy level $I(w)$ is called *ground state level*, and is denoted by $\ell(a, \mu)$.

Let us introduce

$$\mathcal{M} = \mathcal{M}_{a,\mu} := \left\{ u \in S_a : \int_{\mathbb{R}^3} |\nabla u|^2 = \frac{3\mu}{4} \int_{\mathbb{R}^3} u^4 \right\}. \tag{3.3}$$

It is not difficult to modify the proof of Lemma 2.2 (alternatively, one can directly apply the forthcoming Lemma 4.2) to check that $\mathcal{M} \cap S_a^r$ is a $\mathcal{C}^1$-submanifold of S_a^r, so that w is a critical point of I on $\mathcal{M}$.

Lemma 3.1. *The Palais-Smale condition holds for I restricted to $\mathcal{M}$.*

Proof. We refer the reader to the more general Lemma 5.4. □

Proposition 3.2. *The function w is the unique positive radial minimizer for I on $\mathcal{M}$. The set of minimizers for I on $\mathcal{M}$ is $\{w, -w\}$.*

Proof. The minimality of w is proved in [13, Proposition 8.2.4] or [21, Lemma 2.10]. For the uniqueness, by Theorem 1.5 we know that any minimizer v of I on $\mathcal{M}$ yields a solution (ν, v) to (3.1), and hence the uniqueness of w as positive minimizer follows by [9, Proposition 2.2]. Notice that also $-w$ is a minimizer. In order to prove that no-sign-changing minimizer exists, we argue by contradiction noting that if v minimizes I on $\mathcal{M}$, so does $|v|$. Thus, by Theorem 1.5 and the previous lemma, $|v|$ is a non-negative solution to (3.1) for some $\nu \in \mathbb{R}$, and its zero-level set $\{|v| = 0\}$ is not empty. The strong maximum principle implies then that $|v| \equiv 0$, which is impossible since $0 \notin \mathcal{M}$. □

Lemma 3.3. *For any $u \in S_a^r$ there exists a unique $s_u \in \mathbb{R}$ such that $s_u \star u \in \mathcal{M}$. It is defined by the equation*

$$e^{s_u} = \frac{4 \int_{\mathbb{R}^3} |\nabla u|^2}{3 \int_{\mathbb{R}^3} \mu u^4}.$$

The value s_u is the unique (strict) maximum point of the function $s \mapsto I(s \star u)$.

Proof. Existence and uniqueness are contained in [21, Lemma 2.9]. The explicit expression of s_u follows by direct computations, observing that

$$I(s \star u) = \frac{e^{2s}}{2} \int_{\mathbb{R}^3} |\nabla u|^2 - \frac{\mu e^{3s}}{4} \int_{\mathbb{R}^3} u^4.$$ □

We conclude this section with the simple observation that 0 is a critical point of the functional I extended to the whole space $H^1(\mathbb{R}^3, \mathbb{R}^2)$, and the free second differential of I in 0 is positive.

Lemma 3.4. *If we consider I as a functional in $H^1(\mathbb{R}^3, \mathbb{R}^2)$, we have that*

$$dI(0) = 0 \quad \textit{and} \quad d^2 I(0)[\varphi, \varphi] = \int_{\mathbb{R}^3} |\nabla \varphi|^2.$$

for any $\varphi \in H^1(\mathbb{R}^3, \mathbb{R}^2)$.

3.2. **Construction of a Palais-Smale sequence for $J|_{\mathcal{P}}$.** We aim at proving that $J|_{\mathcal{P}}$ satisfies the assumptions of a minimax principle, and more precisely it has a mountain pass geometry. Our argument is somehow inspired by the proof of Theorem 5.4 in [2], even though this result is tailor-made for the case $\beta > 0$. Here several complications arise because we deal with 3 constraints (and not with 1) and with arbitrary $\beta < 0$.

First, having in mind that the compactness of any Palais-Smale sequence would be far from being trivial, we confine ourselves in a radial setting. That is, we work in $S^r_{a_1} \times S^r_{a_2}$ instead of in $S_{a_1} \times S_{a_2}$. Since the problem is rotation-invariant, this is possible as a consequence of the principle of symmetric criticality [34].

For $i = 1, 2$, consider $w_i := w_{a_i,\mu_i}$ as defined in (3.1). We recall that w_i is a minimizer for $I_i := I_{\mu_i}$ in $\mathcal{M}_i := \mathcal{M}_{a_i,\mu_i}$ (see (3.2) and (3.3)). This suggests that $(w_1, 0)$ and $(0, w_2)$ are local minimizers for J on $\mathcal{P}$ (recall (1.4) and (1.6)), so that $J|_{\mathcal{P}}$ has a mountain pass geometry. Of course, such an argument is incorrect in the present setting, since for instance $(w_1, 0)$ and $(0, w_2)$ do not belong to $\mathcal{P} \subset S_{a_1} \times S_{a_2}$, or else the set $\mathcal{P}$ is not necessarily connected by arcs for $\beta < -\sqrt{\mu_1\mu_2}$. On the other hand, in what follows we show how the previous heuristic idea can be adjusted in our context, leading to the following statement:

Proposition 3.5. *There exists a Palais-Smale sequence $\{(\tilde{u}_{1,n}, \tilde{u}_{2,n})\}$ at a mountain pass level*

$$c > \max\{\ell(a_1, \mu_1), \ell(a_2, \mu_2)\}$$

for J restricted to $\mathcal{P}$, satisfying the additional condition $\tilde{u}^-_{i,n} \to 0$ a.e. in $\mathbb{R}^3$ for $i = 1, 2$. As a consequence, there exists a Palais-Smale sequence $\{(u_{1,n}, u_{2,n})\}$ for J on $S_{a_1} \times S_{a_2}$, with $(u_{1,n}, u_{2,n}) \in \mathcal{P}$ for every n, such that $u^-_{i,n} \to 0$ a.e. in $\mathbb{R}^3$ for $i = 1, 2$.
Moreover, there exists $C > 0$ independent of β such that $c < C$.

Recall that $\ell(a_i, \mu_i)$ denotes the ground state energy level $I_i(w_i)$. Without loss of generality, we can suppose that

$$\ell(a_1, \mu_1) \geq \ell(a_2, \mu_2). \tag{3.4}$$

We have already mentioned that $(w_1, 0), (0, w_2) \notin S^r_{a_1} \times S^r_{a_2}$. On the other hand they both belong to the closure, with respect to the $\mathcal{D}^{1,2}$ topology, of $S^r_{a_1} \times S^r_{a_2}$, where as usual

$$\mathcal{D}^{1,2} := \left\{ u \in L^6(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\nabla u|^2 < +\infty \right\}, \quad \|u\|^2_{\mathcal{D}^{1,2}} := \int_{\mathbb{R}^3} |\nabla u|^2.$$

Actually, we can easily check that any family of type $\{(w_1, s \star v) : s \in \mathbb{R}\}$, with $v \in S_{a_2}$, strongly converges in $\mathcal{D}^{1,2}$, as $s \to -\infty$, to $(w_1, 0)$. It is sufficient to observe that

$$\|s \star v\|^2_{\mathcal{D}^{1,2}} = \int_{\mathbb{R}^3} |\nabla(s \star v)|^2 = e^{2s} \int_{\mathbb{R}^3} |\nabla v|^2 \to 0$$

as $s \to -\infty$. In particular, this implies that

$$\left(B(w_1, \rho_1; H^1) \times B(0, \rho_2; \mathcal{D}^{1,2})\right) \cap \left(S^r_{a_1} \times S^r_{a_2}\right) \neq \emptyset$$

for any $\rho_1, \rho_2 > 0$, where $B(u, \rho; F)$ denotes the ball in F (Banach space) with centre u and radius ρ.

Before proceeding, we also emphasize that in the previous example there is no strong convergence in $H^1(\mathbb{R}^3, \mathbb{R}^2)$, since $\|s \star v\|_{L^2(\mathbb{R}^3)} = a_2$ for every s. This phenomenon is related to the fact that weak convergence in $H^1_{\text{rad}}(\mathbb{R}^3)$ does not imply strong convergence in $L^2(\mathbb{R}^3)$, and is a source of complications when dealing with normalization constraints of type (1.2b).

In order to prove Proposition 3.5, we investigate s_{u_1}, defined in Lemma 3.3, for $(u_1, u_2) \in \left(B(w_1, \rho_1; H^1) \times B(0, \rho_2; \mathcal{D}^{1,2})\right) \cap \mathcal{P}$ and determine the asymptotic behaviour when $\rho_1, \rho_2 \to 0$.

Lemma 3.6. *There exist $\delta_1 > 0$ small and $C > 0$ such that*

$$0 < \rho_1, \rho_2 < \delta_1 \quad \Longrightarrow \quad |s_{u_1}| \le C\rho_2^{3/2},$$

for every $(u_1, u_2) \in \big(B(w_1, \rho_1; H^1) \times B(0, \rho_2; \mathcal{D}^{1,2})\big) \cap \mathcal{P}$.

Proof. On one side, as $(u_1, u_2) \in \mathcal{P}$,

$$1 = \frac{4\int_{\mathbb{R}^3} \sum_i |\nabla u_i|^2}{3\int_{\mathbb{R}^3} \sum_{i,j} \beta_{ij} u_i^2 u_j^2}. \tag{3.5}$$

On the other hand, by the Lagrange theorem there exists $\xi \in (0,1)$ such that

$$\begin{aligned} \frac{4\int_{\mathbb{R}^3} \sum_i |\nabla u_i|^2}{3\int_{\mathbb{R}^3} \sum_{i,j} \beta_{ij} u_i^2 u_j^2} &= \frac{4\int_{\mathbb{R}^3} |\nabla u_1|^2}{3\int_{\mathbb{R}^3} \beta_{11} u_1^4} \\ &\quad + \frac{4\int_{\mathbb{R}^3} |\nabla u_2|^2}{3\left(\int_{\mathbb{R}^3} \beta_{11} u_1^4 + \xi \int_{\mathbb{R}^3} 2\beta_{12} u_1^2 u_2^2 + \beta_{22} u_2^4\right)} \\ &\quad - \frac{4\left(\int_{\mathbb{R}^3} |\nabla u_1|^2 + \xi |\nabla u_2|^2\right)\left(\int_{\mathbb{R}^3} 2\beta_{12} u_1^2 u_2^2 + \beta_{22} u_2^4\right)}{3\left(\int_{\mathbb{R}^3} \beta_{11} u_1^4 + \xi \int_{\mathbb{R}^3} 2\beta_{12} u_1^2 u_2^2 + \beta_{22} u_2^4\right)^2} \end{aligned} \tag{3.6}$$

The first term on the right hand side is, by definition, $\exp(s_u)$ (see Lemma 3.3). In order to estimate the remaining terms on the right hand side, we recall the Gagliardo-Nirenberg inequality: there exists $S > 0$ such that

$$\int_{\mathbb{R}^3} w^4 \le S \left(\int_{\mathbb{R}^3} w^2\right)^{1/2} \left(\int_{\mathbb{R}^3} |\nabla w|^2\right)^{3/2} \qquad \text{for all } w \in H^1(\mathbb{R}^3).$$

Let ρ_1 and ρ_2 small so that

$$\rho_2 \le \min\left\{1, \int_{\mathbb{R}^3} |\nabla w_1|^2, \frac{1}{4}\int_{\mathbb{R}^3} w_1^4\right\} \le \frac{1}{2}\int_{\mathbb{R}^3} u_1^4 \le \int_{\mathbb{R}^3} w_1^4 \tag{3.7}$$

for $\|u_1 - w_1\|_{H^1} < \rho_1$. For $(u_1, u_2) \in \big(B(w_1, \rho_1; H^1) \times B(0, \rho_2; \mathcal{D}^{1,2})\big) \cap \mathcal{P}$ and $\xi \in (0,1)$, we have

$$\int_{\mathbb{R}^3} |\nabla u_1|^2 + \xi |\nabla u_2|^2 \le 3\int_{\mathbb{R}^3} |\nabla w_1|^2,$$

and

$$\begin{aligned} \left|\int_{\mathbb{R}^3} \beta_{22} u_2^4 + 2\beta_{12} u_1^2 u_2^2\right| &\le \beta_{22} S a_2 \|u_2\|^3_{\mathcal{D}^{1,2}} + 2|\beta_{12}| \left(\int_{\mathbb{R}^3} u_1^4\right)^{1/2} \left(\int_{\mathbb{R}^3} u_2^4\right)^{1/2} \\ &\le \beta_{22} S a_2 \|u_2\|^3_{\mathcal{D}^{1,2}} + \sqrt{8 S a_2} |\beta_{12}| \|w_1\|^{1/2}_{L^4} \|u_2\|^{3/2}_{\mathcal{D}^{1,2}} \\ &\le C\|u_2\|^{3/2}_{\mathcal{D}^{1,2}}, \end{aligned}$$

hence, replacing ρ_2 with a smaller quantity if necessary,

$$\left|\int_{\mathbb{R}^3} \beta_{11} u_1^4 + \xi \int_{\mathbb{R}^3} 2\beta_{12} u_1^2 u_2^2 + \beta_{22} u_2^4\right| \ge \frac{1}{2}\int_{\mathbb{R}^3} \beta_{11} w_1^4 - C\rho_2^{3/2} \ge \frac{1}{4}\int_{\mathbb{R}^3} \beta_{11} w_1^4.$$

As a consequence we obtain

$$\left|\frac{\int_{\mathbb{R}^3} |\nabla u_2|^2}{\left(\int_{\mathbb{R}^3} \beta_{11} u_1^4 + \xi \int_{\mathbb{R}^3} 2\beta_{12} u_1^2 u_2^2 + \beta_{22} u_2^4\right)}\right| \qquad \le \frac{4\|u_2\|^2_{\mathcal{D}^{1,2}}}{\int_{\mathbb{R}^3} \beta_{11} w_1^4} \le C\rho_2^2, \tag{3.8}$$

and

$$\left| \frac{\left(\int_{\mathbb{R}^3} |\nabla u_1|^2 + \xi |\nabla u_2|^2\right) \left(\int_{\mathbb{R}^3} 2\beta_{12} u_1^2 u_2^2 + \beta_{22} u_2^4\right)}{\left(\int_{\mathbb{R}^3} \beta_{11} u_1^4 + \xi \int_{\mathbb{R}^3} 2\beta_{12} u_1^2 u_2^2 + \beta_{22} u_2^4\right)^2} \right| \leq C \frac{\left(\int_{\mathbb{R}^3} |\nabla w_1|^2\right) \|u_2\|_{\mathcal{D}^{1,2}}^{3/2}}{\int_{\mathbb{R}^3} \beta_{11} w_1^4} \leq C \rho_2^{3/2}. \tag{3.9}$$

Using (3.5), (3.8) and (3.9), we see that (3.6) becomes

$$1 = e^{s_{u_1}} + O\left(\rho_2^{3/2}\right),$$

which implies the lemma. □

Lemma 3.7. *For any $r_1, r_2 > 0$ there exists $\delta_2 \in (0, \delta_1]$ such that if $\rho_1, \rho_2 \in (0, \delta_2)$, then*

$$\|s_{u_1} \star u_1 - w_1\|_{H^1} < r_1 \quad \text{and} \quad \|s_{u_1} \star u_2\|_{\mathcal{D}^{1,2}} < r_2$$

for every $(u_1, u_2) \in \left(B(w_1, \rho_1; H^1) \times B(0, \rho_2; \mathcal{D}^{1,2})\right) \cap \mathcal{P}$.

Proof. Let $0 < \rho_1, \rho_2 < \delta_1$. First of all, by Lemma 3.6 and Lemma 2.7 there exists $\delta' \in (0, \delta_1]$ so that $0 < \rho_2 < \delta'$ implies

$$\|s_{u_1} \star w_1 - w_1\|_{H^1} < \frac{r_1}{2}$$

for every $(u_1, u_2) \in \left(B(w_1, \rho_1; H^1) \times B(0, \rho_2; \mathcal{D}^{1,2})\right) \cap \mathcal{P}$. Now we observe that for $\rho_1 \in (0, \delta_1)$ and $\rho_2 \in (0, \delta')$

$$\begin{aligned} \|s_{u_1} \star u_1 - w_1\|_{H^1} &\leq \|s_{u_1} \star u_1 - s_{u_1} \star w_1\|_{H^1} + \|s_{u_1} \star w_1 - w_1\|_{H^1} \\ &\leq \max\{e^{s_{u_1}}, 1\} \|u_1 - w_1\|_{H^1} + \frac{r_1}{2} \\ &\leq e^{C\rho_2^{3/2}} \rho_1 + \frac{r_1}{2} < r_1 \end{aligned}$$

provided ρ_1 and ρ_2 are small enough, and similarly

$$\|s_{u_1} \star u_2\|_{\mathcal{D}^{1,2}} = e^{s_{u_1}} \|u_2\|_{\mathcal{D}^{1,2}} \leq e^{C\rho_2^{3/2}} \rho_2 < r_2$$

for ρ_2 small. □

Lemma 3.8. *Let $\rho_1 \in (0, \delta_2)$ be fixed. There exists $\delta_3 \in (0, \delta_2]$ (possibly depending on ρ_1) such that*

$$\inf \left\{ \|s_{u_1} \star u_1 - w_1\|_{H^1} \;\middle|\; \begin{array}{l} (u_1, u_2) \in \mathcal{P} \\ u_1 \in \partial B(w_1, \rho_1; H^1) \\ u_2 \in B(0, \rho_2; \mathcal{D}^{1,2}) \end{array} \right\} > \frac{\rho_1}{2},$$

and

$$\inf \left\{ \|s_{u_1} \star u_2\|_{\mathcal{D}^{1,2}} \;\middle|\; \begin{array}{l} (u_1, u_2) \in \mathcal{P} \\ u_1 \in B(w_1, \rho_1; H^1) \\ u_2 \in \partial B(0, \rho_2; \mathcal{D}^{1,2}) \end{array} \right\} > \frac{\rho_2}{2}$$

for every $\rho_2 \in (0, \delta_3)$.

Proof. We start with the first estimate in the thesis. Let us suppose by contradiction that there exist sequences $\rho_{2,n} \to 0$ and

$$(u_{1,n}, u_{2,n}) \in \left(\partial B(w_1, \rho_1; H^1) \times B(0, \rho_{2,n}; \mathcal{D}^{1,2})\right) \cap \mathcal{P},$$

such that $s_n := s_{u_{1,n}}$ satisfies

$$\|s_n \star u_{1,n} - w_1\|_{H^1} \le \rho_1/2 \qquad \forall n.$$

First, from Lemma 3.6 we deduce that $s_n \to 0$ as $n \to \infty$, and hence by Lemma 2.7 we have $s_n \star w_1 \to w_1$ strongly in $H^1(\mathbb{R}^3)$ as $n \to \infty$.

Now it is not difficult to obtain a contradiction, using again the fact that $s_n \to 0$:

$$\begin{aligned}\|s_n \star u_{1,n} - w_1\|_{H^1} &\ge \|s_n \star u_{1,n} - s_n \star w_1\|_{H^1} - \|s_n \star w_1 - w_1\|_{H^1} \\ &= e^{s_n}\|\nabla(u_{1,n} - w_1)\|_{L^2} + \|u_{1,n} - w_1\|_{L^2} - o(1) \to \rho_1\end{aligned}$$

as $n \to \infty$, a contradiction.

For the second estimate in the thesis, we use again Lemma 3.6:

$$\|s_{u_1} \star u_2\|_{\mathcal{D}^{1,2}} = e^{s_{u_1}}\|u_2\|_{\mathcal{D}^{1,2}} \ge e^{-C\rho_2^{3/2}}\rho_2 > \frac{\rho_2}{2}$$

whenever $u_2 \in \partial B(0, \rho_2; \mathcal{D}^{1,2})$, provided $\rho_2 > 0$ is small enough. □

Lemma 3.9. *There exist $\rho_1, \rho_2, \bar{C} > 0$ such that*

$$J(u_1, u_2) \ge \ell(a_1, \mu_1) + \bar{C}$$

for every $(u_1, u_2) \in \partial\left(B(w_1, \rho_1; H^1) \times B(0, \rho_2; \mathcal{D}^{1,2})\right) \cap \mathcal{P}$.

Proof. Let $(u_1, u_2) \in \mathcal{P}$, and recall that $s_{u_1} \star u_1 \in \mathcal{M}_1$, with $\mathcal{M}_1$ defined in (3.3).

As a consequence of Lemma 2.3, and using $\beta \le 0$, we obtain

$$J(u_1, u_2) \ge J(s_{u_1} \star (u_1, u_2)) \ge I_1(s_{u_1} \star u_1) + I_2(s_{u_1} \star u_2). \tag{3.10}$$

To estimate the right hand side, we observe that by Lemma 3.4 there exist $\bar{\rho}_2 > 0$ and $C > 0$ such that

$$I(v_2) \ge C\|v_2\|^2_{\mathcal{D}^{1,2}} \qquad \text{if } v_2 \in B(0, \bar{\rho}_2; \mathcal{D}^{1,2}). \tag{3.11}$$

Let $\bar{\rho}_1 > 0$ be such that $\bar{\rho}_1 < \|w_1\|_{H^1}$. By Lemma 3.7 there exists $\delta_2 > 0$ such that $\rho_1, \rho_2 \in (0, \delta_2)$ implies

$$s_{u_1} \star (u_1, u_2) \in B(w_1, \bar{\rho}_1; H^1) \times B(0, \bar{\rho}_2; \mathcal{D}^{1,2})$$

provided $(u_1, u_2) \in \left(B(w_1, \rho_1; H^1) \times B(0, \rho_2; \mathcal{D}^{1,2})\right) \cap \mathcal{P}$. Now we fix $\rho_1 \in (0, \delta_2)$, and next $\rho_2 \in (0, \delta_3]$, with $\delta_3 > 0$ given by Lemma 3.8. We claim that ρ_1 and ρ_2 are the desired quantities. To prove the claim, we observe first that the boundary of $B(w_1, \rho_1; H^1) \times B(0, \rho_2; \mathcal{D}^{1,2})$ splits as

$$\left[\partial B(w_1, \rho_1; H^1) \times B(0, \rho_2; \mathcal{D}^{1,2})\right] \cup \left[B(w_1, \rho_1; H^1) \times \partial B(0, \rho_2; \mathcal{D}^{1,2})\right].$$

Since $s_{u_1} \star u_1 \in \mathcal{M}_1$ and $\ell(a_1, \mu_1) = \inf_{\mathcal{M}_1} I_1$, we have by (3.11) and Lemma 3.8 (notice that the lemma is applicable in light of our choice of ρ_1 and ρ_2))

$$\begin{aligned}J(u_1, u_2) &\ge I_1(s_{u_1} \star u_1) + I_2(s_{u_1} \star u_2) \\ &\ge \ell(a_1, \mu_1) + C\|s_{u_1} \star u_2\|^2_{\mathcal{D}^{1,2}} \ge \ell(a_1, \mu_1) + C\frac{\rho_2^2}{4}\end{aligned} \tag{3.12}$$

for every $(u_1, u_2) \in \left(B(w_1, \rho_1; H^1) \times \partial B(0, \rho_2; \mathcal{D}^{1,2})\right) \cap \mathcal{P}$. On the other hand, using again (3.11) we deduce also that

$$J(u_1, u_2) \ge I(s_{u_1} \star u_1).$$

for every $(u_1, u_2) \in \left(\partial B(w_1, \rho_1; H^1) \times B(0, \rho_2; \mathcal{D}^{1,2})\right) \cap \mathcal{P}$. We claim that

$$\inf\left\{I_1(s_{u_1} \star u_1) : (u_1, u_2) \in \left(\partial B(w_1, \rho_1; H^1) \times B(0, \rho_2; \mathcal{D}^{1,2})\right) \cap \mathcal{P}\right\} > \ell(a_1, \mu_1). \tag{3.13}$$

Indeed, the Lemmas 3.7 and 3.8 yield

$$\frac{\rho_1}{2} \le \|s_{u_1} \star u_1 - w_1\|_{H^1} < \bar\rho_1$$

for every $(u_1, u_2) \in \big(\partial B(w_1, \rho_1; H^1) \times B(0, \rho_2; \mathcal{D}^{1,2})\big) \cap \mathcal{P}$, so that the left hand side in (3.13) is larger than or equal to

$$\inf \left\{ I_1(u) : \frac{\rho_1}{2} \le \|u - w_1\|_{H^1} < \bar\rho_1,\ u \in \mathcal{M}_1 \right\}.$$

If by contradiction this infimum is $\ell(a_1, \mu_1)$, then there exists a bounded sequence $\{u_n\} \subset \mathcal{M}_1$ with $\|u_n - w_1\|_{H^1} \ge \rho_1/2$ such that $I_1(u_n) \to \ell(a_1, \mu_1)$; that is, $\{u_n\}$ is a bounded minimizing sequence for I_1 restricted to $\mathcal{M}_1$. By Lemma 3.1 we infer that $u_n \to u$ strongly in $H^1(\mathbb{R}^3)$, where by strong convergence u minimizes I_1 on $\mathcal{M}_1$. Notice that

$$\|u - w_1\|_{H^1} \le \bar\rho_1 \le \|w_1\|_{H^1} < \|w_1 - (-w_1)\|_{H^1};$$

this rules out the possibility that $u = -w_1$, so that by Proposition 3.2 necessarily $u = w_1$. But on the other hand, always by strong convergence, $\|u - w_1\|_{H^1} \ge \rho_1/2$, a contradiction. This proves claim (3.13), which together with (3.12) and (3.10) gives the thesis. □

In order to complete the proof of Proposition 3.5, the idea is now to define a convenient minimax class of paths connecting two pairs (u_1, u_2) and (v_1, v_2), sufficiently close to $(w_1, 0)$ and to $(0, w_2)$ respectively. The problem is that, at least for $\beta < -\sqrt{\mu_1\mu_2}$, it is not clear whether the set $\mathcal{P}$ is connected by arcs, and in particular it is not clear if an arc connecting (u_1, u_2) and (v_1, v_2) does exists. In the next lemma we conveniently choose (u_1, u_2) and (v_1, v_2) so that they lie in the same connected component of $\mathcal{P}$.

Lemma 3.10. *Let ρ_1, ρ_2 be defined in Lemma 3.9. For every $\varepsilon > 0$ there exist*

$$(u_1^\varepsilon, u_2^\varepsilon) \in \big(B(w_1, \rho_1; H^1) \times B(0, \rho_2; \mathcal{D}^{1,2})\big) \cap \mathcal{P}$$
$$(v_1^\varepsilon, v_2^\varepsilon) \in \big(B(0, \rho_1; \mathcal{D}^{1,2}) \times B(w_2, \rho_2; H^1)\big) \cap \mathcal{P}$$

with the following properties:

- *(i)* $(u_1^\varepsilon, u_2^\varepsilon), (v_1^\varepsilon, v_2^\varepsilon) \in \mathcal{C}_c^\infty(\mathbb{R}^3, \mathbb{R}^2)$ *and* $u_i^\varepsilon, v_i^\varepsilon \ge 0$ *in* $\mathbb{R}^3$ *for both* $i = 1, 2$.
- *(ii)* $u_1^\varepsilon u_2^\varepsilon \equiv 0$, $v_1^\varepsilon v_2^\varepsilon \equiv 0$, *and* $u_2^\varepsilon v_1^\varepsilon \equiv 0$.
- *(iii)* $J(u_1^\varepsilon, u_2^\varepsilon), J(v_1^\varepsilon, v_2^\varepsilon) \le \ell(a_1, \mu_1) + \varepsilon$.
- *(iv) There exists* $\gamma = (\gamma_1, \gamma_2) : [0, 1] \to \mathcal{P}$, *continuous with respect to the* H^1*-topology, such that* $\gamma(0) = (u_1^\varepsilon, u_2^\varepsilon)$, $\gamma(1) = (v_1^\varepsilon, v_2^\varepsilon)$, *and moreover* $\gamma_1(t)\gamma_2(t) \equiv 0$ *in* $\mathbb{R}^N$ *for every* $t \in [0, 1]$.

Proof. Let $\varepsilon > 0$ be fixed. Arguing as in the proof of Theorem 2.1-(ii), we can check that there exists $\{u_{1,n}\} \subset \mathcal{C}_c^\infty(\mathbb{R}^3) \cap \mathcal{M}_1$ strongly convergent to w_1 in $H^1(\mathbb{R}^3)$ as $n \to \infty$; moreover, since $w_1 > 0$ in $\mathbb{R}^3$, it is not restrictive to suppose that $u_{1,n} \ge 0$ in $\mathbb{R}^3$ for every n sufficiently large. By continuity, we can take $u_{1,\bar n}$ with $\bar n$ very large, so that

$$I_1(u_{1,\bar n}) < \ell(a_1, \mu_1) + \frac{\varepsilon}{2}. \tag{3.14}$$

The support of $u_{1,\bar n}$ is contained in $B_R(0)$ for some positive $R > 0$.

Let us now consider $u \in S^r_{a_2}$, $u \geq 0$ in $\mathbb{R}^3$, with support in $A(0;2,3)$, the annulus of center 0 and radii $2 < 3$, and define

$$u_{2,m}(x) := ((-m) \star u)(x) = e^{-3m/2} u(e^{-m}x).$$

Then $u_{2,m} \to 0$ strongly in $\mathcal{D}^{1,2}$ as $m \to \infty$, and $\operatorname{supp} u_{2,m} \subset A(0; 2e^m, 3e^m)$, as

$$\operatorname{supp} u_{2,m} = \{2 < |e^{-m}x| < 3\} = \{2e^m < |x| < 3e^m\}$$
$$\|u_{2,m}\|^2_{\mathcal{D}^{1,2}} = e^{-2m}\|u\|^2_{\mathcal{D}^{1,2}} \to 0 \qquad \text{as } m \to \infty.$$

In particular, there exists $\bar{m}$ very large so that

$$\operatorname{supp} u_{2,m} \cap B_R(0) = \emptyset \text{ for every } m \geq \bar{m}.$$

Let $s_m := s_{(u_{1,\bar{n}}, u_{2,m})}$ be defined by Lemma 2.3 (we remark that s_m is well defined, since by construction $(u_{1,\bar{n}}, u_{2,m}) \in \mathcal{E}$, with $\mathcal{E}$ defined in (2.4)). Since

$$u_{1,\bar{n}} \in \mathcal{M}_1 \quad \Longrightarrow \quad \frac{4 \int_{\mathbb{R}^3} |\nabla u_{1,\bar{n}}|^2}{3 \int_{\mathbb{R}^3} \beta_{11} u_{1,\bar{n}}^4} = 1$$

and $u_{2,m} \to 0$ in $\mathcal{D}^{1,2}$, it is possible to repeat step by step (with minor changes) the computations between (3.6) and (3.9), obtaining

$$s_m = O(\|u_{2,m}\|^{3/2}_{\mathcal{D}^{1,2}}) \qquad \text{as } m \to \infty.$$

Now, $s_m \star u_{2,m} \to 0$ strongly in $\mathcal{D}^{1,2}$ as $m \to +\infty$, since

$$\|s_m \star u_{2,m}\|_{\mathcal{D}^{1,2}} = e^{s_m}\|u_{2,m}\|_{\mathcal{D}^{1,2}}.$$

Therefore, by continuity (with respect to the $\mathcal{D}^{1,2}$ topology)

$$I_2(s_m \star u_{2,m}) < \frac{\varepsilon}{2}$$

for any $m \geq \tilde{m}$, with $\tilde{m} \geq \bar{m}$ sufficiently large. Observing that by construction $s_m \star u_{1,\bar{n}}$ and $s_m \star u_{2,m}$ have disjoint support, that $I_1(s_m \star u_{1,\bar{n}}) < I_1(u_{1,\bar{n}})$ (see Lemma 3.3), and recalling (3.14), we finally conclude

$$\begin{aligned} J(s_m \star (u_{1,\bar{n}}, u_{2,m})) &= I_1(s_m \star u_{1,\bar{n}}) + I_2(s_m \star u_{2,m}) \\ &< I_1(u_{1,\bar{n}}) + \frac{\varepsilon}{2} < \ell(a_1,\mu_1) + \varepsilon \end{aligned}$$

for any $m \geq \tilde{m}$. We set $(u_1^\varepsilon, u_2^\varepsilon) := \tilde{m} \star (u_{1,\bar{n}}, u_{2,\tilde{m}})$, and for future convenience we denote by R_ε a radius such that $\operatorname{supp} u_{2,\varepsilon} \subset B_{R_\varepsilon}(0)$.

The existence of $(v_1^\varepsilon, v_2^\varepsilon)$ can be proved in a similar way. We first take $v_{2,\bar{n}} \in \mathcal{C}_c^\infty(\mathbb{R}^3) \cap \mathcal{M}_2$ close to w_2 in H^1, supported in $B_{R'}(0)$ for some $R' > 0$. Then we set $v_{1,m} := (-m) \star v$, where $v \in S^r_{a_1}$ is a function with $\operatorname{supp} v \subset A(0;2,3)$. There exists $\bar{m}$ so large that

$$\operatorname{supp} v_{1,m} \cap B_{R'}(0) = \emptyset \quad \text{and also} \quad \operatorname{supp} v_{1,m} \cap B_{R_\varepsilon}(0) = \emptyset$$

for every $m > \bar{m}$. Now we can proceed exactly as before, obtaining in the end a pair $(v_1^\varepsilon, v_2^\varepsilon)$ with

$$J(v_1^\varepsilon, v_2^\varepsilon) < \ell(a_2,\mu_2) + \varepsilon \leq \ell(a_1,\mu_1) + \varepsilon$$

(recall condition (3.4)).

So far we proved the existence of $(u_1^\varepsilon, u_2^\varepsilon)$ and $(v_1^\varepsilon, v_2^\varepsilon)$ satisfying (i)-(iii) of the thesis. It remains to prove (iv). To this purpose, it is sufficient to find a path $\tilde{\gamma} : [0,1] \to \mathcal{E}$ with $\tilde{\gamma}(0) = (u_1^\varepsilon, u_2^\varepsilon)$, $\tilde{\gamma}(1) = (v_1^\varepsilon, v_2^\varepsilon)$, and $\tilde{\gamma}_1(t)\tilde{\gamma}_2(t) \equiv 0$ in $\mathbb{R}^N$ for every $t \in [0,1]$, where $\mathcal{E}$ was defined in (2.4). Indeed if such a $\tilde{\gamma}$ does exist, then the path $\gamma(t) := s_{\tilde\gamma(t)} \star \tilde\gamma(t)$ satisfies all the properties in point (iv) of the lemma. For the continuity, we observe that $(u_{1,n}, u_{2,n}) \to (u_1, u_2)$ in $H^1(\mathbb{R}^3)$ implies $s_n := s_{(u_{1,n},u_{2,n})} \to s_{(u_1,u_2)} =: s_\infty$. Thus, Lemma 2.7 yields

$$\|s_n \star u_{1,n} - s_\infty \star u_1\|_{H^1} \to 0$$

as $n \to \infty$, and the same holds for the second component.

In order to define $\tilde\gamma$, we set

$$\sigma_1(t) := \left(a_1 \frac{(1-t)u_1^\varepsilon + tv_1^\varepsilon}{\|(1-t)u_1^\varepsilon + tv_1^\varepsilon\|_{L^2}}, u_2^\varepsilon \right) \qquad t \in [0,1]$$

Since (ii) of this lemma holds true, $\sigma_1(t) \in \mathcal{E}$ and $\sigma_{1,1}(t)\sigma_{1,2}(t) \equiv 0$ in $\mathbb{R}^N$ for every $t \in [0,1]$. Now we set

$$\sigma_2(t) := \left(v_1^\varepsilon, a_2 \frac{(1-t)u_2^\varepsilon + tv_2^\varepsilon}{\|(1-t)u_2^\varepsilon + tv_2^\varepsilon\|_{L^2}} \right) \qquad t \in [0,1],$$

and again we note that $\sigma_2(t) \in \mathcal{E}$ and $\sigma_{2,1}(t)\sigma_{2,2}(t) \equiv 0$ in $\mathbb{R}^N$ for every $t \in [0,1]$. The path

$$\tilde\gamma(t) := \begin{cases} \sigma_1(2t) & t \in \left[0, \frac12\right] \\ \sigma_2(2t-1) & t \in \left[\frac12, 1\right] \end{cases}$$

is then a continuous path on $\mathcal{E}$ connecting $(u_1^\varepsilon, u_2^\varepsilon)$ with $(v_1^\varepsilon, v_2^\varepsilon)$, and such that $\tilde\gamma_1(t)\tilde\gamma_2(t) \equiv 0$ in $\mathbb{R}^N$ for every $t \in [0,1]$. □

Proof of Proposition 3.5. Let ρ_1, ρ_2 and $\bar C$ be defined in Lemma 3.9, and let $0 < \varepsilon < \bar C/2$. For such an $\varepsilon > 0$, thanks to Lemma 3.10 we find $(u_1^\varepsilon, u_2^\varepsilon)$ and $(v_1^\varepsilon, v_2^\varepsilon)$. Let now $\bar{\mathcal{P}}$ be the connected component of $\mathcal{P}$ containing $(u_1^\varepsilon, u_2^\varepsilon)$ and $(v_1^\varepsilon, v_2^\varepsilon)$ (the existence of $\bar{\mathcal{P}}$ follows by Lemma 3.10-(iv)). Recalling Lemma 2.2, we have that $\bar{\mathcal{P}}$ is a complete connected $\mathcal{C}^1$ manifold without boundary. We introduce the minimax class

$$\Gamma := \left\{ \gamma \in \mathcal{C}\left([0,1], \bar{\mathcal{P}}\right), \gamma(0) = (u_1^\varepsilon, u_2^\varepsilon),\ \gamma(1) = (u_2^\varepsilon, u_2^\varepsilon) \right\},$$

and the associated minimax level

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)).$$

It is clear that for any $\gamma \in \Gamma$ there exists $t \in (0,1)$ such that

$$\gamma(t) \in \partial\left(B(w_1, \rho_1; H^1) \cap B(0, \rho_2; \mathcal{D}^{1,2}) \right) \cap \mathcal{P},$$

so that Lemma 3.9 and the choice of ε permit to apply the minimax principle [19, Theorem 3.2]: we deduce that for every minimizing sequence $\{\gamma_n\} \subset \Gamma$ for c, there exists a Palais-Smale sequence $\{(\tilde u_{1,n}, \tilde u_{2,n})\}$ of J on $\mathcal{P}$ at level c, such that

$$\operatorname{dist}_{H^1}((\tilde u_{1,n}, \tilde u_{2,n}), \gamma_n([0,1])) \to 0 \qquad \text{as } n \to \infty. \tag{3.15}$$

Since J and G (see (1.5)) are even and $(u_1^\varepsilon, u_2^\varepsilon)$ and $(v_1^\varepsilon, v_2^\varepsilon)$ have both non-negative components, we claim that it is not restrictive to suppose that $\gamma_{1,n}(t), \gamma_{2,n}(t) \geq 0$ a.e. in $\mathbb{R}^3$, for every n, for every $t \in [0,1]$. To prove the claim, we show that if $\gamma \in \Gamma$, then also $|\gamma| := (|\gamma_1|, |\gamma_2|) \in \Gamma$. It is clear that $|\gamma|$ is continuous and $|\gamma(t)| \in \mathcal{P}$, but we have to prove the stronger assertion $|\gamma(t)| \in \bar{\mathcal{P}}$. Let us define, for $t \in [0,1]$,

$$\sigma_1(\tau) := \gamma_n((1-\tau)t) \quad \text{and} \quad \sigma_2(\tau) := |\gamma_n(\tau t)|,$$

with $\tau \in [0,1]$. Setting

$$\sigma(\tau, t) := \begin{cases} \sigma_1(\tau t) & \tau \in \left[0, \frac{1}{2}\right] \\ \sigma_2((2\tau - 1)t) & \tau \in \left[\frac{1}{2}, 1\right], \end{cases}$$

we have a continuous path in $\mathcal{P}$ connecting $\gamma(t)$ with $|\gamma(t)|$. Hence $\gamma(t)$ and $|\gamma(t)|$ live in the same connected connected component of $\mathcal{P}$. Since this holds for every t, we conclude that $|\gamma| \in \Gamma$, as desired.

The fact that $\gamma_{1,n}(t), \gamma_{2,n}(t) \geq 0$ a.e. in $\mathbb{R}^3$, together with (3.15), imply that $\tilde{u}_{i,n}^- \to 0$ a.e. in $\mathbb{R}^3$. The rest of the proposition follows now by Theorem 2.1, with the exception of the uniform boundedness of $c = c_\beta$ with respect to β. To this purpose, let us denote by $\mathcal{P}_\beta$ the natural constraint defined in (1.6) for a prescribed value of β, and by J_β the associated energy functional. Let us consider the path γ constructed in Lemma 3.10. Since $\gamma_1(t)\gamma_2(t) \equiv 0$ in $\mathbb{R}^N$ for every $t \in [0,1]$, we have that $\gamma(t) \in \mathcal{P}_\beta$ for every $t \in [0,1]$, for every $\beta < 0$. As a consequence, by definition

$$c_\beta \leq \max_{t \in [0,1]} J_\beta(\gamma(t)) =: C$$

with $C > 0$ independent of β. This completes the proof. □

3.3. **Convergence of the Palais-Smale sequence.** In order to complete the proof of Theorem 1.1, we have to show that the Palais-Smale sequence $\{(u_{1,n}, u_{2,n})\}$ strongly converges in $H^1(\mathbb{R}^3, \mathbb{R}^2)$ to a couple $(\bar{u}_1, \bar{u}_2)$, solution of (1.2a) for suitable $\bar{\lambda}_1, \bar{\lambda}_2 < 0$. Once that this is done, we observe that by strong convergence $(\bar{u}_1, \bar{u}_2)$ fulfills also (1.2b), and hence $(\bar{u}_1, \bar{u}_2, \lambda_1, \lambda_2)$ is a solution to (1.2), as desired.

For the strong convergence, we adapt the argument used in the last part of the proofs of Theorems 1.1 and 1.2 in [9]. Since $(u_{1,n}, u_{2,n}) \in \mathcal{P}$, arguing as in [9, Lemma 3.7] we deduce that $\{(u_{1,n}, u_{2,n})\}$ is bounded in $H^1(\mathbb{R}^3, \mathbb{R}^2)$, and moreover there exists $C > 0$ such that

$$\int_{\mathbb{R}^3} |\nabla u_{1,n}|^2 + |\nabla u_{2,n}|^2 \geq C \qquad \text{for all } n.$$

Hence, up to a subsequence $(u_{1,n}, u_{2,n}) \rightharpoonup (\bar{u}_1, \bar{u}_2)$ weakly in H^1, strongly in L^4, and a.e. in $\mathbb{R}^3$ (we recall that the embedding $H^1_{\text{rad}}(\mathbb{R}^3) \hookrightarrow L^4(\mathbb{R}^3)$ is compact), and in particular $\bar{u}_1, \bar{u}_2 \geq 0$ a.e. in $\mathbb{R}^3$. Since $dJ|_{S_{a_1} \times S_{a_2}}(u_{1,n}, u_{2,n}) \to 0$, by the Lagrange multipliers rule there exist two sequences of real numbers $(\lambda_{1,n})$ and $(\lambda_{2,n})$ such that

$$\sum_i \int_{\mathbb{R}^3} \nabla u_{i,n} \cdot \nabla \varphi_i - \lambda_{i,n} u_{i,n} \varphi - \sum_{i,j} \int_{\mathbb{R}^3} \beta_{ij} u_{i,n} u_{j,n} (u_{i,n} \varphi_j + \varphi_i u_{j,n}) = o(1) \|(\varphi_1, \varphi_2)\|_{H^1} \tag{3.16}$$

for every $(\varphi_1, \varphi_2) \in H^1(\mathbb{R}^3, \mathbb{R}^2)$, with $o(1) \to 0$ as $n \to \infty$. For more details we refer to [8, Lemma 3.2]. In light of (3.16), we can check as in [9, Lemma 3.8] that up to a subsequence $\lambda_{i,n} \to \lambda_i \in \mathbb{R}$ for $i = 1, 2$, and at least one limit value, say λ_1, is strictly negative. Moreover, thanks to [9, Lemma 3.9], we know that if $\lambda_i < 0$, then necessarily $u_{i,n} \to \bar{u}_i$ strongly in H^1. Hence, to complete the proof of the strong convergence $(u_{1,n}, u_{2,n}) \to (\bar{u}_1, \bar{u}_2)$, it remains to show that also λ_2 is negative. In [9] we used in a decisive way the assumption $\beta > 0$, and hence we have to modify our argument in the following way. First, we notice that by (3.16) and weak convergence $(\bar u_1, \bar u_2)$ is a (weak, and by regularity classical) solution to

$$\begin{cases} -\Delta \bar u_i - \lambda_i \bar u_i = \sum_{i,j} \beta_{ij} \bar u_i \bar u_j^2 & \text{in } \mathbb{R}^3 \\ u_i \geq 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{3.17}$$

Being $\lambda_1 < 0$, we deduce the following decay property for $\bar u_1$.

Lemma 3.11. *There exists $\alpha, \gamma > 0$ such that*

$$\bar u_1(x) \leq \alpha e^{-\sqrt{1+\gamma|x|^2}} \qquad \text{for every } x \in \mathbb{R}^3.$$

Proof. It is well known that radially symmetric H^1 continuous functions uniformly converge to 0 as $|x| \to +\infty$, see e.g. [5, Lemma 3.1.2]. Thus, we observe that

$$-\Delta \bar u_1 + q(x) \bar u_1 = 0 \qquad \text{in } \mathbb{R}^3,$$

where

$$q(x) = -\lambda_1 - \beta \bar u_1 \bar u_2^2 - \mu_1 \bar u_1^3 \geq \frac{|\lambda_1|}{2} \qquad \text{for } |x| > M,$$

provided M is sufficiently large. Let $\alpha > 0$ to be determined, $\gamma \in (0, |\lambda_1|/2)$, and

$$z(x) := \alpha e^{-\sqrt{1+\gamma|x|^2}}.$$

By direct computations

$$-\Delta z + \gamma z \geq 0 \qquad \text{in } \mathbb{R}^N,$$

so that

$$-\Delta(z - \bar u_1) + \gamma(z - \bar u_1) \geq \left(\frac{|\lambda_1|}{2} - \gamma \right) \bar u_1 \geq 0 \qquad \text{for } |x| > M.$$

We can also choose α so large that $z \geq \bar u_1$ for $|x| \leq M$. Therefore, testing the previous inequality with $(z - \bar u_1)^-$, we deduce that $\bar u_1 \leq z$ in $\mathbb{R}^N$. $\square$

Now we focus on the equation satisfied by $\bar u_2$:

$$\begin{cases} -\Delta \bar u_2 = \lambda_2 \bar u_2 + \mu_2 \bar u_2^3 + c(x) \bar u_2 & \text{in } \mathbb{R}^3 \\ \bar u_2 \geq 0 & \text{in } \mathbb{R}^3, \end{cases}$$

with $c(x) = \beta \bar u_1^2(x)$.

Lemma 3.12. *If $\lambda_2 \geq 0$, then necessarily $\bar u_2 \equiv 0$.*

Proof. We show that it is possible to apply the very general Theorem 4.1 in [4] (in the rest of the proof we adopt the notation therein). The equation for $\bar u_2$ can be written in the form

$$-Q \bar u_2 = f(\bar u_2, x),$$

where $Q := \Delta$, and

$$f(s, x) := \lambda_2 s + \mu_2 s^3 + c(x) s, \quad \text{with} \quad c(x) = \beta \bar u_1^2(x).$$

First, the Laplacian operator in $\mathbb{R}^3$ satisfies all the assumptions $(H1)$-$(H5)$ in [4], with $\alpha^* = \tilde\alpha^* = 1$ (see [4, page 2027]), $p = 2$, and fundamental solution $\Phi(x) = c_3 |x|^{-1}$ for some $c_3 > 0$. The nonlinearity f clearly satisfies $(f1)$ and $(f4)$ in [4]. The remaining assumptions are:

$(f2)$ $|x|^2 f(s, x) \to +\infty$ as $|x| \to +\infty$ locally uniformly in $s \in (0, +\infty)$;

$(f3)$ there exists $\mu > 0$ such that, if we define

$$\Psi_k(x) := |x|^2 \inf_{k\Phi(x) \le s \le \mu} s^{-1} f(s,x) \quad \text{and} \quad h(k) := \liminf_{|x| \to +\infty} \Psi_k(x),$$

then $0 < h(k) \le +\infty$ for every $k > 0$, and

$$\lim_{k \to +\infty} h(k) = +\infty.$$

Regarding $(f2)$, we have

$$\begin{aligned} \lim_{|x| \to +\infty} |x|^2 f(s,x) &= \lim_{|x| \to +\infty} |x|^2 \left(\lambda_2 s + \mu_2 s^3\right) + \lim_{|x| \to +\infty} |x|^2 c(x) s \\ &= \lim_{|x| \to +\infty} |x|^2 \left(\lambda_2 s + \mu_2 s^3\right) = +\infty \end{aligned}$$

locally uniformly in $s \in (0, +\infty)$, thanks to Lemma 3.11. As far as $(f3)$ is concerned, using again Lemma 3.11 we note that for any k, $\mu_2 > 0$

$$\Psi_k(x) = \lambda_2 |x|^2 + \mu_2 c_3^2 k^2 + |x|^2 c(x) \quad \text{and} \quad h(k) = \begin{cases} +\infty & \text{if } \lambda_2 > 0 \\ \mu_2 c_3^2 k^2 & \text{if } \lambda_2 = 0, \end{cases}$$

so that $(f3)$ holds.

In conclusion, Theorem 4.1 in [4] is applicable, and together with the strong maximum principle implies that $\bar{u}_2 \equiv 0$. □

Conclusion of the proof of Theorem 1.1. We observed that, if $\lambda_2 < 0$, then necessarily $u_{2,n} \to \bar{u}_2$ strongly in $H^1(\mathbb{R}^3)$, and hence the proof is complete. Let us suppose by contradiction that $\lambda_2 \ge 0$. Then by Lemma 3.12 we deduce that $\bar{u}_2 \equiv 0$, so that $\bar{u}_1$ is radial and solves

$$\begin{cases} -\Delta \bar{u}_1 - \lambda_1 \bar{u}_1 = \mu_1 \bar{u}_1^3 & \text{in } \mathbb{R}^3 \\ \bar{u}_1 > 0 & \text{in } \mathbb{R}^3 \\ \int_{\mathbb{R}^3} \bar{u}_1^2 = a_1^2. \end{cases} \tag{3.18}$$

We infer that $\bar{u}_1 \in \mathcal{M}_{a_1,\mu_1}$, and moreover, by the uniqueness of the radial positive solution to (3.18),

$$I_{\mu_1}(\bar{u}_1) = \ell(a_1, \mu_1).$$

Notice also that, as $\bar{u}_1 \in \mathcal{M}_{a_1,\mu_1}$,

$$I_{\mu_1}(\bar{u}_1) = \frac{\mu_1}{8} \int_{\mathbb{R}^3} \bar{u}_1^4.$$

In the same way, since $(u_{1,n}, u_{2,n}) \in \mathcal{P}$,

$$J(u_{1,n}, u_{2,n}) = \frac{1}{8} \sum_{i,j} \int_{\mathbb{R}^3} \beta_{ij} u_{i,n}^2 u_{j,n}^2.$$

Thus, by the strong L^4-convergence $(u_{1,n}, u_{2,n}) \to (\bar{u}_1, 0)$, the level c of the Palais-Smale sequence $\{(u_{1,n}, u_{2,n})\}$ is

$$\begin{aligned} c = \lim_{n \to \infty} J(u_n, v_n) &= \lim_{n \to \infty} \frac{1}{8} \sum_{i,j} \int_{\mathbb{R}^3} \beta_{ij} u_{i,n}^2 u_{j,n}^2 \\ &= \frac{\mu_1}{8} \int_{\mathbb{R}^3} \bar{u}_1^4 = I_{\mu_1}(\bar{u}_1) = \ell(a_1, \mu_1), \end{aligned} \tag{3.19}$$

in contradiction with the fact that $c > \ell(a_1, \mu_1)$ (see Proposition 3.5). □

3.4. **Phase-separation.** In this subsection we prove Theorem 1.2, and we use the subscript β to emphasize the dependence of all the considered quantities and functions with respect to β.

Due to the uniform bound $c_\beta = J_\beta(\bar{u}_{1,\beta}, \bar{u}_{2,\beta}) \le C$ (see Proposition 3.5), the proof follows a well understood scheme. Since $(\bar{u}_{1,\beta}, \bar{u}_{2,\beta}) \in \mathcal{P}_\beta$ for every β, we have

$$J_\beta(\bar{u}_{1,\beta}, \bar{u}_{2,\beta}) = \frac{1}{6}\int_{\mathbb{R}^N} |\nabla \bar{u}_{1,\beta}|^2 + |\nabla \bar{u}_{2,\beta}|^2,$$

and hence $\{(\bar{u}_{1,\beta}, \bar{u}_{2,\beta})\}$ is bounded in $H^1(\mathbb{R}^3, \mathbb{R}^2)$. Testing the equation (1.2a) with $(\bar{u}_{1,\beta}, \bar{u}_{2,\beta})$, this implies the boundedness of the sequences $\{\lambda_{1,\beta}\}$ and $\{\lambda_{2,\beta}\}$. Moreover, observing that

$$-\Delta \bar{u}_{i,\beta} \le \mu_i \bar{u}_{i,\beta}^3 \qquad \text{in } \mathbb{R}^3,$$

through a Brézis-Kato argument we can check that uniform boundedness in $H^1(\mathbb{R}^3, \mathbb{R}^2)$ implies also uniform boundedness in $L^\infty(\mathbb{R}^3, \mathbb{R}^2)$ (see [44, page 124] for a detailed proof, and [11] for the original argument). At this point, the rest of the proof follows directly by the general theory developed in [31, 39, 40, 45].

4. A natural constraint for scalar equations

In this and the next sections we deal with the scalar problem (1.7):

$$\begin{cases} -\Delta u - \lambda u = f(u) & \text{in } \mathbb{R}^N \\ u > 0, u \in H^1(\mathbb{R}^N) \\ \int_{\mathbb{R}^N} u^2 = a^2. \end{cases}$$

Solutions (λ, u) to (1.7) are obtained as critical points of the functional I, defined in (1.8), on S_a. Let us recall the definition (1.10) of G in this context:

$$\begin{aligned} G(u) &:= \int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} \left(\frac{N}{2} f(u)u - NF(u)\right) \\ &= \int_{\mathbb{R}^N} |\nabla u|^2 - \frac{N}{2}\int_{\mathbb{R}^N} \tilde{F}(u). \end{aligned}$$

Then we set $\mathcal{M} := \{u \in S_a : G(u) = 0\}$. It is known that, thanks to the Pohozaev identity, any solution to (1.7) stays in $\mathcal{M}$ (see [21, Lemma 2.7]). The purpose of this section consists in proving a strong version of Theorem 1.5.

Theorem 4.1. *Under (f1)-(f3), the set $\mathcal{M}$ is a $\mathcal{C}^1$ manifold, and moreover:*

- (i) *If $\{u_n\} \subset \mathcal{C}_c^\infty(\mathbb{R}^3) \cap \mathcal{M}$ is a Palais-Smale sequence for I restricted to $\mathcal{M}$ at a certain level $\ell \in \mathbb{R}$, then $\{u_n\}$ is a Palais-Smale sequence for I restricted to S_a.*
- (ii) *If there exists a Palais-Smale sequence $\{\tilde{u}_n\}$ for I restricted to $\mathcal{M}$ at level $\ell \in \mathbb{R}$, then there exists a possibly different Palais-Smale sequence $\{u_n\} \subset \mathcal{C}_c^\infty(\mathbb{R}^3)$ for I restricted to $\mathcal{M}$ at the same level $\ell \in \mathbb{R}$. Moreover $\|\tilde{u}_n - \tilde{u}_{i,n}\|_{H^1} \to 0$ as $n \to \infty$.*
- (iii) *If there exists a Palais-Smale sequence $\{\tilde{u}_n\}$ for I restricted to $\mathcal{M}$ at level $\ell \in \mathbb{R}$, then there exists a possibly different Palais-Smale sequence $\{u_n\} \subset \mathcal{C}_c^\infty(\mathbb{R}^3)$ for I restricted to S_a at the same level $\ell \in \mathbb{R}$. Moreover $\|\tilde{u}_n - \tilde{u}_{i,n}\|_{H^1} \to 0$ as $n \to \infty$.*
- (iv) *Let u be a critical point of I restricted on $\mathcal{M}$. Then u is a critical point of I restricted on S_a, and hence a solution to (1.2).*

The proof of this theorem is divided into several intermediate lemmas.

Lemma 4.2. *The set $\mathcal{M}$ is a $\mathcal{C}^1$ manifold of codimension 1 in S_a, hence a $\mathcal{C}^1$ manifold of codimension 2 in $H^1(\mathbb{R}^N)$.*

Proof. As subset of $H^1(\mathbb{R}^N)$, the constraint $\mathcal{M}$ is defined by the two equations $G(u) = 0$, $G_1(u) = 0$, where

$$G_1(u) := a^2 - \int_{\mathbb{R}^3} u^2,$$

and clearly G and G_1 are of class $\mathcal{C}^1$. We have to check that

$$d(G_1, G) : H^1(\mathbb{R}^N) \to \mathbb{R}^2 \quad \text{is surjective.}$$

If this is not true, $dG(u)$ and $dG_1(u)$ are linearly dependent, i.e. there exist $\nu \in \mathbb{R}$ such that

$$2\int_{\mathbb{R}^N} \nabla u \cdot \nabla \varphi - \frac{N}{2}\int_{\mathbb{R}^N} \tilde{F}'(u)\varphi = 2\nu \int_{\mathbb{R}^N} u\varphi$$

for every $\varphi \in H^1(\mathbb{R}^N)$. This means that u is a solution to

$$\begin{cases} -\Delta u - \nu u = \frac{N}{4}\tilde{F}'(u) \\ \int_{\mathbb{R}^3} u^2 = a^2 \end{cases} \quad \text{in } \mathbb{R}^3,$$

Thanks to [21, Lemma 2.7], we infer that

$$\int_{\mathbb{R}^N} |\nabla u|^2 = \int_{\mathbb{R}^N} \left(\frac{N^2}{8}\tilde{F}'(u)u - \frac{N^2}{4}\tilde{F}(u) \right).$$

Since $u \in \mathcal{M}$, this gives

$$\frac{N}{2}\int_{\mathbb{R}^N} \tilde{F}(u) = \int_{\mathbb{R}^N} \left(\frac{N^2}{8}\tilde{F}'(u)u - \frac{N^2}{4}\tilde{F}(u) \right),$$

in contradiction with $(f3)$ and the fact that $u \in S_a$ (and hence $u \not\equiv 0$ in $\mathbb{R}^N$). $\square$

Let us introduce some notation, similar to that of Section 2. For $u \in S_a$ and $s \in \mathbb{R}$, we define

$$(s \star u)(x) := e^{Ns/2} u(e^s x),$$

so that $\|s \star u\|_{L^2(\mathbb{R}^N)} = \|u\|_{L^2(\mathbb{R}^N)}$. We also consider

$$\Psi_u(s) := I(s \star u) = \frac{e^{2s}}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{1}{e^{Ns}}\int_{\mathbb{R}^N} F(e^{Ns/2}u). \tag{4.1}$$

The study of Ψ_u is the object of the next lemma, for which we refer to [21, Lemma 2.9].

Lemma 4.3. *For any $u \in S_a$, a value $s \in \mathbb{R}$ is a critical point of Ψ_u if and only if $s \star u \in \mathcal{M}$. Moreover, for any $u \in S_a$ the function Ψ_u has a unique critical point s_u, which is a strict maximum.*

Remark 4.4. We observe that assumption $(f3)$ is used in [21] to prove the uniqueness of s_u.

Lemma 4.5. *The map $u \mapsto s_u$ is of class $\mathcal{C}^1$.*

Proof. The value s_u can be found solving the equation $\Psi_u'(s) = 0$, i.e.

$$\int_{\mathbb{R}^N} e^{2s}|\nabla u|^2 - \frac{N}{2}e^{-Ns}\tilde{F}(e^{Ns/2}u) = 0.$$

Thanks to $(f3)$, it is not difficult to check that the assumptions of the implicit function theorem are satisfied. □

In the next lemma we obtain a description of $T_u S_a$ for $u \in \mathcal{M}$, similar to the one in Lemma 2.4.

Lemma 4.6. *For any $u \in \mathcal{M} \cap \mathcal{C}_c^\infty(\mathbb{R}^N)$, we have*

$$T_u S_a = T_u\mathcal{M} \oplus \mathbb{R} \left.\frac{d}{ds}\right|_{s=0} (s \star u).$$

Proof. The proof is similar to that of Lemma 2.4, and hence is sketched. We have to show that

$$\left.\frac{d}{ds}\right|_{s=0} (s \star u) \in T_u S_a \setminus T_u\mathcal{M}.$$

For any $u \in \mathcal{C}_c^\infty(\mathbb{R}^N)$

$$\left.\frac{d}{ds}\right|_{s=0} (s \star u)(x) = \frac{N}{2}u(x) + \nabla u(x) \cdot x \in \mathcal{C}_c^\infty(\mathbb{R}^3),$$

and hence it is easy to check that $\frac{d}{ds}(s \star u)\big|_{s=0} \in T_u S_a$. Using the divergence theorem as in Lemma 2.4, we also obtain

$$\begin{aligned} dG(u)\left[\left.\frac{d}{ds}\right|_{s=0} (s \star u)\right] &= 2\int_{\mathbb{R}^N} \left[\frac{N}{2}|\nabla u|^2 + \nabla u \cdot \nabla(\nabla u \cdot x)\right] \\ &\quad - \frac{N}{2}\int_{\mathbb{R}^N} \tilde{F}'(u)\left(\frac{N}{2}u + \nabla u \cdot x\right) \\ &= 2\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{N^2}{4}\int_{\mathbb{R}^N} \tilde{F}'(u)u + \frac{N^2}{2}\int_{\mathbb{R}^N} \tilde{F}(u). \end{aligned}$$

Since $u \in \mathcal{M}$, this implies that

$$\begin{aligned} &dG(u)\left[\left.\frac{d}{ds}\right|_{s=0} (s \star u)\right] \\ &\qquad = \left(N + \frac{N^2}{2}\right)\int_{\mathbb{R}^N} \tilde{F}(u) - \frac{N^2}{4}\int_{\mathbb{R}^N} \tilde{F}'(u)u < 0, \end{aligned}$$

where we used assumptions $(f2)$, $(f3)$, and the fact that $u \not\equiv 0$. □

Lemma 4.7. *If $u \in \mathcal{C}_c^\infty(\mathbb{R}^N) \cap \mathcal{M}$, then*

$$dI(u)\left[\left.\frac{d}{ds}\right|_{s=0} (s \star u)\right] = 0.$$

The proof is an easy consequence of the definition of $\mathcal{M}$, see Lemma 2.6 for more details.

Conclusion of the proof of Theorem 4.1. We only prove point (i). Let $\{u_n\} \subset \mathcal{C}_c^\infty(\mathbb{R}^N) \cap \mathcal{P}$ be a Palais-Smale sequence for $I|_{\mathcal{M}}$. We denote by $(T_u S_a)^*$ the dual space to $T_u S_a$, and by $\|\cdot\|$ the $H^1(\mathbb{R}^N)$ norm. In view of Lemma 4.6, we have:

$$\begin{aligned}\|dI(u_n)\|_{(T_u S_a)^*} &= \sup\left\{|dI(u_n)[\varphi]| : \varphi \in T_u S_a,\ \|\varphi\| \le 1\right\} \\ &= \sup\left\{ |dI(u_n)[\phi] + dI(u_n)[\psi]| \,\middle|\, \begin{array}{l} \varphi = \phi + \psi, \|\varphi\| \le 1 \\ \phi \in T_u\mathcal{M},\ \psi \in \mathbb{R}\left(\frac{d}{ds}\big|_{s=0} (s \star u_n)\right) \end{array} \right\}.\end{aligned}$$

Now $dI(u_n)[\psi] = 0$ by Lemma 4.7, and hence

$$\begin{aligned}\|dI(u_n)\|_{(T_u(S_a))^*} &= \sup\left\{|dI(u_n)[\phi]| : \phi \in T_u\mathcal{P},\ \|\phi\| \le 1\right\} \\ &= \|dI(u_n)\|_{(T_u\mathcal{M})^*} \to 0\end{aligned}$$

as $n \to \infty$, since $\{u_n\}$ is a Palais-Smale sequence for I restricted to $\mathcal{M}$. □

5. Existence and multiplicity of solutions to (1.7)

This section is devoted to the proof of Theorem 1.4. We are interested in the existence or radial solutions, and hence throughout this section we will always work in S_a^r. This simplifies some compactness issues. As a consequence of Theorem 4.1, the existence of solutions to (1.7) reduces to the existence of critical points for I restricted to $\mathcal{M}$. The main advantage is that, in contrast to I restricted on S_a, the functional I restricted to $\mathcal{M}$ satisfies the Palais-Smale condition, and is bounded from below. Thus, the Lusternik-Schnirelman theory yields infinitely many critical points. This idea is rigorously developed in what follows.

We denote by $\text{cat}_{\mathbb{Z}/2}(\mathcal{M})$ the equivariant Lusternik-Schnirelman category of $\mathcal{M}$ with respect to the antipodal action of $\mathbb{Z}/2$, and by genus$(\mathcal{M})$ the Krasnoselskii genus of $\mathcal{M}$. For the definitions and the properties of $\text{cat}_{\mathbb{Z}/2}$ and genus we refer to [6, Section 2] (there it is considered a much more general setting with respect to the one considered here; an easier reference for the genus is [3]).

Notice that both I, G and G_1 are even functionals, and hence the problem is invariant under the action of $\mathbb{Z}/2$.

Lemma 5.1. *It results that* $\text{cat}_{\mathbb{Z}/2}(\mathcal{M}) = +\infty$.

Proof. It is well known that $\text{cat}_{\mathbb{Z}/2}(\mathcal{M}) \ge \text{genus}(\mathcal{M})$, see for instance [6, Proposition 2.10]. Therefore, we can prove that $\text{genus}(\mathcal{M}) = +\infty$. Let $V \subset H^1(\mathbb{R}^N)$ with $\dim V = n$, and let $SV := V \cap S_a^r$. Notice that $\text{genus}(SV) = \dim V = n$ (this follows for instance by [3, Theorem 10.5]). We show that there exists a map $\psi : SV \to \mathcal{M}$ continuous and odd, whence by [3, Lemma 10.4] we deduce that $\text{genus}(\mathcal{M}) \ge \text{genus}(SV) = n$; since n is arbitrary, the thesis follows.

The explicit expression of $I(s \star u)$ (see (4.1)) and the oddness of f ensure that the map $SV \ni u \mapsto s_u \in \mathbb{R}$ is even: $s_u = s_{-u}$. It is also continuous by Lemma 4.5. The map $\psi(u) = s_u \star u$ is then odd because

$$\psi(-u) = s_{-u} \star (-u) = -s_u \star u = -\psi(u),$$

and it is also continuous due to Lemma 2.7. □

Now we describe the properties of I on $\mathcal{M}$. We shall use many times the following inequalities, which can be easily proved using assumptions $(f1)$ and $(f2)$: for every

$t \in \mathbb{R}$ and $s \geq 0$ there holds

$$\begin{cases} s^\beta F(t) \leq F(ts) \leq s^\alpha F(t) & \text{if } s \leq 1 \\ s^\alpha F(t) \leq F(ts) \leq s^\beta F(t) & \text{if } s \geq 1. \end{cases} \tag{5.1}$$

We also recall the Gagliardo-Nirenberg inequality: There exists a constant S depending on N and on $r \in (2, 2^*)$ such that

$$\|u\|_{L^r} \leq S\|u\|_{L^2}^{1-\gamma}\|\nabla u\|_{L^2}^{\gamma} \qquad \text{for all } u \in H^1(\mathbb{R}^N); \tag{5.2}$$

here $\gamma = N\left(\frac{1}{2} - \frac{1}{r}\right)$.

Lemma 5.2. *There exists $\delta > 0$ such that $\|u\|_{\mathcal{D}^{1,2}} \geq \delta$ for every $u \in \mathcal{M}$.*

Proof. Since $F(s) \geq 0$ for every $s \in \mathbb{R}$ and by $(f2)$, for $u \in \mathcal{M}$ we have

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla u|^2 &\leq \frac{N}{2}\int_{\mathbb{R}^3} f(u)u \leq \frac{N\beta}{2}\int_{\mathbb{R}^3} F(u) \\ &\leq \frac{N\beta}{2}\int_{\{|u|\leq 1\}} F(u) + \frac{N\beta}{2}\int_{\{|u|\geq 1\}} F(u) \\ &\leq \frac{N\beta}{2}\int_{\{|u|\leq 1\}} F(1)|u|^\alpha + \frac{N\beta}{2}\int_{\{|u|\geq 1\}} F(1)|u|^\beta \\ &\leq C\int_{\mathbb{R}^3} |u|^\alpha + |u|^\beta, \end{aligned}$$

where we used (5.1). To estimate the right hand side, we apply (5.2) with $r = \alpha$ and $r = \beta$, obtaining

$$\|\nabla u\|_{L^2}^2 \leq C\|\nabla u\|_{L^2}^{\frac{N}{2}(\alpha-2)} + C\|\nabla u\|_{L^2}^{\frac{N}{2}(\beta-2)}.$$

Now due to $(f2)$ we know that both $N(\alpha-2)/2$ and $N(\beta-2)/2$ are strictly larger than 2, and hence the lemma follows. □

Lemma 5.3. *The functional I restricted to $\mathcal{M}$ is coercive and bounded from below by a positive constant.*

Proof. By $(f2)$, we infer that for any $u \in \mathcal{M}$

$$\int_{\mathbb{R}^3} |\nabla u|^2 \leq \frac{N}{2}\int_{\mathbb{R}^N} f(u)u \leq \frac{N\beta}{2}\int_{\mathbb{R}^N} F(u). \tag{5.3}$$

Therefore, using again $(f2)$

$$\begin{aligned} I(u) = \frac{N}{4}\int_{\mathbb{R}^N} f(u)u - \left(\frac{N+2}{2}\right)\int_{\mathbb{R}^N} F(u) &= \frac{N}{4}\int_{\mathbb{R}^N}\left(f(u)u - \left(2 + \frac{4}{N}\right)F(u)\right) \\ \geq \frac{N}{4}\left(\alpha - 2 - \frac{4}{N}\right)\int_{\mathbb{R}^N} F(u) &\geq \frac{1}{2\beta}\left(\alpha - 2 - \frac{4}{N}\right)\int_{\mathbb{R}^N} |\nabla u|^2 \end{aligned}$$

for any $u \in \mathcal{M}$. Now Lemma 5.3 follows from Lemma 5.2. □

Lemma 5.4. *The Palais-Smale condition is satisfied by I constrained to $\mathcal{M}$.*

Proof. Let $\{\tilde{u}_n\} \subset \mathcal{M}$ be a Palais-Smale sequence for $I|_{\mathcal{M}}$ at some level $c \in \mathbb{R}$ (notice that automatically $c > 0$ by Lemma 5.3), and let $\{u_n\} \subset \mathcal{M} \cap \mathcal{C}_c^\infty(\mathbb{R}^N)$ be the possibly different Palais-Smale sequence given by Theorem 4.1-(ii). It is sufficient to show that $\{u_n\}$ converges strongly in $H^1(\mathbb{R}^N)$ to some limit, up to a subsequence.

By Lemma 5.3 $\{u_n\}$ is bounded, and hence up to a subsequence $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^3)$, for a suitable $u \in H^1(\mathbb{R}^N)$. Moreover, due to Theorem 4.1 and the Lagrange multipliers rule (see also [21, Lemma 2.5] for more details), we have

$$\int_{\mathbb{R}^N} (\nabla u_n \cdot \nabla \varphi - f(u_n)\varphi - \lambda_n u_n \varphi) = o(1)\|\varphi\|_{H^1}$$

for every $\varphi \in H^1(\mathbb{R}^N)$, where $o(1) \to 0$ as $n \to \infty$ and $\lambda_n \in \mathbb{R}$. Taking $\varphi = u_n$ and recalling the definition of $\mathcal{M}$, we deduce that

$$\begin{aligned} \lambda_n a^2 &= \int_{\mathbb{R}^N} (|\nabla u_n|^2 - f(u_n)u_n) + o(1) \\ &\le \int_{\mathbb{R}^N} \left(\left(\frac{N-2}{2} \right) f(u_n)u_n - NF(u_n) \right) + o(1). \end{aligned}$$

Let $N \ge 3$; using assumption $(f2)$, estimate (5.3) and Lemma 5.2, the previous computation gives

$$\begin{aligned} \lambda_n a^2 &\le \int_{\mathbb{R}^N} \left(\frac{N-2}{2N} \right) \left(\beta - \frac{2N}{N-2} \right) F(u_n) + o(1) \\ &\le C \int_{\mathbb{R}^N} |\nabla u_n|^2 \le -C < 0. \end{aligned}$$

If $N = 2$, the same conclusion follows using simply estimate (5.3) and Lemma 5.2.

Notice also that $\{\lambda_n\}$ is bounded (since $\{u_n\}$ is), and hence up to a subsequence $\lambda_n \to \lambda < 0$.

The conclusion of the proof follows from now on exactly as in [21, Section 2.4]. $\square$

Proof of Theorem 1.4. Due to Lemmas 5.3 and 5.4, we can apply the Lusternik-Schnirelman Theorem 2.19 in [6]; this, together with Lemma 5.1, completes the proof of existence and multiplicity. We also observe that the minimizer for $I|_{\mathcal{M}}$ can be taken positive, because $u \in \mathcal{M}$ implies $|u| \in \mathcal{M}$ and $I(u) = I(|u|)$. $\square$

Remark 5.5. Theorem 2.19 in [6] is stated for $\mathcal{C}^1$ functionals on $\mathcal{C}^{2-}$ manifolds, while under our assumption $\mathcal{M}$ is merely $\mathcal{C}^1$. This is not a problem, as observed in [6, page 21], since the Szulkin's approach developed in [43] permits to replace the $\mathcal{C}^{2-}$ assumption in [6] with simple $\mathcal{C}^1$ regularity.

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References

[1] N. Akhmediev and A. Ankiewicz. Partially coherent solitons on a finite background. *Phys. Rev. Lett.*, 82:2661, 1999.

[2] A. Ambrosetti and E. Colorado. Standing waves of some coupled nonlinear Schrödinger equations. *J. Lond. Math. Soc. (2)*, 75(1):67–82, 2007.

[3] A. Ambrosetti and A. Malchiodi. *Nonlinear analysis and semilinear elliptic problems.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2007.

[4] S. N. Armstrong and B. Sirakov. Nonexistence of positive supersolutions of elliptic equations via the maximum principle. *Commun. Partial Differential Equations*, 36(10-12):2011–2047, 2011.

[5] M. Badiale and E. Serra. *Semilinear elliptic equations for beginners. Existence results via the variational approach.* Universitext. Springer, London, 2011.

[6] T. Bartsch. *Topological methods for variational problems with symmetries.* Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1993.

[7] T. Bartsch and S. de Valeriola. Normalized solutions of nonlinear Schrödinger equations. *Archiv der Mathematik*, 100(1):75–83, 2012.

[8] T. Bartsch and L. Jeanjean. Normalized solutions for nonlinear Schrödinger systems. Preprint 2015, arXiv: 1507.04649.

[9] T. Bartsch, L. Jeanjean, and N. Soave. Normalized solutions for a system of coupled cubic Schrödinger equations on $\mathbb{R}^3$. *J. Math. Pures Appl.*, doi:10.1016/j.matpur.2016.03.004.

[10] T. Bartsch and Z.-Q. Wang. Note on ground states of nonlinear Schrödinger systems. *J. Partial Differential Equations*, 19(3):200–207, 2006.

[11] H. Brézis and T. Kato. Remarks on the Schrödinger operator with singular complex potentials. *J. Math. Pures Appl. (9)*, 58(2):137–151, 1979.

[12] D. Cao, I.-L. Chern, J. Wei. On ground state of spinor Bose-Einstein condensates. *NoDEA Nonlinear Differential Equations Appl.*, 18: 427–445, 2011.

[13] T. Cazenave. *Semilinear Schrödinger equations*, volume 10 of *Courant Lecture Notes in Mathematics.* New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.

[14] Z. Chen and W. Zou. An optimal constant for the existence of least energy solutions of a coupled Schrödinger system. *Calc. Var. Partial Differential Equations*, 48(3-4):695–711, 2013.

[15] J.-N. Corvellec, M. Degiovanni and M. Marzocchi. Deformation properties for continuous functionals and critical point theory. *Top. Meth. Nonlin. Anal.*, 1(1):151–171, 1993.

[16] B. D. Esry, C. H. Greene, J. P. Burke Jr. and J. L. Bohn. Hartree-Fock theory for double condensates. *Phys. Rev. Lett.* 78 (1997), 3594.

[17] G. Fibich and F. Merle, Self-focusing on bounded domains. *Phys. D* 155:12 , 132–158, 2001.

[18] D. J. Frantzeskakis. Dark solitons in atomic Bose-Einstein condensates: from theory to experiments. *J. Phys. A: Math. Theor.* 43, 2010.

[19] N. Ghoussoub. *Duality and perturbation methods in critical point theory*, volume 107 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 1993. With appendices by David Robinson.

[20] T. Gou and L. Jeanjean. Existence and orbital stability of standing waves for nonlinear Schrödinger systems. *Nonlinear Analysis: TMA*, doi:10.1016/j.na.2016.05.016.

[21] L. Jeanjean. Existence of solutions with prescribed norm for semilinear elliptic equations. *Nonlinear Anal.*, 28(10):1633–1659, 1997.

[22] T.-C. Lin and J. Wei. Ground state of N coupled nonlinear Schrödinger equations in $\mathbb{R}^n$, $n \leq 3$. *Comm. Math. Phys.*, 255(3):629–653, 2005.

[23] P. L. Lions. The concentration-compactness principle in the calculus of variations. the locally compact case, part 1. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 1(2):109–145, 1984.

[24] P. L. Lions. The concentration-compactness principle in the calculus of variations. the locally compact case, part 2. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 1(4):223–283, 1984.

[25] Z. Liu and Z.-Q. Wang. Ground states and bound states of a nonlinear Schrödinger system. *Adv. Nonlinear Stud.*, 10(1):175–193, 2010.

[26] L. A. Maia, E. Montefusco, and B. Pellacci. Positive solutions for a weakly coupled nonlinear Schrödinger system. *J. Differential Equations*, 229(2):743–767, 2006.

[27] B. Malomed. Multi-component Bose-Einstein condensates: Theory. In: P.G. Kevrekidis, D.J. Frantzeskakis, R. Carretero-Gonzalez (Eds.), Emergent Nonlinear Phenomena in Bose-Einstein Condensation, Springer-Verlag, Berlin, 2008, 287-305.

[28] R. Mandel. Minimal energy solutions for cooperative nonlinear Schrödinger systems. *NoDEA Nonlinear Differential Equations Appl.*, 22(2):239–262, 2015.

[29] N. V. Nguyen and Z.-Q. Wang. Existence and stability of a two-parameter family of solitary waves for a 2-coupled nonlinear Schrödinger system. *Discrete Contin. Dyn. Syst.*, 36(2):1005–1021, 2016.

[30] B. Noris, H. Tavares, S. Terracini, and G. Verzini. Convergence of minimax structures and continuation of critical points for singularly perturbed systems. *J. Eur. Math. Soc. (JEMS)*, 14(4):1245–1273, 2012.

[31] B. Noris, H. Tavares, S. Terracini, and G. Verzini. Uniform Hölder bounds for nonlinear Schrödinger systems with strong competition. *Comm. Pure Appl. Math.*, 63(3):267–302, 2010.
[32] B. Noris, H. Tavares, and G. Verzini. Existence and orbital stability of the ground states with prescribed mass for the L^2-critical and supercritical NLS on bounded domains. *Anal. PDE* 7(8) (2014), 1807–1838.
[33] B. Noris, H. Tavares, and G. Verzini. Stable solitary waves with prescribed L^2-mass for the cubic Schrödinger system with trapping potentials. *Discrete and Continuous Dynamical Systems*, 35(12):6085–6112, 2015.
[34] R. S. Palais. The principle of symmetric criticality. *Commun. Math. Phys.*, 69:19–30, 1979.
[35] Y. Sato and Z.-Q. Wang. Least energy solutions for nonlinear Schrödinger systems with mixed attractive and repulsive couplings. *Adv. Nonlinear Stud.*, 15(1):1–22, 2015.
[36] B. Sirakov. Least energy solitary waves for a system of nonlinear Schrödinger equations in $\mathbb{R}^n$. *Comm. Math. Phys.*, 271(1):199–221, 2007.
[37] N. Soave. On existence and phase separation of solitary waves for nonlinear Schrödinger systems modelling simultaneous cooperation and competition. *Calculus of Variations and Partial Differential Equations*, 53(3):689–718, 2015.
[38] N. Soave and H. Tavares. New existence and symmetry results for least energy positive solutions of Schrödinger systems with mixed competition and cooperation terms. *J. Differential Equations*, 261(1):505537, 2016.
[39] N. Soave, H. Tavares, S. Terracini, and A. Zilio. Hölder bounds and regularity of emerging free boundaries for strongly competing Schrödinger equations with nontrivial grouping. *Nonlinear Anal.*, 138:388-427, 2016.
[40] N. Soave and A. Zilio. Uniform bounds for strongly competing systems: the optimal Lipschitz case. *Arch. Ration. Mech. Anal.*, 218(2):647–697, 2015.
[41] C. A. Stuart. Bifurcation for Dirichlet problems without eigenvalues. *Proceedings of the London Mathematical Society*, 45(1):169–192, 1982.
[42] C. A. Stuart. Bifurcation from the continuous spectrum in L^2-theory of elliptic equations on $\mathbb{R}^N$. Recent Methods in Nonlinear Analysis and Applications, Liguori, Napoli, 1981.
[43] A. Szulkin. Ljusternik-Schnirelmann theory on C^1-manifolds. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 5: 119-139, 1988.
[44] H. Tavares. *Nonlinear elliptic systems with a variational structure: existence, asymptotics and other qualitative properties.* PhD thesis, Universidade de Lisboa, 2010.
[45] H. Tavares and S. Terracini. Regularity of the nodal set of segregated critical configurations under a weak reflection law. *Calc. Var. Partial Differential Equations*, 45(3-4):273–317, 2012.
[46] H. Tavares and S. Terracini. Sign-changing solutions of competition-diffusion elliptic systems and optimal partition problems. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 29(2):279–300, 2012.
[47] S. Terracini and G. Verzini. Multipulse phases in k-mixtures of Bose-Einstein condensates. *Arch. Ration. Mech. Anal.*, 194(3):717–741, 2009.
[48] E. Timmermans. Phase separation of Bose-Einstein condensates. *Phys. Rev. Lett.*, 81:5718–5721, Dec 1998.
[49] J. Wei and T. Weth. Radial solutions and phase separation in a system of two coupled Schrödinger equations. *Arch. Ration. Mech. Anal.*, 190(1):83–106, 2008.

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