Periodic solutions with prescribed minimal period of the 2-vortex problem in domains

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Abstract

We consider the Hamiltonian system

$$\dot{z}_k = J\nabla_{z_k} H_\Omega(z_1, z_2), \quad k = 1, 2,$$

for two point vortices $z_1, z_2 \in \Omega$ in a domain $\Omega \subset \mathbb{R}^2$. The Hamiltonian H_Ω is of the form

$$H_\Omega(z_1, z_2) = -\frac{1}{2\pi}\log|z_1 - z_2| - 2g(z_1, z_2) - h(z_1) - h(z_2),$$

where $g : \Omega \times \Omega \to \mathbb{R}$ is the regular part of a hydrodynamic Green's function in Ω, and $h : \Omega \to \mathbb{R}$ is the Robin function: $h(z) = g(z, z)$. The system is singular and not integrable, except when Ω is a disk or an annulus. We prove the existence of infinitely many periodic solutions with minimal period T which are a superposition of a slow motion of the center of vorticity along a level line of h and of a fast rotation of the two vortices around their center of vorticity. These vortices move in a prescribed subset $\mathcal{A} \subset \Omega$ that has to satisfy a geometric condition. The minimal period can be any T in an interval $I(\mathcal{A}) \subset \mathbb{R}$. Subsets $\mathcal{A}$ to which our results apply can be found in any generic bounded domain. The proofs are based on a recent higher dimensional version of the Poincaré-Birkhoff theorem due to Fonda and Ureña.

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1 Introduction

Given a domain $\Omega \subset \mathbb{R}^2$, the dynamics of N point vortices $z_1(t), \dots, z_N(t) \in \Omega$ with vortex strengths $\kappa_1, \dots, \kappa_N \in \mathbb{R}$ is described by a Hamiltonian system

$$\kappa_k \dot{z}_k = J\nabla_{z_k} H_\Omega(z_1, \dots, z_N), \quad k = 1, \dots, N; \tag{1.1}$$

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here $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the standard symplectic matrix in $\mathbb{R}^2$. The Hamiltonian is of the form

$$H_\Omega(z_1, \ldots, z_N) = -\frac{1}{2\pi} \sum_{\substack{j,k=1 \\ j \neq k}}^{N} \kappa_j \kappa_k \log |z_j - z_k| - F(z_1, \ldots, z_N)$$

where $F : \Omega^N \to \mathbb{R}$ is a function of class $\mathcal{C}^2$. The Hamiltonian is defined on the configuration space

$$\mathcal{F}_N\Omega = \left\{(z_1, \ldots, z_N) \in \Omega^N : z_j \neq z_k \text{ for } j \neq k\right\}.$$

Observe that the system is singular, but of a very different type than the singular second order equations from celestial mechanics.

Systems like (1.1) arise as a singular limit problem in Fluid Mechanics. A model for an incompressible, non viscous fluid is given by the two dimensional Euler equations

$$\begin{cases} v_t + (v \cdot \nabla)v = -\nabla P \\ \nabla \cdot v = 0, \end{cases}$$

in which v represents the velocity of the fluid and P its pressure. Making a point vortex ansatz $\omega = \sum_{k=1}^{N} \kappa_k \delta_{z_k}$, where δ_{z_k} is the Dirac delta, for the scalar vorticity $\omega = \nabla \times v = \partial_1 v_2 - \partial_2 v_1$, one obtains system (1.1); see [22].

Classically the point vortex equations (1.1) were first derived by Kirchhoff in [17], who considered the case where $\Omega = \mathbb{R}^2$ is the whole plane. In this case the function F in the Hamiltonian is identically zero. On the other hand, when $\Omega \neq \mathbb{R}^2$, one has to take account of the boundaries of the domain which leads to

$$F(z_1, \ldots, z_N) = \sum_{j,k=1}^{N} \kappa_j \kappa_k g(z_j, z_k)$$

where $g : \Omega \times \Omega \to \mathbb{R}$ is the regular part of a hydrodynamic Green's function in Ω. An important role plays the Robin function $h : \Omega \to \mathbb{R}$ defined by $h(z) = g(z, z)$. In fact, a single vortex $z(t) \in \Omega$ moves along level lines of h according to the Hamiltonian system $\dot{z} = \kappa J \nabla h(z)$. This goes back to work of Routh [26] and Lin [19, 20]. Except in a few special cases, the Hamiltonian H_Ω is not explicitly known, it is not bounded from above or below, its level sets are not compact, and the system (1.1) is not integrable. We refer the reader to [21, 22, 25, 27] for modern presentations of the point vortex method.

It is worthwhile to mention that systems like (1.1) also arise in other contexts from mathematical physics, e.g. in models from superconductivity (Ginzburg-Landau-Schrödinger equation), or in equations modeling the dynamics of a magnetic vortex system in a thin ferromagnetic film (Landau-Lifshitz-Gilbert equation); see [7] for references to the literature. The domain can also be a subset of a two-dimensional surface.

Many authors worked on this problem, mostly in the case $\Omega = \mathbb{R}^2$ with F=0. In the presence of boundaries much less is known, except in the case of special domains like the half plane or a radially symmetric domain, i.e. disk or annulus, when the Green's function is explicitly known. In the case of two vortices and $\kappa_1 \kappa_2 < 0$ the Hamiltonian is bounded below and satisfies $H_\Omega(z_1, z_2) \to \infty$ as $z = (z_1, z_2) \to \partial\mathcal{F}_N\Omega$. Consequently all level surfaces of H_Ω are compact, and standard results about Hamiltonian systems apply. In particular, by a result of Struwe [28] almost every level surface contains periodic solutions. Another simple setting is the case of Ω being radially symmetric and $N = 2$ whence the system (1.1) is integrable and can be analyzed in detail. For Ω being a disk this has been done in [12].

Except in the above mentioned special cases even the existence of equilibrium solutions of (1.1) is difficult to prove; see [8, 9]. The problem of finding periodic solutions in a general domain has only recently been addressed in the papers [4, 5, 7] where several one parameter families of periodic solutions of the general N-vortex problem (1.1) have been found. These solutions rotate around their center of vorticity, which is situated near a stable critical point of the Robin function h. The periods tend to zero as the solutions approach the critical point of h. Recall that $h(z) \to \infty$ as $z \to \partial\Omega$, hence h always has a minimum. It may have arbitrarily many critical points. For a generic domain all critical points are non-degenerate (see [24]), hence in this case the results from [4, 5, 7] produce many one-parameter families of periodic solutions. Moreover, these solutions lie on global continua that are obtained via an equivariant degree theory for gradient maps. A different type of periodic solutions has been discovered in [6] on a simply connected domain Ω. There the solutions are choreographies where the vortices move near the boundary $\partial\Omega$ almost following a level line $h^{-1}(c)$ with $c \gg 1$.

In the present paper we consider (1.1) in a domain $\Omega \subsetneq \mathbb{R}^2$. We find a new type of solutions that are not (necessarily) located near an equilibrium of h but lie in a prescribed annular shaped region $\mathcal{A}$ whose boundary curves are level lines of h. In order to present our idea in a most simple way we consider the case of two identical vortices, so we may assume without loss of generality that $\kappa_1 = \kappa_2 = 1$. We require assumptions on $\mathcal{A}$ but no further assumptions on Ω, in particular we need not be close to an integrable setting. We find an interval $I = I(\mathcal{A}) \subset \mathbb{R}$ such that for every $T \in I$ the system has infinitely many periodic solutions in $\mathcal{A}$ with minimal period T. The solutions that we obtain are essentially superpositions of a slow motion of the center of vorticity along some level line $h^{-1}(c)$ of h, and of a fast rotation of the two vortices around their center of vorticity. This will be described in detail. These solutions are of a very different nature from those obtained in [4, 5, 7]. We also give several classes of domains Ω for which one can find such regions $\mathcal{A}$. In particular we can find $\mathcal{A}$ in any generic bounded domain. Our proofs are based on a recent generalization of the Poincaré-Birkhoff theorem due to Fonda-Ureña [16].

The paper is organized as follows. In Section 2 we state and discuss our results about the existence and shape of periodic solutions of (1.1). In Section 3 we prove the main Theorem 2.2 about the existence of a periodic solution by an application of [16, Theorem 1.2]. This requires the computation of certain rotation numbers which will be done in Section 4. In the last Section 5 we prove the various consequences of Theorem 2.2 and its proof.

2 Statement of results

We consider the Hamiltonian system

$$\dot{z}_k = J\nabla_{z_k} H_\Omega(z_1, z_2), \quad k = 1, 2, \tag{2.1}$$

on a domain $\Omega \subset \mathbb{R}^2$ with Hamilton function

$$H_\Omega(z_1, z_2) = -\frac{1}{2\pi} \log |z_1 - z_2| - 2g(z_1, z_2) - h(z_1) - h(z_2)$$

where $g : \Omega \times \Omega \to \mathbb{R}$ is the regular part of a hydrodynamic Green's function in Ω, and $h : \Omega \to \mathbb{R}$ is the Robin function: $h(z) = g(z, z)$. For simplicity we assume that Ω satisfies the uniform exterior ball condition. This implies that the flow associated to (2.1) is defined for all $t \in \mathbb{R}$; see Proposition 3.1.

If $C \subset h^{-1}(a)$ is a compact connected component of $h^{-1}(a)$ not containing a critical point of h then the Hamiltonian system

$$\dot{z} = -2J\nabla h(z) \tag{2.2}$$

has a periodic solution with trajectory C. Let $T(C)$ be the minimal period of this solution. Observe that system (2.2) describes the motion of one vortex in Ω with strength $\kappa = 2$.

We need one geometric assumption on h.

Assumption 2.1. *There exists an open bounded annular shaped region $\mathcal{A} \subset \Omega$ bounded by two closed curves Γ_1, Γ_2, each Γ_k being strictly star-shaped with respect to a point $z_0 \in \mathbb{R}^2$, and each being a connected component of some level set of h. Moreover h does not have a critical point in $\partial\mathcal{A} = \Gamma_1 \cup \Gamma_2$.*

Now we can state our main result.

Theorem 2.2. *Suppose that Assumption 2.1 holds and that $T(\Gamma_1) \neq T(\Gamma_2)$. Let $I = I(\mathcal{A}) \subset \mathbb{R}$ be the open interval with end points $T(\Gamma_1), T(\Gamma_2)$. Then for any $T \in I$ and any $a_0 > 0$ there exist $0 < a_1 < b_1 < a_0$ such that system (2.1) has a T-periodic solution satisfying*

$$z_1(t), z_2(t) \in \mathcal{A} \quad \text{for all } t \in \mathbb{R}, \text{ and } \quad |z_1(0) - z_2(0)| \in (a_1, b_1). \tag{2.3}$$

As a consequence we immediately obtain the existence of infinitely many T-periodic solutions.

Corollary 2.3. *Under the assumptions of Theorem 2.2, for every $T \in I$ there exists a sequence $z^{(n)}(t)$ of T-periodic solutions with trajectories in $\mathcal{A}$ and satisfying $z_1^{(n)}(0) - z_2^{(n)}(0) \to 0$ as $n \to \infty$.*

We can also describe the shape of the solutions of Theorem 2.2 in the limit $a_0 \to 0$.

Theorem 2.4. *Let $z^{(n)}(t)$ be a sequence of solutions of (2.1) satisfying $z_1^{(n)}(0), z_2^{(n)}(0) \to C_0 \in \Omega$ and such that the solution of*

$$\dot{C}(t) = -2J\nabla h\big(C(t)\big), \quad C(0) = C_0, \tag{2.4}$$

is non-stationary periodic. Then the following holds.

a) The center of vorticity $C^{(n)}(t) := \frac{1}{2}\left(z_1^{(n)}(t) + z_2^{(n)}(t)\right)$ converges as $n \to \infty$ uniformly in t towards the solution $C(t)$ of (2.4). Setting $\Gamma_0 := \{C(t) : t \in \mathbb{R}\}$ the minimal period of $C^{(n)}(t)$ converges towards $T(\Gamma_0)$ as $n \to \infty$.

b) *Consider the difference* $D^{(n)}(t) := z_1^{(n)}(t) - z_2^{(n)}(t) = \rho^{(n)}(t)(\cos\theta^{(n)}(t), \sin\theta^{(n)}(t))$ *in polar coordinates and set* $d_n = \left|z_1^{(n)}(0) - z_2^{(n)}(0)\right|$. *Then the angular velocity* $\dot\theta^{(n)}$ *satisfies*

$$d_n^2\dot\theta^{(n)}(d_n^2 t) = \frac{1}{\pi} + o(1) \quad \text{as } n\to\infty \qquad \text{uniformly in } t.$$

Remark 2.5. a) This result can be interpreted as follows, using the notation of Theorem 2.4. In the limit $n\to\infty$ the solutions

$$z_1^{(n)}(t) = C^{(n)}(t) + \frac{1}{2}D^{(n)}(t) \quad \text{and} \quad z_2^{(n)}(t) = C^{(n)}(t) - \frac{1}{2}D^{(n)}(t)$$

are superpositions of a slow motion of the center of vorticity along a level line of h with minimal period approaching $T(\Gamma_0)$, and of a fast rotation of the two vortices around their center of vorticity. The angular velocity of the two vortices around their center of vorticity is asymptotic to $\frac{1}{d_n^2\pi}$ as $d_n\to 0$ where d_n is the distance of the initial positions of the two vortices. The rotation number of $z_1^{(n)}(t) - z_2^{(n)}(t)$ in $[0,T]$ is asymptotic to $\frac{T}{2\pi^2 d_n^2}$ and tends to infinity as $d_n\to 0$.

b) Suppose $\mathcal{A} = \bigcup_{c\in(a,b)}\Gamma_c$ is the union of level lines $\Gamma_c = h^{-1}(c)\cap\mathcal{A}$ such that each Γ_c is star-shaped. Suppose moreover that the map $(a,b)\to\mathbb{R}$, $c\mapsto T(\Gamma_c)$, is strictly monotone and that h has no critical points in $\mathcal{A}$, i.e. each Γ_c is a regular level line of h. Then Theorem 2.2 and Theorem 2.4 imply that $\mathcal{A}$ contains infinitely many periodic solutions of (2.1) with *minimal* period $T(\Gamma_c)$, for each $c\in(a,b)$. The corollaries 2.7, 2.8, 2.10 contain several examples for such a situation.

c) If the solution of (2.4) is not periodic then the behavior of $z^{(n)}(t)$ as $n\to\infty$ can be very different from the one described in Theorem 2.4. Of course, if $C_0\in\mathcal{A}$ and if h does not have a critical point in $\mathcal{A}$ then the solution of (2.4) is periodic.

d) Suppose that for some $c_0\in\mathbb{R}$ the level set $h^{-1}(c_0)$ contains a connected component $\Gamma(c_0)\subset h^{-1}(c_0)$ which is strictly star-shaped with respect to some $z_0\in\mathbb{R}^2$, and which does not contain a critical point of h. Then for $c\in[c_0-\delta, c_0+\delta]$ close to c_0 there exists such a component $\Gamma(c)\subset h^{-1}(c)$ close to $\Gamma(c_0)$. Hence assumption 2.1 holds for $\mathcal{A} = \bigcup_{c\in(a,b)}\Gamma(c)$ for any $c_0-\delta\le a<b\le c_0+\delta$. Below we shall produce several examples of this kind.

e) We would like to mention that the theorem can be extended to general symmetric $\mathcal{C}^2$ functions $g:\Omega\times\Omega\to\mathbb{R}$ and $h:\Omega\to\mathbb{R}$, $h(z)=g(z,z)$. The assumption that Ω satisfies the uniform exterior ball condition can also be dropped. We stayed with the explicit setting of vortex dynamics because we use results from [14] that we would otherwise have to reprove in the more general setting. More precisely, we would need a substitute for Proposition 3.1 below. The full strength of this proposition is not necessary, however.

f) It is an interesting problem whether it is possible to weaken or to drop the condition that Γ_1,Γ_2 are strictly star-shaped. We refer the reader to [13,18,23] for results and discussions of this delicate issue in the setting of the Poincaré-Birkhoff fixed point theorem for nonautonomous one degree of freedom Hamiltonian systems. Although star-shapedness is essential for the multidimensional Poincaré-Birkhoff fixed point theorem [16, Theorem 1.2] we believe that it is not essential in our special case; see also [15].

We shall now present several examples where the assumptions of Theorem 2.2 can be verified. Let us begin with the case of a bounded convex domain Ω. Clearly the uniform exterior ball condition is automatically satisfied for convex domains. It is well known that the Robin function $h : \Omega \to \mathbb{R}$ is strictly convex and that it has a unique non-degenerate minimum (see [11]). Moreover $h(z) \to \infty$ as $z \to \partial\Omega$. We may assume without loss of generality that $0 \in \Omega$ and that the minimum of h is at the origin. We set $m := h(0) = \min h$. Obviously the level sets $h^{-1}(c)$ with $c > m$ are connected and strictly star-shaped with respect to the origin. For $c > m$ we may therefore define $T_c = T\left(h^{-1}(c)\right)$ to be the minimal period of the solution of (2.2) with trajectory $h^{-1}(c)$. The following lemma shows that the assumptions of Theorem 2.2 are satisfied for $\mathcal{A} = \mathcal{A}(a,b) = \{z \in \Omega : a \le h(z) \le b\}$, any $m < a < b < \infty$; the boundary of $\mathcal{A}$ consists of the two curves $\Gamma_1 = h^{-1}(a)$ and $\Gamma_2 = h^{-1}(b)$.

Lemma 2.6. *For a bounded convex domain Ω the function $(m,\infty) \to \mathbb{R}$, $c \mapsto T_c$, defined above is strictly decreasing with $T_m := \lim_{c\to m} T_c = \dfrac{\pi}{\sqrt{\det h''(0)}}$ and $T_c \to 0$ as $c \to \infty$.*

The lemma will be proved in Section 5 below. As a consequence of this lemma we can apply Theorem 2.2 in an arbitrary bounded convex domain for any $\mathcal{A} = \mathcal{A}(a,b)$:

Corollary 2.7. *For all $m < a < b < \infty$, for every $T \in (T_b, T_a)$ and for every $a_0 > 0$ there exist $0 < a_1 < b_1 < a_0$ such that system (2.1) has a T-periodic solution satisfying*

$$z_1(t), z_2(t) \in \mathcal{A}(a,b) \quad \text{and} \quad |z_1(0) - z_2(0)| \in (a_1, b_1).$$

There exist infinitely many periodic solutions of (2.1) with minimal period T and with trajectory in $\mathcal{A}(a,b)$.

Now we get back to a general domain Ω. Here we obtain solutions near a non-degenerate local minimum.

Corollary 2.8. *Let z_0 be a non-degenerate local minimum of h and set $m := h(z_0)$, $T_m := \frac{\pi}{\sqrt{\det h''(z_0)}}$. Then for any neighborhood U of z_0 there exists $T(U) < T_m$ such that for any $T(U) < T < T_m$ and for every $a_0 > 0$ there exist $0 < a_1 < b_1 < a_0$ such that system (2.1) has a T-periodic solution satisfying*

$$z_1(t), z_2(t) \in U \quad \text{and} \quad |z_1(0) - z_2(0)| \in (a_1, b_1).$$

There exist infinitely many periodic solutions of (2.1) with minimal period T and with trajectory in U.

Remark 2.9. a) Since the Robin function satisfies $h(z) \to \infty$ as $z \to \partial\Omega$ in a bounded domain there always exists a minimum. It is not difficult to produce examples of domains so that the associated Robin function has many local minima. Moreover, for a generic domain all critical points are non-degenerate; see [24]. Therefore Corollary 2.8 applies to generic domains.

b) Corollary 2.8 in particular yields solutions $z^{(n)}(t)$ approaching the local minimum z_0 of h, i.e. $z_k^{(n)}(t) \to z_0$ as $n \to \infty$, $k = 1,2$. The minimal periods of these solutions converge towards $T_m = \frac{\pi}{\sqrt{\det h''(z_0)}}$. In [4,5,7] the authors also obtained periodic solutions converging towards z_0. More precisely, they produced a family of T_r-periodic solutions

$z^{(r)}(t)$, parameterized over $r \in (0, r_0)$ with $z_k^{(r)}(t) \to z_0$ and $T_r \to 0$ as $r \to 0$. Therefore these solutions are different from those obtained in the present paper. Also the method of proof is very different. In [4,5,7] variational methods or degree methods were used whereas we apply a multidimensional version of the Poincaré-Birkhoff theorem. Therefore here we do not obtain continua of periodic solutions. Instead we obtain infinitely many periodic solutions with prescribed period.

In our last corollary we consider the case when $\partial\Omega$ has a component that is strictly star-shaped.

Corollary 2.10. *Suppose $\partial\Omega$ has a compact component Γ_0 that is of class $\mathcal{C}^2$ and strictly star-shaped with respect to some point $z_0 \in \mathbb{R}^2$. Then for any neighborhood U of Γ_0 there exists $T(U) > 0$ such that for any $T < T(U)$ and for any $a_0 > 0$ there exist $0 < a_1 < b_1 < a_0$ such that system* (2.1) *has a T-periodic solution satisfying*

$$z_1(t), z_2(t) \in U \quad \text{and} \quad |z_1(0) - z_2(0)| \in (a_1, b_1).$$

There exist infinitely many periodic solutions of (2.1) *with minimal period T and with trajectory in U.*

Remark 2.11. In [6] the authors also obtain periodic solutions near the boundary. There Ω has to be bounded and simply connected, hence $\partial\Omega$ consists of just one (connected) curve. On the other hand it is not required that Ω is star-shaped, and the authors could deal with $N \geq 2$ vortices. For $T > 0$ small they obtain T-periodic solutions where the vortices $z_1, \dots, z_N$ all follow the same trajectory $\Gamma = \{z_1(t) : t \in \mathbb{R}\}$ with a time shift: $z_k(t) = z_1(t + \frac{(k-1)T}{N})$. Moreover for $T \to 0$ the trajectory Γ approaches $\partial\Omega$. These solutions are very different from those obtained in Corollary 2.10, however.

3 Proof of Theorem 2.2

We begin with a few known facts about the 2-vortex problem. The following result is a consequence of [14, Theorem 17].

Proposition 3.1. *Consider* (1.1) *for $N = 2$ and suppose that the domain Ω satisfies the uniform exterior ball condition. Then the following hold:*

a) All solutions exist for all times $t \in \mathbb{R}$.

b) There exists a constant C_Ω such that $|z_1(t) - z_2(t)| \leq C_\Omega |z_1(0) - z_2(0)|$ for all solutions and all $t \in \mathbb{R}$.

Remark 3.2. Proposition 3.1 has been proved in [14] for g being the regular part of a hydrodynamic Green's function and h the Robin function. It holds for much more general classes of functions g and associated $h(z) = g(z, z)$. In fact, for our purpose we do not even need the full strength of Proposition 3.1, and we can deal with very general $\mathcal{C}^2$ maps $g : \mathcal{F}_2(\Omega) \to \mathbb{R}$ in H_Ω. We do need that g is symmetric and that $h(z) = g(z, z)$. We leave these generalizations to the interested reader.

For the proof of Theorem 2.2 we may assume that $z_0 = 0$. We may also assume $T(\Gamma_1) < T(\Gamma_2)$. From now on we fix $T \in I = (T(\Gamma_1), T(\Gamma_2))$. The following lemma is an immediate consequence of the assumptions of Theorem 2.2.

Lemma 3.3. *There exists an open annular shaped region $\mathcal{A}' \subset \Omega$ with the following properties.*

(i) $\mathcal{A}'$ is compactly contained in $\mathcal{A}$: $\overline{\mathcal{A}'} \subset \mathcal{A}$.

(ii) The boundary of $\mathcal{A}'$ consists of two closed curves Γ_1', Γ_2' that are strictly star-shaped with respect to $z_0 = 0$, and that are components of level sets of h. Moreover, h does not have a critical point in $\partial\mathcal{A}' = \Gamma_1' \cup \Gamma_2'$.

(iii) $T(\Gamma_1') < T(\Gamma_2')$ where $T(\Gamma_k')$ denotes the minimal period of the solution of (2.2) *with trajectory Γ_k'. Moreover $T \in (T(\Gamma_1'), T(\Gamma_2'))$.*

We apply the canonical transformation $A = \frac{1}{\sqrt{2}}\begin{pmatrix} E_2 & -E_2 \\ E_2 & E_2 \end{pmatrix} \in \mathbb{R}^{4\times 4}$ where E_2 is the 2×2 identity matrix:

$$\begin{cases} w_1 = \dfrac{1}{\sqrt{2}}(z_1 - z_2) \\ w_2 = \dfrac{1}{\sqrt{2}}(z_1 + z_2) \end{cases}$$

with inverse transformation given by

$$\begin{cases} z_1 = \dfrac{1}{\sqrt{2}}(w_1 + w_2) \\ z_2 = \dfrac{1}{\sqrt{2}}(-w_1 + w_2). \end{cases}$$

The system (2.1) transforms to

$$\dot{w}_k = J\nabla_{w_k} H_1(w_1, w_2) \quad \text{for } k = 1, 2, \tag{3.1}$$

with Hamiltonian

$$\begin{aligned} H_1(w_1, w_2) = &-\frac{1}{2\pi}\log|w_1| - 2g\left(\frac{1}{\sqrt{2}}(w_1 + w_2), \frac{1}{\sqrt{2}}(-w_1 + w_2)\right) \\ &- h\left(\frac{1}{\sqrt{2}}(w_1 + w_2)\right) - h\left(\frac{1}{\sqrt{2}}(-w_1 + w_2)\right) \end{aligned}$$

defined on $A\mathcal{F}_2\Omega = A(\mathcal{F}_2\Omega)$. Note that $w_2 \in \sqrt{2}\Omega$ provided $|z_1 - z_2| < \operatorname{dist}(z_2, \partial\Omega)$, and that given a compact subset $K \subset \sqrt{2}\Omega$ there exists $\delta > 0$ so that $\big(B_\delta(0) \setminus \{0\}\big) \times K \subset A\mathcal{F}_2\Omega$. Here $B_\delta(0)$ denotes the closed disk around 0 with radius δ.

Lemma 3.4. *The gradient of H_1 with respect to w_2 satisfies*

$$\nabla_{w_2} H_1(w) = -2\sqrt{2}\nabla h(w_2/\sqrt{2}) + Q(w),$$

with $Q(w) = o(|w_1|)$ as $w_1 \to 0$ uniformly for w_2 in compact subsets of $\sqrt{2}\Omega$.

Proof. Setting $z := A^{-1}w$ we obtain

$$\nabla_{w_2} H_1(w) = -\frac{1}{\sqrt{2}}\big(2\nabla_{z_1} g(z) + 2\nabla_{z_2} g(z) + \nabla h(z_1) + \nabla h(z_2)\big).$$

The Taylor expansion for h near $w_2/\sqrt{2}$ yields

$$\nabla h(z_1) = \nabla h\big(w_2/\sqrt{2}\big) + \frac{1}{\sqrt{2}} h''\big(w_2/\sqrt{2}\big)[w_1] + o(|w_1|) \quad \text{as } w_1 \to 0,$$

and

$$\nabla h(z_2) = \nabla h\big(w_2/\sqrt{2}\big) + \frac{1}{\sqrt{2}} h''\big(w_2/\sqrt{2}\big)[-w_1] + o(|w_1|) \quad \text{as } w_1 \to 0.$$

This implies

$$\nabla h(z_1) + \nabla h(z_2) = 2\nabla h\big(w_2/\sqrt{2}\big) + o(|w_1|) \quad \text{as } w_1 \to 0.$$

Using the symmetry of $g(z_1, z_2)$ and $h(z) = g(z, z)$ we obtain analogously

$$\nabla_{z_1} g(z) + \nabla_{z_2} g(z) = \nabla h(w_2/\sqrt{2}) + o(|w_1|) \quad \text{as } w_1 \to 0.$$

This yields $Q(w) = o(|w_1|)$ as $w_1 \to 0$, and since all functions are of class $\mathcal{C}^2$ the convergence is uniform for w_2 in a compact subset of $\sqrt{2}\Omega$. □

Now let $W(t, w) \in A\mathcal{F}_2\Omega$ be the solution of the initial value problem for (3.1) with initial condition $W(0, w) = w$. Recall that it is defined for all $t \in \mathbb{R}$ by Proposition 3.1. The following lemma concerns $W_2(t, w)$ as $w_1 \to 0$. We use the notation

$$\mathcal{A}_2 = \sqrt{2}\mathcal{A} \qquad \text{and} \qquad \mathcal{A}'_2 = \sqrt{2}\mathcal{A}'.$$

Lemma 3.5. *The solution $W_2(t, w)$ converges towards $Z(t, w_2)$ as $w_1 \to 0$ uniformly in $t \in [0, T]$, $w_2 \in \mathcal{A}'_2$. The function $Z(t, w_2)$ solves the initial value problem*

$$\dot{Z}(t, w_2) = -2\sqrt{2}J\nabla h(Z(t, w)/\sqrt{2}), \quad Z(0, w_2) = w_2. \tag{3.2}$$

Proof. Set $\varepsilon := \frac{1}{2}\operatorname{dist}(\mathcal{A}'_2, \partial\mathcal{A}_2)$, choose $\delta_0 > 0$ such that $\big(B_{\delta_0}(0) \setminus \{0\}\big) \times \overline{\mathcal{A}_2} \subset A\mathcal{F}_2\Omega$ and set

$$C := \sup_{\substack{0<|w_1|\le\delta \\ w_2\in\overline{\mathcal{A}_2}}} |\nabla_{w_2} H_1(w_1, w_2)|\,.$$

Note that $C < \infty$ because $\nabla_{w_2} H_1$ is defined also for $|w_1| = 0$. Let $\mathcal{U}_\varepsilon(\mathcal{A}'_2) = \{w \in \mathcal{A}_2 : \operatorname{dist}(w, \mathcal{A}'_2) < \varepsilon\}$ be the ε-neighborhood of $\mathcal{A}'_2$. Clearly, if $W_2(t, w) \in \partial\mathcal{A}_2$ for some $t > 0$, $0 < |w_1| \le \frac{\delta_0}{C_\Omega}$ and $w_2 \in \overline{\mathcal{U}_\varepsilon(\mathcal{A}'_2)}$, then $t \ge \frac{\varepsilon}{C} =: t_0$.

Step 1: If $w_1^{(n)} \to 0$ and $w_2^{(n)} \in \mathcal{U}_\varepsilon(\mathcal{A}'_2)$ with $w_2^{(n)} \to w_2$, $w_2 \in \overline{\mathcal{U}_\varepsilon(\mathcal{A}'_2)}$, then $W_2(t, w^{(n)}) \to Z(t, w_2)$, uniformly for $t \in [0, t_0]$.

In fact, using the equation for w_2 in integral form we have for $t \in [0, t_0]$:

$$\begin{aligned}
&\big|W_2(t, w^{(n)}) - W_2(t, w^{(m)})\big| \\
&\qquad \le \big|w_2^{(n)} - w_2^{(m)}\big| + \int_0^t \big|\nabla_{w_2} H_1(W(s, w^{(n)})) - \nabla_{w_2} H_1(W(s, w^{(m)}))\big| ds.
\end{aligned}$$

Note that $\Big\{W(t,w) : t \in [0,t_0],\ w \in \big(B_{\delta_0/C_\Omega}(0) \setminus \{0\}\big) \times \overline{\mathcal{U}_\varepsilon(\mathcal{A}_2')}\Big\} \subset A\mathcal{F}_2\Omega$ is a relatively compact subset in $\Omega \times \Omega$. Since $\nabla_{w_2} H_1$ is defined on $\Omega \times \Omega$ and is Lipschitz continuous on compact sets there exists $k > 0$ such that

$$
\begin{aligned}
&\big|W_2(t;w^{(n)}) - W_2(t;w^{(m)})\big| \\
&\quad \le \big|w_2^{(n)} - w_2^{(m)}\big| + k\int_0^t \big|W_1(s,w^{(n)}) - W_1(s,w^{(m)})\big| + \big|W_2(s,w^{(n)}) - W_2(s,w^{(m)})\big| ds \\
&\quad \le \big|w_2^{(n)} - w_2^{(m)}\big| + kC_\Omega t_0 \left(\big|w_1^{(n)}\big| + \big|w_1^{(m)}\big|\right) + k\int_0^t \big|W_2(s,w^{(n)}) - W_2(s,w^{(m)})\big| ds.
\end{aligned}
$$

Now Gronwall's Lemma yields for $t \in [0,t_0]$:

$$
|W_2(t,w^{(n)}) - W_2(t,w^{(m)})| \le \left(\big|w_2^{(n)} - w_2^{(m)}\big| + kC_\Omega t_0 \left(\big|w_1^{(n)}\big| + \big|w_1^{(m)}\big|\right)\right) e^{kt_0}.
$$

This implies that $W_2(t,w^{(n)})$ converges as $n \to \infty$ uniformly for $t \in [0,t_0]$. The limit $Z(t,w_2)$ satisfies the equation (3.2) because

$$
\nabla_{w_2} H_1(W(t,w^{(n)})) \to -2\sqrt{2}\nabla h(Z(t,w_2)/\sqrt{2}) \quad \text{as } n \to \infty;
$$

see Lemma 3.4. This proves Step 1.

Step 2: There exists δ_1 with $0 < \delta_1 \le \frac{\delta_0}{C_\Omega}$ such that if $0 < |w_1| \le \delta_1$ and $w_2 \in \mathcal{A}_2'$ then $W_2(t,w) \in \mathcal{A}_2$, for all $t \in [0,T]$.

Arguing by contradiction, suppose there exist $w_1^{(n)} \to 0$, $w_2^{(n)} \to w_2 \in \overline{\mathcal{A}_2'}$ and $t_n \in [0,T]$ such that $W_2(t_n,w^{(n)}) \in \partial\mathcal{A}_2$. It is immediate to see that $t_n \ge t_0$ for all n. Moreover, by Step 1, $W_2(t,w^{(n)}) \to Z(t,w_2)$ as $n \to \infty$ uniformly on $[0,t_0]$. Then there exists n_1 such that for all $n \ge n_1$ we have $W_2(t_0,w^{(n)}) \in \mathcal{U}_\varepsilon(\mathcal{A}_2')$. This implies that $t_n \ge 2t_0$ for all $n \ge n_1$. So we can apply again Step 1 and obtain that $W_2(t,w^{(n)}) \to Z(t,w_2)$ uniformly on $[0,2t_0]$. Proceeding as before, we can find $n_2 \ge n_1$ such that for all $n \ge n_2$ we have $W_2(2t_0,w^{(n)}) \in \mathcal{U}_\varepsilon(\mathcal{A}_2')$. By induction the procedure continues until we obtain in a finite number of steps that $W_2(t,w^{(n)}) \to Z(t,w_2)$ uniformly on $[0,T]$, which gives the contradiction and proves Step 2.

In order to complete the proof, one argues as in Step 1 using that

$$
\{W(t,w) : t \in [0,T], 0 < |w_1| \le \delta_1, w_2 \in \mathcal{A}_2'\} \subset A\mathcal{F}_2\Omega
$$

is a relatively compact subset of $\Omega \times \Omega$ as a consequence of Step 2. ☐

Corollary 3.6. *There exists* $0 < \delta_1 \le \delta_0$ *such that* $W_2(t,w) \in \mathcal{A}_2 = \sqrt{2}\mathcal{A}$ *for all* $t \in [0,T]$*, provided* $0 < |w_1| \le \delta_1$*,* $w_2 \in \mathcal{A}_2' = \sqrt{2}\mathcal{A}'$*.*

Corollary 3.6 and Proposition 3.1 imply that the first statement in (2.3) of Theorem 2.2 is a consequence of the second provided b_1 is small and provided the initial conditions $z_1(0), z_2(0)$ lie in $\mathcal{A}'$.

Clearly $\mathcal{A}_2' = \sqrt{2}\mathcal{A}'$ is bounded by the strictly star-shaped curves $\sqrt{2}\Gamma_k'$, $k = 1,2$. Now we let δ_1 be as in Corollary 3.6. For $0 < a_1 < b_1$ we define the annulus

$$
\mathcal{A}_1(a_1,b_1) := \{w_1 \in \mathbb{R}^2 : a_1 < |w_1| < b_1\}.
$$

We want to find $0 < a_1 < b_1 < \min\{a_0, \delta_1\}$ and a T-periodic orbit of the map $W(t, w)$ with $w \in \mathcal{A}_1(a_1, b_1) \times \mathcal{A}_2'$.

Observe that $W_1(t, w) \neq 0$ for any $w \in A\mathcal{F}_2\Omega$ and any $t \in \mathbb{R}$ by Proposition 3.1. Therefore there exists a continuous choice of the argument of $W_1(t, w)$ and we may define the rotation number

$$\mathrm{Rot}(W_1(t, w); [0, T]) := \frac{1}{2\pi}\big(\arg(W_1(T, w)) - \arg(w_1)\big) \in \mathbb{R}.$$

Moreover, Corollary 3.6 implies that $W_2(t, w) \neq 0$ for $w \in \mathcal{A}_1(a_1, b_1) \times \mathcal{A}_2'$ and $t \in [0, T]$ provided $0 < a_1 < b_1 < \delta_1$. Thus we may also define the rotation number

$$\mathrm{Rot}(W_2(t, w); [0, T]) := \frac{1}{2\pi}\big(\arg(W_2(T, w)) - \arg(w_2)\big) \in \mathbb{R}.$$

In the next section we shall prove the following result.

Proposition 3.7. *For every $a_0 > 0$ there exist $0 < a_1 < b_1 < \min\{a_0, \delta_1\}$ and $\nu \in \mathbb{Z}$ such that the following holds for $w \in \mathcal{A}_1(a_1, b_1) \times \mathcal{A}_2'$.*

a) $\mathrm{Rot}(W_1(t, w); [0, T]) \begin{cases} > \nu, & \textit{if } |w_1| = a_1 \\ < \nu, & \textit{if } |w_1| = b_1. \end{cases}$

b) $\mathrm{Rot}(W_2(t, w); [0, T]) \begin{cases} > 1, & \textit{if } w_2/\sqrt{2} \in \Gamma_1' \\ < 1, & \textit{if } w_2/\sqrt{2} \in \Gamma_2'. \end{cases}$

Thus for any $w_2 \in \mathcal{A}_2'$ the rotation number of $W_1(t, w)$ in the interval $[0, T]$ changes from bigger than ν to less than ν as w_1 passes from the inner boundary of $\mathcal{A}_1(a_1, b_1)$ to the outer boundary of $\mathcal{A}_1(a_1, b_1)$. Similarly, for any $w_1 \in \mathcal{A}_1(a_1, b_1)$ the rotation number of $W_2(t, w)$ in the interval $[0, T]$ changes from bigger than 1 to less than 1 as w_2 passes from the boundary curve $\sqrt{2}\Gamma_1'$ of $\mathcal{A}_2'$ to the boundary curve $\sqrt{2}\Gamma_2'$ of $\mathcal{A}_2'$.

This is precisely the setting of the generalized Poincaré-Birkhoff theorem [16, Theorem 1.2]. As a consequence we deduce that the Hamiltonian system (3.1) has a T-periodic solution with initial conditions $w \in \mathcal{A}_1(a_1, b_1) \times \mathcal{A}_2'$. For the proof of Theorem 2.2 it therefore remains to prove Proposition 3.7.

4 Proof of Proposition 3.7

It will be useful to introduce polar coordinates for W_1, W_2. Recall that any solution of (3.1) with initial condition $w \in \mathcal{A}_1(a_1, b_1) \times \mathcal{A}_2'$ satisfies $W_k(t, w) \neq 0$ for $t \in [0, T]$, $k = 1, 2$. We set

$$e(\theta) = (\cos\theta, \sin\theta) \tag{4.1}$$

and fix initial conditions $w_1 = \rho_1 e(\theta_1)$, $w_2 = \rho_2 e(\theta_2)$. Then setting $\rho = (\rho_1, \rho_2)$ and $\theta = (\theta_1, \theta_2)$ we define $R_k(t, \rho, \theta) = \big|W_k(t, \rho_1 e(\theta_1), \rho_2 e(\theta_2))\big|$ and let $\Theta_k(t, \rho, \theta)$ be a continuous choice of the argument of $W_k(t, \rho_1 e(\theta_1), \rho_2 e(\theta_2))$. Thus we can write $W_k(t, w) =$

$R_k(t,\rho,\theta)e(\Theta_k(t,\rho,\theta))$ for $k=1,2$. We will also write $R(t,\rho,\theta)=(R_1,R_2)(t,\rho,\theta)$ and $\Theta(t,\rho,\theta)=(\Theta_1,\Theta_2)(t,\rho,\theta)$.

Next we describe the radial component of the boundary curves of $\mathcal{A}_2'$ as a function of the angle, obtaining functions $r_k:\mathbb{R}\to(0,\infty)$ defined by the equation $h\left(r_k(\theta)e(\theta)\right)\in\sqrt{2}\Gamma_k'$. Since Γ_k' is strictly star-shaped with respect to the origin, r_k is well defined. Clearly r_k is 2π-periodic and there holds

$$\sqrt{2}\Gamma_k'=\{r_k(\theta)e(\theta):\theta\in\mathbb{R}\}.$$

We also set

$$\mathcal{A}_2^{pol}:=\{(\rho_2,\theta_2)\in\mathbb{R}^+\times\mathbb{R}:\rho_2e(\theta_2)\in\mathcal{A}_2'\}.$$

Proposition 3.7 is now equivalent to the following result.

Proposition 4.1. *For every $a_0>0$ there exist $0<a_1<b_1<a_0$ and $\nu\in\mathbb{Z}$ such that the following holds for $w\in\mathcal{A}_1(a_1,b_1)\times\mathcal{A}_2'$.*

$$a)\ \Theta_1(T,\rho_1,\rho_2,\theta_1,\theta_2)-\theta_1\begin{cases}>2\pi\nu, & \text{if } \rho_1=a_1,\ (\rho_2,\theta_2)\in\mathcal{A}_2^{pol},\\ <2\pi\nu, & \text{if } \rho_1=b_1,\ (\rho_2,\theta_2)\in\mathcal{A}_2^{pol}.\end{cases}$$

$$b)\ \Theta_2(T,\rho_1,\rho_2,\theta_1,\theta_2)-\theta_2\begin{cases}>2\pi, & \text{if } \rho_1\in[a_1,b_1],\ \rho_2=r_1(\theta_2),\\ <2\pi, & \text{if } \rho_1\in[a_1,b_1],\ \rho_2=r_2(\theta_2).\end{cases}$$

Proof. We begin with the proof of part b) because this determines the choice of b_1 which will then be used in the proof of part a) where we choose a_1. For $\rho_2=r_1(\theta_2)$, that is

$$w_2=\rho_2e(\theta_2)\in\sqrt{2}\Gamma_1'\subset\partial\mathcal{A}_2'=\sqrt{2}\partial\mathcal{A}',$$

the solution $Z(t,w_2)$ of the initial value problem (3.2) has the period $T(\Gamma_1')$. Now Corollary 3.6 implies that $W_2(T,w)\to Z(T,w_2)$ as $w_1\to0$. Since $T(\Gamma_1')<T$ the argument Θ_2 of W_2 satisfies

$$\Theta_2(T,\rho_1,\rho_2,\theta_1,\theta_2)-\theta_2>2\pi \tag{4.2}$$

for $\rho_1=|w_1|$ small. Similarly, for $\rho_2=r_2(\theta_2)$, that is

$$w_2=\rho_2e(\theta_2)\in\sqrt{2}\Gamma_2'\subset\partial\mathcal{A}_2'=\sqrt{2}\partial\mathcal{A}',$$

the solution $Z(t,w_2)$ of the initial value problem (3.2) has the period $T(\Gamma_2')>T$, so $W_2(T,w)\to Z(T,w_2)$ as $w_1\to0$ implies

$$\Theta_2(T,\rho_1,\rho_2,\theta_1,\theta_2)-\theta_2<2\pi \tag{4.3}$$

for $\rho_1=|w_1|$ small. Part b) follows provided we choose b_1 so small that (4.2) and (4.3) hold for $\rho_1=|w_1|<b_1$.

Now we can prove part a). The proof of this part is similar to the proof of the main result in [10]. With b_1 determined above we choose $\nu\in\mathbb{Z}$ satisfying

$$2\pi\nu>\max\left\{\Theta_1(T;b_1,\rho_2,\theta_1,\theta_2)-\theta_1:\theta_1\in[0,2\pi],\ (\rho_2,\theta_2)\in\overline{\mathcal{A}_2^{pol}}\right\}. \tag{4.4}$$

Setting

$$z_1(R,\Theta) = \frac{R_1}{\sqrt{2}}e(\Theta_1) + \frac{R_2}{\sqrt{2}}e(\Theta_2) \quad \text{and} \quad z_2(R,\Theta) = -\frac{R_1}{\sqrt{2}}e(\Theta_1) + \frac{R_2}{\sqrt{2}}e(\Theta_2)$$

and

$$k(R,\Theta) = 2\left(\nabla_{z_1} - \nabla_{z_2}\right) g(z_1(R,\Theta), z_2(R,\Theta)) + \nabla h(z_1(R,\Theta)) - \nabla h(z_2(R,\Theta)),$$

the equations for R_1, Θ_1 are given by

$$\begin{cases} \dot{R}_1 = \dfrac{1}{\sqrt{2}} \langle -Jk(R,\Theta), e(\Theta_1)\rangle \\ \dot{\Theta}_1 = \dfrac{1}{2\pi R_1^2} + \dfrac{1}{\sqrt{2}R_1} \langle k(R,\Theta), e(\Theta_1)\rangle =: f(R_1, R_2, \Theta_1, \Theta_2). \end{cases} \tag{4.5}$$

Observe that

$$\lim_{R_1 \to 0} f(R_1, R_2, \Theta_1, \Theta_2) = +\infty$$

because

$$\lim_{R_1\to 0} \frac{1}{\sqrt{2}R_1} \langle k(R,\Theta), e(\Theta_1)\rangle = \left\langle D^2 h\left(\frac{R_2}{\sqrt{2}}e(\Theta_2)\right) e(\Theta_1), e(\Theta_1)\right\rangle.$$

Thus we can choose $0 < \tilde{a}_1 < b_1$ such that

$$f(R,\Theta) > \frac{2\pi\nu}{T} \quad \text{for every } 0 < R_1 \le \tilde{a}_1,\ \Theta_1 \in \mathbb{R},\ (R_2,\Theta_2) \in \overline{\mathcal{A}_2^{pol}}. \tag{4.6}$$

Then, by Proposition 3.1, there exists $0 < a_1 < \tilde{a}_1$ such that

$$R_1(t; a_1, \rho_2, \theta_1, \theta_2) \le \tilde{a}_1 \quad \text{for every } t \in [0,T],\ \theta_1 \in \mathbb{R},\ (\rho_2, \theta_2) \in \overline{\mathcal{A}_2^{pol}}.$$

Now integrating (4.6) on $[0,T]$ gives

$$\Theta_1(T; a_1, \rho_2, \theta_1, \theta_2) - \theta_1 = \int_0^T f(R(t, a_1, \rho_2, \theta_1, \theta_2), \Theta(t, a_1, \rho_2, \theta_1, \theta_2))dt > 2\pi\nu \tag{4.7}$$

for all $\theta_1 \in \mathbb{R}$, all $(\rho_2, \theta_2) \in \mathcal{A}_2^{pol}$. Now (4.4) and (4.7) imply a). $\square$

5 Proof of the remaining results

Proof of Theorem 2.4. Consider solutions $z^{(n)}(t)$ with $z_1^{(n)}(0), z_2^{(n)}(0) \to C_0 \in \Omega$ and such that the solution of (2.4) is non-stationary periodic. It follows from Proposition 3.1 that

$$w_1^{(n)}(t) = \frac{1}{\sqrt{2}}\left(z_1^{(n)}(t) - z_2^{(n)}(t)\right) \to 0 \quad \text{as } n \to \infty \text{ uniformly in } t \in \mathbb{R}.$$

Lemma 3.5 now implies that

$$w_2^{(n)}(t) = \frac{1}{\sqrt{2}}\left(z_1^{(n)}(t) + z_2^{(n)}(t)\right) \to Z(t, \sqrt{2}C_0) \quad \text{as } n \to \infty \text{ uniformly in } t \in \mathbb{R}$$

where $Z(t, \sqrt{2}C_0)$ solves the initial value problem (3.2) with initial condition $w_2 = \sqrt{2}C_0$. This is equivalent to part a) from Theorem 2.4 because the centers of vorticity satisfies $C^{(n)}(t) = \frac{1}{\sqrt{2}} w_2^{(n)}(t)$ and $C(t) = \frac{1}{\sqrt{2}} Z(t)$.

For the proof of part b) we define

$$u_n(s) := \frac{1}{d_n} D^{(n)}(d_n^2 s) = \rho^{(n)}\big(e(\theta^{(n)}(d_n^2 s)\big)$$

where $d_n = \big|z_1^{(n)}(0) - z_2^{(n)}(0)\big|$ and $e(\theta)$ is as in (4.1). Then u_n satisfies

$$\dot{u}_n = -\frac{1}{\pi} J \frac{u_n}{|u_n|^2} - o(1) \qquad \text{as } n \to \infty, \text{ uniformly in } [0, T].$$

Note that $|u_n(0)| = 1$ for all n, so up to a subsequence $u_n(0) \to \bar{u}$ with $|\bar{u}| = 1$. By a straightforward calculation we obtain that $\frac{d}{ds}|u_n(s)|^2 = o(1)$ as $n \to \infty$, uniformly in $[0, T]$. Thus there exists $\varepsilon > 0$ such that for n sufficiently large we have $|u_n(s)| \geq \varepsilon$ uniformly for $s \in [0, T]$. Next let u_∞ be the solution of the initial value problem

$$\begin{cases} \dot{u}_\infty = -\frac{1}{\pi} J \frac{u_\infty}{|u_\infty|^2} \\ u_\infty(0) = \bar{u}. \end{cases}$$

We now deduce easily that $u_n \to u_\infty$ uniformly on $[0, T]$. Note that $\frac{d}{ds} arg(u_\infty(s)) = \frac{1}{\pi}$, which implies $d_n^2 \dot{\theta}^{(n)}(d_n^2 s) \to \frac{1}{\pi}$. □

Proof of Lemma 2.6. First we transform the equation (2.2) using the canonical coordinate change $(\rho, \theta) \mapsto \sqrt{2\rho}e(\theta)$. Setting $h_1(\rho, \theta) = h(\sqrt{2\rho}e(\theta))$ this leads to the system

$$\begin{cases} \dot{\rho} = -\frac{\partial}{\partial \theta} h_1(\rho, \theta) \\ \dot{\theta} = \frac{\partial}{\partial \rho} h_1(\rho, \theta). \end{cases}$$

For any fixed θ the function $\rho \mapsto \frac{\partial}{\partial \rho} h_1(\rho, \theta)$ is strictly increasing because h is strictly convex by [11]. This means that the angular velocity in any fixed radial direction is strictly increasing with respect to the radius, hence T_c is strictly decreasing. Moreover, $T_c \to 0$ as $c \to \infty$ is a consequence of $|\nabla h(z)| \to \infty$ as $z \to \partial\Omega$. Finally, since the origin is a nondegenerate critical point of h the Taylor expansion $\nabla h(z) = h''(0)[z] + o(|z|)$ at 0 implies that

$$T_c \to T_m := \frac{\pi}{\sqrt{\det h''(0)}} \qquad \text{as } c \to m$$

because T_m is the minimal period of the nontrivial solutions of $\dot{z} = 2Jh''(0)[z]$. □

Proof of 2.7. The corollary is an immediate consequence of Lemma 2.6, Theorem 2.2 and Remark 2.5 b). □

Proof of 2.8. Since $h''(z_0)$ is positive definite the Robin function is strictly convex in a neighborhood U of z_0. Therefore the level lines $h^{-1}(c) \cap U$ for $c > c_0 = h(z_0)$ close to c_0 are convex. As in the proof of Lemma 2.6 the period T_c of the solution of (2.2) with trajectory $h^{-1}(c) \cap U$ is strictly decreasing in c. The corollary follows now from Theorem 2.2 and Remark 2.5 b). $\square$

Proof of 2.10. Let $U \subset \mathbb{R}^2$ be a tubular neighborhood of Γ_0 and $p : U \to \Gamma_0$ be the orthogonal projection. Moreover let $\nu : \Gamma_0 \to \mathbb{R}^2$ be the exterior normal. It is well known that

$$\nabla h(z) = \frac{\nu(p(z))}{2\pi d(z, \Gamma_0)} + O(1) \quad \text{as } d(z, \Gamma_0) = \operatorname{dist}(z, \Gamma_0) \to 0; \tag{5.1}$$

see [3]. Therefore the level lines $h^{-1}(c) \cap U$ for $c > c_0$ are also strictly star-shaped with respect to z_0, if c_0 is large enough. Moreover the period T_c of the solution of (2.2) with trajectory $h^{-1}(c) \cap U$ is strictly decreasing in c due to (5.1). Consequently the corollary follows from Theorem 2.2 and Remark 2.5 b). $\square$

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