Nodal blow-up solutions to slightly subcritical elliptic problems with Hardy-critical term

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Dedicated to the memory of Abbas Bahri.

Abstract The paper is concerned with the slightly subcritical elliptic problem with Hardy-critical term

$$\begin{cases} -\Delta u - \mu \dfrac{u}{|x|^2} = |u|^{2^*-2-\varepsilon}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$ with $0 \in \Omega$, in dimensions $N \geq 7$. We investigate the possible blow-up behavior of solutions as $\mu, \varepsilon \to 0$. In particular, we prove the existence of nodal solutions that blow up positively at the origin and negatively at a different point as $\mu, \varepsilon \to 0^+$. The location of the negative blow-up point is determined by the geometry of Ω. Moreover, the asymptotic shape of the solutions is described in detail. An interesting new consequence of our results is that the type of blow-up solutions considered here exists for $\mu = O(\varepsilon^\alpha)$ with $\alpha > \frac{N-4}{N-2}$. The bound $\frac{N-4}{N-2}$ is sharp.

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Key words: Hardy term; critical exponent; slightly subcritical problems; Hardy term; nodal solutions; blow-up solutions; multi-bubble solutions; singular perturbation methods.

1 Introduction

The paper is concerned with the semilinear singular problem

$$\begin{cases} -\Delta u - \mu \dfrac{u}{|x|^2} = |u|^{2^*-2-\varepsilon}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 7$, is a smooth bounded domain with $0 \in \Omega$; $2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent. Using variational methods Ghoussoub and Yuan [23, Theorem 1.2] proved that this problem has infinitely many solutions provided $0 < \varepsilon < 2^* - 2$ and $0 < \mu < \overline{\mu}$ where $\overline{\mu}$ is the best constant in the Hardy inequality, i.e.

$$\overline{\mu} = \inf_{0 \neq u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2/|x|^2}.$$

The main goal of this paper and subsequent work is to understand the possible blow-up behavior of solutions as $\mu, \varepsilon \to 0$. In particular we construct nodal (i.e. sign changing) solutions that blow-up as $\mu, \varepsilon \to 0$ at precisely two points: one is the origin, the second blow-up point is away from the origin and determined by the geometry of the domain. The shape of the solutions will be described in detail. We use singular perturbation techniques and a Lyapunov-Schmidt type reduction. An interesting and new feature of our type of solutions is that their existence depends on the relative speed of $\varepsilon \to 0$ and $\mu \to 0$. This will be made precise below.

It is well known that the blow-up of solutions near singular parameter values is closely related to Bahri's [2] theory of critical points at infinity. The existence of a positive solution in domains with nontrivial homology and when $\mu = \varepsilon = 0$ has been shown in the seminal work [3] of Bahri and Coron. Around the same time the blow-up phenomenon for positive and for nodal solutions to problem (1.1) has been studied extensively in the case $\mu = 0$, $\varepsilon \to 0$. It was proved in [10, 22, 26, 32, 33] that as $\varepsilon \to 0^+$, the positive solution u_ε blows up and concentrates at a critical point of the Robin's function of Ω. In [4, 34], the existence of positive solutions with multiple bubbles was considered. In convex domains a positive solution cannot have multiple bubbles, see [24]. The existence of nodal solutions with k bubbles at k different points was proved in [6] in the case $k = 2$, in [7] in the case $k = 4$ when Ω is convex and satisfies a certain symmetry, and in [8] in the case $k = 3$ when Ω is a ball. Bubble tower solutions, i.e. solutions with multiple bubbles concentrating at the same point, were obtained in [29, 31], based on an idea from [16]. All these papers only treat the regular case $\mu = 0$.

When $\mu \neq 0$, the Hardy-critical potential $\frac{1}{|x|^2}$ cannot be regarded as a lower order perturbation because it has the same homogeneity as the Laplace operator and because it does not belong to the Kato class. This makes the analysis much more complicated compared with the case $\mu = 0$. For the existence of positive and nodal solutions for the problem with Hardy type potentials and possibly critical Sobolev exponent we refer the reader to [11, 12, 18, 21, 23, 25, 27, 35, 36, 38] and the references therein. However, concerning blow-up solutions to the problem involving Hardy type potentials very few results are known. We are only aware of the papers [19, 20], dealing with the problem

$$\begin{cases} -\Delta u - \dfrac{\mu}{|x|^2} u = k(x) u^{2^*-1}, \\ u \in D^{1,2}(\mathbb{R}^N), \quad u > 0 \text{ in } \mathbb{R}^N \setminus \{0\}; \end{cases} \tag{1.2}$$

here $D^{1,2}(\mathbb{R}^N) := \{u \in L^{2^*}(\mathbb{R}^N) |\ |\nabla u| \in L^2(\mathbb{R}^N)\}$. In [19] the existence of a positive solution to (1.2) blowing up at a critical point of $k(x)$ was obtained as $\mu \to 0^+$. In [20] the existence of positive bubble tower solutions to (1.2), blowing up at the origin, was proved for $k(x) = 1 + \varepsilon K(x)$ as $\varepsilon \to 0$; here $K(x)$ is a continuous bounded function. These solutions, called fountain-like in [20], are superpositions of positive bubbles. On the other hand, for fixed $0 < \mu < \overline{\mu} = \frac{(N-2)^2}{4}$, Musso and Wei [30] considered (1.2) when $k(x) = 1$ and proved the existence of entire sign changing solutions by gluing together a large number of positive and negative bubbles distributed along the vertices of a regular polygon. In [13] Cao and Peng investigated the asymptotic behavior of positive solutions to (1.1) in a ball.

In this paper we investigate the existence of nodal solutions to problem (1.1) that blow up, as $\mu, \varepsilon \to 0$, positively at the origin and negatively at a point staying away from the origin. Compared with [19, 20] the location of the bow-up points does not depend on the shape of a coefficient function $k(x)$ but on the more subtle influence of the geometry of the domain. Since we look for solutions of a special form we make a corresponding ansatz and derive a reduced finite-dimensional variational problem via a Lyapunov-Schmidt reduction scheme. The reduced functional depends on k different points in $\Omega \setminus \{0\}$, the blow-up points, and on $k+1$ real parameters. We shall then prove the existence of a critical point of the reduced functional in the case $k=1$. The case of $k>1$ bubbles outside the origin will be treated in subsequent work. Though these singular perturbation techniques have of course been used in a variety of other problems we would like to emphasize that there are not only major technical difficulties due to the Hardy-critical term. We also discover a new phenomenon concerning the existence of solutions of the special shape we are looking for. Namely, as mentioned above this depends on the speed $\mu \to 0$ and $\varepsilon \to 0$. More precisely, fixing $\mu_0 > 0$ and setting $\mu = \mu_0 \varepsilon^\alpha$ the solutions exist for $\alpha > \frac{N-4}{N-2}$, and do not exist for $0 < \alpha \le \frac{N-4}{N-2}$. Thus μ has to converge to 0 like a power of ε (or faster), and we have a precise threshold value for this power.

The paper is organized as follows. In Section 2, we state and discuss our main theorems. Then in Section 3 we collect some notations and preliminary results, in particular concerning the limit problem when $\varepsilon = 0$ on $\mathbb{R}^N$. In Section 4 we perform the finite-dimensional reduction. This will be done for multiple blow-up points (not just two). Section 5 is devoted to the proof of the main theorems, that is, the existence of nodal solutions with two bubbles blowing up at two different points. Finally, various useful technical lemmas are collected in the appendix. Section 4 and Proposition 5.1 together with the computations in the appendix form the core of the paper. We point out that the reduction results in Section 4 and the lemmas in the appendix will also be used in future work on solutions with more than two blow-up points and solutions with bubble towers.

Throughout this paper $\Omega \subset \mathbb{R}^N$, $N \ge 7$, is a smooth bounded domain. The results can be extended to the case $N = 6$, but this requires a separate treatment which we avoid in order to not to make the presentation too heavy. We do not know whether our results hold in dimensions $N \le 5$.

2 Statement of results

In order to state our results we introduce some notations. By Hardy's inequality, the norm

$$\|u\|_\mu := \left(\int_\Omega (|\nabla u|^2 - \mu \frac{u^2}{|x|^2}) dx \right)^{\frac{1}{2}}$$

is equivalent to the norm $\|u\|_0 = \left(\int_\Omega |\nabla u|^2 dx \right)^{1/2}$ on $H_0^1(\Omega)$ provided $0 \le \mu < \overline{\mu} = \frac{(N-2)^2}{4}$. This will of course be the case for $\mu \to 0$. As in [21] we write $H_\mu(\Omega)$ for the Hilbert space consisting of $H_0^1(\Omega)$ functions with the inner product

$$(u, v)_\mu := \int_\Omega \left(\nabla u \nabla v - \mu \frac{uv}{|x|^2} \right) dx.$$

It is known that the nonzero critical points of the energy functional

$$J_{\mu,\varepsilon}(u) := \frac{1}{2} \int_\Omega \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2} \right) dx - \frac{1}{2^* - \varepsilon} \int_\Omega |u|^{2^* - \varepsilon} dx$$

defined on $H_\mu(\Omega)$ are precisely the nontrivial weak solutions of problem (1.1).

Next we introduce two limiting problems. The first one is

$$\begin{cases} -\Delta u = |u|^{2^*-2} u & \text{in } \mathbb{R}^N, \\ u \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{2.1}$$

It is well known that the nontrivial least energy (positive) solutions to (2.1) are the instantons

$$U_{\delta,\xi}(x) := C_0\left(\frac{\delta}{\delta^2+|x-\xi|^2}\right)^{\frac{N-2}{2}}$$

with $\delta>0$, $\xi\in\mathbb{R}^N$ and $C_0 := (N(N-2))^{\frac{N-2}{4}}$, cf. [1, 37]. These solutions minimize the Rayleigh quotient

$$S_0 := \min_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}|\nabla u|^2dx}{(\int_{\mathbb{R}^N}|u|^{2^*}dx)^{2/2^*}}.$$

Moreover there holds

$$\int_{\mathbb{R}^N}|\nabla U_{\delta,\xi}|^2dx = \int_{\mathbb{R}^N}|U_{\delta,\xi}|^{2^*}dx = S_0^{\frac{N}{2}}.$$

The second limiting problem, dealing with $\mu>0$, is

$$\begin{cases} -\Delta u - \mu\dfrac{u}{|x|^2} = |u|^{2^*-2}u & \text{in } \mathbb{R}^N,\\[2ex] \qquad\qquad u\to 0 & \text{as } |x|\to\infty. \end{cases} \tag{2.2}$$

For $0<\mu<\overline{\mu}$ we know from [14, 38] that all positive solutions to (2.2) are given by

$$V_{\mu,\sigma}(x) = V_\sigma(x) = C_\mu\left(\frac{\sigma}{\sigma^2|x|^{\beta_1}+|x|^{\beta_2}}\right)^{\frac{N-2}{2}}$$

with $\sigma>0$, $\beta_1 := (\sqrt{\overline{\mu}}-\sqrt{\overline{\mu}-\mu})/\sqrt{\overline{\mu}}$, $\beta_2 := (\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-\mu})/\sqrt{\overline{\mu}}$, and $C_\mu := \left(\frac{4N(\overline{\mu}-\mu)}{N-2}\right)^{\frac{N-2}{4}}$. We drop the index μ if it is clear from the context. These solutions are minimizers of

$$S_\mu := \min_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}(|\nabla u|^2-\mu\frac{u^2}{|x|^2})dx}{(\int_{\mathbb{R}^N}|u|^{2^*}dx)^{2/2^*}},$$

and they satisfy

$$\int_{\mathbb{R}^N}\left(|\nabla V_{\mu,\sigma}|^2-\mu\frac{|V_{\mu,\sigma}|^2}{|x|^2}\right)dx = \int_{\mathbb{R}^N}|V_{\mu,\sigma}|^{2^*}dx = S_\mu^{\frac{N}{2}}.$$

Clearly $V_{\mu,\sigma}\to U_{\sigma,0}$ as $\mu\to 0$.

Now we can formulate the goal of this paper: We investigate the existence of solutions u_ε of (1.1) close to $V_{\mu,\sigma}-U_{\delta,\xi}$ with μ,σ,δ,ξ all depending on ε, and $\mu,\sigma,\delta\to 0$ as $\varepsilon\to 0$. Thus u_ε blows up positively at the origin and negatively at ξ. The precise blow-up rate is determined as is the location of ξ in the limit $\varepsilon\to 0$.

The Green's function of the Dirichlet Laplacian can be written as $G(x,y) = \frac{1}{|x-y|^{N-2}} - H(x,y)$, for $x,y\in\Omega$, where H is the regular part. The regular part is symmetric, i.e. $H(x,y)=H(y,x)$, and satisfies $H(x,x)\to\infty$ as $x\to\partial\Omega$. An important ingredient of our results will be the map

$$\varphi:\Omega\setminus\{0\}\to\mathbb{R},\quad \varphi(x) := H^{\frac12}(0,0)H^{\frac12}(x,x)+G(x,0).$$

Observe that $0<\varphi(x)\to\infty$ as $x\to 0$ or $x\to\partial\Omega$, hence φ has a minimum in $\Omega\setminus\{0\}$.

Theorem 2.1. *Suppose $\xi^*\in\Omega\setminus\{0\}$ is an isolated stable critical point of φ. Then for fixed $\alpha>\frac{N-4}{N-2}$ and $\mu_0>0$ the following holds. For $\varepsilon>0$ small the problem (1.1) with $\mu=\mu_0\varepsilon^\alpha$ has a solution u_ε of the form*

$$u_\varepsilon(x) = C_\mu\left(\frac{\sigma^\varepsilon}{(\sigma^\varepsilon)^2|x|^{\beta_1}+|x|^{\beta_2}}\right)^{\frac{N-2}{2}} - C_0\left(\frac{\delta^\varepsilon}{(\delta^\varepsilon)^2+|x-\xi^\varepsilon|^2}\right)^{\frac{N-2}{2}} + o(1) \qquad \text{as } \varepsilon\to 0. \tag{2.3}$$

Here $\delta^\varepsilon=\lambda^\varepsilon\varepsilon^{\frac{1}{N-2}}$ and $\sigma^\varepsilon=\overline{\lambda}^\varepsilon\varepsilon^{\frac{1}{N-2}}$ with λ^ε and $\overline{\lambda}^\varepsilon$ bounded away from 0 and ∞, i.e. $\lambda^\varepsilon,\overline{\lambda}^\varepsilon\in(\eta,\frac{1}{\eta})$ for some $\eta\in(0,1)$. Moreover, $\xi^\varepsilon\in\Omega\setminus\{0\}$ converges as $\varepsilon\to 0$ towards $\xi^*\in\Omega\setminus\{0\}$

Remark 2.2. a) Here ξ^* being a stable critical point means that a C^1-function $\psi : B_\rho(\xi^*) \to \mathbb{R}$ that is close to φ in the C^1-norm on a neighborhood $B_\rho(\xi^*)$ must have a critical point close to ξ^*. This is clearly the case if ξ^* is a non-degenerate critical point of φ. It is also the case if the index of ξ^* or, more generally, the critical groups of ξ^* as critical point of φ are not trivial. We expect that for a generic domain all critical points of φ are non-degenerate. We also expect that if ξ^* is a non-degenerate critical point of φ with Morse index $m(\xi^*)$ then the corresponding solution u_ε has Morse index $k+2$ as critical point of $J_{\mu,\varepsilon}$. Below we state more results where we do not require any a-priori knowledge about the existence of isolated stable critical points of φ.

b) The assumption $\alpha > \frac{N-4}{N-2}$ is essential for the result to hold. From a technical point of view it implies that the interaction between the bubbles $U_{\delta,\zeta}$, V_σ, and the Hardy potential is negligible. This is not just technical, however. Setting $\mu = \mu_0\varepsilon^\alpha$, $\mu_0 > 0$ arbitrary, and making the ansatz for u_ε as in Theorem 2.1 leads to a reduced function $\psi = \psi(\lambda, \overline{\lambda}, \xi)$ defined on $\mathbb{R}^+ \times \mathbb{R}^+ \times (\Omega \setminus \{0\})$. Then problem (1.1) has a solution u_ε with a limiting behavior for $\varepsilon \to 0$ as in Theorem 2.1 if, and only if, ψ has a critical point. We shall see that critical points of ψ correspond to critical points of φ if $\alpha > \frac{N-4}{N-2}$. On the other hand, for $0 \le \alpha \le \frac{N-4}{N-2}$ critical points of ψ do not exist; see Remark 5.2.

The next two theorems apply to any bounded smooth domain.

Theorem 2.3. *For fixed $\alpha > \frac{N-4}{N-2}$ and $\mu_0 > 0$ the following holds. For $\varepsilon > 0$ small the problem* (1.1) *with $\mu = \mu_0\varepsilon^\alpha$ has a solution u_ε of the form* (2.3)*. Moreover, $\xi^\varepsilon \in \Omega \setminus \{0\}$ converges as $\varepsilon \to 0$ towards a minimizer $\xi^* \in \Omega \setminus \{0\}$ of φ.*

In Theorem 2.3 it is not required that φ has an isolated minimizer. Next we state a multiplicity for solutions of the form (2.3) in terms of the Lusternik-Schnirelmann category of $\Omega \setminus \{0\}$.

Theorem 2.4. *For fixed $\alpha > \frac{N-4}{N-2}$ and $\mu_0 > 0$ the following holds. For $\varepsilon > 0$ small problem* (1.1) *with $\mu = \mu_0\varepsilon^\alpha$, has at least* $\mathrm{cat}(\Omega \setminus \{0\})$ *solutions $u_\varepsilon^{(i)}$ of the form* (2.3)*. The parameters $\lambda^\varepsilon = \lambda_i^\varepsilon, \overline{\lambda}^\varepsilon = \overline{\lambda}_i^\varepsilon \in (\eta, \frac{1}{\eta})$ and the blow-up points $\xi_i^\varepsilon \in \Omega \setminus \{0\}$ depend on $i \in \{1, \dots, \mathrm{cat}(\Omega \setminus \{0\})\}$. Moreover, $\xi_i^\varepsilon \to \xi_i^*$ with $\xi_i^* \in \Omega \setminus \{0\}$ being a critical point of φ.*

More can be said when the domain is symmetric in the following sense.

(S_1) Ω is invariant under a compact Lie group $\Gamma \subset O(N)$, i.e. $g\Omega = \Omega$ for all $g \in \Gamma$.

A simple example is when $\Omega = -\Omega$; here $\Gamma = \{\pm \mathrm{id}\}$. If (S_1) holds then any solution generates a Γ-orbit of solutions in the following way. If u solves (1.1) then for any $g \in \Gamma$ the function $g * u$ defined by $g * u(x) = u(g^{-1}x)$ solves (1.1). In that case one can use the equivariant category cat_Γ in Theorem 2.4; see [5, 15] for definitions and properties.

Theorem 2.5. *If (S_1) holds then in the setting of Theorem 2.4 for $\varepsilon > 0$ small there exist at least* $\mathrm{cat}_\Gamma(\Omega \setminus \{0\})$ *Γ-orbits $\Gamma * u_\varepsilon^{(i)}$ of solutions of* (1.1) *of the form* (2.3)*. The parameters and the blow-up points depend on i as in Theorem 2.4.*

Remark 2.6. It is well known that $2 \le \mathrm{cat}(\Omega \setminus \{0\}) \le N$ for any smooth bounded domain $\Omega \subset \mathbb{R}^N$ with $0 \in \Omega$. For Ω diffeomorphic to a ball one has $\mathrm{cat}(\Omega \setminus \{0\}) = 2$. On the other hand, if $\Omega = -\Omega$, so that (S_1) holds with $\Gamma = \{\pm \mathrm{id}\}$, then $\mathrm{cat}_\Gamma(\Omega \setminus \{0\}) = N$.

In the symmetric case one can also say something about the localization of the blow-up point $\xi^* \in \Omega \setminus \{0\}$.

Theorem 2.7. *Suppose* (S_1) *holds and let* $\Sigma \subset \Gamma$ *be a subgroup such that* $\Omega^\Sigma = \{x \in \Omega : gx = x \text{ for all } g \in \Sigma\} \neq \{0\}$*. Then there exist solutions* u_ε *as in Theorem 2.3 with* $\xi^\varepsilon \in \Omega^\Sigma$ *and* $\xi^\varepsilon \to \xi^*$ *as* $\varepsilon \to 0$ *where* ξ^* *is a minimum of* $\varphi|\Omega^\Sigma$*. Moreover,* $g * u_\varepsilon = u_\varepsilon$ *for all* $g \in \Sigma$*.*

Remark 2.8. a) In the setting of Theorem 2.7 one can also formulate a multiplicity result as in Theorems 2.4 and 2.5. Let $N\Sigma$ be the normalizer of Σ in Γ and $W\Sigma = N\Sigma/\Sigma$ the Weyl group. Observe that $W\Sigma$ acts on Ω^Σ so that the equivariant category $\mathrm{cat}_{W\Sigma}(\Omega^\Sigma \setminus \{0\})$ is defined. Then one obtains at least $\mathrm{cat}_{W\Sigma}(\Omega^\Sigma \setminus \{0\})$ different orbits of solutions as in Theorem 2.7 with $\xi^\varepsilon \in \Omega^\Sigma$. The blow-up points converge towards critical points of $\varphi|\Omega^\Sigma$. We leave details to the reader.

b) In order to illustrate Theorem 2.7 suppose Ω is invariant under the reflection $T_1 : (x_1, x') \mapsto (x_1, -x')$; here $x_1 \in \mathbb{R}$ and $x' \in \mathbb{R}^{N-1}$. Then setting $\Gamma = \Sigma = \{\mathrm{id}, T_1\}$ Theorem 2.7 yields a solution u_ε with $\xi^\varepsilon \in \Omega \cap (\mathbb{R} \times \{0\})$ and such that $\xi^\varepsilon \to \xi^*$ where ξ^* is a minimizer of φ in $\Omega^\Sigma \setminus \{0\} = (\Omega \setminus \{0\}) \cap (\mathbb{R} \times \{0\})$. Moreover, $u_\varepsilon(T_1 x) = u_\varepsilon(x)$. In fact, there are at least $\mathrm{cat}_{W\Sigma}(\Omega^\Sigma \setminus \{0\})$ such solution with blow up points $\xi_i^\varepsilon \to \xi_i^*$. Here $\mathrm{cat}_{W\Sigma}(\Omega^\Sigma \setminus \{0\}) = \mathrm{cat}(\Omega^\Sigma \setminus \{0\})$ is simply the number of components of $\Omega^\Sigma \setminus \{0\}$. The point ξ_i^* is a minimizer of φ constrained to the i-th component of $(\Omega \setminus \{0\}) \cap (\mathbb{R} \times \{0\})$.

c) Suppose Ω is invariant under T_1 as in b) and under T_2, where T_2 is the reflection at the x_2-axis. Then one can set $\Gamma = \{\mathrm{id}, T_1, T_2, T_1T_2\}$ and $\Sigma_k = \{\mathrm{id}, T_k\}$ for $k = 1, 2$. Then $N\Sigma_k = \Gamma$ and $W\Sigma_k \cong \mathbb{Z}/2$, and one can count the number of solutions using a). Details and further examples are left to the reader.

3 Notations and preliminary results

Throughout this paper, positive constants will be denoted by C, c. Let $\iota_\mu^* : L^{2N/(N+2)}(\Omega) \to H_\mu(\Omega)$ be the adjoint operator of the inclusion $\iota_\mu : H_\mu(\Omega) \to L^{2N/(N-2)}(\Omega)$, that is,

$$\iota_\mu^*(u) = v \qquad \Longleftrightarrow \qquad (v, \phi)_\mu = \int_\Omega u(x)\phi(x)dx, \quad \text{for all } \phi \in H_\mu(\Omega). \tag{3.1}$$

There exists $c > 0$ such that

$$\|\iota^*(u)\|_\mu \leq c\|u\|_{2N/(N+2)}. \tag{3.2}$$

Then problem (1.1) is equivalent to the fixed point problem

$$u = \iota_\mu^*(f_\varepsilon(u)),\ u \in H_\mu(\Omega), \tag{3.3}$$

where $f_\varepsilon(s) = |s|^{2^*-2-\varepsilon}s$.

In order to continue, we first solve an eigenvalue problem.

Proposition 3.1. *Let* $0 < \mu < \overline{\mu}$ *be fixed, and let* Λ_i*,* $i = 1, 2, \ldots,$ *be the eigenvalues of*

$$\begin{cases} -\Delta u - \mu\frac{u}{|x|^2} = \Lambda|V_\sigma|^{2^*-2}u & \text{in } \mathbb{R}^N, \\ |u| \to 0 & \text{as } |x| \to +\infty, \end{cases} \tag{3.4}$$

in increasing order. Then $\Lambda_1 = 1$ *with eigenfunction* V_σ*,* $\Lambda_2 = 2^* - 1$ *with eigenfunction* $\frac{\partial V_\sigma}{\partial \sigma}$*.*

Proof. Direct computations give that V_σ and $\frac{\partial V_\sigma}{\partial \sigma}$ are eigenfunctions corresponding to 1 and 2^*-1, respectively. Now as in [39], it is enough to prove that the eigenfunction u corresponding to the eigenvalue $\Lambda \le 2^*-1$ has to be radial.

Denote by ψ_i, $i \in \mathbb{N}_0$, the sequence of spherical harmonics, i.e. the eigenfunctions of the Laplace-Beltrami operator on S^{N-1}:

$$-\Delta_{S^{N-1}}\psi_i = \tau_i \psi_i.$$

It is well known that $\tau_0 = 0$, $\tau_1, \dots, \tau_N = N-1$, $\tau_{N+1} > \tau_N$. We prove that for every $i \ge 1$,

$$\int_{S^{N-1}} u(r,\theta)\psi_i(\theta)d\theta = 0.$$

Setting $\varphi_i(r) = \int_{S^{N-1}} u(r,\theta)\psi_i(\theta)d\theta$ we have:

$$\begin{aligned}
\Delta\varphi_i &= \Delta_r\varphi_i = \int_{S^{N-1}} \Delta_r u(r,\theta)\psi_i(\theta)d\theta \\
&= -\int_{S^{N-1}} \frac{\Delta_\theta u(r,\theta)}{r^2}\psi_i(\theta)d\theta - \int_{S^{N-1}} \left(\frac{\mu u(r,\theta)}{r^2} + \Lambda V_\sigma^{2^*-2}u(r,\theta)\right)\psi_i(\theta)d\theta \\
&= \int_{S^{N-1}} \frac{\tau_i u(r,\theta)}{r^2}\psi_i(\theta)d\theta - \int_{S^{N-1}} \left(\frac{\mu}{r^2} + \Lambda V_\sigma^{2^*-2}\right)u(r,\theta)\psi_i(\theta)d\theta \\
&= \left(\frac{\tau_i}{r^2} - \left(\frac{\mu}{r^2} + \Lambda V_\sigma^{2^*-2}\right)\right)\varphi_i(r).
\end{aligned}$$

This implies for any $R > 0$:

$$\begin{aligned}
0 &= \int_{B_R(0)} \Delta\varphi_i \frac{\partial V_\sigma}{\partial r} + \left(\frac{\mu}{r^2} + \Lambda V_\sigma^{2^*-2} - \frac{\tau_i}{r^2}\right)\varphi_i \frac{\partial V_\sigma}{\partial r} \\
&= \int_{B_R(0)} \varphi_i \Delta\left(\frac{\partial V_\sigma}{\partial r}\right) + \left(\frac{\mu}{r^2} + \Lambda V_\sigma^{2^*-2} - \frac{\tau_i}{r^2}\right)\varphi_i \frac{\partial V_\sigma}{\partial r} + \int_{\partial B_R(0)} \left(\frac{\partial V_\sigma}{\partial r}\cdot\frac{\partial \varphi_i}{\partial r} - \varphi_i \frac{\partial^2 V_\sigma}{\partial r^2}\right) \\
&= \int_{B_R(0)} \frac{N-1}{r^2}\varphi_i \frac{\partial V_\sigma}{\partial r} + \varphi_i \frac{\partial}{\partial r}\left(-\mu\frac{V_\sigma}{r^2} - V_\sigma^{2^*-1}\right) + \left(\frac{\mu}{r^2} + \Lambda V_\sigma^{2^*-2} - \frac{\tau_i}{r^2}\right)\varphi_i \frac{\partial V_\sigma}{\partial r} \\
&\quad + \int_{\partial B_R(0)} \left(\frac{\partial V_\sigma}{\partial r}\cdot\frac{\partial \varphi_i}{\partial r} - \varphi_i \frac{\partial^2 V_\sigma}{\partial r^2}\right) \\
&= \int_{B_R(0)} \frac{N-1-\tau_i}{r^2}\varphi_i \frac{\partial V_\sigma}{\partial r} + (\Lambda - (2^*-1))V_\sigma^{2^*-2}\frac{\partial V_\sigma}{\partial r}\varphi_i + \frac{2\mu V_\sigma}{r^3}\varphi_i \\
&\quad + \int_{\partial B_R(0)} \left(\frac{\partial V_\sigma}{\partial r}\cdot\frac{\partial \varphi_i}{\partial r} - \varphi_i \frac{\partial^2 V_\sigma}{\partial r^2}\right).
\end{aligned}$$

Now let R be the first zero of φ_i; $R := +\infty$ if φ_i is never zero. Without loss of generality we assume $\varphi_i(r) > 0$ for $r \in (0,R)$. Then $\frac{\partial \varphi_i}{\partial r}(R) \le 0$, and we finish the proof. $\square$

Let us define the projection $P : H^1(\mathbb{R}^N) \to H_0^1(\Omega)$, that is, $\Delta Pu = \Delta u$ in Ω, $Pu = 0$ on $\partial\Omega$. Consider the function H satisfying

$$\begin{cases} \Delta H(0,x) = 0 & \text{in } \Omega \setminus \{0\}, \\ H(0,x) = \frac{1}{|x|^{N-2}} & \text{on } \partial\Omega. \end{cases}$$

Finally set $d_{\inf} := \inf\{|x| : x \in \partial\Omega\}$ and $d_{\sup} := \sup\{|x| : x \in \partial\Omega\}$.

Proposition 3.2. *Let $0 < \mu < \overline{\mu}$ be fixed. Then for $\sigma > 0$ the function $\varphi_\sigma := V_\sigma - PV_\sigma$ satisfies*

$$0 \le \varphi_\sigma \le V_\sigma \quad \textit{and} \quad \varphi_\sigma(x) = C_\mu\big(\overline{d}(x)\big)^{\sqrt{\overline{\mu}}-\sqrt{\overline{\mu}-\mu}} H(0,x)\sigma^{\frac{N-2}{2}} + h_\sigma(x), \tag{3.5}$$

with

$$d_{\inf} \le \overline{d}(x) \le d_{\sup}, \quad \textit{and} \quad h_\sigma = o\big(\sigma^{\frac{N+2}{2}}\big), \ \frac{\partial h_\sigma}{\partial \sigma} = o\big(\sigma^{\frac{N}{2}}\big) \quad \textit{as } \sigma \to 0. \tag{3.6}$$

Proof. It is easy to see that φ_σ satisfies

$$\begin{cases} \Delta \varphi_\sigma(x) = 0 & \text{in } \Omega \setminus \{0\}, \\ \varphi_\sigma(x) = V_\sigma(x) = C_\mu \left(\frac{\sigma}{\sigma^2 |x|^{\beta_1} + |x|^{\beta_2}} \right)^{\frac{N-2}{2}} & \text{on } \partial\Omega. \end{cases}$$

Then the first part of (3.5) holds by the maximum principle. It also follows that $\varphi_\sigma \sigma^{-\frac{N-2}{2}}$ is increasing. Now we estimate for $x \in \partial\Omega$ and for $\sigma > 0$ small:

$$\begin{aligned} \frac{C_\mu d_{\inf}^{\sqrt{\overline{\mu}}-\sqrt{\overline{\mu}-\mu}} H(0,x)\sigma^{\frac{N-2}{2}} - \varphi_\sigma(x)}{\sigma^{\frac{N+2}{2}}} &= C_\mu \sigma^{-2} \left(\frac{d_{\inf}^{\sqrt{\overline{\mu}}-\sqrt{\overline{\mu}-\mu}}}{|x|^{N-2}} - \frac{1}{(\sigma^2|x|^{\beta_1} + |x|^{\beta_2})^{\frac{N-2}{2}}} \right) \\ &\le C_1(\mu, d_{\inf}, d_{\sup})\sigma^{-2} \left(d_{\inf}^{\sqrt{\overline{\mu}}-\sqrt{\overline{\mu}-\mu}} \left(\sigma^2|x|^{\beta_1} + |x|^{\beta_2}\right)^{\frac{N-2}{2}} - |x|^{N-2} \right) \\ &\le C_2(\mu, d_{\inf}, d_{\sup})\sigma^{-2} \left(|x|^{\sqrt{\overline{\mu}}-\sqrt{\overline{\mu}-\mu}} \left(\sigma^2|x|^{\beta_1} + |x|^{\beta_2}\right)^{\frac{N-2}{2}} - |x|^{N-2} \right) \\ &\le C_3(\mu, d_{\inf}, d_{\sup})\sigma^{-2} \left(|x|^{\sqrt{\overline{\mu}}-\sqrt{\overline{\mu}-\mu}} (|x|^{\beta_2(\frac{N-2}{2})} + C_4(\mu, d_{\inf}, d_{\sup})\sigma^2) - |x|^{N-2} \right) \\ &\le C = C(\mu, d_{\inf}, d_{\sup}). \end{aligned}$$

The constant $C(\mu, d_{\inf}, d_{\sup})$ can be chosen independent of $x \in \partial\Omega$ and of $\sigma \in (0, \sigma_0)$, some $\sigma_0 > 0$. Similary we have for $\sigma > 0$ small and $x \in \partial\Omega$:

$$\begin{aligned} \frac{C_\mu d_{\sup}^{\sqrt{\overline{\mu}}-\sqrt{\overline{\mu}-\mu}} H(0,x)\sigma^{\frac{N-2}{2}} - \varphi_\sigma(x)}{\sigma^{\frac{N+2}{2}}} &= C_\mu \sigma^{-2} \left(\frac{d_{\sup}^{\sqrt{\overline{\mu}}-\sqrt{\overline{\mu}-\mu}}}{|x|^{N-2}} - \frac{1}{(\sigma^2|x|^{\beta_1} + |x|^{\beta_2})^{\frac{N-2}{2}}} \right) \\ &\ge C_1'(\mu, d_{\inf}, d_{\sup})\sigma^{-2} \left(d_{\sup}^{\sqrt{\overline{\mu}}-\sqrt{\overline{\mu}-\mu}} \left(\sigma^2|x|^{\beta_1} + |x|^{\beta_2}\right)^{\frac{N-2}{2}} - |x|^{N-2} \right) \\ &\ge C_2'(\mu, d_{\inf}, d_{\sup})\sigma^{-2} \left(|x|^{\sqrt{\overline{\mu}}-\sqrt{\overline{\mu}-\mu}} \left(\sigma^2|x|^{\beta_1} + |x|^{\beta_2}\right)^{\frac{N-2}{2}} - |x|^{N-2} \right) \\ &\ge C_3'(\mu, d_{\inf}, d_{\sup})\sigma^{-2} \left(|x|^{\sqrt{\overline{\mu}}-\sqrt{\overline{\mu}-\mu}} (|x|^{\beta_2(\frac{N-2}{2})} + C_4'(\mu, d_{\inf}, d_{\sup})\sigma^2) - |x|^{N-2} \right) \\ &\ge C' = C'(\mu, d_{\inf}, d_{\sup}). \end{aligned}$$

Thus we have:

$$\frac{C_\mu d_{\inf}^{\sqrt{\overline{\mu}}-\sqrt{\overline{\mu}-\mu}} H(0,x)\sigma^{\frac{N-2}{2}}}{\sigma^{\frac{N+2}{2}}} - C \le \frac{\varphi_\sigma(x)}{\sigma^{\frac{N+2}{2}}} \le \frac{C_\mu d_{\sup}^{\sqrt{\overline{\mu}}-\sqrt{\overline{\mu}-\mu}} H(0,x)\sigma^{\frac{N-2}{2}}}{\sigma^{\frac{N+2}{2}}} - C'.$$

Defining $\overline{d}(x)$ by the equation

$$\overline{d}(x)^{\sqrt{\overline{\mu}}-\sqrt{\overline{\mu}-\mu}} = \min\left\{ \max\left\{ \frac{\varphi_\sigma(x)}{C_\mu H(0,x)\sigma^{\frac{N-2}{2}}}, d_{\inf} \right\}, d_{\sup} \right\}$$

and using the maximum principle we deduce $d_{\inf} \le \overline{d}(x) \le d_{\sup}$ and $\hbar_\sigma = o(\sigma^{\frac{N-2}{2}})$ as $\sigma \to 0$. The estimate $\frac{\partial h_\sigma}{\partial \sigma} = o(\sigma^{\frac{N-4}{2}})$ as $\sigma \to 0$ follows by direct similarly. $\square$

Remark 3.3. a) If $\mu \to 0^+$, then

$$\varphi_{\mu,\sigma}(x) = C_0 H(0,x)\sigma^{\frac{N-2}{2}} + O(\mu\sigma^{\frac{N-2}{2}}) + h_{\mu,\sigma} \tag{3.7}$$

where $h_{\mu,\sigma}$ satisfies (3.6) uniformly in μ.

b) A similar result has been obtained in [33] for $U_{\delta,\xi}$:

$$0 \le \varphi_{\delta,\xi} := U_{\delta,\xi} - PU_{\delta,\xi} \le U_{\delta,\xi} \quad \text{and} \quad \varphi_{\delta,\xi} = C_0 H(\xi,\cdot)\delta^{\frac{N-2}{2}} + O(\delta^{\frac{N+2}{2}}) \tag{3.8}$$

as $\delta \to 0$, uniformly in compact subsets of Ω.

4 The finite dimensional reduction

This section is more general than necessary for the proofs of the results in this paper in that we deal with arbitrarily many blow-up points. This is needed for subsequent work. We fix an integer $k \ge 0$, the case $k = 1$ corresponds to our main theorems. Throughout this section we assume $\mu = \mu_0 \varepsilon^\alpha$. For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k, \overline{\lambda}) \in \mathbb{R}_+^{k+1}$ we set

$$\delta_i = \lambda_i \varepsilon^{\frac{1}{N-2}} \quad \text{and} \quad \sigma = \overline{\lambda}\varepsilon^{\frac{1}{N-2}}. \tag{4.1}$$

For $\lambda \in \mathbb{R}_+^{k+1}$ and $\xi = (\xi_1, \dots, \xi_k) \in \Omega^k$ we now define

$$W_{\varepsilon,\lambda,\xi} := \sum_{i=1}^{k} \mathrm{Ker}\left(-\Delta - (2^*-1)U_{\delta_i,\xi_i}^{2^*-2}\right) + \mathrm{Ker}\left(-\Delta - \frac{\mu}{|x|^2} - (2^*-1)V_\sigma^{2^*-2}\right),$$

where. According to [9], the kernel of the operator $-\Delta - (2^*-1)U_{\delta_i,\xi_i}^{2^*-2}$ on $L^2(\mathbb{R}^N)$ has dimension $N+1$ and is spanned by $\frac{\partial U_{\delta_i,\xi_i}}{\partial \delta_i}$, $\frac{\partial U_{\delta_i,\xi_i}}{\partial (\xi_i)^j}$, $j = 1, 2, \dots, N$, where $(\xi_i)^j$ is the $j-$th component of ξ_i. Combining this with Proposition 3.1, we have

$$W_{\varepsilon,\lambda,\xi} = \mathrm{span}\left\{\Psi_i^j,\ \Psi_i^0,\ \overline{\Psi},\ i = 1, 2, \dots, k,\ j = 1, 2, \dots, N\right\},$$

where for $i = 1, 2, \dots, k$ and $j = 1, 2, \dots, N$:

$$\Psi_i^j := \frac{\partial U_{\delta_i,\xi_i}}{\partial (\xi_i)^j}, \quad \Psi_i^0 := \frac{\partial U_{\delta_i,\xi_i}}{\partial \delta_i}, \quad \overline{\Psi} := \frac{\partial V_\sigma}{\partial \sigma}. \tag{4.2}$$

For simplicity of notation here we dropped the dependance on the parameters. Next we define for $\eta \in (0,1)$:

$$\mathcal{O}_\eta := \big\{(\lambda,\xi) \in \mathbb{R}_+^{k+1} \times \Omega^k : \lambda_i \in (\eta, \eta^{-1}), \overline{\lambda} \in (\eta, \eta^{-1}),\ \mathrm{dist}(\xi_i, \partial\Omega) > \eta,$$
$$|\xi_i| > \eta,\ |\xi_{i_1} - \xi_{i_2}| > \eta,\ i, i_1, i_2 = 1, 2, \dots, k,\ i_1 \ne i_2\big\}.$$

Let us introduce the spaces

$$K_{\varepsilon,\lambda,\xi} := PW_{\varepsilon,\lambda,\xi},$$

and

$$K_{\varepsilon,\lambda,\xi}^\perp := \big\{\phi \in H_\mu(\Omega) : (\phi, P\Psi) = 0,\ \text{for all } \Psi \in W_{\varepsilon,\lambda,\xi}\big\},$$

as well as the $(\cdot,\cdot)_\mu$-orthogonal projections

$$\Pi_{\varepsilon,\lambda,\xi} : H_\mu(\Omega) \to K_{\varepsilon,\lambda,\xi},$$

and

$$\Pi_{\varepsilon,\lambda,\xi}^\perp := Id - \Pi_{\varepsilon,\lambda,\xi} : H_\mu(\Omega) \to K_{\varepsilon,\lambda,\xi}^\perp.$$

We want to find solutions of (1.1) close to

$$V_{\varepsilon,\lambda,\xi} = \sum_{i=1}^{k} \tau_i PU_{\delta_i,\xi_i} + PV_\sigma, \tag{4.3}$$

where $(\lambda,\xi) \in \mathcal{O}_\eta$ for some $\eta \in (0,1)$, $\tau_i = 1$ or -1. This is equivalent to finding $\eta > 0$, $(\lambda,\xi) \in \mathcal{O}_\eta$ and $\phi_{\varepsilon,\lambda,\xi} \in K^{\perp}_{\varepsilon,\lambda,\xi}$ such that $V_{\varepsilon,\lambda,\xi} + \phi_{\varepsilon,\lambda,\xi}$ solves (3.3), hence solving:

$$\Pi^{\perp}_{\varepsilon,\lambda,\xi}\left(V_{\varepsilon,\lambda,\xi} + \phi_{\varepsilon,\lambda,\xi} - \iota^*_\mu\big(f_\varepsilon(V_{\varepsilon,\lambda,\xi} + \phi_{\varepsilon,\lambda,\xi})\big)\right) = 0, \tag{4.4}$$

and

$$\Pi_{\varepsilon,\lambda,\xi}\left(V_{\varepsilon,\lambda,\xi} + \phi_{\varepsilon,\lambda,\xi} - \iota^*_\mu\big(f_\varepsilon(V_{\varepsilon,\lambda,\xi} + \phi_{\varepsilon,\lambda,\xi})\big)\right) = 0, \tag{4.5}$$

We solve (4.4) first for $\phi_{\varepsilon,\lambda,\xi}$. Let us introduce the operator $L_{\varepsilon,\lambda,\xi} : K^{\perp}_{\varepsilon,\lambda,\xi} \to K^{\perp}_{\varepsilon,\lambda,\xi}$ defined by

$$L_{\varepsilon,\lambda,\xi}(\phi) = \Pi^{\perp}_{\varepsilon,\lambda,\xi}\iota^*_\mu\big(f_0'(V_{\varepsilon,\lambda,\xi})\phi\big) = \phi - \Pi_{\varepsilon,\lambda,\xi}\iota^*_\mu\big(f_0'(V_{\varepsilon,\lambda,\xi})\phi\big).$$

Proposition 4.1. *For any $\eta \in (0,1)$, there exist $\varepsilon_0 > 0$ and $c > 0$ such that for every $(\lambda,\xi) \in \mathcal{O}_\eta$ and for every $\varepsilon \in (0,\varepsilon_0)$:*

$$\|L_{\varepsilon,\lambda,\xi}(\phi)\|_\mu \geq c\|\phi\|_\mu, \quad \textit{for all } \phi \in K^{\perp}_{\varepsilon,\lambda,\xi}.$$

In particular, $L_{\varepsilon,\lambda,\xi}$ is invertible with continuous inverse.

Proof. We argue by contradiction, following the same line as in [28]. Suppose there exist $\eta > 0$, sequences $\varepsilon_n > 0$, $(\lambda^n,\xi^n) \in \mathcal{O}_\eta$, $\phi_n \in H_\mu(\Omega)$ satisfying $\varepsilon_n \to 0$, $\lambda^n = \big(\lambda^n_1,\dots,\lambda^n_k,\overline{\lambda}^n\big) \to \big(\lambda_1,\dots\lambda_k,\overline{\lambda}\big)$, $\xi^n = \big(\xi^n_1,\dots,\xi^n_k\big) \to (\xi_1,\dots\xi_k)$, as $n \to \infty$, and such that

$$\phi_n \in K^{\perp}_{\varepsilon_n,\lambda^n,\xi^n},\ \|\phi_n\|_\mu = 1, \tag{4.6}$$

and

$$L_{\varepsilon_n,\lambda^n,\xi^n}(\phi_n) = h_n, \quad \text{with } \|h_n\|_\mu \to 0. \tag{4.7}$$

Thus we have

$$\phi_n - \iota^*_\mu(f_0'(V_{\varepsilon_n,\lambda^n,\xi^n})\phi_n) = h_n - \Pi_{\varepsilon_n,\lambda^n,\xi^n}\big(\iota^*_\mu(f_0'(V_{\varepsilon_n,\lambda^n,\xi^n})\phi_n)\big). \tag{4.8}$$

Setting

$$\delta^n_i = \lambda^n_i \varepsilon_n^{\frac{1}{N-2}}, \quad \sigma^n = \overline{\lambda}^n \varepsilon_n^{\frac{1}{N-2}}$$

as in (4.1) and

$$(\Psi^j_i)_n := \frac{\partial U_{\delta^n_i,\xi^n_i}}{\partial(\xi^n_i)^j} \quad \text{for } j = 1,2,\dots,N, \quad (\Psi^0_i)_n := \frac{\partial U_{\delta^n_i,\xi^n_i}}{\partial \delta^n_i}, \quad (\overline{\Psi})_n := \frac{\partial V_{\sigma^n}}{\partial \sigma^n},$$

where $(\xi^n_i)^j$ is the $j-$th component of ξ^n_i, we obtain

$$w^n := -\Pi_{\varepsilon_n,\lambda^n,\xi^n}(\iota^*_\mu(f_0'(V_{\varepsilon_n,\lambda^n,\xi^n})\phi_n)) = \sum_{i=1}^{k}\sum_{j=0}^{N} c^n_{i,j} P(\Psi^j_i)_n + c^n_0 P(\overline{\Psi})_n$$

for some coefficients $c^n_{i,j}, c^n_0$. Now we argue in three steps.

Step 1. We prove $\lim_{n\to\infty}\|w^n\|_\mu = 0$.

Multiplying (4.8) by $\Delta P(\Psi_l^h)_n + \mu \frac{P(\Psi_l^h)_n}{|x|^2}$, we get

$$\int_\Omega \phi_n \left(\Delta P(\Psi_l^h)_n + \mu \frac{P(\Psi_l^h)_n}{|x|^2} \right) - \int_\Omega \iota_\mu^* (f_0'(V_{\varepsilon_n,\lambda^n,\xi^n})\phi_n) \left(\Delta P(\Psi_l^h)_n + \mu \frac{P(\Psi_l^h)_n}{|x|^2} \right)$$
$$= - \int_\Omega h_n \left(-\Delta P(\Psi_l^h)_n - \mu \frac{P(\Psi_l^h)_n}{|x|^2} \right) + \int_\Omega w^n \left(\Delta P(\Psi_l^h)_n + \mu \frac{P(\Psi_l^h)_n}{|x|^2} \right)$$

and then

$$\sum_{i=1}^k \sum_{j=0}^N c_{i,j}^n \big(P(\Psi_i^j)_n, P(\Psi_l^h)_n \big)_\mu + c_0^n \big(P(\overline{\Psi})_n, P(\Psi_l^h)_n \big)_\mu$$
$$= \big(\phi_n, P(\Psi_l^h)_n \big)_\mu - \big(\iota_\mu^* (f_0'(V_{\varepsilon_n,\lambda^n,\xi^n})\phi_n), P(\Psi_l^h)_n \big)_\mu - \big(h_n, P(\Psi_l^h)_n \big)_\mu.$$

From Lemma A.1 we deduce:

$$c_{l,h}^n \widetilde{c}_{l,h}^n \frac{1}{(\delta_l^n)^2} + o\left(\frac{1}{(\delta_l^n)^2} \right) = -\big(\iota_\mu^* (f_0'(V_{\varepsilon_n,\lambda^n,\xi^n})\phi_n), P(\Psi_l^h)_n \big)_\mu, \tag{4.9}$$

where $\widetilde{c}_{l,h}^n > 0$ is a constant. Next Proposition 3.2 implies

$$0 = (\phi_n, P(\Psi_l^h)_n)_\mu = \int_\Omega \nabla \phi_n \nabla P(\Psi_l^h)_n - \mu \frac{\phi_n P(\Psi_l^h)_n}{|x|^2} = \int_\Omega \nabla \phi_n \nabla (\Psi_l^h)_n - \mu \frac{\phi_n (\Psi_l^h)_n}{|x|^2} + o(1)$$
$$= \int_\Omega f_0'(U_{\delta_l^n,\xi_l^n})(\Psi_l^h)_n \phi_n + o(1),$$

and then

$$- (\iota_\mu^* (f_0'(V_{\varepsilon_n,\lambda^n,\xi^n})\phi_n), P(\Psi_l^h)_n)_\mu = - \int_\Omega f_0'(V_{\varepsilon_n,\lambda^n,\xi^n})\phi_n P(\Psi_l^h)_n$$
$$\le \left| \int_\Omega \big(f_0'(V_{\varepsilon_n,\lambda^n,\xi^n}) - f_0'(U_{\delta_l^n,\xi_l^n}) \big) \phi_n (\Psi_l^h)_n \right| + \left| \int_\Omega f_0'(V_{\varepsilon_n,\lambda^n,\xi^n}) \phi_n \big(P(\Psi_l^h)_n - (\Psi_l^h)_n \big) \right| + o(1)$$
$$= o(1)$$

by Lemma A.2 and Lemma A.3.

Combining the above inequality with (4.9) yields $c_{l,h}^n \to 0$ as $n \to \infty$. Similar arguments show that $c_0^n \to 0$ as $n \to \infty$, and $\lim_{n\to\infty} \|w^n\|_\mu = 0$ follows.

STEP 2. Let $\chi : \mathbb{R}^N \to [0,1]$ be a smooth cut-off function, such that $\chi(x) = 1$ if $|x| \le \eta/4$, $\chi(x) = 0$ if $|x| \ge \eta/2$, and $|\nabla \chi(x)| \le \frac{C}{\eta}$. Let α_1, α_2 be positive constants to be determined later. We set

$$\phi_i^n(x) := \big((\varepsilon_n)^{\alpha_1} \big)^{\frac{N-2}{2}} \phi_n \big((\varepsilon_n)^{\alpha_1} x + \xi_i^n \big) \chi \big((\varepsilon_n)^{\alpha_1} x \big), \quad x \in \Omega_i^n := \frac{\Omega - \xi_i^n}{(\varepsilon_n)^{\alpha_1}},\ i = 1, \ldots, k,$$

and

$$\phi_0^n(x) := \big((\varepsilon_n)^{\alpha_2} \big)^{\frac{N-2}{2}} \phi_n \big((\varepsilon_n)^{\alpha_2} x \big) \chi \big((\varepsilon_n)^{\alpha_2} x \big), \quad \text{for } x \in \Omega_0^n := \frac{\Omega}{(\varepsilon_n)^{\alpha_2}}.$$

Since ϕ_i^n is bounded in $D^{1,2}(\mathbb{R}^N)$, we may assume, up to a subsequence,

$$\phi_i^n \rightharpoonup \phi_i^\infty \quad \text{weakly in } D^{1,2}(\mathbb{R}^N),\ i = 0, 1, \ldots, k.$$

Now we claim that

$$\phi_i^\infty(x) = 0, \quad i = 0, 1, \ldots, k. \tag{4.10}$$

Firstly we prove (4.10) for $i=1,\dots,k$. Notice that $\big|\nabla\chi\big((\varepsilon_n)^{\alpha_1}x\big)\big| = \big|(\varepsilon_n)^{\alpha_1}\nabla\chi(\cdot)\big| \le \frac{C(\varepsilon_n)^{\alpha_1}}{\eta} = o(1)$. Thus we have for any $\psi\in C_0^\infty(\mathbb{R}^N)$:

$$\big((\varepsilon_n)^{\alpha_1}\big)^{\frac{N-2}{2}}\int_{\Omega_i^n}\nabla\chi\big((\varepsilon_n)^{\alpha_1}x\big)\big(\phi_n((\varepsilon_n)^{\alpha_1}x+\xi_i^n)\nabla\psi-\psi\nabla\phi_n((\varepsilon_n)^{\alpha_1}x+\xi_i^n)\big)=o(1). \tag{4.11}$$

On the other hand, taking $\alpha_1=\frac{1}{N-2}$ and noticing $N\ge 7$, we get:

$$\big((\varepsilon_n)^{\alpha_1}\big)^{\frac{2-N}{2}}\mu\int_\Omega\frac{\iota_\mu^*\big(f_0'(V_{\varepsilon_n,\lambda^n,\xi^n}(y))\phi_n(y)\big)\chi(y-\xi_i^n)\psi\left(\frac{y-\xi_i^n}{(\varepsilon_n)^{\alpha_1}}\right)}{|y|^2}=o(1). \tag{4.12}$$

By (4.11) and (4.8), we have for any $\psi\in C_0^\infty(\mathbb{R}^N)$:

$$\begin{aligned}\int_{\Omega_i^n}\nabla\phi_i^n\nabla\psi &= \big((\varepsilon_n)^{\alpha_1}\big)^{\frac{N-2}{2}}\int_{\Omega_i^n}\Big(\nabla\phi_n\big((\varepsilon_n)^{\alpha_1}x+\xi_i^n\big)\nabla(\chi((\varepsilon_n)^{\alpha_1}x)\psi)\\
&\qquad+\nabla\chi\big((\varepsilon_n)^{\alpha_1}x\big)\big(\phi_n\big((\varepsilon_n)^{\alpha_1}x+\xi_i^n\big)\nabla\psi-\psi\nabla\phi_n\big((\varepsilon_n)^{\alpha_1}x+\xi_i^n\big)\big)\Big)\\
&=\big((\varepsilon_n)^{\alpha_1}\big)^{\frac{N-2}{2}}\int_{\Omega_i^n}\nabla\phi_n\big((\varepsilon_n)^{\alpha_1}x+\xi_i^n\big)\nabla\big(\chi((\varepsilon_n)^{\alpha_1}x)\psi\big)+o(1)\\
&=\big((\varepsilon_n)^{\alpha_1}\big)^{\frac{N-2}{2}}\int_{\Omega_i^n}\nabla\iota_\mu^*\big(f_0'\big(V_{\varepsilon_n,\lambda^n,\xi^n}((\varepsilon_n)^{\alpha_1}x+\xi_i^n)\big)\phi_n\big((\varepsilon_n)^{\alpha_1}x+\xi_i^n\big)\big)\nabla\big(\chi((\varepsilon_n)^{\alpha_1}x)\psi\big)\\
&\quad+\big((\varepsilon_n)^{\alpha_1}\big)^{\frac{N-2}{2}}\int_{\Omega_i^n}\nabla h_n((\varepsilon_n)^{\alpha_1}x+\xi_i^n)\nabla\big(\chi((\varepsilon_n)^{\alpha_1}x)\psi\big)\\
&\quad+\big((\varepsilon_n)^{\alpha_1}\big)^{\frac{N-2}{2}}\int_{\Omega_i^n}\nabla w_n\big((\varepsilon_n)^{\alpha_1}x+\xi_i^n\big)\nabla(\chi((\varepsilon_n)^{\alpha_1}x)\psi)+o(1)\\
&=: I_1+I_2+I_3+o(1).\end{aligned} \tag{4.13}$$

By (4.7) and STEP 1 it is easy to see that $I_2=o(1)$ and $I_3=o(1)$. On the other hand, (4.12) and (3.8) imply

$$\begin{aligned}I_1&=\big((\varepsilon_n)^{\alpha_1}\big)^{\frac{2-N}{2}}\int_\Omega\nabla\iota_\mu^*\big(f_0'(V_{\varepsilon_n,\lambda^n,\xi^n}(y))\phi_n(y)\big)\nabla\big(\chi(y-\xi_i^n)\psi(\frac{y-\xi_i^n}{(\varepsilon_n)^{\alpha_1}})\big)+o(1)\\
&=\big((\varepsilon_n)^{\alpha_1}\big)^{\frac{2-N}{2}}\int_\Omega f_0'\big(V_{\varepsilon_n,\lambda^n,\xi^n}(y)\big)\phi_n(y)\chi(y-\xi_i^n)\psi\left(\frac{y-\xi_i^n}{(\varepsilon_n)^{\alpha_1}}\right)+o(1)\\
&=\big((\varepsilon_n)^{\alpha_1}\big)^{\frac{2-N}{2}}\int_{|y-\xi_i^n|\le\eta/2}f_0'\left(\sum_{j=1}^k\tau_jPU_{\delta_j^n,\xi_j^n}(y)+PV_{\sigma^n}(y)\right)\phi_n(y)\chi(y-\xi_i^n)\psi\left(\frac{y-\xi_i^n}{(\varepsilon_n)^{\alpha_1}}\right)\\
&\qquad+o(1)\\
&=\big((\varepsilon_n)^{\alpha_1}\big)^{\frac{2-N}{2}}\int_{|y-\xi_i^n|\le\eta/2}f_0'\big(U_{\delta_i^n,\xi_i^n}(y)\big)\phi_n(y)\chi(y-\xi_i^n)\psi(\frac{y-\xi_i^n}{(\varepsilon_n)^{\alpha_1}})+o(1)\\
&=\int_{|(\varepsilon_n)^{\alpha_1}x|\le\eta/2}f_0'\big(U_{\lambda_i^n,0}(x)\big)\big((\varepsilon_n)^{\alpha_1}\big)^{\frac{N-2}{2}}\phi_n\big((\varepsilon_n)^{\alpha_1}x+\xi_i^n\big)\chi\big((\varepsilon_n)^{\alpha_1}x\big)\psi(x)+o(1)\\
&=\int_{\mathbb{R}^N}f_0'(U_{\lambda_i,0}(x))\phi_i^\infty(x)\psi(x)+o(1).\end{aligned} \tag{4.14}$$

Therefore we have

$$\int_{\Omega_i^n}\nabla\phi_i^n\nabla\psi=\int_{\mathbb{R}^N}f_0'(U_{\lambda_i,0}(x))\phi_i^\infty(x)\psi(x)+o(1),$$

which implies that ϕ_i^∞ is a weak solution of

$$-\Delta\phi_i^\infty=f_0'(U_{\lambda_i,0})\phi_i^\infty\quad\text{in } D^{1,2}(\mathbb{R}^N). \tag{4.15}$$

In order to continue we denote $\Psi^j_{\lambda_i,0} := \frac{\partial U_{\lambda_i,0}}{\partial x^j}$ for $j = 1,\dots,N$, and $\Psi^0_{\lambda_i,0} := \frac{\partial U_{\lambda_i,0}}{\partial \lambda_i}$. Now we claim that

$$\int_{\mathbb{R}^N} \nabla\phi_i^\infty(x)\nabla\Psi^j_{\lambda_i,0}(x) = 0, \quad j = 0,1,\dots,N. \tag{4.16}$$

In fact,

$$\begin{aligned}
&\left|\int_{\Omega_i^n} f_0'\big(U_{\lambda_i^n,0}(x)\big)\phi_i^n(x)\Psi^j_{\lambda_i^n,0}(x)\right| \qquad (4.17)\\
&= \left|\int_{\Omega_i^n} f_0'\big(U_{\lambda_i^n,0}(x)\big)\big((\varepsilon_n)^{\alpha_1}\big)^{\frac{N-2}{2}}\phi_n\big((\varepsilon_n)^{\alpha_1}x+\xi_i^n\big)\chi((\varepsilon_n)^{\alpha_1}x)\Psi^j_{\lambda_i^n,0}(x)\right|\\
&= \left|\int_{(\varepsilon_n)^{-\alpha_1}\Omega} f_0'\left(U_{\lambda_i^n,\frac{\xi_i^n}{(\varepsilon_n)^{\alpha_1}}}(x)\right)\big((\varepsilon_n)^{\alpha_1}\big)^{\frac{N-2}{2}}\phi_n\big((\varepsilon_n)^{\alpha_1}x\big)\chi\big((\varepsilon_n)^{\alpha_1}x-\xi_i^n\big)\times\right.\\
&\qquad\left.\times\Psi^j_{\lambda_i^n,0}\big(x-\frac{\xi_i^n}{(\varepsilon_n)^{\alpha_1}}\big)\right|.
\end{aligned}$$

Noticing that

$$\begin{aligned}
&\int_{(\varepsilon_n)^{-\alpha_1}\Omega} f_0'\left(U_{\lambda_i^n,\frac{\xi_i^n}{(\varepsilon_n)^{\alpha_1}}}(x)\right)\big((\varepsilon_n)^{\alpha_1}\big)^{\frac{N-2}{2}}\phi_n\big((\varepsilon_n)^{\alpha_1}x\big)\Psi^j_{\lambda_i^n,0}\big(x-\frac{\xi_i^n}{(\varepsilon_n)^{\alpha_1}}\big)\\
&\qquad = -(\varepsilon_n)^{\alpha_1}\int_\Omega f_0'\big(U_{\delta_i^n,\xi_i^n}(y)\big)\phi_n(y)\big(\Psi^j_i\big)_n(y) = o(1),
\end{aligned}$$

then

$$\begin{aligned}
(4.17) &= \left|\int_{(\varepsilon_n)^{-\alpha_1}\Omega} f_0'\left(U_{\lambda_i^n,\frac{\xi_i^n}{(\varepsilon_n)^{\alpha_1}}}(x)\right)\big((\varepsilon_n)^{\alpha_1}\big)^{\frac{N-2}{2}}\phi_n\big((\varepsilon_n)^{\alpha_1}x\big)\big(\chi((\varepsilon_n)^{\alpha_1}x-\xi_i^n)-1\big)\right.\\
&\qquad\left.\times\,\Psi^j_{\lambda_i^n,0}\left(x-\frac{\xi_i^n}{(\varepsilon_n)^{\alpha_1}}\right)\right| + o(1)\\
&\le \left|\int_{\left|x-\frac{\xi_i^n}{(\varepsilon_n)^{\alpha_1}}\right|\ge\frac{\eta/4}{(\varepsilon_n)^{\alpha_1}}} f_0'\left(U_{\lambda_i^n,\frac{\xi_i^n}{(\varepsilon_n)^{\alpha_1}}}(x)\right)\big((\varepsilon_n)^{\alpha_1}\big)^{\frac{N-2}{2}}\phi_n\big((\varepsilon_n)^{\alpha_1}x\big)\Psi^j_{\lambda_i^n,0}\left(x-\frac{\xi_i^n}{(\varepsilon_n)^{\alpha_1}}\right)\right|\\
&\qquad + o(1)\\
&\le C\|\phi_n\|_{\frac{2N}{N-2}}\left(\int_{\left|x-\frac{\xi_i^n}{(\varepsilon_n)^{\alpha_1}}\right|\ge\frac{\eta/4}{(\varepsilon_n)^{\alpha_1}}}\left(U_{\lambda_i^n,\frac{\xi_i^n}{(\varepsilon_n)^{\alpha_1}}}(x)\right)^{\frac{2N}{N-2}}\right)^{\frac{2}{N}}\\
&\qquad\times\left(\int_{\left|x-\frac{\xi_i^n}{(\varepsilon_n)^{\alpha_1}}\right|\ge\frac{\eta/4}{(\varepsilon_n)^{\alpha_1}}}\left(\Psi^j_{\lambda_i^n,0}\left(x-\frac{\xi_i^n}{(\varepsilon_n)^{\alpha_1}}\right)\right)^{\frac{2N}{N-2}}\right)^{\frac{N-2}{2N}}\\
&= o(1).
\end{aligned}$$

Therefore (4.16) holds. Using this and (4.15) we conclude that (4.10) holds for $i = 1,\dots,k$.

Now we turn to the proof of $\phi_0^\infty=0$. Setting $\alpha_2=\frac{1}{N-2}$ we obtain as in (4.13) and (4.14) :

$$\begin{aligned}
\int_{\Omega_0^n}\nabla\phi_0^n\nabla\psi &= \big((\varepsilon_n)^{\alpha_2}\big)^{\frac{2-N}{2}}\int_\Omega f_0'\big(V_{\varepsilon_n,\lambda^n,\xi^n}(y)\big)\phi_n(y)\chi(y)\psi\big((\varepsilon_n)^{-\alpha_2}y\big)+o(1)\\
&= \big((\varepsilon_n)^{\alpha_2}\big)^{\frac{2-N}{2}}\int_{|y|\le\eta/2} f_0'\left(\sum_{j=1}^k\tau_jPU_{\delta_j^n,\xi_j^n}(y)+PV_{\sigma^n}(y)\right)\phi_n(y)\chi(y)\psi((\varepsilon_n)^{-\alpha_2}y)+o(1)\\
&= \big((\varepsilon_n)^{\alpha_2}\big)^{\frac{2-N}{2}}\int_{|y|\le\eta/2} f_0'\big(V_{\sigma^n}(y)\big)\phi_n(y)\chi(y)\psi((\varepsilon_n)^{-\alpha_2}y)+o(1)\\
&= \int_{|(\varepsilon_n)^{\alpha_2}x|\le\eta/2} f_0'\big(U_{\overline{\lambda}^n,0}(x)\big)\big((\varepsilon_n)^{\alpha_2}\big)^{\frac{N-2}{2}}\phi_n\big((\varepsilon_n)^{\alpha_2}x\big)\chi\big((\varepsilon_n)^{\alpha_2}x\big)\psi(x)+o(1)\\
&= \int_{\mathbb{R}^N} f_0'\big(U_{\overline{\lambda},0}\big)\phi_0^\infty\psi(x)+o(1).
\end{aligned}$$

Therefore ϕ_0^∞ is a weak solution of

$$-\Delta\phi_0^\infty=f_0'(U_{\overline{\lambda},0})\phi_0^\infty,\quad \text{in } D^{1,2}(\mathbb{R}^N).$$

Similarly to (4.16) there holds

$$\int_{\mathbb{R}^N}\nabla\phi_0^\infty(x)\nabla\Psi^j_{\overline{\lambda},0}(x)=0,\quad \text{for } j=0,1,\dots,N,$$

where $\Psi^j_{\overline{\lambda},0}:=\frac{\partial U_{\overline{\lambda},0}}{\partial x^j}$, for $j=1,\dots,N$, and $\Psi^0_{\overline{\lambda},0}:=\frac{\partial U_{\overline{\lambda},0}}{\partial\overline{\lambda}}$. This shows that $\phi_0^\infty=0$ as claimed.

Step 3. We obtain a contradiction.

Firstly we claim that

$$\lim_{n\to\infty}\int_\Omega f_0'\big(V_{\varepsilon_n,\lambda^n,\xi^n}(y)\big)(\phi_n(y))^2=0. \tag{4.18}$$

In fact, (3.7) and (3.8) imply:

$$\begin{aligned}
&\int_\Omega f_0'(V_{\varepsilon_n,\lambda^n,\xi^n}(y))(\phi_n(y))^2\\
&\qquad=\int_{B(0,\frac{\eta}{4})\cup\bigcup\limits_{i=1}^k B(\xi_i,\frac{\eta}{4})} f_0'\left(\sum_{j=1}^k\tau_jU_{\delta_j^n,\xi_j^n}(y)+V_{\sigma^n}(y)\right)(\phi_n(y))^2+o(1).
\end{aligned}$$

Notice that $f_0'(U_{\lambda_i^n,0})\in L^{\frac{N}{2}}(\mathbb{R}^N)$ and (4.10) imply

$$\begin{aligned}
\int_{B(\xi_i,\frac{\eta}{4})} f_0'\left(\sum_{j=1}^k\tau_jU_{\delta_j^n,\xi_j^n}(y)+V_{\sigma^n}(y)\right)\big(\phi_n(y)\big)^2 &= \int_{B(\xi_i,\frac{\eta}{4})} f_0'\big(U_{\delta_i^n,\xi_i^n}(y)\big)(\phi_n(y))^2+o(1)\\
&= \int_{|(\varepsilon_n)^{\alpha_1}x|\le\frac{\eta}{4}} f_0'\big(U_{\lambda_i^n,0}(x)\big)(\phi_i^n(x))^2+o(1)\\
&= o(1).
\end{aligned}\tag{4.19}$$

Similarly we obtain:

$$\int_{B(0,\frac{\eta}{4})} f_0'\left(\sum_{j=1}^k\tau_jU_{\delta_j^n,\xi_j^n}(y)+V_{\sigma^n}(y)\right)(\phi_n(y))^2=o(1). \tag{4.20}$$

Now we obtain (4.18) from (4.19) and (4.20).

On the other hand, (4.8), (4.7), and STEP 1 imply:

$$\begin{aligned}\int_\Omega |\nabla\phi_n|^2 &= \int_\Omega \nabla\iota_\mu^*(f_0'(V_{\varepsilon_n,\lambda^n,\xi^n})\phi_n)\nabla\phi_n + \int_\Omega \nabla h_n\nabla\phi_n + \int_\Omega \nabla w^n\nabla\phi_n\\ &= \int_\Omega \nabla\iota_\mu^*\big(f_0'(V_{\varepsilon_n,\lambda^n,\xi^n})\phi_n\big)\nabla\phi_n - \mu\int_\Omega \frac{\iota_\mu^*(f_0'(V_{\varepsilon_n,\lambda^n,\xi^n})\phi_n)\phi_n}{|x|^2} + o(1)\\ &= \int_\Omega f_0'\big(V_{\varepsilon_n,\lambda^n,\xi^n}(y)\big)(\phi_n(y))^2 + o(1),\end{aligned}$$

which contradicts (4.18) using (4.6). □

Proposition 4.2. *For every $\eta\in(0,1)$, there exist $\varepsilon_0>0$ and $c_0>0$ with the following property: for every $(\lambda,\xi)\in\mathcal{O}_\eta$ and for every $\varepsilon\in(0,\varepsilon_0)$ there exists a unique solution $\phi_{\varepsilon,\lambda,\xi}\in K^\perp_{\varepsilon,\lambda,\xi}$ of equation (4.4) satisfying*

$$\|\phi_{\varepsilon,\lambda,\xi}\|_\mu \le c_0\left(\varepsilon^{\frac{N+2}{2(N-2)}} + \varepsilon^{\frac{1+2\alpha}{4}}\right).$$

Moreover, $\Phi_\varepsilon:\mathcal{O}_\eta\to K^\perp_{\varepsilon,\lambda,\xi}$ defined by $\Phi_\varepsilon(\lambda,\xi):=\phi_{\varepsilon,\lambda,\xi}$ is C^1.

Proof. As in [6] solving (4.4) is equivalent to finding a fixed point of the operator $T_{\varepsilon,\lambda,\xi}:K^\perp_{\varepsilon,\lambda,\xi}\to K^\perp_{\varepsilon,\lambda,\xi}$ defined by

$$T_{\varepsilon,\lambda,\xi}(\phi) = L^{-1}_{\varepsilon,\lambda,\xi}\Pi^\perp_{\varepsilon,\lambda,\xi}\big(\iota_\mu^*\big(f_\varepsilon(V_{\varepsilon,\lambda,\xi}+\phi) - f_0'(V_{\varepsilon,\lambda,\xi})\phi\big) - V_{\varepsilon,\lambda,\xi}\big).$$

We claim that $T_{\varepsilon,\lambda,\xi}$ is a contraction mapping.

First of all, Proposition 4.1, Lemma A.4 and (3.2) imply

$$\begin{aligned}\|T_{\varepsilon,\lambda,\xi}(\phi)\|_\mu &\le C\|\iota_\mu^*(f_\varepsilon(V_{\varepsilon,\lambda,\xi}+\phi) - f_0'(V_{\varepsilon,\lambda,\xi})\phi) - V_{\varepsilon,\lambda,\xi}\|_\mu\\ &\le C\left(\left\|\iota_\mu^*\left(f_\varepsilon(V_{\varepsilon,\lambda,\xi}+\phi) - f_0'(V_{\varepsilon,\lambda,\xi})\phi - \left(\sum_{i=1}^k \tau_i f_0(U_{\delta_i,\xi_i}) + f_0(V_\sigma)\right)\right)\right\|_\mu\right.\\ &\qquad\left. + \left\|\iota_\mu^*\left(\sum_{i=1}^k \tau_i f_0(U_{\delta_i,\xi_i}) + f_0(V_\sigma)\right) - V_{\varepsilon,\lambda,\xi}\right\|_\mu\right)\\ &\le C\left(\left\|f_\varepsilon(V_{\varepsilon,\lambda,\xi}+\phi) - f_0'(V_{\varepsilon,\lambda,\xi})\phi - \left(\sum_{i=1}^k \tau_i f_0(U_{\delta_i,\xi_i}) + f_0(V_\sigma)\right)\right\|_{2N/(N+2)}\right.\\ &\qquad\left. + \sum_{i=1}^k O(\mu\delta_i) + O\left((\mu\sigma^{\frac{N-2}{2}})^{\frac12}\right)\right)\\ &\le C\big\|f_\varepsilon(V_{\varepsilon,\lambda,\xi}+\phi) - f_\varepsilon(V_{\varepsilon,\lambda,\xi}) - f_\varepsilon'(V_{\varepsilon,\lambda,\xi})\phi\big\|_{2N/(N+2)}\\ &\qquad + C\big\|(f_\varepsilon'(V_{\varepsilon,\lambda,\xi}) - f_0'(V_{\varepsilon,\lambda,\xi}))\phi\big\|_{2N/(N+2)}\\ &\qquad + C\big\|f_\varepsilon(V_{\varepsilon,\lambda,\xi}) - f_0(V_{\varepsilon,\lambda,\xi})\big\|_{2N/(N+2)}\\ &\qquad + C\left\|f_0(V_{\varepsilon,\lambda,\xi}) - \left(\sum_{i=1}^k \tau_i f_0(U_{\delta_i,\xi_i}) + f_0(V_\sigma)\right)\right\|_{2N/(N+2)}\\ &\qquad + \sum_{i=1}^k O(\mu\delta_i) + O\left((\mu\sigma^{\frac{N-2}{2}})^{\frac12}\right).\end{aligned}$$

By using Lemma A.5 and noticing that

$$\|f_\varepsilon(V_{\varepsilon,\lambda,\xi}+\phi) - f_\varepsilon(V_{\varepsilon,\lambda,\xi}) - f_\varepsilon'(V_{\varepsilon,\lambda,\xi})\phi\|_{2N/(N+2)} \le C\|\phi\|_\mu^{2^*-1},$$

we deduce

$$\begin{aligned}\|T_{\varepsilon,\lambda,\xi}(\phi)\|_\mu &\le C\|\phi\|_\mu^{2^*-1} + C\varepsilon\|\phi\|_\mu + C\varepsilon\\ &\quad + O(\sigma^{\frac{N+2}{2}}) + \sum_{i=1}^k O\big(\delta_i^{\frac{N+2}{2}}\big) + \sum_{i=1}^k O(\mu\delta_i) + O\big((\mu\sigma^{\frac{N-2}{2}})^{\frac12}\big)\\ &= C\|\phi\|_\mu^{2^*-1} + C\varepsilon\|\phi\|_\mu + O\big(\varepsilon^{\frac{N+2}{2(N-2)}}\big) + O\big(\varepsilon^{\frac{1+2\alpha}{4}}\big).\end{aligned}$$

The remaining argument is standard, see e.g. [6]. □

Now we consider the reduced functional

$$I_\varepsilon(\lambda,\xi) = J_\varepsilon(V_{\varepsilon,\lambda,\xi} + \phi_{\varepsilon,\lambda,\xi}).$$

Proposition 4.3. *If $(\lambda,\xi)\in\mathcal{O}_\eta$ is a critical point of I_ε then $V_{\varepsilon,\lambda,\xi}+\phi_{\varepsilon,\lambda,\xi}$ is a solution of problem* (1.1) *for $\varepsilon>0$ small.*

Proof. It is enough to prove that $V_{\varepsilon,\lambda,\xi}+\phi_{\varepsilon,\lambda,\xi}$ satisfies (4.5). As in [29], equation (4.4) implies that there exist constants $c_{i,j}$, $i=1,\dots,k$ and $j=0,\dots,N$, and c_0 so that:

$$\nabla J_\varepsilon(V_{\varepsilon,\lambda,\xi}+\phi_{\varepsilon,\lambda,\xi})[\omega] = \sum_{i=1}^k\sum_{j=0}^N c_{i,j}P\Psi_i^j + c_0P\overline{\Psi}.$$

It remains to prove that $c_{i,j}=0$ and $c_0=0$, provided $\varepsilon>0$ is small enough.

Let ∂_s denote one of ∂_{λ_i}, $\partial_{\overline{\lambda}}$, $\partial_{(\xi_i)^j}$, $i=1,\dots,k$, $j=1,\dots,N$. If (λ,ξ) is a critical point of $I_\varepsilon(\lambda,\xi)$, then

$$\sum_{i=1}^k\sum_{j=0}^N c_{i,j}\big(P\Psi_i^j, \partial_sV_{\varepsilon,\lambda,\xi}+\partial_s\phi_{\varepsilon,\lambda,\xi}\big)_\mu + c_0\big(P\overline{\Psi}, \partial_sV_{\varepsilon,\lambda,\xi}+\partial_s\phi_{\varepsilon,\lambda,\xi}\big)_\mu = 0. \tag{4.21}$$

Observe that

$$\partial_{\lambda_i}V_{\varepsilon,\lambda,\xi} = \tau_i\varepsilon^{\frac{1}{N-2}}P\Psi_i^0,\quad \partial_{\overline{\lambda}}V_{\varepsilon,\lambda,\xi} = \varepsilon^{\frac{1}{N-2}}P\overline{\Psi},\quad \partial_{(\xi_i)^j}V_{\varepsilon,\lambda,\xi} = \tau_iP\Psi_i^j,\quad j=1,\dots,N. \tag{4.22}$$

On the other hand, $(P\Psi_i^j,\phi_{\varepsilon,\lambda,\xi})_\mu = 0$ for $j=0,1,\dots,N$, Proposition 4.2 and Lemma A.6 imply

$$(P\Psi_i^j,\partial_s\phi_{\varepsilon,\lambda,\xi})_\mu = -(\partial_sP\Psi_i^j,\phi_{\varepsilon,\lambda,\xi}) = O(\|\partial_sP\Psi_i^j\|_\mu\|\phi_{\varepsilon,\lambda,\xi}\|_\mu) = o(\|\partial_sP\Psi_i^j\|_\mu) = o(\delta_i^{-2}).$$

Similarly we have

$$(P\overline{\Psi},\partial_s\phi_{\varepsilon,\lambda,\xi})_\mu = o(\|\partial_sP\overline{\Psi}\|_\mu) = o(\varepsilon^{\frac{1}{N-2}}\sigma^{-2}).$$

Now Lemma A.1, (4.21) and (4.22) yield

$$\begin{aligned}0 &= \sum_{i=1}^k\sum_{j=0}^N c_{i,j}(P\Psi_i^j,\partial_{\overline{\lambda}}V_{\varepsilon,\lambda,\xi}) + c_0(P\overline{\Psi},\partial_{\overline{\lambda}}V_{\varepsilon,\lambda,\xi}) + o\big(\varepsilon^{\frac{1}{N-2}}\sigma^{-2}\big)\\ &= \varepsilon^{\frac{1}{N-2}}\left(\sum_{i=1}^k\sum_{j=0}^N c_{i,j}(P\Psi_i^j,P\overline{\Psi}) + c_0(P\overline{\Psi},P\overline{\Psi})\right) + o\big(\varepsilon^{\frac{1}{N-2}}\sigma^{-2}\big)\\ &= c_0\widetilde{c}_0\varepsilon^{\frac{1}{N-2}}\sigma^{-2}\big(1+o(1)\big),\end{aligned}$$

which implies $c_0=0$. Similar arguments show that $c_{i,j}=0$ for $i=1,\dots,k$, $j=0,1,\dots,N$. □

5 Proof of the main results

As in Section 4 we assume $\mu = \mu_0\varepsilon^\alpha$ and we use the notation $\delta_i = \lambda_i\varepsilon^{\frac{1}{N-2}}$, $\sigma = \overline{\lambda}\varepsilon^{\frac{1}{N-2}}$ from (4.1). We continue to consider $V_{\varepsilon,\lambda,\xi} = -\sum\limits_{i=1}^{k} PU_{\delta_i,\xi_i} + PV_\sigma$ as in (4.3) . The reduced energy is expanded as follows.

Proposition 5.1. *For $\varepsilon \to 0^+$ there holds*

$$I_\varepsilon(\lambda,\xi) = a_1 + a_2\varepsilon - a_3\varepsilon^\alpha - a_4\varepsilon\ln\varepsilon + \psi(\lambda,\xi)\varepsilon + o(\varepsilon) \tag{5.1}$$

C^1-uniformly with respect to (λ,ξ) in compact sets of $\mathcal{O}_\eta$. The constants are given by $a_1 = \frac{1}{N}(k+1)S_0^{\frac{N}{2}}$, $a_2 = \frac{(k+1)}{2^*}\int_{\mathbb{R}^N} U_{1,0}^{2^*}\ln U_{1,0} - \frac{k+1}{(2^*)^2}S_0^{\frac{N}{2}}$, $a_3 = \frac{1}{2}S_0^{\frac{N-2}{2}}\overline{S}\mu_0$, and $a_4 = \frac{k+1}{2\cdot 2^*}\int_{\mathbb{R}^N} U_{1,0}^{2^*}$. The function ψ is given by

$$\begin{aligned}\psi(\lambda,\xi) = b_1\Bigg(&H(0,0)\overline{\lambda}^{N-2} + \sum_{i=1}^{k} H(\xi_i,\xi_i)\lambda_i^{N-2} - 2\sum_{i=1}^{k}\tau_i G(\xi_i,0)\lambda_i^{\frac{N-2}{2}}\overline{\lambda}^{\frac{N-2}{2}}\\ &- 2\sum_{\substack{i,j=1\\ i<j}}^{k}\tau_i\tau_j G(\xi_i,\xi_j)\lambda_i^{\frac{N-2}{2}}\lambda_j^{\frac{N-2}{2}}\Bigg) - b_2\ln(\lambda_1\lambda_2\ldots\lambda_k\overline{\lambda})^{\frac{N-2}{2}}\end{aligned}$$

with $b_1 = \frac{1}{2}C_0\int_{\mathbb{R}^N} U_{1,0}^{2^*-1}$ and $b_2 = \frac{1}{2^*}\int_{\mathbb{R}^N} U_{1,0}^{2^*}$.

Proof. Observe that

$$\begin{aligned}J_\varepsilon(V_{\varepsilon,\lambda,\xi}) &= \frac{1}{2}\int_\Omega\left(|\nabla V_{\varepsilon,\lambda,\xi}|^2 - \mu\frac{|V_{\varepsilon,\lambda,\xi}|^2}{|x|^2}\right) - \frac{1}{2^*}\int_\Omega |V_{\varepsilon,\lambda,\xi}|^{2^*}\\ &\quad + \left(\frac{1}{2^*}\int_\Omega |V_{\varepsilon,\lambda,\xi}|^{2^*} - \frac{1}{2^*-\varepsilon}\int_\Omega |V_{\varepsilon,\lambda,\xi}|^{2^*-\varepsilon}\right)\\ &= I_1 + I_2 + I_3.\end{aligned}$$

By Lemma A.7, Lemma A.10, and noticing $\mu = \mu_0\varepsilon^\alpha$, $\varepsilon \to 0^+$, we obtain

$$\begin{aligned}I_1 &= \frac{1}{2}\int_\Omega\left(\left(|\nabla PV_\sigma|^2 - \mu\frac{|PV_\sigma|^2}{|x|^2}\right) + \sum_{i=1}^{k}\left(|\nabla PU_{\delta_i,\xi_i}|^2 - \mu\frac{|PU_{\delta_i,\xi_i}|^2}{|x|^2}\right)\right.\\ &\quad + \sum_{i=1}^{k}\tau_i\int_\Omega\left(\nabla PV_\sigma\nabla PU_{\delta_i,\xi_i} - \mu\frac{PV_\sigma PU_{\delta_i,\xi_i}}{|x|^2}\right)\\ &\quad + \sum_{\substack{i,j=1\\ i<j}}^{k}\tau_i\tau_j\int_\Omega\left(\nabla PU_{\delta_i,\xi_i}\nabla PU_{\delta_j,\xi_j} - \mu\frac{PU_{\delta_i,\xi_i}PU_{\delta_j,\xi_j}}{|x|^2}\right)\\ &= \frac{1}{2}(k+1)S_0^{\frac{N}{2}} - \frac{N}{4}S_0^{\frac{N-2}{2}}\overline{S}\mu_0\varepsilon^\alpha + \frac{1}{2}C_0\int_{\mathbb{R}^N} U_{1,0}^{2^*-1}\Bigg(-H(0,0)\sigma^{N-2} - \sum_{i=1}^{k} H(\xi_i,\xi_i)\delta_i^{N-2}\\ &\quad + 2\sum_{i=1}^{k}\tau_i\sigma^{\frac{N-2}{2}}\delta_i^{\frac{N-2}{2}}G(\xi_i,0) + 2\sum_{\substack{i,j=1\\ i<j}}^{k}\tau_i\tau_j G(\xi_i,\xi_j)\delta_i^{\frac{N-2}{2}}\delta_j^{\frac{N-2}{2}}\Bigg) + o(\varepsilon).\end{aligned} \tag{5.2}$$

By Lemma A.8 and Lemma A.10, and again using $\mu=\mu_0\varepsilon^\alpha$, $\varepsilon\to 0^+$, we obtain:

$$
\begin{aligned}
I_2 = &-\frac{1}{2^*}(k+1)S_0^{\frac{N}{2}}+\frac{N-2}{4}S_0^{\frac{N-2}{2}}\overline{S}\mu_0\varepsilon^\alpha + C_0\int_{\mathbb{R}^N}U_{1,0}^{2^*-1}\Big(H(0,0)\sigma^{N-2}\\
&+\sum_{i=1}^k H(\xi_i,\xi_i)\delta_i^{N-2}-2\sum_{i=1}^k\tau_i\sigma^{\frac{N-2}{2}}\delta_i^{\frac{N-2}{2}}G(\xi_i,0)-2\sum_{\substack{i,j=1\\ i<j}}^k\tau_i\tau_jG(\xi_i,\xi_j)\delta_i^{\frac{N-2}{2}}\delta_j^{\frac{N-2}{2}}\Big)\\
&+o(\varepsilon).
\end{aligned}
\tag{5.3}
$$

Next Lemma A.8, Lemma A.9 and Lemma A.10 yield:

$$
\begin{aligned}
I_3 &= -\frac{\varepsilon}{(2^*)^2}\int_\Omega |V_{\varepsilon,\lambda,\xi}|^{2^*}+\frac{\varepsilon}{2^*}\int_\Omega |V_{\varepsilon,\lambda,\xi}|^{2^*}\ln|V_{\varepsilon,\lambda,\xi}|+o(\varepsilon)\\
&= -\frac{\varepsilon}{(2^*)^2}(k+1)S_0^{\frac{N}{2}}+\frac{\varepsilon}{2^*}\Big(-\frac{N-2}{2}\ln\sigma\cdot\int_{\mathbb{R}^N}V_1^{2^*}-\frac{N-2}{2}\ln(\delta_1\dots\delta_k)\cdot\int_{\mathbb{R}^N}U_{1,0}^{2^*}\\
&\quad+\int_{\mathbb{R}^N}V_1^{2^*}\ln V_1+k\int_{\mathbb{R}^N}U_{1,0}^{2^*}\ln U_{1,0}\Big)+o(\varepsilon)\\
&= -\frac{\varepsilon}{(2^*)^2}(k+1)S_0^{\frac{N}{2}}-\frac{(N-2)\varepsilon}{2\cdot 2^*}\int_{\mathbb{R}^N}U_{1,0}^{2^*}\cdot\ln(\delta_1\dots\delta_k\sigma)\\
&\quad+\frac{(k+1)\varepsilon}{2^*}\int_{\mathbb{R}^N}U_{1,0}^{2^*}\ln U_{1,0}+o(\varepsilon).
\end{aligned}
\tag{5.4}
$$

Arguing similarly to Lemma 6.1 in [29], we deduce from Proposition 4.2, (3.7), (3.8), and Lemma A.5, that

$$
\begin{aligned}
J_\varepsilon(V_{\varepsilon,\lambda,\xi}+\phi_{\varepsilon,\lambda,\xi})-J_\varepsilon(V_{\varepsilon,\lambda,\xi}) &= \frac12\|\phi_{\varepsilon,\lambda,\xi}\|_\mu^2+\int_\Omega(\nabla V_{\varepsilon,\lambda,\xi}\nabla\phi_{\varepsilon,\lambda,\xi}-\mu\frac{V_{\varepsilon,\lambda,\xi}\phi_{\varepsilon,\lambda,\xi}}{|x|^2})\\
&\quad-\frac{1}{2^*-\varepsilon}(\int_\Omega|V_{\varepsilon,\lambda,\xi}+\phi_{\varepsilon,\lambda,\xi}|^{2^*-\varepsilon}-|V_{\varepsilon,\lambda,\xi}|^{2^*-\varepsilon})\\
&= o(\varepsilon).
\end{aligned}
\tag{5.5}
$$

Now (5.2)–(5.5) imply (5.1). That (5.1) holds C^1-uniformly with respect to (λ,ξ) in compact sets of $\mathcal{O}_\eta$ can be seen as in [29, Lemma 7.1]. We omit the details here. $\square$

Remark 5.2. If $0<\alpha<\frac{N-4}{N-2}$ then $\alpha+\frac{2}{N-2}<1$. From the proof of the above lemma we can see that in that case

$$I_\varepsilon(\lambda,\xi)=a_1+a_2\varepsilon-a_3\varepsilon^\alpha-a_4\varepsilon\ln\varepsilon+\psi(\lambda,\xi)\varepsilon^{\alpha+\frac{2}{N-2}}+o(\varepsilon^{\alpha+\frac{2}{N-2}})$$

with

$$\psi(\lambda,\xi)=-b_3\sum_{i=1}^k\frac{\lambda_i^2}{|\xi_i|^2}.$$

where $b_3=\mu_0C_0^2\int_{\mathbb{R}^N}U_{1,0}^2$ and a_1,a_2,a_3,a_4 are the same as above. Clearly ψ has no critical points, hence I_ε has no critical points for ε small, as stated in Remark 2.2. If $\alpha=\frac{N-4}{N-2}$ then $\alpha+\frac{2}{N-2}=1$, and we have:

$$I_\varepsilon(\lambda,\xi)=a_1+a_2\varepsilon-a_3\varepsilon^\alpha-a_4\varepsilon\ln\varepsilon+\psi(\lambda,\xi)\varepsilon+o(\varepsilon)$$

with

$$
\begin{aligned}
\psi(\lambda,\xi)=b_1\Big(&H(0,0)\overline{\lambda}^{N-2}+\sum_{i=1}^kH(\xi_i,\xi_i)\lambda_i^{N-2}-2\sum_{i=1}^k\tau_iG(\xi_i,0)\lambda_i^{\frac{N-2}{2}}\overline{\lambda}^{\frac{N-2}{2}}\\
&-2\sum_{\substack{i,j=1\\ i<j}}^k\tau_i\tau_jG(\xi_i,\xi_j)\lambda_i^{\frac{N-2}{2}}\lambda_j^{\frac{N-2}{2}}\Big)-b_2\ln(\lambda_1\lambda_2\dots\lambda_k\overline{\lambda})^{\frac{N-2}{2}}-b_3\sum_{i=1}^k\frac{\lambda_i^2}{|\xi_i|^2},
\end{aligned}
$$

Also in this case ψ does not have critical points, hence the functional I_ε does not admit critical points for $\varepsilon > 0$ small.

As a corollary of Proposition 5.1 we obtain the following.

Corollary 5.3. *If (λ, ξ) is a stable (non-degenerate) critical point of ψ then I_ε has for $\varepsilon > 0$ small a critical point $(\lambda_\varepsilon, \xi_\varepsilon)$ that converges towards (λ, ξ) as $\varepsilon \to 0$.*

Proof of Theorem 2.1. The reduced function $\psi(\lambda, \xi)$ from Proposition 5.1 becomes when $k = 1, \tau_1 = -1$, so $\xi = \xi_1$:

$$\psi(\lambda, \xi) = b_1 \left(H(0,0)\overline{\lambda}^{N-2} + H(\xi,\xi)\lambda_1^{N-2} + 2G(\xi,0)\lambda_1^{\frac{N-2}{2}}\overline{\lambda}^{\frac{N-2}{2}} \right) - b_2 \ln(\lambda_1 \overline{\lambda})^{\frac{N-2}{2}}.$$

Observe that ψ is coercive, that is $\psi(\lambda, \xi) \to \infty$ as $(\lambda, \xi) \to \partial(\mathbb{R}^+ \times \mathbb{R}^+ \times (\Omega \setminus \{0\}))$; here $\partial\mathbb{R}^+ = \{0, \infty\}$. From

$$\lambda_1 \frac{\partial \psi(\lambda,\xi)}{\partial \lambda_1} = (N-2)b_1 \left(H(\xi,\xi)\lambda_1^{N-2} + G(\xi,0)\lambda_1^{\frac{N-2}{2}}\overline{\lambda}^{\frac{N-2}{2}} \right) - \frac{(N-2)b_2}{2} \tag{5.6}$$

and

$$\overline{\lambda} \frac{\partial \psi(\lambda,\xi)}{\partial \overline{\lambda}} = (N-2)b_1 \left(H(0,0)\overline{\lambda}^{N-2} + G(\xi,0)\lambda_1^{\frac{N-2}{2}}\overline{\lambda}^{\frac{N-2}{2}} \right) - \frac{(N-2)b_2}{2}, \tag{5.7}$$

we deduce that $\nabla_\lambda \psi(\lambda, \xi) = 0$ implies

$$\lambda_1^{N-2} H(\xi,\xi) = \overline{\lambda}^{N-2} H(0,0),$$

and then

$$\lambda_1 = \lambda_1(\xi) = \left(\frac{b_2}{2b_1} \cdot \frac{1}{H(\xi,\xi) + G(\xi,0)\left(\frac{h(t,t)}{h(0,0)}\right)^{\frac{1}{2}}} \right)^{\frac{1}{N-2}}$$

and

$$\overline{\lambda} = \overline{\lambda}(\xi) = \left(\frac{b_2}{2b_1} \cdot \frac{1}{H(0,0) + G(\xi,0)\left(\frac{H(0,0)}{H(\xi,\xi)}\right)^{\frac{1}{2}}} \right)^{\frac{1}{N-2}}.$$

Thus for fixed $\xi \in \Omega \setminus \{0\}$ the function $\psi(\,\cdot\,, \xi)$ has a unique critical point $\lambda(\xi) = (\lambda_1(\xi), \overline{\lambda}(\xi))$, which must be its global minimum. An elementary computation shows that $\lambda(\xi)$ is a non-degenerate minimum of $\psi(\,\cdot\,, \xi)$.

Now we consider the reduced function $\nu : \Omega \setminus \{0\} \to \mathbb{R}$ defined by $\nu(\xi) = \psi(\lambda(\xi), \xi)$. The above considerations show that (λ, ξ) is a critical point of ψ if, and only if, $\lambda = \lambda(\xi)$ and ξ is a critical point of ν. Moreover, (λ, ξ) is a stable or non-degenerate critical point of ψ iff ξ is stable or non-degenerate critical point of ν. Also the critical groups are isomorphic, the Morse indices, nullities are the same. A direct computation gives:

$$\nu(\xi) = b_2 - b_2 \ln \frac{b_2}{2b_1} + b_2 \ln \varphi(\xi).$$

Consequently, ν and φ have the same critical points with the same Morse indices, nullities, critical groups. Theorem 2.1 now follows from Corollary 5.3. □

Proof of Theorem 2.3. Since ψ is coercive one can minimize I_ε, for $\varepsilon > 0$ small, as in [6, Theorem 1.1(i)]. □

Proof of Theorem 2.4. The idea is as in the proof of [6, Theorem 1.2]. There exists a compact subset

$C \subset \mathbb{R}^+ \times \mathbb{R}^+ \times (\Omega \setminus \{0\})$ with $\operatorname{cat}(C) = \operatorname{cat}(\Omega \setminus \{0\}) =: k$. Since ψ is coercive there exists a compact neighborhood U of C in $\mathbb{R}^+ \times \mathbb{R}^+ \times (\Omega \setminus \{0\})$ such that $\min\limits_{\partial U} \psi > \max\limits_{C} \psi$. Then $\min\limits_{\partial U} I_\varepsilon > \max\limits_{C} I_\varepsilon =: c$ for $\varepsilon > 0$ small. Now standard Lusternik-Schnirelmann theory implies that I_ε has at least k critical points in the sublevel set I_ε^c. Theorem 2.4 follows now from Proposition 4.3. $\square$

Proof of Theorem 2.5. The invariance of Ω under the action of $\Gamma \subset O(N)$ implies $J_\varepsilon(g * u) = J_\varepsilon(u)$ and $g * V_{\varepsilon,\lambda,x} = V_{\varepsilon,\lambda,gx}$, for all $g \in \Gamma$. Then Proposition 4.2 yields $g * \phi_{\varepsilon,\lambda,x} = V_{\varepsilon,\lambda,gx}$, and consequently $I_\varepsilon(\lambda, gx) = I_\varepsilon(\lambda, x)$, for all $g \in \Gamma$. Now the proof proceeds as the one of Theorem 2.4 using $\operatorname{cat}_\Gamma$ instead of cat. $\square$

Proof of Theorem 2.7. Using the equivariance properties proved above the principle of symmetric criticality implies that a critical point (λ, ξ) of I_ε constrained to the fixed point set $\mathbb{R}^+ \times \mathbb{R}^+ \times (\Omega^\Sigma \setminus \{0\})$ is a critical point of I_ε, hence induces a critical point $u_\varepsilon = V_{\varepsilon,\lambda,\xi} + \phi_{\varepsilon,\lambda,\xi}$ of J_ε. Clearly we have $V_{\varepsilon,\lambda,\xi} \in (H_\mu)^\Sigma = \{u \in H_\mu : g * u = u \text{ for all } g \in \Sigma\}$ because $\xi \in \Omega^\Sigma$. This implies $u_\varepsilon \in (H_\mu)^\Sigma$. $\square$

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A Appendix

In this appendix we prove several lemmas that were used in the proofs of our main results. The lemmas are more general than needed and will be of use also in subsequent work. We fix $0 < \eta < \min\{|\xi_i|,\ \operatorname{dist}(\xi_i, \partial\Omega),\ |\xi_{i_1} - \xi_{i_2}|,\ i, i_1, i_2 = 1, \ldots, k\}$. Recall the functions $\overline{\Psi}$, Ψ_i^j defined in (4.2) and their dependence on $\mu, \sigma, \delta_i \in \mathbb{R}^+$, $\xi_i \in \Omega$.

Lemma A.1. *For $i, l = 1, \ldots, k$, and $j, h = 0, 1, \ldots, N$, with $i \neq l$ or $j \neq h$, there are constants $\widetilde{c}_0 > 0$, $\widetilde{c}_{i,j} > 0$ such that the following estimates hold uniformly for $0 < \mu < \overline{\mu}$.*

$$P\overline{\Psi}, P\overline{\Psi})_\mu = \widetilde{c}_0 \frac{1}{\sigma^2} + o(\sigma^{-2}) \quad \textit{for } \sigma \to 0. \tag{A.1}$$

$$(P\overline{\Psi}, P\Psi_i^j)_\mu = o(\sigma^{-2}) o(\delta_i^{-2}) \quad \textit{as } \sigma \to 0,\ \delta_i \to 0, \textit{ uniformly for } \xi_i \textit{ in a compact subset of } \Omega. \tag{A.2}$$

$$(P\Psi_i^j, P\Psi_i^j)_\mu = \widetilde{c}_{i,j} \frac{1}{\delta_i^2} + o(\delta_i^{-2}) \quad \textit{as } \delta_i \to 0, \textit{ uniformly for } \xi_i \textit{ in a compact subset of } \Omega. \tag{A.3}$$

$$(P\Psi_i^j, P\Psi_l^h)_\mu = o(\delta_i^{-2}) \quad \textit{as } \delta_i \to 0, \textit{ uniformly for } \xi_i, \xi_l \textit{ in a compact subset of } \Omega. \tag{A.4}$$

Proof. We only prove (A.1), and (A.2) with $j = 0$; b) with $j \neq 0$ is similar. Parts (A.3) and (A.4) are from Lemma A.5 in [29]. In order to prove (A.1), recall that $\overline{\Psi}$ is an eigenfunction to (3.4) with

$\Lambda = 2^* - 1$. Then by Proposition (3.2) we have

$$
\begin{aligned}
(P\overline{\Psi}, P\overline{\Psi})_\mu &= \int_\Omega |\nabla P\overline{\Psi}|^2 - \mu \frac{|P\overline{\Psi}|^2}{|x|^2} = \int_\Omega \nabla\overline{\Psi}\nabla P\overline{\Psi} - \mu\frac{\overline{\Psi}P\overline{\Psi}}{|x|^2} - \mu\frac{(P\overline{\Psi}-\overline{\Psi})P\overline{\Psi}}{|x|^2} \\
&= (2^*-1)\int_\Omega V_\sigma^{2^*-2}\overline{\Psi}^2 - (2^*-1)\int_\Omega V_\sigma^{2^*-2}\overline{\Psi}(\overline{\Psi}-P\overline{\Psi}) - \mu\frac{(P\overline{\Psi}-\overline{\Psi})P\overline{\Psi}}{|x|^2} \\
&= (2^*-1)\int_\Omega V_\sigma^{2^*-2}\overline{\Psi}^2 + O(\sigma^{\frac{N-2}{2}}) \\
&= \frac{(N^2-4)C_\mu^{2^*}}{4}\int_\Omega \frac{\sigma^2}{(\sigma^2|x|^{\beta_1}+|x|^{\beta_2})^2}\cdot\sigma^{N-4}\cdot\frac{(|x|^{\beta_2}-\sigma^2|x|^{\beta_1})^2}{(\sigma^2|x|^{\beta_1}+|x|^{\beta_2})^N} + O(\sigma^{\frac{N-2}{2}}) \\
&= \frac{(N^2-4)C_\mu^{2^*}}{4}\int_{\Omega_{\sigma,\mu}} \frac{1}{\sigma^2}\cdot\frac{(|y|^{\beta_2}-|y|^{\beta_1})^2}{(|y|^{\beta_1}+|y|^{\beta_2})^{2+N}} + O(\sigma^{\frac{N-2}{2}}) \\
&= \frac{(N^2-4)C_\mu^{2^*}}{4}\int_{\mathbb{R}^N} \frac{1}{\sigma^2}\cdot\frac{(|y|^{\beta_2}-|y|^{\beta_1})^2}{(|y|^{\beta_1}+|y|^{\beta_2})^{2+N}} + o(\sigma^{-2}) \\
&= \widetilde{c}_0\frac{1}{\sigma^2} + o(\sigma^{-2}),
\end{aligned}
$$

for a positive constant $\widetilde{c}_0$. Here $\Omega_{\sigma,\mu} := \left\{ y = \sigma^{-\frac{\sqrt{\overline{\mu}}}{\sqrt{\overline{\mu}-\mu}}}x : x \in \Omega \right\}$. Similarly we compute:

$$
\begin{aligned}
(P\overline{\Psi}, P\Psi_i^0)_\mu &= \int_\Omega \nabla P\overline{\Psi}\cdot\nabla P\Psi_i^0 - \mu\frac{P\overline{\Psi}\cdot P\Psi_i^0}{|x|^2} = \int_\Omega \nabla\overline{\Psi}\nabla P\Psi_i^0 - \mu\frac{\overline{\Psi}P\Psi_i^0}{|x|^2} - \mu\frac{(P\overline{\Psi}-\overline{\Psi})P\Psi_i^0}{|x|^2} \\
&= (2^*-1)\int_\Omega V_\sigma^{2^*-2}\overline{\Psi}\Psi_i^0 + O(\sigma^{\frac{N-2}{2}}) \\
&= C_0C_\mu^{2^*-1}(2^*-1)\int_\Omega \frac{\sigma^2}{(\sigma^2|x|^{\beta_1}+|x|^{\beta_2})^2}\cdot\frac{N-2}{2}\cdot\sigma^{\frac{N-4}{2}}\cdot\frac{(|x|^{\beta_2}-\sigma^2|x|^{\beta_1})}{(\sigma^2|x|^{\beta_1}+|x|^{\beta_2})^{\frac{N}{2}}} \\
&\quad \cdot\frac{N-2}{2}\cdot\delta_i^{\frac{N-4}{2}}\cdot\frac{(|x-\xi_i|^2-\delta_i^2)}{(\delta_i^2+|x-\xi_i|^2)^{\frac{N}{2}}} + O(\sigma^{\frac{N-2}{2}}) \\
&= o(\sigma^{-2})o(\delta_i^{-2})
\end{aligned}
$$

□

Lemma A.2. *a) For $i = 1,\dots,k$, and $j = 0,1,\dots,N$, there holds:*

$$
\|P\Psi_i^j - \Psi_i^j\|_{2N/(N-2)} = \begin{cases} O\big(\delta_i^{\frac{N-2}{2}}\big) & \text{if } j = 1,2,\dots,N, \\ O\big(\delta_i^{\frac{N-4}{2}}\big) & \text{if } j = 0 \end{cases}
$$

as $\delta_i \to 0$ uniformly for ξ_i in a compact subset of Ω.

b)

$$
\|P\overline{\Psi} - \overline{\Psi}\|_{2N/(N-2)} = O\big(\sigma^{\frac{N-4}{2}}\big)
$$

as $\sigma \to 0$, uniformly for $0 < \mu < \overline{\mu}$.

Proof. a) can be proved as Lemma B.4 in [28], and b) can be obtained similarly using Proposition 3.2.
□

Lemma A.3. *a) For $i,l=1,\dots,k$ there holds*

$$\left\|(f_0'(\sum_{i=1}^k\tau_iPU_{\delta_i,\xi_i}+PV_\sigma)-f_0'(U_{\delta_l,\xi_l}))\Psi_l^h\right\|_{2N/(N+2)}$$
$$\le\begin{cases}O\big(\sigma^{\frac{N-2}{2}}\big)+\sum_{i=1}^kO\big(\delta_i^{\frac{N-2}{2}}\big) & \text{if } h=1,\dots,N,\\ O\big(\sigma^{\frac{N-2}{2}}\big)+\sum\limits_{\substack{i=1\\i\neq l}}^kO\big(\delta_i^{\frac{N-2}{2}}\big)+O\big(\delta_l^{\frac{N-4}{2}}\big) & \text{if } h=0;\end{cases}$$

as $\sigma,\delta_i,\delta_l\to0$ uniformly for $0<\mu<\overline{\mu}$ and ξ in a compact subset of Ω_η.

b)
$$\left\|\left(f_0'\left(\sum_{i=1}^k\tau_iPU_{\delta_i,\xi_i}+PV_\sigma\right)-f_0'(V_\sigma)\right)\overline{\Psi}\right\|_{2N/(N+2)}\le O(\sigma^{\frac{N-4}{2}})+\sum_{i=1}^kO(\delta_i^{\frac{N-2}{2}})$$

as $\sigma,\delta_i\to0$ uniformly for $0<\mu<\overline{\mu}$ and ξ in Ω_η.

Proof. We only prove b).

$$\int_\Omega\left|\left(f_0'\left(\sum_{i=1}^k\tau_iPU_{\delta_i,\xi_i}+PV_\sigma\right)-f_0'(U_{\delta_l,\xi_l})\right)\Psi_l^h\right|^{2N/(N+2)}$$
$$=\int_{B(\xi_l,\frac{\eta}{2})}\left|\left(f_0'\left(\sum_{i=1}^k\tau_iPU_{\delta_i,\xi_i}+PV_\sigma\right)-f_0'(U_{\delta_l,\xi_l})\right)\Psi_l^h\right|^{2N/(N+2)}$$
$$+\int_{B(0,\frac{\eta}{2})\bigcup\bigcup_{\substack{i=1\\i\neq l}}^kB(\xi_i,\frac{\eta}{2})}\left|\left(f_0'\left(\sum_{i=1}^k\tau_iPU_{\delta_i,\xi_i}+PV_\sigma\right)-f_0'(U_{\delta_l,\xi_l})\right)\Psi_l^h\right|^{2N/(N+2)}$$
$$+\int_{\Omega\setminus(B(0,\frac{\eta}{2})\bigcup\bigcup_{i=1}^kB(\xi_i,\frac{\eta}{2}))}\left|\left(f_0'\left(\sum_{i=1}^k\tau_iPU_{\delta_i,\xi_i}+PV_\sigma\right)-f_0'(U_{\delta_l,\xi_l})\right)\Psi_l^h\right|^{2N/(N+2)}.$$

First of all, (3.5) and (3.8) yield

$$\int_{B(\xi_l,\frac{\eta}{2})}\left|\left(f_0'\left(\sum_{i=1}^k\tau_iPU_{\delta_i,\xi_i}+PV_\sigma\right)-f_0'(U_{\delta_l,\xi_l})\right)\Psi_l^h\right|^{2N/(N+2)}$$
$$\le\int_{B(\xi_l,\frac{\eta}{2})}\left|(f_0'(PU_{\delta_l,\xi_l})-f_0'(U_{\delta_l,\xi_l}))\Psi_l^h\right|^{2N/(N+2)}+O(\sigma^{\frac{N(N-2)}{N+2}})+\sum_{\substack{i=1\\i\neq l}}^kO(\delta_i^{\frac{N(N-2)}{N+2}})$$
$$\le O\big(\sigma^{\frac{N(N-2)}{N+2}}\big)+\sum_{i=1}^kO\big(\delta_i^{\frac{N(N-2)}{N+2}}\big).$$

For $i\neq l$, we have

$$\int_{B(\xi_i,\frac{\eta}{2})}|(f_0'(\sum_{i=1}^k\tau_iPU_{\delta_i,\xi_i}+PV_\sigma)-f_0'(U_{\delta_l,\xi_l}))\Psi_l^h|^{2N/(N+2)}$$
$$=\int_{B(\xi_i,\frac{\eta}{2})}\left|\left(f_0'\left(PU_{\delta_i,\xi_i}\right)+O\big(\sigma^{\frac{N-2}{2}}\big)+\sum_{\substack{j=1\\j\neq i,j\neq l}}^kO\big(\delta_j^{\frac{N-2}{2}}\big)+O\big(\delta_l^2\big)\right)\Psi_l^h\right|^{2N/(N+2)}$$
$$=\begin{cases}O\big(\delta_l^{\frac{N(N-2)}{N+2}}\big) & \text{if } h=1,\dots,N,\\ O\big(\delta_l^{\frac{N(N-4)}{N+2}}\big) & \text{if } h=0.\end{cases}$$

At last,

$$\int_{\Omega\setminus\left(B(0,\frac{\eta}{2})\bigcup_{i=1}^k B(\xi_i,\frac{\eta}{2})\right)}\left|\left(f_0'\left(\sum_{i=1}^k \tau_i PU_{\delta_i,\xi_i}+PV_\sigma\right)-f_0'(U_{\delta_l,\xi_l})\right)\Psi_l^h\right|^{2N/(N+2)}$$
$$\leq \begin{cases} O\big(\delta_l^{\frac{N(N-2)}{N+2}}\big)\left(O\big(\sigma^{\frac{4N}{N+2}}\big)+\sum_{i=1}^k O\big(\delta_i^{\frac{4N}{N+2}}\big)\right) & \text{if } h=1,\dots,N,\\ O\big(\delta_l^{\frac{N(N-4)}{N+2}}\big)\left(O\big(\sigma^{\frac{4N}{N+2}}\big)+\sum_{i=1}^k O\big(\delta_i^{\frac{4N}{N+2}}\big)\right) & \text{if } h=0.\end{cases}$$

Now b) follows. □

Lemma A.4.

$$\left\|\iota_\mu^*\left(\sum_{i=1}^k \tau_i f_0(U_{\delta_i,\xi_i})+f_0(V_\sigma)\right)-V_{\varepsilon,\lambda,\xi}\right\|_\mu \leq \sum_{i=1}^k O(\mu\delta_i)+O\big((\mu\sigma^{\frac{N-2}{2}})^{\frac12}\big)$$

as $\mu,\sigma,\delta_i\to 0$ uniformly for ξ in Ω_η.

Proof. By (3.1), there holds

$$\begin{aligned}\int_\Omega \nabla\iota_\mu^*(f_0(V_\sigma))\nabla\big(\iota_\mu^*(f_0(V_\sigma))-PV_\sigma\big)-\mu\int_\Omega \frac{\iota_\mu^*(f_0(V_\sigma))\big(\iota_\mu^*(f_0(V_\sigma))-PV_\sigma\big)}{|x|^2}\\ =\int_\Omega f_0(V_\sigma)(\iota_\mu^*(f_0(V_\sigma))-PV_\sigma).\end{aligned}\tag{A.5}$$

We also have

$$\begin{cases}-\Delta PV_\sigma=-\Delta V_\sigma=\mu\frac{V_\sigma}{|x|^2}+f_0(V_\sigma) & \text{in } \Omega,\\ PV_\sigma=0 & \text{on } \partial\Omega.\end{cases}$$

Now we obtain

$$\begin{aligned}\int_\Omega \nabla PV_\sigma\nabla\big(\iota_\mu^*(f_0(V_\sigma))-PV_\sigma\big)-\mu\int_\Omega\frac{V_\sigma(\iota_\mu^*(f_0(V_\sigma))-PV_\sigma)}{|x|^2}\\ =\int_\Omega f_0(V_\sigma)\big(\iota_\mu^*(f_0(V_\sigma))-PV_\sigma\big).\end{aligned}\tag{A.6}$$

Combining (A.5) and (A.6) yields

$$\int_\Omega|\nabla\big(\iota_\mu^*(f_0(V_\sigma))-PV_\sigma\big)|^2=\mu\int_\Omega\frac{\big(\iota_\mu^*(f_0(V_\sigma))-V_\sigma\big)\big(\iota_\mu^*(f_0(V_\sigma))-PV_\sigma\big)}{|x|^2}.$$

Next (3.5) implies

$$\|\iota_\mu^*(f_0(V_\sigma))-PV_\sigma\|_\mu=(\mu\int_\Omega\frac{|(V_\sigma-PV_\sigma)(\iota_\mu^*(f_0(V_\sigma))-PV_\sigma)|}{|x|^2})^{\frac12}\leq O((\mu\sigma^{\frac{N-2}{2}})^{\frac12}).\tag{A.7}$$

Similarly to (A.5) and (A.6) we also have

$$\begin{aligned}\int_\Omega\nabla\iota_\mu^*(f_0(U_{\delta_i,\xi_i}))\nabla(\iota_\mu^*(f_0(U_{\delta_i,\xi_i}))-PU_{\delta_i,\xi_i})-\mu\int_\Omega\frac{\iota_\mu^*(f_0(U_{\delta_i,\xi_i}))(\iota_\mu^*(f_0(U_{\delta_i,\xi_i}))-PU_{\delta_i,\xi_i})}{|x|^2}\\ =\int_\Omega f_0(U_{\delta_i,\xi_i})(\iota_\mu^*(f_0(U_{\delta_i,\xi_i}))-PU_{\delta_i,\xi_i})\end{aligned}$$

and

$$\int_\Omega\nabla PU_{\delta_i,\xi_i}\nabla(\iota_\mu^*(f_0(U_{\delta_i,\xi_i}))-PU_{\delta_i,\xi_i})=\int_\Omega f_0(U_{\delta_i,\xi_i})(\iota_\mu^*(f_0(U_{\delta_i,\xi_i}))-PU_{\delta_i,\xi_i}).$$

Then

$$\int_\Omega|\nabla(\iota_\mu^*(f_0(U_{\delta_i,\xi_i}))-PU_{\delta_i,\xi_i})|^2=\mu\int_\Omega\frac{\iota_\mu^*(f_0(U_{\delta_i,\xi_i}))(\iota_\mu^*(f_0(U_{\delta_i,\xi_i}))-PU_{\delta_i,\xi_i})}{|x|^2}.$$

Therefore, by the Hölder and Hardy inequalities,

$$
\begin{aligned}
\|\iota_\mu^*(f_0(U_{\delta_i,\xi_i}))-PU_{\delta_i,\xi_i}\|_\mu &= \left(\mu\int_\Omega \frac{PU_{\delta_i,\xi_i}(\iota_\mu^*(f_0(U_{\delta_i,\xi_i}))-PU_{\delta_i,\xi_i})}{|x|^2}\right)^{\frac{1}{2}}\\
&\le \mu^{\frac{1}{2}}\left(\int_\Omega \frac{(PU_{\delta_i,\xi_i})^2}{|x|^2}\right)^{\frac{1}{4}}\left(\int_\Omega \frac{(\iota_\mu^*(f_0(U_{\delta_i,\xi_i}))-PU_{\delta_i,\xi_i})^2}{|x|^2}\right)^{\frac{1}{4}}\\
&\le C(\mu\delta_i\|\iota_\mu^*(f_0(U_{\delta_i,\xi_i}))-PU_{\delta_i,\xi_i}\|_\mu)^{\frac{1}{2}},
\end{aligned}
$$

which implies

$$
\|\iota_\mu^*(f_0(U_{\delta_i,\xi_i}))-PU_{\delta_i,\xi_i}\|_\mu \le O(\mu\delta_i). \tag{A.8}
$$

Hence, Lemma A.4 follows from (A.7) and (A.8). $\square$

Lemma A.5. *The following estimates hold uniformly for $0<\mu<\overline{\mu}$ and $(\lambda,\xi)\in\Omega_\eta$.*

$$
\|(f_\varepsilon'(V_{\varepsilon,\lambda,\xi})-f_0'(V_{\varepsilon,\lambda,\xi}))\phi\|_{2N/(N+2)} = O(\varepsilon)\|\phi\|_\mu \quad \text{as } \varepsilon\to 0. \tag{A.9}
$$

$$
|f_\varepsilon(V_{\varepsilon,\lambda,\xi})-f_0(V_{\varepsilon,\lambda,\xi})\|_{2N/(N+2)} = O(\varepsilon) \quad \text{as } \varepsilon\to 0. \tag{A.10}
$$

$$
\left\|f_0(V_{\varepsilon,\lambda,\xi})-\left(\sum_{i=1}^k \tau_i f_0(U_{\delta_i,\xi_i})+f_0(V_\sigma)\right)\right\|_{2N/(N+2)} = O\big(\sigma^{\frac{N+2}{2}}\big)+\sum_{i=1}^k O\big(\delta_i^{\frac{N+2}{2}}\big) \quad \text{as } \sigma,\delta_i\to 0. \tag{A.11}
$$

Proof. We only prove (A.11), since (A.9) and (A.10) are easier. Using (3.7) and (3.8) we can estimate:

$$
\begin{aligned}
&\left(\int_{B(0,\frac{\eta}{2})}\left|f_0(V_{\varepsilon,\lambda,\xi})-\left(\sum_{i=1}^k \tau_i f_0(U_{\delta_i,\xi_i})+f_0(V_\sigma)\right)\right|^{2N/(N+2)}\right)^{(N+2)/2N}\\
&\le C\left(\int_{B(0,\frac{\eta}{2})}|(PV_\sigma)^{2^*-1}-V_\sigma^{2^*-1}|^{2N/(N+2)}\right)^{(N+2)/2N}+\sum_{i=1}^k O(\delta_i^{\frac{N+2}{2}})\\
&\le C\sigma^{\frac{N-2}{2}}\left(\int_{B(0,\frac{\eta}{2})}|(V_\sigma)^{2^*-2}|^{2N/(N+2)}\right)^{(N+2)/2N}+\sum_{i=1}^k O(\delta_i^{\frac{N+2}{2}})\\
&= O(\sigma^{\frac{N+2}{2}})+\sum_{i=1}^k O(\delta_i^{\frac{N+2}{2}}),
\end{aligned}
$$

We also have:

$$
\begin{aligned}
&\left(\int_{B(\xi_i,\frac{\eta}{2})}\left|f_0(V_{\varepsilon,\lambda,\xi})-\left(\sum_{i=1}^k \tau_i f_0(U_{\delta_i,\xi_i})+f_0(V_\sigma)\right)\right|^{2N/(N+2)}\right)^{(N+2)/2N}\\
&\le C\left(\int_{B(\xi_i,\frac{\eta}{2})}|(PU_{\delta_i,\xi_i})^{2^*-1}-U_{\delta_i,\xi_i}^{2^*-1}|^{2N/(N+2)}\right)^{(N+2)/2N}+O(\sigma^{\frac{N+2}{2}})+\sum_{\substack{j=1\\ j\neq i}}^k O(\delta_j^{\frac{N+2}{2}})\\
&= O(\sigma^{\frac{N+2}{2}})+\sum_{i=1}^k O(\delta_i^{\frac{N+2}{2}}),
\end{aligned}
$$

and

$$
\Big(\int_{\Omega\setminus\left(B(0,\frac{\eta}{2})\bigcup_{i=1}^{k}B(\xi_i,\frac{\eta}{2})\right)}\left|f_0(V_{\varepsilon,\lambda,\xi})-\left(\sum_{i=1}^{k}\tau_i f_0(U_{\delta_i,\xi_i})+f_0(V_\sigma)\right)\right|^{2N/(N+2)}\Big)^{(N+2)/2N}
$$

$$
=O(\sigma^{\frac{N+2}{2}})+\sum_{i=1}^{k}O(\delta_i^{\frac{N+2}{2}}).
$$

Now (A.11) follows immediately. □

Lemma A.6. *The following estimates hold for $i=1,\dots,k$, $j,l=1,\dots,N$ with $j\neq l$, uniformly for $0<\mu<\overline{\mu}$ and $(\lambda,\xi)\in\Omega_\eta$.*

$$
\|\partial_{\lambda_i}P\Psi_i^j\|_\mu=O(\varepsilon^{\frac{1}{N-2}}\delta_i^{-2})\quad as\ \varepsilon,\delta_i\to 0. \tag{A.12}
$$

$$
\|\partial_{(\xi_i)^j}P\Psi_i^j\|_\mu=O(\delta_i^{-2})\quad as\ \delta_i\to 0. \tag{A.13}
$$

$$
\|\partial_{(\xi_i)^l}P\Psi_i^j\|_\mu=O(\delta_i^{-2})\quad as\ \delta_i\to 0. \tag{A.14}
$$

$$
\|\partial_{\lambda_i}P\Psi_i^0\|_\mu=O\big(\varepsilon^{\frac{1}{N-2}}\delta_i^{-2}\big)\quad as\ \varepsilon,\delta_i\to 0. \tag{A.15}
$$

$$
\|\partial_{(\xi_i)^j}P\Psi_i^0\|_\mu=O(\delta_i^{-2})\quad as\ \delta_i\to 0. \tag{A.16}
$$

$$
\|\partial_{\overline{\lambda}}P\overline{\Psi}\|_\mu=O(\varepsilon^{\frac{1}{N-2}}\sigma^{-2})\quad as\ \varepsilon,\sigma\to 0. \tag{A.17}
$$

Proof. We only prove (A.12) and (A.17) here because the proofs of the other parts are analogous. For (A.12) we compute:

$$
\begin{aligned}
\|\partial_{\lambda_i}P\Psi_i^j\|_\mu^2&=\varepsilon^{\frac{2}{N-2}}\|\partial_{\delta_i}P\Psi_i^j\|_\mu^2\\
&\le C\varepsilon^{\frac{2}{N-2}}\int_\Omega\nabla\partial_{\delta_i}\Psi_i^j\nabla\partial_{\delta_i}P\Psi_i^j\\
&=C\varepsilon^{\frac{2}{N-2}}\int_\Omega\Big((2^*-1)(2^*-2)U_{\delta_i,\xi_i}^{2^*-3}\Psi_i^j\Psi_i^0+(2^*-1)U_{\delta_i,\xi_i}^{2^*-2}\partial_{\delta_i}\Psi_i^j\Big)\partial_{\delta_i}P\Psi_i^j\\
&=O(\varepsilon^{\frac{2}{N-2}}\delta_i^{-4}).
\end{aligned}
$$

And (A.17) is obtained as follows.

$$
\begin{aligned}
\|\partial_{\overline{\lambda}}P\overline{\Psi}\|_\mu^2&=\varepsilon^{\frac{2}{N-2}}\|\partial_\sigma P\overline{\Psi}\|_\mu^2\\
&=\varepsilon^{\frac{2}{N-2}}\int_\Omega\nabla\partial_\sigma\overline{\Psi}\nabla\partial_\sigma P\overline{\Psi}-\mu\frac{\partial_\sigma\overline{\Psi}\partial_\sigma P\overline{\Psi}}{|x|^2}+\mu\frac{\partial_\sigma P\overline{\Psi}(\partial_\sigma\overline{\Psi}-\partial_\sigma P\overline{\Psi})}{|x|^2}\\
&=\varepsilon^{\frac{2}{N-2}}\left(\int_\Omega((2^*-1)(2^*-2)V_\sigma^{2^*-3}\overline{\Psi}^2+(2^*-1)V_\sigma^{2^*-2}\partial_\sigma\overline{\Psi})\partial_\sigma P\overline{\Psi}+o(1)\right)\\
&=O(\varepsilon^{\frac{2}{N-2}}\sigma^{-4}).
\end{aligned}
$$

□

Lemma A.7. *For $i,j=1,\dots,k$, $i\neq j$, the following estimates hold uniformly for $(\lambda,\xi)\in\Omega_\eta$.*

a) For $\mu,\sigma\to 0$:

$$
\begin{aligned}
&\int_\Omega|\nabla PV_{\mu,\sigma}|^2-\mu\frac{|PV_{\mu,\sigma}|^2}{|x|^2}\\
&\quad=S_\mu^{\frac{N}{2}}-C_0C_\mu^{2^*-1}H(0,0)\sigma^{N-2}\int_{\mathbb{R}^N}\frac{1}{(|z|^{\beta_1}+|z|^{\beta_2})^{\frac{N+2}{2}}}+O(\mu\sigma^{N-2})+O(\sigma^N).
\end{aligned}\tag{A.18}
$$

b) For $\mu, \delta_i, \sigma \to 0$:

$$\begin{aligned}&\int_\Omega \nabla PV_{\mu,\sigma} \nabla PU_{\delta_i,\xi_i} - \mu \frac{PV_{\mu,\sigma} PU_{\delta_i,\xi_i}}{|x|^2} \\ &\quad = C_0 C_\mu^{2^*-1} \sigma^{\frac{N-2}{2}} \delta_i^{\frac{N-2}{2}} \int_{\mathbb{R}^N} \frac{G(\xi_i, 0)}{(|z|^{\beta_1} + |z|^{\beta_2})^{\frac{N+2}{2}}} + O\big(\mu \sigma^{\frac{N-2}{2}} \delta_i^{\frac{N-2}{2}}\big) + o\big(\sigma^{\frac{N-2}{2}} \delta_i^{\frac{N-2}{2}}\big).\end{aligned} \tag{A.19}$$

c) For $\delta_i \to 0$:

$$\int_\Omega \frac{|PU_{\delta_i,\xi_i}|^2}{|x|^2} = \frac{C_0^2 \delta_i^2}{|\xi_i|^2} \int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^{N-2}} + O(\delta_i^4). \tag{A.20}$$

d) For $\delta_i \to 0$:

$$\int_\Omega \frac{PU_{\delta_i,\xi_i} PU_{\delta_j,\xi_j}}{|x|^2} = O\big(\delta_i^{\frac{N-2}{2}} \delta_j^{\frac{N-2}{2}}\big). \tag{A.21}$$

e) For $\delta_i, \delta_j, \sigma \to 0$:

$$\int_\Omega |\nabla PU_{\delta_i,\xi_i}|^2 = S_0^{\frac{N}{2}} - C_0^{2^*} H(\xi_i, \xi_i) \delta_i^{N-2} \int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^{\frac{N+2}{2}}} + o(\delta_i^{N-2}). \tag{A.22}$$

f) For $\mu, \delta_i, \delta_j, \sigma \to 0$:

$$\int_\Omega \nabla PU_{\delta_i,\xi_i} \nabla PU_{\delta_j,\xi_j} = C_0^{2^*} G(\xi_i, \xi_j) \delta_i^{\frac{N-2}{2}} \delta_j^{\frac{N-2}{2}} \int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^{\frac{N+2}{2}}} + o\big(\delta_i^{\frac{N-2}{2}} \delta_j^{\frac{N-2}{2}}\big). \tag{A.23}$$

Proof. The proofs of (A.22) and (A.23) are from [3]. We prove the remaining estimates.

Proof of (A.18). Integration by parts yields

$$\begin{aligned}\int_\Omega |\nabla PV_\sigma|^2 - \mu \frac{|PV_\sigma|^2}{|x|^2} &= \int_\Omega (-\Delta V_\sigma) PV_\sigma - \mu \frac{|PV_\sigma|^2}{|x|^2} \\ &= \int_\Omega V_\sigma^{2^*-1} PV_\sigma + \mu \frac{V_\sigma PV_\sigma - |PV_\sigma|^2}{|x|^2} \\ &= \int_\Omega V_\sigma^{2^*} - \int_\Omega V_\sigma^{2^*-1} \varphi_\sigma + \mu \int_\Omega \frac{\varphi_\sigma (V_\sigma - \varphi_\sigma)}{|x|^2}.\end{aligned}$$

Next we compute:

$$\begin{aligned}\int_\Omega V_\sigma^{2^*-1} H(0,x) &= \int_{B(0,\frac{\eta}{2})} V_\sigma^{2^*-1} H(0,x) + O(\sigma^{\frac{N+2}{2}}) \\ &= H(0,0) \int_{B(0,\frac{\eta}{2})} V_\sigma^{2^*-1} + O(\sigma^{\frac{N+2}{2}}) \\ &= H(0,0) C_\mu^{2^*-1} \int_{B(0,\frac{\eta}{2})} \frac{\sigma^{\frac{N+2}{2}}}{(\sigma^2 |x|^{\beta_1} + |x|^{\beta_2})^{\frac{N+2}{2}}} + O(\sigma^{\frac{N+2}{2}}) \\ &= H(0,0) C_\mu^{2^*-1} \int_{B(0,\rho)} \frac{\sigma^{\frac{\overline{\mu}}{\sqrt{\overline{\mu}-\mu}}}}{(|z|^{\beta_1} + |z|^{\beta_2})^{\frac{N+2}{2}}} + O(\sigma^{\frac{N+2}{2}}) \\ &= H(0,0) C_\mu^{2^*-1} \sigma^{\frac{\overline{\mu}}{\sqrt{\overline{\mu}-\mu}}} \int_{\mathbb{R}^N} \frac{1}{(|z|^{\beta_1} + |z|^{\beta_2})^{\frac{N+2}{2}}} + O(\sigma^{\frac{N+2}{2}}).\end{aligned}$$

Here $\rho = \frac{\eta}{2} \cdot \sigma^{-\frac{\sqrt{\overline{\mu}}}{\sqrt{\overline{\mu}-\mu}}}$ in the second to last line. Now, using (3.7) yields:

$$\begin{aligned}\int_\Omega V_\sigma^{2^*-1} \varphi_\sigma &= C_0 C_\mu^{2^*-1} H(0,0) \sigma^{\frac{\sqrt{\overline{\mu}}(\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-\mu})}{\sqrt{\overline{\mu}-\mu}}} \int_{\mathbb{R}^N} \frac{1}{(|z|^{\beta_1} + |z|^{\beta_2})^{\frac{N+2}{2}}} \\ &\quad + O(\mu \sigma^{\frac{\sqrt{\overline{\mu}}(\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-\mu})}{\sqrt{\overline{\mu}-\mu}}}) + O(\sigma^N) \\ &= C_0 C_\mu^{2^*-1} H(0,0) \sigma^{N-2} \int_{\mathbb{R}^N} \frac{1}{(|z|^{\beta_1} + |z|^{\beta_2})^{\frac{N+2}{2}}} + O(\mu \sigma^{N-2}) + O(\sigma^N).\end{aligned} \tag{A.24}$$

Moreover, we have

$$
\begin{aligned}
&\left|\int_\Omega \frac{\overline{d}^{\sqrt{\mu}-\sqrt{\mu-\mu}}(x)H(0,x)}{|x|^2}\left(\frac{1}{(\sigma^2|x|^{\beta_1}+|x|^{\beta_2})^{\frac{N-2}{2}}}-\frac{1}{|x|^{\sqrt{\mu}+\sqrt{\mu-\mu}}}\right)\right| \\
&\le C\int_{B(0,\rho)}\frac{1}{|x|^2}\cdot\frac{\left||x|^{\sqrt{\mu}+\sqrt{\mu-\mu}}-(\sigma^2|x|^{\beta_1}+|x|^{\beta_2})^{\frac{N-2}{2}}\right|}{(\sigma^2|x|^{\beta_1}+|x|^{\beta_2})^{\frac{N-2}{2}}|x|^{\sqrt{\mu}+\sqrt{\mu-\mu}}} \\
&\quad + C\int_{\Omega\setminus B(0,\rho)}\frac{1}{|x|^2}\cdot\frac{\left||x|^{\sqrt{\mu}+\sqrt{\mu-\mu}}-(\sigma^2|x|^{\beta_1}+|x|^{\beta_2})^{\frac{N-2}{2}}\right|}{(\sigma^2|x|^{\beta_1}+|x|^{\beta_2})^{\frac{N-2}{2}}|x|^{\sqrt{\mu}+\sqrt{\mu-\mu}}} \\
&\le C\int_{B(0,\rho)}\frac{1}{|x|^2}\frac{\sigma^{N-2}|x|^{\sqrt{\mu}-\sqrt{\mu-\mu}}}{\sigma^{N-2}|x|^{\sqrt{\mu}-\sqrt{\mu-\mu}}|x|^{\sqrt{\mu}+\sqrt{\mu-\mu}}} \\
&\quad + C\int_{\Omega\setminus B(0,\rho)}\frac{1}{|x|^2}\frac{|x|^{\sqrt{\mu}+\sqrt{\mu-\mu}}\sigma^2|x|^{\frac{-2\sqrt{\mu-\mu}}{\sqrt{\mu}}}}{|x|^{\sqrt{\mu}+\sqrt{\mu-\mu}}|x|^{\sqrt{\mu}+\sqrt{\mu-\mu}}} \\
&= O(\sigma^{\frac{\sqrt{\mu}(\sqrt{\mu}-\sqrt{\mu-\mu})}{\sqrt{\mu-\mu}}})+O(\sigma^2),
\end{aligned}
\tag{A.25}
$$

where $\rho=\sigma^{\frac{\sqrt{\mu}}{\sqrt{\mu-\mu}}}$. It follows that

$$
\begin{aligned}
\mu\int_\Omega\frac{\varphi_\sigma V_\sigma}{|x|^2} &= (1+o(1))\mu C_\mu^2\sigma^{N-2}\int_\Omega\frac{(\overline{d}(x))^{\sqrt{\mu}-\sqrt{\mu-\mu}}H(0,x)}{|x|^2|x|^{\sqrt{\mu}+\sqrt{\mu-\mu}}} \\
&= O(\mu\sigma^{N-2}).
\end{aligned}
\tag{A.26}
$$

It is also easy to see that

$$
\mu\int_\Omega\frac{\varphi_\sigma^2}{|x|^2}=O(\mu\sigma^{N-2})
\tag{A.27}
$$

and

$$
\int_\Omega V_\sigma^{2^*}=S_\mu^{\frac{N}{2}}+O(\sigma^N).
\tag{A.28}
$$

Now (A.24), (A.26), (A.27), and (A.28) yield (A.18).

Proof of (A.19). Using (3.8) integration by parts yields

$$
\begin{aligned}
&\int_\Omega \nabla PV_\sigma\nabla PU_{\delta_i,\xi_i}-\mu\frac{PV_\sigma PU_{\delta_i,\xi_i}}{|x|^2} \\
&\qquad=\int_\Omega V_\sigma^{2^*-1}U_{\delta_i,\xi_i}-\int_\Omega V_\sigma^{2^*-1}\varphi_{\delta_i,\xi_i}+\mu\int_\Omega\frac{\varphi_\sigma(U_{\delta_i,\xi_i}-\varphi_{\delta_i,\xi_i})}{|x|^2}.
\end{aligned}
\tag{A.29}
$$

Now we estimate these three summands.

$$
\begin{aligned}
\int_\Omega V_\sigma^{2^*-1}U_{\delta_i,\xi_i} &= \left(\int_{B(0,\frac{\eta}{4})}+\int_{B(\xi_i,\frac{\eta}{4})}+\int_{\Omega\setminus\left(B(0,\frac{\eta}{4})\cup B(\xi_i,\frac{\eta}{4})\right)}\right)V_\sigma^{2^*-1}U_{\delta_i,\xi_i} \\
&= \left(\int_{B(0,\frac{\eta}{4})}+\int_{B(\xi_i,\frac{\eta}{4})}\right)V_\sigma^{2^*-1}U_{\delta_i,\xi_i}+O(\sigma^{\frac{N+2}{2}}\delta_i^{\frac{N-2}{2}}).
\end{aligned}
\tag{A.30}
$$

For $\mu \to 0^+$ we have:

$$
\begin{aligned}
&\int_{B(0,\frac{\eta}{4})} V_\sigma^{2^*-1} U_{\delta_i,\xi_i} \\
&\quad = C_0 C_\mu^{2^*-1} \sigma^{\frac{N+2}{2}} \delta_i^{\frac{N-2}{2}} \int_{B(0,\frac{\eta}{4})} \frac{1}{(\sigma^2|x|^{\beta_1} + |x|^{\beta_2})^{\frac{N+2}{2}}} \cdot \frac{1}{(\delta_i^2 + |x-\xi_i|^2)^{\frac{N-2}{2}}} \\
&\quad = C_0 C_\mu^{2^*-1} \sigma^{\frac{N+2}{2}} \delta_i^{\frac{N-2}{2}} \int_{B(0,\frac{\eta}{4})} \frac{1}{(\sigma^2|x|^{\beta_1} + |x|^{\beta_2})^{\frac{N+2}{2}}} \left(\frac{1}{(\delta_i^2 + |\xi_i|^2)^{\frac{N-2}{2}}} + O(|x|^2) \right) \\
&\quad = C_0 C_\mu^{2^*-1} \sigma^{\frac{\mu}{\sqrt{\bar\mu - \mu}}} \delta_i^{\frac{N-2}{2}} \int_{\mathbb{R}^N} \frac{1}{(|z|^{\beta_1} + |z|^{\beta_2})^{\frac{N+2}{2}}} \frac{1}{|\xi_i|^{N-2}} + o(\sigma^{\frac{\mu}{\sqrt{\bar\mu - \mu}}} \delta_i^{\frac{N-2}{2}})
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{B(\xi_i,\frac{\eta}{4})} V_\sigma^{2^*-1} U_{\delta_i,\xi_i} \\
&\quad = C_0 C_\mu^{2^*-1} \sigma^{\frac{N+2}{2}} \delta_i^{\frac{N-2}{2}} \int_{B(0,\frac{\eta}{4})} \frac{1}{(\sigma^2|x+\xi_i|^{\beta_1} + |x+\xi_i|^{\beta_2})^{\frac{N+2}{2}}} \cdot \frac{1}{(\delta_i^2 + |x|^2)^{\frac{N-2}{2}}} \\
&\quad \le O(\sigma^{\frac{N+2}{2}} \delta_i^{\frac{N-2}{2}}).
\end{aligned}
$$

Therefore (A.30) gives

$$
\int_\Omega V_\sigma^{2^*-1} U_{\delta_i,\xi_i} \quad = C_0 C_\mu^{2^*-1} \sigma^{\frac{\mu}{\sqrt{\bar\mu - \mu}}} \delta_i^{\frac{N-2}{2}} \int_{\mathbb{R}^N} \frac{1}{(|z|^{\beta_1}|z|^{\beta_2})^{\frac{N+2}{2}}} \frac{1}{|\xi_i|^{N-2}} + o(\sigma^{\frac{\mu}{\sqrt{\bar\mu - \mu}}} \delta_i^{\frac{N-2}{2}}).
$$

Next we treat the second summand in (A.29).

$$
\begin{aligned}
\int_\Omega V_\sigma^{2^*-1} \varphi_{\delta_i,\xi_i} &= \int_{B(0,\frac{\eta}{2})} V_\sigma^{2^*-1} \varphi_{\delta_i,\xi_i} + O(\sigma^{\frac{N+2}{2}} \delta_i^{\frac{N-2}{2}}) \\
&= C_0 C_\mu^{2^*-1} \sigma^{\frac{N+2}{2}} \delta_i^{\frac{N-2}{2}} \int_{B(0,\frac{\eta}{2})} \frac{H(\xi_i, x)}{(\sigma^2|x|^{\beta_1} + |x|^{\beta_2})^{\frac{N+2}{2}}} + O(\sigma^{\frac{N+2}{2}} \delta_i^{\frac{N-2}{2}}) \\
&= C_0 C_\mu^{2^*-1} \sigma^{\frac{N+2}{2}} \delta_i^{\frac{N-2}{2}} \int_{B(0,\frac{\eta}{2})} \frac{H(\xi_i, 0)}{(\sigma^2|x|^{\beta_1} + |x|^{\beta_2})^{\frac{N+2}{2}}} + O(\sigma^{\frac{N+2}{2}} \delta_i^{\frac{N-2}{2}}) \\
&= C_0 C_\mu^{2^*-1} \sigma^{\frac{\mu}{\sqrt{\bar\mu - \mu}}} \delta_i^{\frac{N-2}{2}} \int_{\mathbb{R}^N} \frac{H(\xi_i, 0)}{(|z|^{\beta_1} + |z|^{\beta_2})^{\frac{N+2}{2}}} + o(\sigma^{\frac{\mu}{\sqrt{\bar\mu - \mu}}} \delta_i^{\frac{N-2}{2}}).
\end{aligned} \tag{A.31}
$$

Concerning the third summand in (A.29) we observe that similarly to (A.25)

$$
\left| \int_\Omega \frac{\overline{d}^{\sqrt{\bar\mu} - \sqrt{\bar\mu - \mu}}(x) H(0,x)}{|x|^2} \left(\frac{1}{(\delta_i^2 + |x-\xi_i|^2)^{\frac{N-2}{2}}} - \frac{1}{(|x-\xi_i|^2)^{\frac{N-2}{2}}} \right) \right| \le O(\delta_i^2),
$$

which yields

$$
\begin{aligned}
\mu \int_\Omega \frac{\varphi_\sigma U_{\delta_i,\xi_i}}{|x|^2} &= \mu C_\mu C_0 \sigma^{\frac{N-2}{2}} \delta_i^{\frac{N-2}{2}} \int_\Omega \frac{\overline{d}^{\sqrt{\bar\mu} - \sqrt{\bar\mu - \mu}}(x) H(0,x)}{|x|^2 |x-\xi_i|^{N-2}} + o(\mu \sigma^{\frac{N-2}{2}} \delta_i^{\frac{N-2}{2}}) \\
&= O(\mu \sigma^{\frac{N-2}{2}} \delta_i^{\frac{N-2}{2}}).
\end{aligned} \tag{A.32}
$$

It is also easy to see that

$$
\mu \int_\Omega \frac{\varphi_\sigma \varphi_{\delta_i,\xi_i}}{|x|^2} = O(\mu \sigma^{\frac{N-2}{2}} \delta_i^{\frac{N-2}{2}}). \tag{A.33}
$$

Now (A.30), (A.31), (A.32), (A.33) imply (A.19).

Proof of (A.20).

$$
\begin{aligned}
\int_\Omega \frac{U_{\delta_i,\xi_i}^2}{|x|^2} &= C_0^2\delta_i^{N-2}\int_\Omega \frac{1}{|x|^2(\delta_i^2+|x-\xi_i|^2)^{N-2}}\\
&= C_0^2\delta_i^{N-2}\left(C+\int_{B(\xi_i,\frac{\eta}{4})}\frac{1}{|x|^2(\delta_i^2+|x-\xi_i|^2)^{N-2}}\right)\\
&= C_0^2\delta_i^{N-2}\left(C+\int_{B(0,\frac{\eta}{4})}\frac{1}{|x+\xi_i|^2(\delta_i^2+|x|^2)^{N-2}}\right)\\
&= C_0^2\delta_i^{N-2}\left(C+\int_{B(0,\frac{\eta}{4})}\frac{1+O(|x|^2)}{|\xi_i|^2(\delta_i^2+|x|^2)^{N-2}}\right)\\
&= C_0^2\delta_i^{N-2}\left(C+\frac{1}{|\xi_i|^2}\left(-\int_{\mathbb{R}^N\setminus B(0,\frac{\eta}{4\delta_i})}+\int_{\mathbb{R}^N}\right)\frac{\delta_i^{4-N}}{(1+|z|^2)^{N-2}}+\frac{1}{|\xi_i|^2}\int_{B(0,\frac{\eta}{4})}\frac{O(|x|^2)}{(\delta_i^2+|x|^2)^{N-2}}\right)\\
&= \frac{C_0^2}{|\xi_i|^2}\delta_i^2\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^{N-2}}+O(\delta_i^4).
\end{aligned}
\tag{A.34}
$$

On the other hand, there holds

$$
\int_\Omega \frac{\varphi_{\delta_i,\xi_i}^2}{|x|^2}=O(\delta_i^{N-2}) \quad \text{and} \quad \int_\Omega \frac{\varphi_{\delta_i,\xi_i}U_{\delta_i,\xi_i}}{|x|^2}=O(\delta_i^{N-2}).
$$

which together with (A.34) implies (A.20).

Proof of (A.21). This is similar to the proof of (A.20), and will therefore be omitted. □

Lemma A.8. *a) For $\mu,\sigma\to 0$ there holds:*

$$
\int_\Omega |PV_{\mu,\sigma}|^{2^*} = S_\mu^{\frac{N}{2}} - 2^*C_0C_\mu^{2^*-1}H(0,0)\sigma^{N-2}\int_{\mathbb{R}^N}\frac{1}{(|z|^{\beta_1}+|z|^{\beta_2})^{\frac{N+2}{2}}}
$$
$$
+O(\mu\sigma^{N-2})+O(\sigma^N).
$$

b) As $\delta_i\to 0$ there holds uniformly for ξ_i in compact subsets of Ω:

$$
\int_\Omega |PU_{\delta_i,\xi_i}|^{2^*} = S_0^{\frac{N}{2}} - 2^*C_0^{2^*}H(\xi_i,\xi_i)\delta_i^{\frac{N-2}{2}}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^{\frac{N+2}{2}}}+O(\delta_i^N).
$$

c) For $\mu, \sigma, \delta_i \to 0$ there holds uniformly in compact subsets of Ω:

$$\begin{aligned}
&\int_\Omega \left| \sum_{i=1}^k \tau_i PU_{\delta_i,\xi_i} + PV_{\mu,\sigma} \right|^{2^*} \\
&\quad = S_\mu^{\frac{N}{2}} - 2^* C_0 C_\mu^{2^*-1} H(0,0)\sigma^{N-2} \int_{\mathbb{R}^N} \frac{1}{(|z|^{\beta_1} + |z|^{\beta_2})^{\frac{N+2}{2}}} \\
&\qquad + 2^* \sum_{i=1}^k C_0 C_\mu^{2^*-1} \sigma^{\frac{N-2}{2}} \delta_i^{\frac{N-2}{2}} \int_{\mathbb{R}^N} \frac{\tau_i G(\xi_i, 0)}{(|z|^{\beta_1} + |z|^{\beta_2})^{\frac{N+2}{2}}} \\
&\qquad + \sum_{i=1}^k \Bigg(S_0^{\frac{N}{2}} - 2^* C_0^{2^*} H(\xi_i,\xi_i)\delta_i^{N-2} \int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^{\frac{N+2}{2}}} \\
&\qquad + 2^* \sum_{\substack{j=1\\ j\neq i}}^k C_0^{2^*} \delta_i^{\frac{N-2}{2}} \delta_j^{\frac{N-2}{2}} \int_{\mathbb{R}^N} \frac{\tau_i \tau_j G(\xi_i,\xi_j)}{(1+|z|^2)^{\frac{N+2}{2}}} \\
&\qquad + 2^* C_\mu C_0^{2^*-1} \sigma^{\frac{N-2}{2}} \delta_i^{\frac{N-2}{2}} \int_{\mathbb{R}^N} \frac{\tau_i G(\xi_i, 0)}{(1+|z|^2)^{\frac{N+2}{2}}} \Bigg) \\
&\qquad + \sum_{i=1}^k \Bigg(o(\sigma^{\frac{N-2}{2}} \delta_i^{\frac{N-2}{2}}) + \sum_{\substack{j=1\\ j\neq i}}^k o(\delta_j^{\frac{N-2}{2}} \delta_i^{\frac{N-2}{2}}) + O(\delta_i^N) \Bigg) + O(\mu\sigma^{N-2}) + O(\sigma^N).
\end{aligned}$$

Proof. a) By (A.24) we have:

$$\begin{aligned}
\int_\Omega |PV_{\mu,\sigma}|^{2^*} &= \int_\Omega V_{\mu,\sigma}^{2^*} - 2^* \int_\Omega V_{\mu,\sigma}^{2^*-1} \varphi_\sigma + O(\sigma^N) \\
&= S_\mu^{\frac{N}{2}} - 2^* C_0 C_\mu^{2^*-1} H(0,0) \sigma^{\frac{\sqrt{\bar\mu}(\sqrt{\bar\mu} + \sqrt{\bar\mu-\mu})}{\sqrt{\bar\mu-\mu}}} \int_{\mathbb{R}^N} \frac{1}{(|z|^{\beta_1} + |z|^{\beta_2})^{\frac{N+2}{2}}} \\
&\qquad + O(\mu \sigma^{\frac{\sqrt{\bar\mu}(\sqrt{\bar\mu} + \sqrt{\bar\mu-\mu})}{\sqrt{\bar\mu-\mu}}}) + O(\sigma^N).
\end{aligned}$$

b) is from [3].

c) Here (3.7), (3.8), (A.30), (A.31) imply:

$$\begin{aligned}
\int_{B(0,\frac{\eta}{2})} (PV_{\mu,\sigma})^{2^*-1} PU_{\delta_i,\xi_i} &= \int_{B(0,\frac{\eta}{2})} \left(V_{\mu,\sigma}^{2^*-1} + O(V_{\mu,\sigma}^{2^*-2}\varphi_\sigma) \right) (U_{\delta_i,\xi_i} - \varphi_{\delta_i,\xi_i}) \\
&= \int_{B(0,\frac{\eta}{2})} V_{\mu,\sigma}^{2^*-1} (U_{\delta_i,\xi_i} - \varphi_{\delta_i,\xi_i}) + O(\sigma^{\frac{\sqrt{\bar\mu}(N-2+\sqrt{\bar\mu-\mu})}{\sqrt{\bar\mu-\mu}}} \delta_i^{\frac{N-2}{2}}) \\
&= C_0 C_\mu^{2^*-1} \sigma^{\frac{N-2}{2}} \delta_i^{\frac{N-2}{2}} \int_{\mathbb{R}^N} \frac{G(\xi_i, 0)}{(|z|^{\beta_1} + |z|^{\beta_2})^{\frac{N+2}{2}}} + o(\sigma^{\frac{N-2}{2}} \delta_i^{\frac{N-2}{2}}).
\end{aligned}$$

Then using part a) we deduce:

$$\begin{aligned}
&\int_{B(0,\frac{\eta}{2})}\left|\sum_{i=1}^{k}\tau_i PU_{\delta_i,\xi_i}+PV_{\mu,\sigma}\right|^{2^*}\\
&=\int_{B(0,\frac{\eta}{2})}(PV_{\mu,\sigma})^{2^*}+2^*\sum_{i=1}^{k}\int_{B(0,\frac{\eta}{2})}\tau_i(PV_{\mu,\sigma})^{2^*-1}PU_{\delta_i,\xi_i}+\sum_{i=1}^{k}O\left(\int_{B(0,\frac{\eta}{2})}(PV_{\mu,\sigma})^{2^*-2}(PU_{\delta_i,\xi_i})^2\right)\\
&=S_\mu^{\frac{N}{2}}-2^*C_0C_\mu^{2^*-1}H(0,0)\sigma^{N-2}\int_{\mathbb{R}^N}\frac{1}{(|z|^{\beta_1}+|z|^{\beta_2})^{\frac{N+2}{2}}}\\
&\qquad+2^*\sum_{i=1}^{k}C_0C_\mu^{2^*-1}\sigma^{\frac{N-2}{2}}\delta_i^{\frac{N-2}{2}}\int_{\mathbb{R}^N}\frac{\tau_i G(\xi_i,0)}{(|z|^{\beta_1}+|z|^{\beta_2})^{\frac{N+2}{2}}}\\
&\qquad+\sum_{i=1}^{k}o(\sigma^{\frac{N-2}{2}}\delta_i^{\frac{N-2}{2}})+O(\mu\sigma^{N-2})+O(\sigma^N).
\end{aligned}\tag{A.35}$$

We also have

$$\begin{aligned}
&\int_{B(\xi_i,\frac{\eta}{2})}(PU_{\delta_i,\xi_i})^{2^*-1}PV_{\mu,\sigma}=\int_{B(\xi_i,\frac{\eta}{2})}U_{\delta_i,\xi_i}^{2^*-1}(V_{\mu,\sigma}-\varphi_\sigma)+o(\delta_i^{\frac{N-2}{2}}\sigma^{\frac{N-2}{2}})\\
&\quad=C_\mu C_0^{2^*-1}\sigma^{\frac{N-2}{2}}\delta_i^{\frac{N-2}{2}}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^{\frac{N+2}{2}}}\left(\frac{1}{|\xi_i|^{\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}}}-|\bar d(x)|^{\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}}H(0,\xi_i)\right)\\
&\qquad+o(\sigma^{\frac{N-2}{2}}\delta_i^{\frac{N-2}{2}})\\
&\quad=C_\mu C_0^{2^*-1}\sigma^{\frac{N-2}{2}}\delta_i^{\frac{N-2}{2}}\int_{\mathbb{R}^N}\frac{G(\xi_i,0)}{(1+|z|^2)^{\frac{N+2}{2}}}+o(\sigma^{\frac{N-2}{2}}\delta_i^{\frac{N-2}{2}}),
\end{aligned}$$

hence

$$\begin{aligned}
&\int_{B(\xi_i,\frac{\eta}{2})}\left|\sum_{i=1}^{k}\tau_i PU_{\delta_i,\xi_i}+PV_{\mu,\sigma}\right|^{2^*}=\int_{B(\xi_i,\frac{\eta}{2})}\left|PU_{\delta_i,\xi_i}+\sum_{\substack{j=1\\ j\neq i}}^{k}\tau_i\tau_j PU_{\delta_j,\xi_j}+\tau_i PV_{\mu,\sigma}\right|^{2^*}\\
&\quad=\int_{B(\xi_i,\frac{\eta}{2})}(PU_{\delta_i,\xi_i})^{2^*}+2^*\sum_{j=1,j\neq i}^{k}\tau_i\tau_j(PU_{\delta_i,\xi_i})^{2^*-1}PU_{\delta_j,\xi_j}+2^*\tau_i(PU_{\delta_i,\xi_i})^{2^*-1}PV_{\mu,\sigma}\\
&\qquad+O\left(\int_{B(\xi_i,\frac{\eta}{2})}(PU_{\delta_i,\xi_i})^{2^*-2}\left(\sum_{\substack{j=1\\ j\neq i}}^{k}\tau_j PU_{\delta_j,\xi_j}+PV_{\mu,\sigma}\right)^2\right)\\
&\quad=S_0^{\frac{N}{2}}-2^*C_0^{2^*}H(\xi_i,\xi_i)\delta_i^{N-2}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^{\frac{N+2}{2}}}+O(\delta_i^N)\\
&\qquad+2^*\sum_{\substack{j=1\\ j\neq i}}^{k}C_0^{2^*}\delta_i^{\frac{N-2}{2}}\delta_j^{\frac{N-2}{2}}\int_{\mathbb{R}^N}\frac{\tau_i\tau_j G(\xi_i,\xi_j)}{(1+|z|^2)^{\frac{N+2}{2}}}+\sum_{\substack{j=1\\ j\neq i}}^{k}o(\delta_j^{\frac{N-2}{2}}\delta_i^{\frac{N-2}{2}})\\
&\qquad+2^*C_\mu C_0^{2^*-1}\sigma^{\frac{N-2}{2}}\delta_i^{\frac{N-2}{2}}\int_{\mathbb{R}^N}\frac{\tau_i G(\xi_i,0)}{(1+|z|^2)^{\frac{N+2}{2}}}+o(\sigma^{\frac{N-2}{2}}\delta_i^{\frac{N-2}{2}}).
\end{aligned}\tag{A.36}$$

where the last equality was obtained by the results in [3]. Finally we have:

$$\int_{\Omega\setminus\left(B(0,\frac{\eta}{2})\cup\bigcup_{-}i=1^k B(\xi_i,\frac{\eta}{2})\right)}\left|\sum_{i=1}^{k}\tau_i PU_{\delta_i,\xi_i}+PV_{\mu,\sigma}\right|^{2^*}\le\sum_{i=1}^{k}O(\delta_i^N)+O(\sigma^N).\tag{A.37}$$

Now (A.35), (A.36), (A.37) yield c). $\square$

Lemma A.9. *For $\mu, \sigma, \delta_i \to 0$ there holds uniformly in compact subsets of Ω:*

$$\int_\Omega \left| \sum_{i=1}^k \tau_i PU_{\delta_i,\xi_i} + PV_{\mu,\sigma} \right|^{2^*} \ln \left| \sum_{i=1}^k \tau_i PU_{\delta_i,\xi_i} + PV_{\mu,\sigma} \right|$$
$$= -\frac{N-2}{2} \ln \sigma \cdot \int_{\mathbb{R}^N} V_{\mu,1}^{2^*} - \frac{N-2}{2} \ln(\delta_1 \ldots \delta_k) \cdot \int_{\mathbb{R}^N} U_{1,0}^{2^*}$$
$$+ \int_{\mathbb{R}^N} V_{\mu,1}^{2^*} \ln V_{\mu,1} + k \int_{\mathbb{R}^N} U_{1,0}^{2^*} \ln U_{1,0} + o(1).$$

Proof. Similarly to [17] we obtain:

$$\int_{B(0,\frac{\eta}{2})} \left| \sum_{i=1}^k \tau_i PU_{\delta_i,\xi_i} + PV_{\mu,\sigma} \right|^{2^*} \ln \left| \sum_{i=1}^k \tau_i PU_{\delta_i,\xi_i} + PV_{\mu,\sigma} \right|$$
$$= -\frac{\overline{\mu}}{\sqrt{\overline{\mu} - \mu}} \ln \sigma \cdot \int_{\mathbb{R}^N} V_{\mu,1}^{2^*} + \int_{\mathbb{R}^N} V_{\mu,1}^{2^*} \ln V_{\mu,1} + o(1)$$
$$= -\frac{N-2}{2} \ln \sigma \cdot \int_{\mathbb{R}^N} V_{\mu,1}^{2^*} + \int_{\mathbb{R}^N} V_{\mu,1}^{2^*} \ln V_{\mu,1} + o(1),$$

and

$$\int_{B(\xi_i,\frac{\eta}{2})} \left| \sum_{i=1}^k \tau_i PU_{\delta_i,\xi_i} + PV_{\mu,\sigma} \right|^{2^*} \ln \left| \sum_{i=1}^k \tau_i PU_{\delta_i,\xi_i} + PV_{\mu,\sigma} \right|$$
$$= -\frac{N-2}{2} \ln \delta_i \cdot \int_{\mathbb{R}^N} U_{1,0}^{2^*} + \int_{\mathbb{R}^N} U_{1,0}^{2^*} \ln U_{1,0} + o(1)$$

and

$$\int_{\Omega \setminus \left(B(0,\frac{\eta}{2}) \cup \bigcup_{i=1}^k B(\xi_i,\frac{\eta}{2})\right)} \left| \sum_{i=1}^k \tau_i PU_{\delta_i,\xi_i} + PV_{\mu,\sigma} \right|^{2^*} \ln \left| \sum_{i=1}^k \tau_i PU_{\delta_i,\xi_i} + PV_{\mu,\sigma} \right| = o(1).$$

The Lemma follows now. $\square$

Lemma A.10. *For $\mu \to 0^+$ there holds:*

$$\int_{\mathbb{R}^N} V_{\mu,1}^p = \int_{\mathbb{R}^N} U_{1,0}^p + o(1) \quad \textit{and} \quad \int_{\mathbb{R}^N} V_{\mu,1}^p \ln V_{\mu,1} = \int_{\mathbb{R}^N} U_{1,0}^p \ln U_{1,0} + o(1)$$

for $p > 1$ as well as

$$C_\mu = C_0 - \frac{C_0}{N-2}\mu + O(\mu^2) \quad \textit{and} \quad S_\mu = S_0 - \overline{S}\mu + O(\mu^2),$$

for some positive constant $\overline{S}$ independent of μ.

Proof. These equalities can be obtained by direct computations. $\square$

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