Normalized solutions for a coupled Schrödinger system*

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Abstract

In the present paper, we prove the existence of solutions $(\lambda_1, \lambda_2, u, v) \in \mathbb{R}^2 \times H^1(\mathbb{R}^3, \mathbb{R}^2)$ to systems of coupled Schrödinger equations

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 u^3 + \beta u v^2 & \text{in } \mathbb{R}^3 \\ -\Delta v + \lambda_2 v = \mu_2 v^3 + \beta u^2 v & \text{in } \mathbb{R}^3 \\ u, v > 0 & \text{in } \mathbb{R}^3 \end{cases}$$

satisfying the normalization constraint $\int_{\mathbb{R}^3} u^2 = a^2$ and $\int_{\mathbb{R}^3} v^2 = b^2$, which appear in binary mixtures of Bose-Einstein condensates or in nonlinear optics. The parameters $\mu_1, \mu_2, \beta > 0$ are prescribed as are the masses $a, b > 0$. The system has been considered mostly in the fixed frequency case. And when the masses are prescribed, the standard approach to this problem is variational with λ_1, λ_2 appearing as Lagrange multipliers. Here we present a new approach based on bifurcation theory and the continuation method. We obtain the existence of normalized solutions for any given $a, b > 0$ for β in a large range. We also give a result about the nonexistence of positive solutions. From which one can see that our existence theorem is almost the best. Especially, if $\mu_1 = \mu_2$ we prove that normalized solutions exist for all $\beta > 0$ and all $a, b > 0$.

Key words: Schrödinger system; self-focusing; attractive interaction; solitary wave; normalized solution; global bifurcation.

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1 Introduction

The time-dependent system of coupled nonlinear Schrödinger equations

$$\begin{cases} -i\frac{\partial}{\partial t}\Phi_1 = \Delta\Phi_1 + \mu_1|\Phi_1|^2\Phi_1 + \beta|\Phi_2|^2\Phi_1, \\ -i\frac{\partial}{\partial t}\Phi_2 = \Delta\Phi_2 + \mu_2|\Phi_2|^2\Phi_1 + \beta|\Phi_1|^2\Phi_2, \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R}, \\ \Phi_j = \Phi_j(x,t) \in \mathbb{C}, j = 1,2,\ N \le 3, \end{cases} \tag{1.1}$$

is used as model for various physical phenomena, for instance binary mixtures of Bose-Einstein condensates, or the propagation of mutually incoherent wave packets in nonlinear optics; see e.g. [1, 18, 19, 33]. In the models, i is the imaginary unit, Φ_j is the wave function of the j-th component, and the real numbers μ_j and β represent the intra-spaces and inter-species scattering length, describing respectively the interaction between particles of the same component or of different components. In particular, the positive sign of μ_j (and of β) stays for attractive interaction, while the negative sign stays for repulsive interaction. In present paper, we consider the case of positive parameters $\mu_1, \mu_2, \beta > 0$, i.e. the self-focusing and attractive case. An important, and of course well known, feature of (1.1) is conservation of masses: the L^2-norms $|\Phi_1(\cdot,t)|_2, |\Phi_2(\cdot,t)|_2$ of solutions are independent of $t \in \mathbb{R}$. These norms have a clear physical meaning. In the aforementioned contexts, they represent the number of particles of each component in Bose-Einstein condensates, or the power supply in the nonlinear optics framework.

The ansatz $\Phi_1(x,t) = e^{i\lambda_1 t}u(x)$ and $\Phi_2(x,t) = e^{i\lambda_2 t}v(x)$ for solitary wave solutions leads to the elliptic system:

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 u^3 + \beta u v^2, \\ -\Delta v + \lambda_2 v = \mu_2 v^3 + \beta v u^2, \end{cases} \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

This system has been investigated by many authors since about 2005, mainly in the fixed frequency case where $\lambda_1, \lambda_2 > 0$ are prescribed; see e.g. [4, 11, 12, 14, 24, 25, 26, 29, 30, 31, 32, 34] and the references therein.

Much less is known when the L^2-norms $|u|_2, |v|_2$ are prescribed, in spite of the physical relevance of normalized solutions. A natural approach to finding solutions of (1.2) satisfying the normalization constraints

$$\int_{\mathbb{R}^N} u^2 = a^2 \quad \text{and} \quad \int_{\mathbb{R}^N} v^2 = b^2 \tag{1.3}$$

consists in finding critical points $(u,v) \in H^1(\mathbb{R}^N, \mathbb{R}^2)$ of the energy

$$J(u,v) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2\right) - \frac{1}{4}\int_{\mathbb{R}^N} \left(\mu_1 u^4 + \mu_2 v^4 + 2\beta u^2 v^2\right)$$

under the constraints (1.3). Then the parameters λ_1, λ_2 appear as Lagrange multipliers. All papers on normalized solutions of (1.2) are based on this approach; see [7, 8, 9, 10, 21] and the references therein. Only the papers [8, 21] deal with (1.2)-(1.3) with $\beta > 0$. The existence of normalized solutions for systems of nonlinear Schrödinger equations with trapping potential has been proved in [27], and on bounded domains in [28], also by variational methods. In [27, 28] the masses a^2, b^2 have to be small.

In the present paper we propose a different approach based on bifurcation theory applied to (1.2) with $\lambda_2 = 1$, taking λ_1 as parameter. There are two families of semitrivial solutions of (1.2) where either $u = 0$ or $v = 0$. The bifurcation of global continua of positive solutions of (1.2) from these semitrivial solutions has been proved in [12]. We shall investigate the global behavior of these continua, and the L^2-norms of the solutions along them, in order to obtain the existence of solutions of (1.2)-(1.3). A major tool will be the fixed point index in cones.

In this paper we deal with the case $N = 3$ when the growth of the nonlinearity is mass-supercritical. In dimension $N = 1$ the growth of the nonlinearity is mass-subcritical so that J is bounded from below on the constraint and normalized solutions can be obtained by minimization. In dimension $N = 2$ the growth of the nonlinearity in (1.2) is mass-critical making the existence of normalized solutions a very subtle issue, heavily depending on the prescribed masses a^2, b^2, as can already be seen in the scalar case.

The paper is organized as follows. In the next section we state and discuss our results, in particular we compare them with existing results on normalized solutions. We also state and discuss some new non-existence and uniqueness theorems for (1.2) that will enter in the proofs of our results on normalized solutions. Then in Section 3 we collect and prove a few basic facts about (1.2). Section 4 contains the main idea of our approach. There we reduce the proofs of our results on normalized solutions to the problem of controlling the L^2-norms along continua of solutions of (1.2), and we describe the bifurcating continua. An important part of our proof is to understand the behavior of the L^2-norms as $\lambda \to 0$ or $\lambda \to \infty$. We investigate this in Section 5 where we also prove the non-existence and uniqueness theorems for (1.2). The main results about normalized solutions will be proved in section 6.

2 Statement of results

We are concerned with the existence of real numbers $\lambda_1, \lambda_2 \in \mathbb{R}$ and of radial functions $u, v \in H^1_{rad}(\mathbb{R}^3)$ that solve

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 u^3 + \beta u v^2, & \text{in } \mathbb{R}^3, \\ -\Delta v + \lambda_2 v = \mu_2 v^3 + \beta u^2 v, & \text{in } \mathbb{R}^3, \\ u, v > 0, & \text{in } \mathbb{R}^3, \\ |u|_2 = a \quad \text{and} \quad |v|_2 = b, \end{cases} \tag{2.1}$$

where $\mu_1, \mu_2, \beta, a, b > 0$ are prescribed positive real numbers. In order to state our results we define

$$\tau_0 := \inf_{\phi \in \mathcal{D}_0^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^3} |\nabla \phi|^2 dx}{\int_{\mathbb{R}^3} U^2 \phi^2 dx}, \tag{2.2}$$

where U is the unique positive radial solution to

$$-\Delta u + u = u^3 \text{ in } \mathbb{R}^N; \quad u(x) \to 0 \text{ as } |x| \to \infty; \tag{2.3}$$

cf. [23]. We shall see that $\tau_0 \in (0,1)$.

Theorem 2.1. *Let $\mu_1, \mu_2 > 0$. Then we have the following conclusions.*

a) If $\beta \in (0, \tau_0 \min\{\mu_1, \mu_2\}] \cup (\tau_0 \max\{\mu_1, \mu_2\}, +\infty)$ then for any $a, b > 0$, the problem (2.1) *has a solution $(\lambda_1, \lambda_2, u, v)$ with $\lambda_1 > 0, \lambda_2 > 0$ and $u, v \in H^1_{rad}(\mathbb{R}^3)$.*

b) If $\beta \in (\tau_0 \min\{\mu_1, \mu_2\}, \tau_0 \max\{\mu_1, \mu_2\}]$ then there exists $\delta > 0$ such that for any $a, b > 0$ satisfying

$$\begin{cases} \frac{a}{b} \le \delta & \text{if } \mu_2 < \mu_1; \\ \frac{a}{b} \ge \frac{1}{\delta} & \text{if } \mu_2 > \mu_1, \end{cases}$$

the problem (2.1) *has a solution $(\lambda_1, \lambda_2, u, v)$ with $\lambda_1 > 0, \lambda_2 > 0$ and $u, v \in H^1_{rad}(\mathbb{R}^3)$. If in addition $\beta \in (\tau_0 \min\{\mu_1, \mu_2\}, \min\{\mu_1, \mu_2\})$ then*

$$\delta \ge \sqrt{\frac{\beta - \min\{\mu_1, \mu_2\}}{\beta - \max\{\mu_1, \mu_2\}}}.$$

Of course it is natural to ask whether (2.1) has a solution without any conditions on $\mu_1, \mu_2, \beta, a, b$. This is not true however, as the next result shows.

Proposition 2.2. *If $\mu_2 \le \beta \le \tau_0 \mu_1$, then there exists $q > 0$ such that* (2.1) *has no solution for $\frac{a}{b} > q$. If $\mu_1 \le \beta \le \tau_0 \mu_2$, then there exists $\tilde{q} > 0$ such that* (2.1) *has no solution for $\frac{a}{b} < \tilde{q}$.*

Theorem 2.1 and Proposition 2.2 will be proved in Section 6.

Remark 2.3. *As mentioned in the introduction, only the papers [8, 21] deal with* (1.2)-(1.3) *in the case $\beta > 0$. Theorem 2.1 significantly improves and complements the results of [8]. There the authors obtain a solution $(\lambda_1, \lambda_2, u, v)$ of* (2.1) *as in Theorem 2.1 for $0 < \beta < \beta_1$ and for $\beta > \beta_2$ where $\beta_1, \beta_2 > 0$ are defined implicitely by*

$$\max\left\{\frac{1}{a^2\mu_1^2}, \frac{1}{b^2\mu_2^2}\right\} = \frac{1}{a^2(\mu_1 + \beta_1)^2} + \frac{1}{b^2(\mu_2 + \beta_1)^2}.$$

and

$$\frac{(a^2 + b^2)^3}{(\mu_1 a^4 + \mu_2 b^4 + 2\beta_2 a^2 b^2)^2} = \min\left\{\frac{1}{a^2\mu_1^2}, \frac{1}{b^2\mu_2^2}\right\}.$$

Clearly the bounds β_1, β_2 depend on the masses $a, b > 0$ and

$$\beta_1 \to 0,\ \beta_2 \to \infty \quad \text{as } \frac{a}{b} \to 0 \text{ or } \frac{a}{b} \to \infty.$$

In particular there is no value of β so that the results from [8] yield a solution for all masses.

In [21] the authors consider more general (but still homogeneous) nonlinearities and interaction terms. Specialized to (1.2)-(1.3) *their results recover those of [8]. Our new approach via bifurcation theory and continuaton can also be applied to the systems considered in [21] and to improve the results in that paper.*

We now add a few results on (1.2) which enter in the proofs of Theorem 2.1 and which have some interest in itself. Below we assume $\lambda_1, \lambda_2 > 0$. This is no restriction because we shall prove that positive solutions of (1.2) with $\mu_1, \mu_2, \beta > 0$ can only exist if $\lambda_1, \lambda_2 > 0$; see Lemma 3.3.

Theorem 2.4. a) *For $\beta \geq \mu_1$ there exists $\eta_1(\beta) > 0$ such that* (1.2) *has no positive solution if $\frac{\lambda_1}{\lambda_2} > \eta_1(\beta)$.*

b) *For $\beta \geq \mu_2$ there exists $\eta_2(\beta) > 0$ such that* (1.2) *has no positive solution if $\frac{\lambda_1}{\lambda_2} < \eta_2(\beta)$.*

The next theorem makes some progress towards uniqueness of positive solutions of (1.2).

Theorem 2.5. a) *Problem* (1.2) *with $N = 3$ has at most one positive solution for $\frac{\lambda_1}{\lambda_2} > 0$ small or for $\frac{\lambda_1}{\lambda_2}$ large.*

b) *If $\beta \leq \tau_0\mu_2$ then* (1.2)*with $N = 3$ has a unique positive solution for $\frac{\lambda_1}{\lambda_2} > 0$ small. If $\beta \leq \tau_0\mu_1$ then* (1.2) *with $N = 3$ has a unique positive solution for $\frac{\lambda_1}{\lambda_2}$ large.*

Theorems 2.4 and 2.5 will be proved in Section 5.

Remark 2.6. *It is known and easy to see (cf. [11, 29]) that the problem*

$$\begin{cases} -\Delta u + u = \mu_1 u^3 + \beta u v^2, & \text{in } \mathbb{R}^3, \\ -\Delta v + v = \mu_2 v^3 + \beta u^2 v, & \text{in } \mathbb{R}^3, \\ u, v > 0, & \text{in } \mathbb{R}^3. \end{cases} \tag{2.4}$$

has no solution in the regime $\beta \in [\min\{\mu_1, \mu_2\}, \max\{\mu_1, \mu_2\}]$, if $\mu_1 \neq \mu_2$. On the other hand, for $\beta \in (0, \min\{\mu_1, \mu_2\}) \cup (\max\{\mu_1, \mu_2\}, +\infty)$ it is also easy to see that

$$u_\beta(x) = \sqrt{\frac{\beta - \mu_2}{\beta^2 - \mu_1\mu_2}} U(x), \quad v_\beta(x) = \sqrt{\frac{\beta - \mu_1}{\beta^2 - \mu_1\mu_2}} U(x)$$

solve (2.4)*. The solution (u_β, v_β) is nondegenerate in the space $H^1_{rad}(\mathbb{R}^3, \mathbb{R}^2)$; see [17, Lemma 2.2]. Sirakov [29, Remark 2]) conjectured that, up to translations, (u_β, v_β) is the unique positive solution of* (2.4)*. Wei and Yao [35, Theorem 4.1, Theorem 4.2] proved this conjecture for $\beta > \max\{\mu_1, \mu_2\}$ and for $0 < \beta < \beta_0$ close to 0. Chen and Zou [14, Theorem 1.1] proved the conjecture in case $\beta_0' < \beta < \min\{\mu_1, \mu_2\}$ close to $\min\{\mu_1, \mu_2\}$. The remaining range $\beta \in [\beta_0, \beta_0']$ is open up to now.*

3 Some Preliminaries

In this section we collect results that hold for more general N, not only for $N = 3$. We write $|u|_p$ for the L^p-norm. Let us first recall two results from [9].

Lemma 3.1. *Let (u, v) be a solution to*

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 u^3 + \beta u v^2 & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \mu_2 v^3 + \beta u^2 v & \text{in } \mathbb{R}^N, \\ u \geq 0, v \geq 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{3.1}$$

with $N \leq 3$. If $\lambda_1 > 0$ then there exists $\alpha, \gamma > 0$ such that

$$u(x) \leq \alpha e^{-\sqrt{1+\gamma|x|^2}} \text{ for every } x \in \mathbb{R}^N.$$

Although only the case $N = 3$ has been considered in [9, Lemma 3.11] the proof works verbatim for $N \leq 3$. The second result [9, Lemma 3.12] is a Liouville-type theorem.

Lemma 3.2. *If $0 \leq u \in H^1(\mathbb{R}^N)$ satisfies*

$$-\Delta u + c(x) u \geq 0 \text{ in } \mathbb{R}^N, N \leq 3,$$

with $0 \leq c(x) \leq Ce^{-C|x|}$ for some $C > 0$, then $u \equiv 0$.

Proof. The proof in [9, Lemma 3.12] for $N = 3$ can be modified to cover $N \leq 2$ as follows. Suppose by contradiction that $u \not\equiv 0$, hence $u > 0$ by the strong maximum principle. Setting $v(x) := |x|^{-\alpha}$ for some $0 < \alpha \leq \frac{1}{2}$ there holds

$$\begin{aligned} -\Delta v + c(x) v &= \alpha(-\alpha + N - 2)|x|^{-\alpha-2} + c(x) v \\ &\leq \alpha(-\alpha + N - 2)|x|^{-\alpha-2} + Ce^{-C|x|}|x|^{-\alpha} < 0 \end{aligned}$$

for every $|x| > r_0$ with r_0 large enough. Since $u > 0$ in $\mathbb{R}^N$, there exists $C_0 > 0$ such that $u(x) \geq C_0 r_0^{-\alpha}$ for $|x| = r_0$. Now the comparison principle implies $u > C_0|x|^{-\alpha}$ in $\mathbb{R}^N \setminus B_{r_0}(0)$, hence $|u|_2 = \infty$, contradicting $u \in H^1(\mathbb{R}^N)$. □

Lemma 3.3. *Assume that $u, v \in H^1(\mathbb{R}^3)$ are positive and solve (1.2) with $\mu_1, \mu_2 > 0$ and $\beta \neq 0$. If in addition*

$$\int_{\mathbb{R}^N} \left(\mu_1 u^4 + \mu_2 v^4 + 2\beta u^2 v^2\right) > 0$$

then $\lambda_1, \lambda_2 > 0$. Moreover, u, v are radial functions (up to translation) and strictly radially decreasing if $\beta > 0$.

Proof. We first observe that

$$|\nabla u|_2^2 + \lambda_1 |u|_2^2 = \mu_1 |u|_4^4 + \beta |uv|_2^2, \quad |\nabla v|_2^2 + \lambda_2 |v|_2^2 = \mu_2 |v|_4^4 + \beta |uv|_2^2,$$

hence

$$|\nabla u|_2^2 + |\nabla v|_2^2 = -(\lambda_1 |u|_2^2 + \lambda_2 |v|_2^2) + \left(\mu_1 |u|_4^4 + \mu_2 |v|_4^4 + 2\beta |uv|_2^2\right).$$

Now the Pohozaev identity

$$(N-2)\big(|\nabla u|_2^2+|\nabla v|_2^2\big)\\ =-N\big(\lambda_1|u|_2^2+\lambda_2|v|_2^2\big)+\frac{N}{2}\big(\mu_1|u|_4^4+\mu_2|v|_4^4+2\beta|uv|_2^2\big)$$

implies

$$\big(\lambda_1|u|_2^2+\lambda_2|v|_2^2\big)=\frac{4-N}{4}\big(\mu_1|u|_4^4+\mu_2|v|_4^4+2\beta|uv|_2^2\big)>0.$$

Therefore without loss of generality we may assume $\lambda_1>0$. Then $u(x)$ decays exponentially at infinity according to Lemma 3.1. If $\lambda_2\le 0$ we distinguish by the sign of β. In the case $\beta<0$, we have

$$-\Delta v+(-\beta u^2)v=\mu_2 v^3-\lambda_2 v\ge 0.$$

Then $0\le c(x):=-\beta u^2\le Ce^{-C|x|}$ and $-\Delta v+c(x)v\ge 0$, hence $v\equiv 0$ by Lemma 3.2. In the case $\beta\ge 0$, we have

$$-\Delta v\ge \mu_2 v^3 \text{ in } \mathbb{R}^N \text{ and } v\ge 0.$$

Now the classical Liouville-type theorem from [20] yields $v\equiv 0$, a contradiction. The last statement is due to [13, Theorem 1]. □

Let S be the sharp constant for the embedding $H^1(\mathbb{R}^N)\hookrightarrow L^4(\mathbb{R}^N)$, i.e.

$$S|u|_4^2\le\big(|\nabla u|_2^2+|u|_2^2\big)\quad\text{for all } u\in H^1(\mathbb{R}^N), \tag{3.2}$$

and

$$S=\big(|\nabla U|_2^2+|U|_2^2\big)^{\frac12}=|U|_4^2 \tag{3.3}$$

where U is the positive radial solution of (2.3). As in [12, (1.6)] we introduce the function $\tau:\mathbb{R}^+\to\mathbb{R}^+$ defined by

$$\tau(s):=\inf_{\phi\in H^1(\mathbb{R}^N)\setminus\{0\}}\frac{\int_{\mathbb{R}^N}\big(|\nabla\phi|^2+s\phi^2\big)}{\int_{\mathbb{R}^N}U^2\phi^2}. \tag{3.4}$$

Lemma 3.4. a) *The infimum τ_0 in* (2.2) *and the infimum in* (3.4) *are achieved by unique positive radial functions (and their scalar multiples).*

b) $\tau\in\mathcal{C}^0(\mathbb{R}^+,\mathbb{R}^+)$ *is strictly increasing and satisfies: $\tau(1)=1$, $\tau(s)\to\tau_0$ as $s\to 0$, $\tau(s)\to\infty$ as $s\to\infty$.*

Proof. a) follows in a standard way from the compactness of the embedding $\mathcal{D}^{1,2}_{0,rad}\hookrightarrow L^2(U^2dx)$ and symmetrization. The positive radial minimizer ϕ_s, $s\ge 0$, is the first eigenfunction of the eigenvalue problem $-\Delta\phi+s\phi=\lambda U^2\phi$. We choose ϕ_s to be normalized in $L^2(U^2dx)$.

b) We have for $s_1>s_2>0$:

$$\tau(s_2)<|\nabla\phi_{s_1}|_2^2+s_2|\phi_{s_1}|_2^2<|\nabla\phi_{s_1}|_2^2+s_1|\phi_{s_1}|_2^2=\tau(s_1),$$

hence $\tau(s)$ is strictly increasing.

In order to prove the continuity consider a sequence $s_n \to s > 0$. Clearly the minimizers ϕ_{s_n} are bounded, hence up to a subsequence $\phi_{s_n} \rightharpoonup \phi$ in $H^1(\mathbb{R}^N)$, and $\phi_{s_n} \to \phi$ in $L^2(U^2dx)$. This implies:

$$\begin{aligned}\tau(s) &\le |\nabla\phi|_2^2 + s|\phi|_2^2 \le \liminf_{n\to\infty}\left(|\nabla\phi_{s_n}|_2^2 + s|\phi_{s_n}|_2^2\right) = \liminf_{n\to\infty}\tau(s_n)\\ &\le \limsup_{n\to\infty}\tau(s_n) \le \limsup_{n\to\infty}|\nabla\phi_s|_2^2 + s_n|\phi_s|_2^2 = |\nabla\phi_s|_2^2 + s|\phi_s|_2^2 = \tau(s)\end{aligned}$$

Thus, $\tau(s_n) \to \tau(s)$ and $\phi = \phi_s$, so τ is continuous. Moreover, for $s > 0$ we have $\phi_{s_n} \to \phi_s$ in $H^1(\mathbb{R}^N)$ because

$$|\nabla\phi_{s_n}|_2^2 + s|\phi_{s_n}|_2^2 = \tau(s_n) + o(1) \to \tau(s) = |\nabla\phi_s|_2^2 + s|\phi_s|_2^2.$$

The identity $\tau(1) = 1$ is obvious because by definition $U > 0$ is an eigenfunction of $-\Delta\phi + \phi = \lambda U^2\phi$ associated to the eigenvalue $\lambda = 1$.

Next we observe that $\int_{\mathbb{R}^N} U^2\phi_s^2dx = 1$ and $U \in L^\infty(\mathbb{R}^N)$ imply $|\phi_s|_2 \ge \kappa > 0$ uniformly in s, hence

$$\tau(s) = |\nabla\phi_s|_2^2 + s|\phi_s|_2^2 \ge s\kappa^2 \to \infty \quad \text{as } s \to \infty.$$

In order to prove $\tau(s) \to \tau_0$ as $s \to 0$ assume to the contrary that there exists $\delta > 0$ so that

$$\tau(s) \ge \tau_0 + \delta, \quad \text{for all } s > 0.$$

We choose a smooth cut-off function $\chi : \mathbb{R} \to [0,1]$ that is decreasing and satisfies

$$\chi(r) = \begin{cases} 1 & \text{if } r \le 1;\\ 0 & \text{if } r \ge 2.\end{cases}$$

Setting $\chi_R : \mathbb{R}^N \to \mathbb{R}$, $\chi_R(x) = \chi(|x|/R)$ we have for $R > 0$ large that

$$\frac{|\nabla(\phi_0\chi_R)|_2^2}{\int_{\mathbb{R}^N} U^2(\phi_0\chi_R)^2dx} < \tau_0 + \frac{1}{2}\delta.$$

This implies for s close to 0 the contradiction:

$$\tau(s) \le \frac{|\nabla(\phi_0\chi_R)|_2^2 + s|\phi_0\chi_R|_2^2}{\int_{\mathbb{R}^N} U^2(\psi_0\chi_R)^2dx} < \tau_0 + \delta$$

□

4 Global branches of solutions

We consider a special case of (1.2) , namely

$$\begin{cases} -\Delta u + \lambda u = \mu_1 u^3 + \beta v^2 u & \text{in } \mathbb{R}^3,\\ -\Delta v + v = \mu_2 v^3 + \beta u^2 v & \text{in } \mathbb{R}^3.\end{cases} \tag{4.1}$$

A straightforward computation shows the relation to (2.1).

Lemma 4.1. *If* (u_λ, v_λ) *is a solution of* (4.1) *with*

$$\frac{|u_\lambda|_2}{a} = \frac{|v_\lambda|_2}{b} =: \alpha \tag{4.2}$$

then

$$u(x) = \alpha^2 u_\lambda(\alpha^2 x) \quad \textit{and} \quad v(x) = \alpha^2 v_\lambda(\alpha^2 x)$$

solve (2.1) *with* $\lambda_1 = \lambda\alpha^4$ *and* $\lambda_2 = \alpha^4$.

Remark 4.2. *Clearly the converse holds in Lemma 4.1. If* (u, v) *solves* (2.1) *then*

$$u_\lambda(x) = \sqrt{\lambda_2}u(\sqrt{\lambda_2}x) \quad \textit{and} \quad v_\lambda(x) = \sqrt{\lambda_2}v(\sqrt{\lambda_2}x)$$

solve (4.1) *with* $\lambda = \frac{\lambda_1}{\lambda_2}$ *and such that* (4.2) *holds.*

Recall the solution U of (2.3). Setting

$$U_{\lambda,\mu}(x) = \frac{\sqrt{\lambda}}{\sqrt{\mu}}U(\sqrt{\lambda}x)$$

one easily checks that $(U_{\lambda,\mu_1}, 0)$ and $(0, U_{1,\mu_2})$ solve (4.1). These are called semitrivial solutions in the literature. We fix $\mu_1, \mu_2 > 0$ and consider λ and β as parameters in (4.1). Then we have two families of semitrivial solutions of (4.1):

$$\mathcal{T}_1 = \{(\lambda, \beta, U_{\lambda,\mu_1}, 0) : \ \lambda, \beta > 0\} \quad \text{and} \quad \mathcal{T}_2 = \{(\lambda, \beta, 0, U_{1,\mu_2}) : \ \lambda, \beta > 0\}.$$

Clearly we also have the family $\mathcal{T}_0 := \{(\lambda, \beta, 0, 0) : \ \lambda, \beta > 0\}$ of trivial solutions. Setting $E = H^1_{rad}(\mathbb{R}^3, \mathbb{R}^2)$ and $\mathbb{P} = \{(u, v) \in E : u, v \geq 0\}$ for the positive cone, there holds $\mathcal{T}_1, \mathcal{T}_2 \subset X := (\mathbb{R}^+)^2 \times \mathbb{P}$; here $\mathbb{R}^+ = (0, \infty)$.

We are interested in the set

$$\mathcal{S} = \{(\lambda, \beta, u, v) \in X : (\lambda, \beta, u, v) \text{ solves (4.1)}, \ u, v > 0\}$$

of nontrivial positive solutions. Let us introduce the function

$$\rho : \mathcal{S} \to \mathbb{R}^+, \quad (\lambda, \beta, u.v) \mapsto \frac{|u|_2}{|v|_2}. \tag{4.3}$$

Lemma 4.1 implies the following corollary which is the basic tool of our approach to finding normalized solutions.

Corollary 4.3. *If* $\frac{a}{b} \in \rho(\mathcal{S}^\beta)$ *then* (2.1) *has a solution.*

For the proof of Theorem 2.1 it remains to get information about the image $\rho(\mathcal{S}^\beta)$. We shall approach this using continuation methods and bifurcation theory. First we investigate the solutions bifurcating from $\mathcal{T}_1$ and $\mathcal{T}_2$. Since we are interested in global bifurcation we reformulate (4.1). For $\lambda, \beta > 0$ we define a map $\mathbb{A}_{\lambda,\beta} : \mathbb{P} \to \mathbb{P}$ by

$$\mathbb{A}_{\lambda,\beta}(u, v) := \left((-\Delta + \lambda)^{-1}(\mu_1 u^3 + \beta v^2 u), (-\Delta + 1)^{-1}(\mu_2 v^3 + \beta u^2 v)\right).$$

As a consequence of the compact embedding $H^1_{rad}(\mathbb{R}^3) \hookrightarrow L^4(\mathbb{R}^3)$ the map

$$\mathbb{A} : X \to \mathbb{P},\ \mathbb{A}(\lambda, \beta, u, v) = \mathbb{A}_{\lambda,\beta}(u, v),$$

is completely continuous. Clearly fixed points of $\mathbb{A}_{\lambda,\beta}$ correspond to solutions of (4.1). The set of bifurcation points can be explicitly determined. In order to describe it we define the functions

$$\beta_1(\lambda) = \mu_1\tau(1/\lambda) \quad \text{and} \quad \beta_2(\lambda) = \mu_2\tau(\lambda) \quad \text{for } \lambda > 0 \tag{4.4}$$

with τ from (3.4). Using the fixed point index in the cone $\mathbb{P}$, denoted by $\mathrm{ind}_{\mathbb{P}}$, the following results have been proved in [12].

Proposition 4.4. a) *The map $\mathcal{S} \to \mathbb{R}^+ \times \mathbb{R}^+$, $(\lambda, \beta, u, v) \mapsto (\lambda, \beta)$ is proper, i.e. inverse images of compact sets are compact.*

b) $\overline{\mathcal{S}} \cap \mathcal{T}_1 = \big\{(\lambda, \beta, U_{\lambda,\mu_1}, 0) :\ \lambda > 0,\ \beta = \beta_1(\lambda)\big\} =: \mathcal{B}_1$

c) $\overline{\mathcal{S}} \cap \mathcal{T}_2 = \big\{(\lambda, \beta, 0, U_{1,\mu_2}) :\ \lambda > 0,\ \beta = \beta_2(\lambda)\big\} =: \mathcal{B}_2$

d) *For $\lambda, \beta > 0$ fixed we have*

$$\mathrm{ind}_{\mathbb{P}}\big(\mathbb{A}_{\lambda,\beta}, (U_{\lambda,\mu_1}, 0)\big) = \begin{cases} -1 & \beta < \beta_1(\lambda) \\ 0 & \beta > \beta_1(\lambda) \end{cases}$$

and

$$\mathrm{ind}_{\mathbb{P}}\big(\mathbb{A}_{\lambda,\beta}, (0, U_{1,\mu_2})\big) = \begin{cases} -1 & \beta < \beta_2(\lambda) \\ 0 & \beta > \beta_2(\lambda) \end{cases}$$

In fact, in [12] problem (1.2) has been treated as a 5-parameter problem with parameters $(\lambda_1, \lambda_2, \mu_1, \mu_2, \beta) \in (\mathbb{R}^+)^5$. The statement in [12, Theorem 1.1] about which part of $(\mathbb{R}^+)^5$ is covered by $\mathcal{S}$ is not correct.

As a consequence of Proposition 4.4 there exist global two-dimensional continua $\mathcal{S}_i \subset \mathcal{S}$ bifurcating from $\mathcal{T}_i$ so that $\overline{\mathcal{S}}_i \cap \mathcal{T}_i = \mathcal{B}_i$, $i = 1, 2$. Using the analyticity of $\mathbb{A}$ it can be proved that $\mathcal{S}$ and $\mathcal{S}_i$ are two-dimensional manifolds except for one-dimensional subsets where secondary bifurcation takes place, but we do not need this. The global property of $\mathcal{S}_i$ can be formulated as in [2]. This is somewhat technical and not needed here because we are interested in the case of prescribed $\beta > 0$. We will only use the standard Rabinowitz alternative for one-parameter global bifurcation.

As a corollary of Lemma 3.4 we obtain the following properties of the functions β_i defined in (4.4).

Corollary 4.5. a) *The function β_1 is strictly decreasing and β_2 is strictly increasing in $\lambda \in \mathbb{R}^+$.*

b) $\beta_1(\lambda) \to \begin{cases} \infty & \lambda \to 0 \\ \mu_1\tau_0 & \lambda \to \infty \end{cases}$

c) $\beta_2(\lambda) \to \begin{cases} \mu_2\tau_0 & \lambda \to 0 \\ \infty & \lambda \to \infty \end{cases}$

d) *There exists a unique* $\lambda^* > 0$ *such that* $\beta_1(\lambda^*) = \beta_2(\lambda^*) =: \beta^*$.

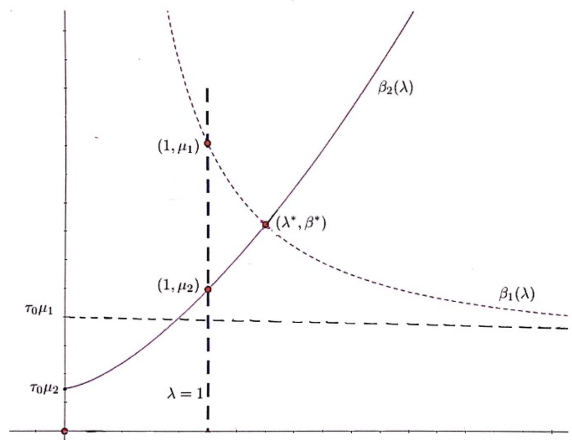

FIGURE 1. The sketches of $\beta_1(\lambda)$ and $\beta_2(\lambda)$ for the case $\mu_2 < \mu_1$.

Now we deduce the global properties of the solutions bifurcating from $\mathcal{T}_i$ that we need for $\beta > 0$ fixed. We set $\ell_i = \beta_i^{-1} : (\mu_i\tau_0, \infty) \to \mathbb{R}^+$ for $i = 1, 2$, define $X^\beta := \mathbb{R}^+ \times \{\beta\} \times \mathbb{P}$ for $\beta > 0$, and write $P_1 : X \to \mathbb{R}^+$ for the projection onto the λ-component. For subsets $M \subset X$ we use the notation $M^\beta := M \cap X^\beta$. The closure $\overline{M}$ of $M \subset X$ has to be understood in the relative topology of X.

Proposition 4.6. a) *There is no bifurcation from the set* $\mathcal{T}_0 = (\mathbb{R}^+)^2 \times \{(0,0)\}$ *of trivial solutions, i.e.* $\overline{\mathcal{S}} \cap \mathcal{T}_0 = \emptyset$.

b) *If* $\beta \le \tau_0 \min\{\mu_1, \mu_2\}$ *then* $\overline{\mathcal{S}^\beta} \cap \mathcal{T}_i^\beta = \emptyset$, $i = 1, 2$.

c) *If* $\mu_1\tau_0 < \beta \le \mu_2\tau_0$ *then there exists a connected component* $\mathcal{S}_1^\beta \subset \mathcal{S}^\beta$ *with* $\overline{\mathcal{S}_1^\beta} \cap \mathcal{T}_1^\beta = \{(\ell_1(\beta), \beta, U_{\lambda,\mu_1}, 0)\}$. *The projection* $P_1(\mathcal{S}_1^\beta)$ *contains the interval* $(0, \ell_1(\beta))$ *or the interval* $(\ell_1(\beta), \infty)$. *There is no bifurcation from* $\mathcal{T}_2^\beta$ *in* X^β.

d) *If* $\mu_2\tau_0 < \beta \le \mu_1\tau_0$ *then there exists a connected component* $\mathcal{S}_2^\beta \subset \mathcal{S}^\beta$ *with* $\overline{\mathcal{S}_2^\beta} \cap \mathcal{T}_2^\beta = \{(\ell_2(\beta), \beta, 0, U_{1,\mu_2})\}$. *The projection* $P_1(\mathcal{S}_2^\beta)$ *contains the interval* $(0, \ell_2(\beta))$ *or the interval* $(\ell_2(\beta), \infty)$. *There is no bifurcation from* $\mathcal{T}_1^\beta$ *in* X^β.

e) *If* $\beta > \tau_0 \max\{\mu_1, \mu_2\}$ *then there exist connected sets* $\mathcal{S}_i^\beta \subset \mathcal{S}^\beta$, $i = 1, 2$, *with* $\overline{\mathcal{S}_1^\beta} \cap \mathcal{T}_1^\beta = \{(\ell_1(\beta), \beta, U_{\lambda,\mu_1}, 0)\}$ *and* $\overline{\mathcal{S}_2^\beta} \cap \mathcal{T}_2^\beta = \{(\ell_2(\beta), \beta, 0, U_{1,\mu_2})\}$. *If*

$\mathcal{S}_1^\beta \cap \mathcal{S}_2^\beta \neq \emptyset$ then $\mathcal{S}_1^\beta = \mathcal{S}_2^\beta$. If this is not the case then $P_1(\mathcal{S}_1^\beta)$ contains the interval $(0, \ell_1(\beta))$ or the interval $(\ell_1(\beta), \infty)$, and $P_1(\mathcal{S}_2^\beta)$ contains the interval $(0, \ell_2(\beta))$ or the interval $(\ell_2(\beta), \infty)$.

Proof. a) This is clear since $(0,0)$ is a nondegenerate solution of (4.1) for all $(\lambda, \beta) \in (\mathbb{R}^+)^2$.

b) As a consequence of Corollary 4.5 there is no $\lambda > 0$ with $\beta_1(\lambda) = \beta$ or $\beta_2(\lambda) = \beta$.

c) Here Corollary 4.5 implies that there exists $\lambda_1 = \ell_1(\beta) > 0$ with $\beta_1(\lambda_1) = \beta$ but there is no $\lambda_2 > 0$ with $\beta_2(\lambda_2) = \beta$. Therefore there exists a connected set $\mathcal{S}_1^\beta \subset \big((id - \mathbb{A})^{-1}(0) \cap X^\beta\big) \setminus \mathcal{T}_1$ with $\overline{\mathcal{S}_1^\beta} \cap \mathcal{T}_1^\beta = \{(\ell_1(\beta), \beta, U_{\lambda,\mu_1}, 0)\}$ and which satisfies the classical Rabinowitz alternative. It cannot return to $\mathcal{T}_1^\beta$ because there is no second bifurcation point on $\mathcal{T}_1^\beta$. Therefore it must be unbounded. Since there is no bifurcation from $\mathcal{T}_0$ and $\mathcal{T}_2$ we deduce that $\overline{\mathcal{S}_1^\beta} \cap \mathcal{T}_i^\beta = \emptyset$, $i = 0, 2$, hence $\mathcal{S}_1^\beta \subset \mathcal{S}$. Now Proposition 4.4 a) implies that the only way for $\mathcal{S}_1^\beta$ to be unbounded is that $P_1(\mathcal{S}_1^\beta)$ contains the interval $(0, \ell_1(\beta))$ or the interval $(\ell_1(\beta), \infty)$. To be careful, if $P_1(\mathcal{S}_1^\beta)$ contains the interval $(0, \ell_1(\beta))$ then $\mathcal{S}_1^\beta$ is already unbounded in the sense of the Rabinowitz alternative because we only consider the parameter range $\lambda \in \mathbb{R}^+$. It is not necessary that the (u,v)-component becomes unbounded in $\mathcal{S}_1^\beta$.

d) The proof is analogous to the one of c).

e) As in the proof of c) and d) there exist connected sets $\widetilde{\mathcal{S}}_i^\beta \subset \big((id - \mathbb{A})^{-1}(0) \cap X^\beta\big) \setminus \mathcal{T}_i$ bifurcating from $\mathcal{T}_i$ which satisfy the Rabinowitz alternative. If the closure of $\widetilde{\mathcal{S}}_1^\beta$ intersects $\mathcal{T}_2^\beta$ then $\widetilde{\mathcal{S}}_1^\beta$ contains $\mathcal{T}_2$ and the connected set of nontrivial solutions bifurcating from $\mathcal{T}_2$. This implies that

$$\mathcal{S}_1^\beta := \widetilde{\mathcal{S}}_1^\beta \cap \mathcal{S} = \widetilde{\mathcal{S}}_1^\beta \setminus \mathcal{T}_2^\beta = \widetilde{\mathcal{S}}_2^\beta \setminus \mathcal{T}_1^\beta = \widetilde{\mathcal{S}}_2^\beta \cap \mathcal{S} =: \mathcal{S}_2^\beta$$

connects $\mathcal{T}_1^\beta$ and$\mathcal{T}_2^\beta$. Analogously this holds if the closure of $\widetilde{\mathcal{S}}_2^\beta$ intersects $\mathcal{T}_1^\beta$.

It remains to consider the case where the closure of $\widetilde{\mathcal{S}}_i^\beta$ does not intersect $\mathcal{T}_{3-i}^\beta$ for $i = 1, 2$. Then $\mathcal{S}_i^\beta := \widetilde{\mathcal{S}}_i^\beta \subset \mathcal{S}^\beta$ is unbounded in the sense of c) and d), i.e. $P_1(\mathcal{S}_i^\beta)$ contains the interval $(0, \ell_i(\beta))$ or the interval $(\ell_i(\beta), \infty)$, $i = 1, 2$. $\square$

Remark 4.7. *Using analytic bifurcation theory one can prove that the sets $\mathcal{S}_i^\beta$ are smooth curves except for a discrete subset of singular points. One can also apply the Crandall-Rabinowitz theorem about bifurcation from simple eigenvalues to see that $\mathcal{S}_i^\beta$ is a curve near the bifurcation point. These results are not needed here.*

As a corollary we obtain a first major building block of the proof of Theorem 2.1.

Corollary 4.8. *If $\beta > \max\{\mu_1\tau_0, \mu_2\tau_0\}$ and $\mathcal{S}_1^\beta \cap \mathcal{S}_2^\beta \neq \emptyset$ then problem* (2.1) *has a solution for every $a, b > 0$.*

Proof. Recall the function ρ from (4.3). By definition there exist $(\lambda_n, \beta, u_n, v_n) \in \mathcal{S}_1^\beta$ such that $(\lambda_n, \beta, u_n, v_n) \to (\ell_1(\beta), \beta, U_{\ell_1(\beta),\mu_1}, 0)\}$, hence $\rho(\lambda_n, \beta, u_n, v_n) \to \infty$ as $n \to \infty$. And as a consequence of Proposition 4.6 e) there exist $(\lambda_n', \beta, u_n', v_n') \in \mathcal{S}_1^\beta$ such that $(\lambda_n', \beta, u_n', v_n') \to (\ell_2(\beta), \beta, 0, U_{1,\mu_2})$, hence $\rho(\lambda_n', \beta, u_n', v_n') \to 0$ as

$n \to \infty$. Since $\mathcal{S}_1^\beta$ is connected it follows that ρ is onto. Now the result follows from Corollary 4.3. $\square$

In addition to the global continua bifurcating from $\mathcal{T}_1$ and $\mathcal{T}_2$ there exists a third global continuum $\widetilde{\mathcal{S}} \subset \mathcal{S}$. In order to see this recall that for $\lambda = 1$ and $\beta \in (0, \beta_0)$ close to 0 the problem (4.1) has precisely four solutions in $\mathbb{P}$: the trivial solution $(0,0)$, the semitrivial solutions $(U_{1,\mu_1}, 0)$, $(0, U_{1,\mu_2})$, and a unique nontrivial solution (u_β, v_β) which satisfies $(u_\beta, v_\beta) \to (U_{1,\mu_1}, U_{1,\mu_2})$ as $\beta \to 0$; see Remark 2.6. The map

$$(0, \beta_0) \to \mathbb{P}, \quad \beta \mapsto (u_\beta, v_\beta),$$

is smooth by the implicit function theorem applied at $(U_{1,\mu_1}, U_{1,\mu_2})$.

Proposition 4.9. *For $\beta \in (0, \beta_0)$ there holds* $\mathrm{ind}_{\mathbb{P}}(\mathbb{A}_{1,\beta}, (u_\beta, v_\beta)) = 1$.

Proof. The solution $(U_{1,\mu_1}, U_{1,\mu_2})$ of (4.1) with $\lambda = 1$ and $\beta = 0$ has Morse index 2 as critical point of J, with negative eigenspace spanned by $(U_{1,\mu_1}, 0), (0, U_{1,\mu_2}) \in \mathbb{P}$. The Poincaré-Hopf theorem in convex sets [5, Theorem 1.5] implies

$$\mathrm{ind}_{\mathbb{P}}(\mathbb{A}_{1,0}, (U_{1,\mu_1}, U_{1,\mu_2}) = (-1)^2 = 1.$$

Now the proposition follows from the homotopy invariance of the fixed point index. $\square$

The homotopy invariance of the fixed point index allows to continue the solutions (u_β, v_β) to other parameter values in $(\mathbb{R}^+)^2$. We define $\widetilde{\mathcal{S}} \subset \mathcal{S}$ to be the connected component of $\mathcal{S}$ containing the nontrivial solutions $(1, \beta, u_\beta, v_\beta)$ for $\beta > 0$ small. As a corollary of Proposition 4.9 we obtain the following.

Corollary 4.10. *If $\beta \le \tau_0 \min\{\mu_1, \mu_2\}$ then there exists a connected set $\mathcal{S}_0^\beta \subset \mathcal{S}^\beta \cap \widetilde{\mathcal{S}}$ such that $P_1(\mathcal{S}_0^\beta) = \mathbb{R}^+$.*

Proof. Let $\mathcal{O} \subset X \setminus (\mathcal{S} \cup \mathcal{B}_1 \cup \mathcal{B}_2)$ be an open neighborhood of

$$\mathcal{T}_0 \cup (\mathcal{T}_1 \setminus \mathcal{B}_1) \cup (\mathcal{T}_2 \setminus \mathcal{B}_2) \subset X \setminus (\mathcal{S} \cup \mathcal{B}_1 \cup \mathcal{B}_2)$$

such that $\mathcal{S} \cap \overline{\mathcal{O}} = \emptyset$. For $\lambda, \beta > 0$ we set $\mathcal{O}_{\lambda,\beta} := \{(u,v) \in \mathbb{P} : (\lambda, \beta, u, v) \in \mathcal{O}\}$. By definition the nontrivial fixed points of $\mathbb{A}_{\lambda,\beta}$ are contained in $\Omega_{\lambda,\beta} := B_R(0) \setminus \overline{\mathcal{O}_{\lambda,\beta}}$ for $R > R(\lambda, \beta)$ large. This a bounded and open subset of $\mathbb{P}$. Proposition 4.9 and the homotopy invariance of the fixed point index imply for $\beta \le \min\{\tau_0\mu_1, \tau_0\mu_2\}$ and $\beta' \in (0, \beta_0)$:

$$\mathrm{ind}_{\mathbb{P}}(\mathbb{A}_{\lambda,\beta}, \Omega_{\lambda,\beta}) = \mathrm{ind}_{\mathbb{P}}(\mathbb{A}_{\lambda,\beta'}, \Omega_{\lambda,\beta'}) = \mathrm{ind}_{\mathbb{P}}(\mathbb{A}_{1,\beta'}, \Omega_{1,\beta'}) = 1$$

The result follows from the continuation principle. $\square$

Observe that $\mathcal{S}_0^\beta$ may differ from $\widetilde{\mathcal{S}}^\beta = \widetilde{\mathcal{S}} \cap X^\beta$ because the latter may not be connected.

We may also use Proposition 4.9 to compute the global fixed point index of all positive solutions of (4.1), for each $\lambda, \beta > 0$. Observe that according to Proposition 4.4 a) for $\lambda, \beta > 0$ there exists $R(\lambda, \beta) > 0$ such that the positive solutions of (4.1) are bounded by $R(\lambda, \beta)$. Therefore the fixed point index

$$i_\infty(\lambda, \beta) = \mathrm{ind}_{\mathbb{P}}(\mathbb{A}_{\lambda,\beta}, B_R(0))$$

is well defined and independent of $R > R(\lambda, \beta)$. Applying the homotopy invariance of the fixed point index and Proposition 4.4 a) again, we also see that $i_\infty := i_\infty(\lambda, \beta)$ is independent of $\lambda, \beta > 0$.

Proposition 4.11. $i_\infty = 0$

Proof. We compute $i_\infty(\lambda, \beta)$ for $\lambda = 1$ and $\beta \in (0, \beta_0)$. Then $i_\infty = i_\infty(1, \beta)$ is the sum of the local indices at the four solutions $(0, 0)$, $(U_{1,\mu_1}, 0)$, $(0, U_{1,\mu_2})$, (u_β, v_β). From [5, Theorem 1.5] it follows that

$$\mathrm{ind}_{\mathbb{P}}\big(\mathbb{A}_{1,0}, (0, 0)\big) = 1.$$

Propositions 4.4 and 4.9 imply for $\beta \in (0, \beta_0)$:

$$\begin{aligned} i_\infty = \mathrm{ind}_{\mathbb{P}}\big(\mathbb{A}_{1,\beta}, (0, 0)\big) + \mathrm{ind}_{\mathbb{P}}\big(\mathbb{A}_{1,\beta}, (U_{1,\mu_1}, 0)\big) + \mathrm{ind}_{\mathbb{P}}\big(\mathbb{A}_{1,\beta}, (0, U_{1,\mu_1})\big) \\ + \mathrm{ind}_{\mathbb{P}}\big(\mathbb{A}_{1,\beta}, (u_\beta, v_\beta)\big) = 1 - 1 - 1 + 1 = 0 \end{aligned}$$

□

5 Asymptotic behavior of positive solutions for $\lambda \to 0$ or $\lambda \to \infty$

In this section we investigate the function

$$\rho : \mathcal{S} \to \mathbb{R}^+, \quad \rho(\lambda, \beta, u, v) = \frac{|u|_2}{|v|_2},$$

from (4.3) as $\lambda \to 0$ or $\lambda \to \infty$.

Lemma 5.1. *Let (u_n, v_n), $n \in \mathbb{N}$, be positive radial solutions to equation* (4.1) *with $\lambda = \lambda_n \to 0$. Then the following conclusions hold up to a subsequence.*

a) $u_n(x) + v_n(x) \to 0$ *as $|x| \to \infty$ uniformly in n.*

b) $|u_n|_\infty \to 0$, $|v_n|_\infty \leq C$*, and* $(u_n, v_n) \to (0, U_{1,\mu_2})$ *in* $\mathcal{C}^2_{loc}(\mathbb{R}^N) \times \mathcal{C}^2_{loc}(\mathbb{R}^N)$.

c) $v_n \to U_{1,\mu_2}$ *in* $H^1(\mathbb{R}^N)$

d) $|\nabla u_n|_2 = O(1)|u_n|_2$*; if u_n is unbounded in $H^1(\mathbb{R}^N)$, then $\rho(\lambda_n, \beta, u_n, v_n) \to \infty$.*

Proof. a) The proof in [14, Step 2 in the proof of Theorem 1.1] is valid here.

b) A standard blow up argument as in [17, Lemma 2.4] shows that $|u_n|_\infty + |v_n|_\infty$ is bounded. If $\alpha := \liminf_{n\to\infty} u_n(0) > 0$ we consider

$$-\Delta \frac{u_n}{u_n(0)} + \lambda_n \frac{u_n}{u_n(0)} = \mu_1 u_n(0)^2 \left(\frac{u_n}{u_n(0)}\right)^3 + \beta v_n^2 \frac{u_n}{u_n(0)}.$$

Then $\frac{u_n}{u_n(0)} \to \tilde{u}$ as $n \to \infty$ along a subsequence, which is a nonnegative radial function satisfying

$$-\Delta \tilde{u} \geq \mu_1 \varepsilon_0^2 \tilde{u}^3.$$

Now [20] implies $\tilde{u} \equiv 0$, contradicting $\tilde{u}(0) = 1$. Therefore $|u_n|_\infty \to 0$, hence $u_n \to 0$ in $C^2_{loc}(\mathbb{R}^N)$ along a subsequence. Since $v_n = (-\Delta + 1)^{-1}(\mu_2 v_n^3 + \beta u_n^2 v_n)$ and $|u_n|_\infty \to 0$, we see that $|v_n|_\infty$ is bounded away from 0. Then $\tilde{v} := \lim_{n\to\infty} v_n$ is a positive radial solution to

$$-\Delta v + v = \mu_2 v^3\,, \quad v(x) \to 0 \text{ as } |x| \to \infty,$$

which implies $\tilde{v} = U_{1,\mu_2}$ and $v_n \to U_{1,\mu_2}$ in $C^2_{loc}(\mathbb{R}^N)$.

c) It is standard to prove that $v_n(x) \to 0$ exponentially and uniformly in n, so there exist $C, R > 0$, independent of n such that

$$v_n(x) \leq Ce^{-\frac{1}{2}|x|} \quad \text{for all } |x| > R, \text{ all } n \in \mathbb{N}.$$

As in b), or [14, Step 3 in the proof of Theorem 1.1], one sees that v_n is bounded in $H^1(\mathbb{R}^N)$. Observe that this argument is not valid for u_n because $\lambda_n \to 0$. Then we have, up to a subsequence:

$$v_n \rightharpoonup v \text{ in } H^1(\mathbb{R}^N),\, v_n \to v \text{ in } L^4(\mathbb{R}^N), \text{ and } v_n \to v \text{ a.e. in } \mathbb{R}^N,$$

which implies $v = U_{1,\mu_2}$. Now we recall that $|u_n|_\infty \to 0$, hence $\beta|u_n v_n|_2^2 \to 0$. Using

$$|\nabla v_n|_2^2 + |v_n|_2^2 = \mu_2 |v_n|_4^4 + \beta |u_n v_n|_2^2$$

and $v_n \to U_{1,\mu_2}$ in $L^4(\mathbb{R}^N)$, we deduce

$$|\nabla v_n|_2^2 + |v_n|_2^2 \to \mu_2 |U_{1,\mu_2}|_4^4 = |\nabla U_{1,\mu_2}|_2^2 + |U_{1,\mu_2}|_2^2.$$

This yields $v_n \to U_{1,\mu_2}$ in $H^1(\mathbb{R}^N)$.

d) Setting $|\nabla u_n|_2^2 = \sigma_n |u_n|_2^2$ we have

$$(\sigma_n + \lambda_n)|u_n|_2^2 = \mu_1 |u_n|_4^4 + \beta |u_n v_n|_2^2.$$

Now a) and b) imply $\mu_1 |u_n|_4^4 + \beta |u_n v_n|_2^2 = O(1)|u_n|_2^2$, hence $|\nabla u_n|_2^2 = O(1)|u_n|_2^2$. Thus if u_n is unbounded in $H^1(\mathbb{R}^N)$ then u_n must be unbounded in $L^2(\mathbb{R}^N)$ and $\rho(\lambda_n, \beta, u_n, v_n) = \frac{|u_n|_2}{|v_n|_2} \to \infty$. $\square$

Lemma 5.2. *Let* (u_n, v_n)*,* $n \in \mathbb{N}$*, be positive radial solutions to equation* (4.1) *with* $\lambda = \lambda_n \to \infty$*. Then* $\bar{u}_n(x) := \frac{1}{\sqrt{\lambda_n}} v_n\left(x/\sqrt{\lambda_n}\right)$ *and* $\bar{v}_n(x) := \frac{1}{\sqrt{\lambda_n}} u_n\left(x/\sqrt{\lambda_n}\right)$ *satisfy (along a subsequence):*

a) $\bar{u}_n(x) + \bar{v}_n(x) \to 0$ *as* $|x| \to \infty$ *uniformly in* n.

b) $|\bar{u}_n|_\infty \to 0$, $|\bar{v}_n|_\infty \leq C$, *and* $(\bar{u}_n, \bar{v}_n) \to (0, U_{1,\mu_1})$ *in* $\mathcal{C}^2_{loc}(\mathbb{R}^N) \times \mathcal{C}^2_{loc}(\mathbb{R}^N)$.

c) $\bar{v}_n \to U_{1,\mu_1}$ *in* $H^1(\mathbb{R}^N)$

d) $|\nabla \bar{u}_n|_2 = O(1)|\bar{u}_n|_2$; *if* $\bar{u}_n$ *is unbounded in* $H^1(\mathbb{R}^N)$ *then* $\rho(\lambda_n, \beta, u_n, v_n) \to \infty$.

Proof. A direct computation shows that $(\bar{u}_n, \bar{v}_n)$ solve

$$\begin{cases} -\Delta u + \frac{1}{\lambda_n} u = \mu_2 u^3 + \beta u v^2 & \text{in } \mathbb{R}^N, \\ -\Delta v + v = \mu_1 v^3 + \beta v u^2 & \text{in } \mathbb{R}^N. \end{cases}$$

The result follows from Lemma 5.1 and

$$\rho(\lambda_n, \beta, u_n, v_n) = \frac{|u_n|_2}{|v_n|_2} = \frac{|\bar{v}_n|_2}{|\bar{u}_n|_2} \to 0.$$

□

Now we prove Theorems 2.4 and 2.5. Observe that (u, v) is a positive solution to (1.2) if and only if

$$\bar{u}(x) := \frac{1}{\sqrt{\lambda_2}} u\left(x/\sqrt{\lambda_2}\right), \quad \bar{v}(x) := \frac{1}{\sqrt{\lambda_2}} v\left(x/\sqrt{\lambda_2}\right),$$

solve (1.2) with $\lambda_1 = \lambda$ and $\lambda_2 = 1$, i.e. (4.1). Therefore ist is sufficient to consider this case.

Proof of Theorem 2.4. a) Arguing by contradiction suppose that for fixed $\beta \geq \mu_2$ there exist a sequence $\lambda_n \to 0$ and positive solutions (u_n, v_n) to (4.1) with $\lambda = \lambda_n$. Then we have

$$\langle \nabla u_n, \nabla v_n \rangle + \lambda_n \int_{\mathbb{R}^N} u_n v_n = \mu_1 \int_{\mathbb{R}^N} u_n^3 v_n + \beta \int_{\mathbb{R}^N} u_n v_n^3$$

and

$$\langle \nabla u_n, \nabla v_n \rangle + \int_{\mathbb{R}^N} u_n v_n = \mu_2 \int_{\mathbb{R}^N} v_n^3 u_n + \beta \int_{\mathbb{R}^N} v_n u_n^3.$$

These identities yield

$$(1 - \lambda_n)\langle \nabla u_n, \nabla v_n \rangle = \int_{\mathbb{R}^N} [(\beta - \lambda_n \mu_2) v_n^3 u_n + (\mu_1 - \lambda_n \beta) v_n u_n^3],$$

which implies $\langle \nabla u_n, \nabla v_n \rangle > 0$ for n large enough. On the other hand, we also have

$$(1 - \frac{\beta}{\mu_2})\langle \nabla u_n, \nabla v_n \rangle + (\lambda_n - \frac{\beta}{\mu_2}) \int_{\mathbb{R}^N} u_n v_n = \int_{\mathbb{R}^N} (\mu_1 - \frac{\beta^2}{\mu_2}) v_n u_n^3.$$

Now $|u_n|_\infty \to 0$ by Lemma 5.1, so that

$$\int_{\mathbb{R}^N} (\mu_1 - \frac{\beta^2}{\mu_2}) v_n u_n^3 = o(1) \int_{\mathbb{R}^N} u_n v_n.$$

In the case $\beta = \mu_2$, we deduce

$$\frac{\beta}{\mu_2} \int_{\mathbb{R}^N} u_n v_n = o(1) \int_{\mathbb{R}^N} u_n v_n,$$

a contradiction. And if $\beta > \mu_2$ we obtain

$$(1 - \frac{\beta}{\mu_2}) \langle \nabla u_n, \nabla v_n \rangle = (\frac{\beta}{\mu_2} + o(1)) \int_{\mathbb{R}^N} u_n v_n > 0,$$

which implies $\langle \nabla u_n, \nabla v_n \rangle < 0$ for n large enough, a contradiction again.

b) This follows from a) using the transformation from the proof of Lemma 5.2. □

Now we recall [17, Lemma 2.3].

Lemma 5.3. *The linearized problem*

$$\begin{cases} \Delta\phi - \lambda\phi + 3\mu_1 u^2 \phi + \beta v^2 \varphi + 2\beta u v \psi = 0, & x \in \mathbb{R}^N, \\ \Delta\psi - \psi + 3\mu_2 v^2 \psi + \beta u^2 \psi + 2\beta u v \phi = 0, & x \in \mathbb{R}^N, \\ \varphi = \varphi(r), \phi = \phi(r), \end{cases}$$

has exactly a one-dimensional set of solutions for $\lambda > 0$ and $\beta = \beta_1(\lambda)$, $(u, v) = (U_{\lambda,\mu_1}, 0)$ or $\beta = \beta_2(\lambda)$, $(u, v) = (0, U_{1,\mu_2})$.

We have a similar result for $\lambda = 0$.

Lemma 5.4. *The linearized problem*

$$\begin{cases} -\Delta\phi = \beta U_{1,\mu_2}^2 \phi, & x \in \mathbb{R}^N, \\ \Delta\psi - \psi + 3\mu_2 U_{1,\mu_2}^2 \psi = 0, & x \in \mathbb{R}^N, \\ \phi = \phi(r), \psi = \psi(r). \end{cases}$$

has only the zero solution if $0 < \beta \neq \tau_0\mu_2$. If $\beta = \tau_0\mu_2$ then the set of solutions has dimension one.

Proof. It is well known that the eigenvalue problem

$$-\Delta\phi + \phi = \nu\mu_2\omega_{1,\mu_2}^2 \phi = \nu\omega_{1,1}^2 \phi$$

has eigenvalues $\nu_1 = 1$, $\nu_2 = \cdots = \nu_{N+1} = 3$,$\nu_k > 3$ for $k \geq N+2$, and that the eigenfunctions corresponding to $\nu = 3$ are not radial. It follows that $\psi = 0$. If $\phi \not\equiv 0$ then $\phi > 0$ by the maximum principle, and ϕ is a minimizer of $\beta_2(0) = \mu_2\tau_0$. The result follows from Lemma 3.4. □

Now we return to study the asymptotic behavior of the positive solution for λ small or large and improve on Lemmas 5.1 and 5.2. And then give the proof of Theorem 2.5 to end this section.

Lemma 5.5. *a) Let* (u_n, v_n)*,* $n \in \mathbb{N}$*, be positive radial solutions of equation* (4.1) *with* $\lambda = \lambda_n \to 0$*. Then*

$$\left(\frac{1}{\sqrt{\lambda_n}} u_n\left(x/\sqrt{\lambda_n}\right), v_n(x)\right) \to \left(U_{1,\mu_1}(x), U_{1,\mu_2}(x)\right) \quad \text{in } \mathcal{C}^2_{loc}(\mathbb{R}^N) \times \mathcal{C}^2_{loc}(\mathbb{R}^N).$$

b) Let (u_n, v_n)*,* $n \in \mathbb{N}$*, be positive radial solutions of equation* (4.1) *with* $\lambda = \lambda_n \to \infty$*. Then*

$$\left(\frac{1}{\sqrt{\lambda_n}} u_n\left(x/\sqrt{\lambda_n}\right), v_n(x)\right) \to \left(U_{1,\mu_1}(x), U_{1,\mu_2}(x)\right) \quad \text{in } \mathcal{C}^2_{loc}(\mathbb{R}^N) \times \mathcal{C}^2_{loc}(\mathbb{R}^N).$$

Proof. a) We first consider the case $\lambda_n \to 0$.

STEP 1: $\liminf_{n\to\infty} \frac{1}{\sqrt{\lambda_n}} u_n(0) > 0$.

We argue by contradiction and assume that $u_n(0) = o(1)\sqrt{\lambda_n}$, after passing to a subsequence. The function

$$\bar{u}_n(x) := \frac{1}{u_n(0)} u_n\left(x/\sqrt{\lambda_n}\right)$$

solves

$$-\Delta \bar{u}_n(x) + \bar{u}_n(x) = \frac{u_n(0)^2}{\lambda_n} \mu_1 \bar{u}_n(x)^3 + \beta \bar{u}_n(x) \bar{v}_n(x)^2 \tag{5.1}$$

with

$$\bar{v}_n(x) := \frac{1}{\sqrt{\lambda_n}} v_n\left(x/\sqrt{\lambda_n}\right).$$

Observe that $\bar{u}_n \to \bar{u}$ in $\mathcal{C}^0_{loc}(\mathbb{R}^N)$ along a subsequence and $\bar{u}(0) = 1$ because $|\bar{u}_n|_\infty = \bar{u}_n(0) = 1$. By Lemma 5.1 we have $v_n \to U_{1,\mu_2}$ both in $H^1(\mathbb{R}^N)$ and in C^2_{loc}, and $v_n(x) \to 0$ as $|x| \to \infty$ uniformly in n. It follows that $\bar{v}_n \to 0$ uniformly outside an arbitrary neighborhood of 0. For a test function $h \in \mathcal{D}(\mathbb{R}^N)$ and $\varepsilon > 0$, there exists r_ε such that

$$\int_{|x|\le r_0} \left|\bar{u}_n \bar{v}_n^2(x) h(x)\right| dx \le |v_n|_3^2 \left(\int_{|x|\le r_\varepsilon} |h(x)|^3 dx\right)^{\frac{1}{3}} < \frac{\varepsilon}{2}.$$

Therefore $\int_{\mathbb{R}^N} \bar{u}_n \bar{v}_n^2 h\, dx \to 0$. Testing (5.1) with h we see that $\bar{u}_n \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$, contradicting $\bar{u}_n \to \bar{u}$ in $\mathcal{C}^0_{loc}(\mathbb{R}^N)$.

STEP 2: $\limsup_{n\to\infty} \frac{1}{\sqrt{\lambda_n}} u_n(0) < \infty$.

Assume by contradiction that $\sqrt{\lambda_n} = o(1) u_n(0)$, after passing to a subsequence. The function

$$\widetilde{u}_n(x) = \frac{1}{u_n(0)} u_n\left(\sqrt{\lambda_n} x / u_n(0)\right)$$

satisfies $|\widetilde{u}_n|_\infty = \widetilde{u}_n(0) = 1$ and

$$-\Delta\widetilde{u}_n + \frac{\sqrt{\lambda_n}}{u_n(0)}\widetilde{u}_n \geq \mu_1 \widetilde{u}_n^3 \quad \text{in } \mathbb{R}^N.$$

Then $\tilde{u}_n \to \tilde{u} \geq 0$ in $C^2_{loc}(\mathbb{R}^N)$, along a subsequence, with $\tilde{u}(0) = 1$, and $\tilde{u}$ satisfies

$$-\Delta\tilde{u} \geq \mu_1 \tilde{u}^3 \quad \text{in } \mathbb{R}^N.$$

This implies $\tilde{u} \equiv 0$, a contradiction.

The conclusion about $v_n(x)$ has already been proved in Lemma 5.1.

STEP 3: $\bar{u}_n(x) := \dfrac{1}{\sqrt{\lambda_n}} u_n\left(x/\sqrt{\lambda_n}\right) \to U_{1,\mu_1}(x)$ in $\mathcal{C}^2_{loc}(\mathbb{R}^N)$

Observe that

$$\begin{cases} -\Delta\bar{u}_n + \bar{u}_n = \mu_1 \bar{u}_n^3 + \dfrac{\beta}{\lambda_n}\bar{u}_n v_n^2\left(\,\cdot\,/\sqrt{\lambda_n}\right) & \text{in } \mathbb{R}^N \\ -\Delta v_n + v_n = \mu_2 v_n^3 + \beta v_n\left(\sqrt{\lambda_n}\bar{u}_n\left(\sqrt{\lambda_n}\,\cdot\right)\right)^2 & \text{in } \mathbb{R}^N. \end{cases}$$

By STEP 1 and STEP 2 we may assume that $\bar{u}_n \to \bar{u} \geq 0$ in $C^2_{loc}(\mathbb{R}^N)$ and $\bar{u}(0) > 0$, hence $\bar{u} > 0$ in $\mathbb{R}^N$. By $\lambda_n \to 0$, we may assume that $\lambda_n < 1$ for all n. Recalling that there exist $C, R > 0$, independent of n such that

$$v_n(x) \leq Ce^{-\frac{1}{2}|x|} \text{ for all } |x| > R, \text{ all } n \in \mathbb{N},$$

we have that

$$\frac{\beta}{\lambda_n} v_n^2\left(x/\sqrt{\lambda_n}\right) \leq \beta C^2 \frac{1}{\lambda_n} e^{-|x|/\sqrt{\lambda_n}} \text{ for all } |x| > R, \text{ all } n \in \mathbb{N}.$$

Fix $R > 0$, then $\beta C^2 \frac{1}{\lambda_n} e^{-R/\sqrt{\lambda_n}} \to 0$ as $n \to \infty$, which implies that

$$\frac{\beta}{\lambda_n} v_n^2\left(x/\sqrt{\lambda_n}\right) < \frac{1}{2} \text{ for all } |x| > R, \text{ and large } n.$$

Then it is standard to prove that $\bar{u}_n(x) \to 0$ exponentially and uniformly in large n. Thus, $\lim\limits_{x\to\infty} \bar{u}(x) = 0$. A similar argument as that in STEP 1 implies that $\bar{u}$ is a weak solution of

$$-\Delta\bar{u} + \bar{u} = \mu_1 \bar{u}^3, \quad \bar{u}(x) \to 0 \text{ as } |x| \to \infty.$$

So we obtain that $\bar{u} = U_{1,\mu_1}$ and thus $\bar{u}_n(x) \to U_{1,\mu_1}(x)$ in $C^2_{loc}(\mathbb{R}^N)$.

b) Using the transformations $\bar{\lambda}_n := \frac{1}{\lambda_n} \to 0$, $\bar{u}_n(x) := \frac{1}{\sqrt{\lambda_n}} v_n\left(x/\sqrt{\lambda_n}\right)$ and $\bar{v}_n(x) := \frac{1}{\sqrt{\lambda_n}} u_n\left(x/\sqrt{\lambda_n}\right)$, we see that (u_n, v_n) is a solution to

$$\begin{cases} -\Delta u + \lambda_n u = \mu_1 u^3 + \beta u v^2 & \text{in } \mathbb{R}^N \\ -\Delta v + v = \mu_2 v^3 + \beta v u^2 & \text{in } \mathbb{R}^N \end{cases}$$

if and only if $(\bar{u}_n, \bar{v}_n)$ is a solution to

$$\begin{cases} -\Delta u + \bar{\lambda}_n u = \mu_2 u^3 + \beta u v^2 & \text{in } \mathbb{R}^N, \\ -\Delta v + v = \mu_1 v^3 + \beta v u^2 & \text{in } \mathbb{R}^N. \end{cases} \tag{5.2}$$

We can apply the conclusion of a) to system (5.2) and obtain that

$$\left(\frac{1}{\sqrt{\bar{\lambda}_n}} \bar{u}_n \left(x/\sqrt{\bar{\lambda}_n} \right), \bar{v}_n(x) \right) \to (U_{1,\mu_2}(x), U_{1,\mu_1}(x)) \quad \text{in } C^2_{loc}(\mathbb{R}^N) \times C^2_{loc}(\mathbb{R}^N),$$

that is,

$$\left(\frac{1}{\sqrt{\lambda_n}} u_n \left(x/\sqrt{\lambda_n} \right), v_n(x) \right) \to (U_{1,\mu_1}(x), U_{1,\mu_2}(x)) \quad \text{in } C^2_{loc}(\mathbb{R}^N) \times C^2_{loc}(\mathbb{R}^N).$$

□

Corollary 5.6. *a) If (u_n, v_n) is a positive radial solution to equation* (4.1) *with $\lambda = \lambda_n$ and $\lambda_n \to 0$ then $\rho(\lambda_n, \beta, u_n, v_n) \to +\infty$.*

b) If (u_n, v_n) is a positive radial solution to equation (4.1) *with $\lambda = \lambda_n$ and $\lambda_n \to \infty$ then $\rho(\lambda_n, \beta, u_n, v_n) \to 0$.*

Proof. a) Lemma 5.5 $\bar{u}_n(x) := \frac{1}{\sqrt{\lambda_n}} u_n(\frac{x}{\sqrt{\lambda_n}}) \to U_{1,\mu_1}(x)$. So we have that

$$|u_n|_2^2 = \lambda_n^{-\frac{1}{2}} |\bar{u}_n|_2^2 \to +\infty$$

and

$$|v_n|_2^2 \to |U_{1,\mu_2}|_2^2.$$

Hence, $\rho(\lambda_n, \beta, u_n, v_n) \to +\infty$.

b) Apply a similar argument as in a), and note that $\lambda_n \to \infty$, we have that

$$|u_n|_2^2 = \lambda_n^{-\frac{1}{2}} |\bar{u}_n|_2^2 \to 0.$$

□

Proof of Theorem 2.5. a) Suppose there exists two families of positive solutions $(u_\lambda^{(1)}, v_\lambda^{(1)})$ and $(u_\lambda^{(2)}, v_\lambda^{(2)})$ to problem (4.1) with $\lambda \to 0^+$. Let

$$\left(\bar{u}_\lambda^{(i)}(x), \bar{v}_\lambda^{(i)}(x)\right) := \left(\frac{1}{\sqrt{\lambda}} u_\lambda^{(i)}(\frac{x}{\sqrt{\lambda}}), v_\lambda^{(i)}(x) \right), \quad i = 1, 2.$$

Then $\left(\bar{u}_\lambda^{(1)}(x), \bar{v}_\lambda^{(1)}(x)\right), \left(\bar{u}_\lambda^{(2)}(x), \bar{v}_\lambda^{(2)}(x)\right) \in E$ are two families of positive solutions to problem

$$\begin{cases} -\Delta u(x) + u(x) = \mu_1 u(x)^3 + \beta u(x) \left(\frac{1}{\sqrt{\lambda}} v(\frac{x}{\sqrt{\lambda}}) \right)^2 & \text{in } \mathbb{R}^N, \\ -\Delta v(x) + v(x) = \mu_2 v(x)^3 + \beta v(x) \left(\sqrt{\lambda} u(\sqrt{\lambda} x) \right)^2 & \text{in } \mathbb{R}^N, \\ 0 < u, v \in H^1(\mathbb{R}^N), N = 3. \end{cases} \tag{P_λ}$$

By Lemma 5.5,

$$\left(\bar{u}_\lambda^{(i)}(x), \bar{v}_\lambda^{(i)}(x)\right) \to (U_{1,\mu_1}, U_{1,\mu_2}) \text{ in } C^2_{loc}(\mathbb{R}^N) \times C^2_{loc}(\mathbb{R}^N), i = 1, 2.$$

Indeed, one can prove that this convergence also holds in E due to the fact that $\bar{u}^i_\lambda(x) \to 0$ exponentially and uniformly in small λ.

Case 1: $\limsup\limits_{\lambda\to 0^+} \dfrac{|\bar{v}_\lambda^{(1)} - \bar{v}_\lambda^{(2)}|_{L^\infty(\mathbb{R}^N)}}{\lambda |\bar{u}_\lambda^{(1)} - \bar{u}_\lambda^{(2)}|_{L^\infty(\mathbb{R}^N)}} < \infty$

We study the normalization

$$\xi_\lambda := \frac{\bar{u}_\lambda^{(1)} - \bar{u}_\lambda^{(2)}}{|\bar{u}_\lambda^{(1)} - \bar{u}_\lambda^{(2)}|_{L^\infty(\mathbb{R}^N)}},$$

Then up to a subsequence $\xi_\lambda \to \xi$ in $C^2_{loc}(\mathbb{R}^N)$. Then we have

$$\begin{aligned}
&\frac{1}{|\bar{u}_\lambda^{(1)} - \bar{u}_\lambda^{(2)}|_{L^\infty(\mathbb{R}^N)}} \left[\mu_1 \left(\bar{u}_\lambda^{(1)}\right)^3 - \mu_1 \left(\bar{u}_\lambda^{(2)}\right)^3\right] \\
=&\mu_1 \xi_\lambda \left[\left(\bar{u}_\lambda^{(1)}\right)^2 + \bar{u}_\lambda^{(1)}\bar{u}_\lambda^{(2)} + \left(\bar{u}_\lambda^{(2)}\right)^2\right] \\
\to& 3\mu_1 U^2_{1,\mu_1}\xi \text{ in } C^2_{loc}(\mathbb{R}^N) \text{ as } \lambda \to 0,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{|\bar{u}_\lambda^{(1)} - \bar{u}_\lambda^{(2)}|_{L^\infty(\mathbb{R}^N)}} \left[\beta \bar{u}_\lambda^{(1)}(x) \left(\frac{1}{\sqrt{\lambda}}\bar{v}_\lambda^{(1)}(\frac{x}{\sqrt{\lambda}})\right)^2 - \beta \bar{u}_\lambda^{(2)}(x) \left(\frac{1}{\sqrt{\lambda}}\bar{v}_\lambda^{(2)}(\frac{x}{\sqrt{\lambda}})\right)^2\right] \\
=&\frac{1}{|\bar{u}_\lambda^{(1)} - \bar{u}_\lambda^{(2)}|_{L^\infty(\mathbb{R}^N)}} \left[\beta \bar{u}_\lambda^{(1)}(x) \left(\frac{1}{\sqrt{\lambda}}\bar{v}_\lambda^{(1)}(\frac{x}{\sqrt{\lambda}})\right)^2 - \beta \bar{u}_\lambda^{(2)}(x) \left(\frac{1}{\sqrt{\lambda}}\bar{v}_\lambda^{(1)}(\frac{x}{\sqrt{\lambda}})\right)^2\right] \\
&+ \frac{1}{|\bar{u}_\lambda^{(1)} - \bar{u}_\lambda^{(2)}|_{L^\infty(\mathbb{R}^N)}} \left[\beta \bar{u}_\lambda^{(2)}(x) \left(\frac{1}{\sqrt{\lambda}}\bar{v}_\lambda^{(1)}(\frac{x}{\sqrt{\lambda}})\right)^2 - \beta \bar{u}_\lambda^{(2)}(x) \left(\frac{1}{\sqrt{\lambda}}\bar{v}_\lambda^{(2)}(\frac{x}{\sqrt{\lambda}})\right)^2\right] \\
=&\beta\xi_\lambda \left(\frac{1}{\sqrt{\lambda}}\bar{v}_\lambda^{(1)}(\frac{x}{\sqrt{\lambda}})\right)^2 + \beta \bar{u}_\lambda^{(2)}(x) \left(\bar{v}_\lambda^{(1)}(\frac{x}{\sqrt{\lambda}}) + \bar{v}_\lambda^{(2)}(\frac{x}{\sqrt{\lambda}})\right) \frac{\bar{v}_\lambda^{(1)}(\frac{x}{\sqrt{\lambda}}) - \bar{v}_\lambda^{(2)}(\frac{x}{\sqrt{\lambda}})}{\lambda|\bar{u}_\lambda^{(1)} - \bar{u}_\lambda^{(2)}|_{L^\infty(\mathbb{R}^N)}}.
\end{aligned}$$

For any $h \in H^1(\mathbb{R}^3)$, one can prove that

$$\lim_{\lambda\to 0} \int_{\mathbb{R}^3} \beta\xi_\lambda \left(\frac{1}{\sqrt{\lambda}}\bar{v}_\lambda^{(1)}(\frac{x}{\sqrt{\lambda}})\right)^2 h dx = 0 \tag{5.3}$$

and

$$\lim_{\lambda\to 0} \int_{\mathbb{R}^3} \bar{u}_\lambda^{(2)}(x)\bar{v}_\lambda^{(i)}(\frac{x}{\sqrt{\lambda}})h(x)dx = 0, i = 1, 2.$$

So we see that ξ is a weak solution to

$$-\Delta\xi + \xi = 3\mu_1 U^2_{1,\mu_1}\xi. \tag{5.4}$$

By $|\xi|_{L^\infty} = 1$, a standard elliptic estimation indicate that ξ is a strong solution. Then by the decay of U_{1,μ_1}, applying the comparison principle, we can obtain that ξ exponentially decay to 0 as $|x| \to \infty$. Hence, $\xi \in H^1(\mathbb{R}^3)$ and then (5.4) implies that

$$\xi = \sum_{i=1}^{3} b_i \frac{\partial U_{1,\mu_1}}{\partial x_i}$$

for some suitable $b_i \in \mathbb{R}$. On the other hand, by the definition, we see that ξ is of radial, and thus $b_i = 0, i = 1, 2, 3$. So $\xi = 0$, a contradiction. Hence,

$$\bar{u}_\lambda^{(1)} \equiv \bar{u}_\lambda^{(2)} \text{ for small } \lambda,$$

and then we also have

$$\bar{v}_\lambda^{(1)} \equiv \bar{v}_\lambda^{(2)} \text{ for small } \lambda$$

due to that

$$\frac{1}{\sqrt{\lambda}} v_\lambda^{(i)}(\frac{x}{\sqrt{\lambda}}) = \left(\frac{-\Delta \bar{u}_\lambda^{(i)} + \bar{u}_\lambda^{(i)} - \mu_1 \left(\bar{u}_\lambda^{(i)}\right)^3}{\beta \bar{u}_\lambda^{(i)}} \right)^{\frac{1}{2}}, \quad i = 1, 2.$$

Case 2: $\limsup_{\lambda \to 0^+} \dfrac{|\bar{v}_\lambda^{(1)} - \bar{v}_\lambda^{(2)}|_{L^\infty(\mathbb{R}^N)}}{\lambda |\bar{u}_\lambda^{(1)} - \bar{u}_\lambda^{(2)}|_{L^\infty(\mathbb{R}^N)}} = \infty$

In this case, we study the normalization

$$\eta_\lambda := \frac{\bar{v}_\lambda^{(1)} - \bar{v}_\lambda^{(2)}}{|\bar{v}_\lambda^{(1)} - \bar{v}_\lambda^{(2)}|_{L^\infty(\mathbb{R}^N)}},$$

And up to a subsequence, $\eta_\lambda \to \eta$ in $C^2_{loc}(\mathbb{R}^N)$. Apply a similar argument as above, we obtain that

$$-\Delta \eta + \eta = 3U_{1,\mu_2}^2 \eta.$$

By η is a radial function, we also obtain that

$$\bar{v}_\lambda^{(1)} \equiv \bar{v}_\lambda^{(2)} \text{ for small } \lambda,$$

and

$$\bar{u}_\lambda^{(1)} \equiv \bar{u}_\lambda^{(2)} \text{ for small } \lambda$$

by

$$\sqrt{\lambda} \bar{u}_\lambda^{(i)}(\sqrt{\lambda} x) = \left(\frac{-\Delta \bar{v}_\lambda^{(i)} + \bar{v}_\lambda^{(i)} - \mu_2 \left(\bar{v}_\lambda^{(i)}\right)^3}{\beta \bar{v}_\lambda^{(i)}} \right)^{\frac{1}{2}}, \quad i = 1, 2.$$

Combining the cases 1 and 2, we see that (4.1) has at most one positive solution for λ small enough. And using the transformation in Lemma 5.2, one can prove the case of λ large.

b) It is well known that (1.2) has a mountain pass type solution for $\beta \le \mu_2 \tau_0 < \beta_2(\lambda) = \min\{\beta_1(\lambda), \beta_2(\lambda)\}$ for $\lambda > 0$ small. It follows from a) that this is unique. The second statement in Theorem 2.5 b) for $\beta \le \mu_1 \tau_0$ follows by applying a transformation as in the proof of Lemma 5.2. $\square$

6 Proof of Theorem 2.1 and Proposition 2.2

Due to Lemma 4.1 it is sufficient to consider the case $\lambda_1 = \lambda$ and $\lambda_2 = 1$, i.e. system (4.1).

Proof of Theorem 2.1. a) For $\beta \le \tau_0 \min\{\mu_1, \mu_2\}$ the existence of normalized solutions for every $a, b > 0$ follows from Corollaries 4.10 and 5.6. For $\beta \ge \tau_0 \max\{\mu_1, \mu_2\}$ let $\mathcal{S}_i^\beta$, $i = 1, 2$, be the connected sets of positive solutions from Proposition 4.6 e). If $\mathcal{S}_1^\beta \cap \mathcal{S}_2^\beta \neq \emptyset$ then the existence of normalized solutions for every $a, b > 0$ follows from Corollary 4.8. Now we suppose $\mathcal{S}_1^\beta \cap \mathcal{S}_2^\beta = \emptyset$. Then Proposition 4.6 e) yields that $P_1(\mathcal{S}_i^\beta)$ contains one of the intervals $(0, \ell_i(\beta))$ or $(\ell_i(\beta), \infty)$, $i = 1, 2$. If $(\ell_1(\beta), \infty) \subset P_1(\mathcal{S}_1^\beta)$ then the existence of normalized solutions for every $a, b > 0$ follows from Corollary 5.6. The same argument applies if $(0, \ell_2(\beta)) \subset P_1(\mathcal{S}_2^\beta)$. Now we show that the case $\mathcal{S}_1^\beta \cap \mathcal{S}_2^\beta = \emptyset$ and $(0, \ell_2(\beta)) \not\subset P_1(\mathcal{S}_2^\beta)$ cannot happen, concluding the proof of a). Similarly one can show that $\mathcal{S}_1^\beta \cap \mathcal{S}_2^\beta = \emptyset$ and $(\ell_1(\beta), \infty) \not\subset P_1(\mathcal{S}_1^\beta)$ leads to a contradiction.

Suppose by contradiction that $\mathcal{S}_1^\beta \cap \mathcal{S}_2^\beta = \emptyset$ and $(0, \ell_2(\beta)) \not\subset P_1(\mathcal{S}_2^\beta)$. Then $(\ell_2(\beta), \infty) \subset P_1(\mathcal{S}_2^\beta)$. Recall from Theorem 2.5 a) that (4.1) has at most one solution for λ large. It follows that there exists a family $(\lambda, \beta, u_{\lambda,\beta}, v_{\lambda,\beta}) \in X$, $\lambda \ge \tilde{\lambda}(\beta)$, so that

$$\mathcal{S}^\beta \cap \big([\tilde{\lambda}(\beta), \infty) \times \mathbb{P}\big) = \mathcal{S}_1^\beta \cap \big([\tilde{\lambda}(\beta), \infty) \times \mathbb{P}\big) = \{(\lambda, \beta, u_{\lambda,\beta}, v_{\lambda,\beta}) : \lambda \ge \tilde{\lambda}(\beta)\}.$$

The fixed point index computations in Section 4, in particular Propositions 4.4, 4.11 and Corollary 4.5, imply for $\lambda \ge \tilde{\lambda}(\beta)$:

$$\begin{aligned} \mathrm{ind}_{\mathbb{P}}\big(\mathbb{A}_{\lambda,\beta}, (u_{\lambda,\beta}, v_{\lambda,\beta})\big) &= i_\infty - \mathrm{ind}_{\mathbb{P}}\big(\mathbb{A}_{\lambda,\beta}, (U_{\lambda,\mu_1}, 0)\big) \\ &\qquad - \mathrm{ind}_{\mathbb{P}}\big(\mathbb{A}_{\lambda,\beta}, (0, U_{1,\mu_2})\big) - \mathrm{ind}_{\mathbb{P}}\big(\mathbb{A}_{\lambda,\beta}, (0,0)\big) \\ &= 0 + 0 + 1 - 1 = 0 \end{aligned} \tag{6.1}$$

Observe that $\mathcal{T}_2^\beta \cup \mathcal{S}_2^\beta$ is a connected component of the set $\mathcal{Z} = \mathcal{T}_0 \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{S}$ of all solutions because $\mathcal{S}_1^\beta \cap \mathcal{S}_2^\beta = \emptyset$. Then there exists an open set $\mathcal{O} \subset X^\beta$ with the following properties:

(i) $\mathcal{T}_2^\beta \cup \mathcal{S}_2^\beta \subset \mathcal{O}$

(ii) $\mathcal{Z} \cap \partial\mathcal{O} = \emptyset$

(iii) There exists $\delta > 0$ so that

$$\mathcal{O} \cap \big((0, \delta] \times \{\beta\} \times \mathbb{P}\big) = \big\{(\lambda, \beta, u, v) : \lambda \in (0, \delta],\ (u, v) \in B_\delta(0, U_{1,\mu_2})\big\}$$

The last property (iii) can be achieved because $(0, \ell_2(\beta)) \not\subset P_1(\mathcal{S}_2^\beta)$, hence $\mathcal{S}_2^\beta \subset [\delta, \infty) \times \{\beta\} \times \mathbb{P}$ for some small $\delta > 0$. Using the notation $\mathcal{O}_{\lambda,\beta} := \{(u, v) \in \mathbb{P} :$

$(\lambda, \beta, u, v) \in \mathcal{O}\}$ it follows for $\lambda \geq \tilde{\lambda}(\beta)$ that:

$$\begin{aligned} \mathrm{ind}_{\mathbb{P}}\big(\mathbb{A}_{\lambda,\beta}, (u_{\lambda,\beta}, v_{\lambda,\beta})\big) &= \mathrm{ind}_{\mathbb{P}}\big(\mathbb{A}_{\lambda,\beta}, \mathcal{O}_{\lambda,\beta}\big) - \mathrm{ind}_{\mathbb{P}}\big(\mathbb{A}_{\lambda,\beta}, (0, U_{1,\mu_2})\big) \\ &= \mathrm{ind}_{\mathbb{P}}\big(\mathbb{A}_{\delta,\beta}, \mathcal{O}_{\delta,\beta}\big) - \mathrm{ind}_{\mathbb{P}}\big(\mathbb{A}_{\lambda,\beta}, (0, U_{1,\mu_2})\big) \\ &= \mathrm{ind}_{\mathbb{P}}\big(\mathbb{A}_{\delta,\beta}, (0, U_{1,\mu_2})\big) - \mathrm{ind}_{\mathbb{P}}\big(\mathbb{A}_{\lambda,\beta}, (0, U_{1,\mu_2})\big) \\ &= 0 + 1 = 1 \end{aligned}$$

This contradicts (6.1).

b) We only prove the case $\mu_2 < \mu_1$. The case $\mu_1 < \mu_2$ can then be deduced using the transformation from the proof of Lemma 5.2. Let $\mathcal{S}_2^\beta$ be the connected set of positive solutions from Proposition 4.6 d). Then Proposition 4.6 d) yields that $P_1(\mathcal{S}_2^\beta)$ contains one of the intervals $(0, \ell_2(\beta))$ or $(\ell_2(\beta), \infty)$. If $(0, \ell_2(\beta)) \subset P_1(\mathcal{S}_2^\beta)$ then the existence of normalized solutions for every $a, b > 0$ follows from Corollary 5.6. If $(\ell_2(\beta), \infty) \subset P_1(\mathcal{S}_2^\beta)$ then

$$\delta := \max_{(\lambda,\beta,u,v) \in \mathcal{S}_2^\beta} \rho(\lambda, \beta, u, v) > 0.$$

Since $\rho(\lambda, \beta, u, v) \to 0$ as $\lambda \to \infty$, and as $\lambda \to \ell_2(\beta)$ on $\mathcal{S}_2^\beta$, we see that $\rho(\mathcal{S}) \supset (0, \delta]$.

Finally, if $\beta \in (\tau_0 \mu_2, \mu_2)$ then there exists the solution $(1, \beta, u_\beta, v_\beta) \in \mathcal{S}$ from Remark 2.6, which has fixed point index 1. Let $\mathcal{S}_0^\beta \subset \mathcal{S}^\beta$ be the connected component of $(1, \beta, u_\beta, v_\beta)$ in $\mathcal{S}^\beta$. An index count as above yields that $P_1(\mathcal{S}_0^\beta) \subset \mathbb{R}^+$ is bounded away from 0. Since it cannot bifurcate from $\mathcal{T}_1$ it must bifurcate from $\mathcal{T}_2$, i.e. $\mathcal{S}_3^\beta = \mathcal{S}_2^\beta$. This implies

$$\delta \geq \rho(1, \beta, u_\beta, v_\beta) = \sqrt{\frac{\beta - \min\{\mu_1, \mu_2\}}{\beta - \max\{\mu_1, \mu_2\}}}.$$

□

Proof of Proposition 2.2. We only prove the case of $\mu_2 \leq \beta \leq \tau\mu_1$, the second part result is easy by using the transformation from the proof of Lemma 5.2. By Theorem 2.4 b), there exists $\eta_2(\beta) > 0$ such that problem (4.1) has no positive solution provided $\lambda < \eta_2(\beta)$. On the other hand, by Theorem 2.5 b), problem (4.1) has a unique positive solution (u_λ, v_λ), which is of mountain pass type, for $\lambda \geq \tilde{\lambda}(\beta)$ large enough. By Corollary 5.6, we have that $\rho(\lambda, \beta, u_\lambda, v_\lambda) \to 0$ as $\lambda \to \infty$. So

$$q_1 := \{\rho(\lambda, \beta, u_\lambda, v_\lambda), \lambda \geq \tilde{\lambda}(\beta)\ \} < \infty.$$

Observe that according to Proposition 4.4 a), see also [12, Lemma 2.1],

$$\sup_{(\lambda,\beta,u,v) \in \mathcal{S}^\beta, \eta_2(\beta) \leq \lambda \leq \tilde{\lambda}(\beta)} \big(|u|_2^2 + |v|_2^2\big) < \infty.$$

Then we have that

$$q_2 := \sup\{\rho(\lambda, \beta, u, v) : (\lambda, \beta, u, v) \in \mathcal{S}^\beta,\ \eta_2(\beta) \leq \lambda \leq \tilde{\lambda}_\beta\} < \infty.$$

Indeed, if there exists a sequence $(\lambda_n, \beta, u_n, v_n)$ with $\lambda_n \to \lambda \in [\eta_2(\beta), \tilde{\lambda}_\beta]$ such that $\rho(\lambda_n, \beta, u_n, v_n) \to \infty$. Then we see that $|v_n|_2^2 \to 0$ and it is standard to prove that $(u_n, v_n) \to (U_{\lambda,\mu_1}, 0)$ in $H^1(\mathbb{R}^N)$. And thus, $\beta = \beta_1(\lambda) > \lim\limits_{\lambda\to\infty} \beta_1(\lambda) = \tau_0\mu_1$, a contradiction. Then $q := \max\{q_1, q_2\}$ is the required bound.

□

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