Strongly localized semiclassical states for nonlinear Dirac equations

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Abstract

We study semiclassical states of the nonlinear Dirac equation

$$-i\hbar\partial_t\psi = ic\hbar\sum_{k=1}^{3}\alpha_k\partial_k\psi - mc^2\beta\psi - M(x)\psi + f(|\psi|)\psi, \quad t\in\mathbb{R},\ x\in\mathbb{R}^3,$$

where V is a bounded continuous potential function and the nonlinear term $f(|\psi|)\psi$ is superlinear, possibly of critical growth. Our main result deals with standing wave solutions that concentrate near a critical point of the potential. Standard methods applicable to nonlinear Schrödinger equations, like Lyapunov-Schmidt reduction or penalization, do not work, not even for the homogeneous nonlinearity $f(s)=s^p$. We develop a variational method for the strongly indefinite functional associated to the problem.

Keywords. Dirac equation, semiclassical states, standing waves, concentration, strongly indefinite functional

1 Introduction

Standing wave solutions for the nonlinear Schrödinger equation

$$-i\hbar\partial_t\psi = -\Delta\psi + V(x)\psi + f(|\psi|)\psi$$

a non-relativistic wave equation, have been in the focus of nonlinear analysis since decades. In particular, semiclassical states that concentrate near a critical point of the potential V

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Mathematics Subject Classification (2010): Primary 35Q40; Secondary 49J35 have been widely investigated ever since the influential paper [24] by Floer and Weinstein who treated the cubic nonlinearity $|\psi|^2\psi$ in one-dimension.

Much less is known for the nonlinear Dirac equation

$$-i\hbar\partial_t\psi = ic\hbar\sum_{k=1}^{3}\alpha_k\partial_k\psi - mc^2\beta\psi - M(x)\psi + f(x,|\psi|)\psi, \qquad t\in\mathbb{R},\ x\in\mathbb{R}^3,$$

a relativistic wave equation and a spinor generalization of the nonlinear Schrödinger equation, not even in the case of f being a pure power with subcritical nonlinearity. Here $\psi(t,x)\in\mathbb{C}^4$, c is the speed of light, $\hbar$ is Planck's constant, m is the mass of the particle and α_1, α_2, α_3 and β are the 4×4 complex Pauli matrices:

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1,2,3,$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The external field $M(x)$ represents an arbitrary electric potential depending only upon $x\in\mathbb{R}^3$. The nonlinear coupling $f(x,|\psi|)\psi$ describes a self-interaction. Typical examples for nonlinear couplings can be found in the self-interacting scalar theories; see [22, 23, 31] and more recently [7, 20, 21, 25, 26, 37, 43]. Usually, in Quantum electrodynamics nonlinear Dirac equations have to satisfy symmetry constraints, in particular the Poincaré covariance. Nonlinear Dirac equations modeling Bose-Einstein condensates break this symmetry, and often the nonlinearity is a *power-type* function that depends only on the local condensate density (see [28–30] for more background from physics).

The ansatz $\psi(t,x) = e^{i\omega t/\hbar}u(x)$ for a standing wave solution and a change of notation (in particular ε instead of $\hbar$) leads to an equation of the form

$$-i\varepsilon\sum_{k=1}^{3}\alpha_k\partial_k u + a\beta u + V(x)u = f(x,|u|)u, \quad u\in H^1(\mathbb{R}^3,\mathbb{C}^4). \tag{1.1}$$

This type of particle-like solution does not change its shape as it evolves in time, hence has a soliton-like behavior. In this paper we investigate the existence of semiclassical states, i.e. solutions u_ε of (1.1) for small $\varepsilon>0$, that concentrate as $\varepsilon\to 0$ at a critical point x_0 of the potential V. There are many results of this type for nonlinear Schrödinger equations

$$-\varepsilon^2\Delta u + V(x)u = g(u), \quad u\in H^1(\mathbb{R}^N), \tag{1.2}$$

beginning with the pioneering work by Floer and Weinstein [24] and then continued by Oh [38, 39] and many others, e.g. [2–6, 8–11, 40]. It has been proved that there exists a family of semiclassical solutions to (1.2) for small ε which concentrate around stable critical points of the potential V as $\varepsilon \to 0$. The proofs are based on Lyapunov-Schmidt type methods, penalization, and variational techniques.

Very few results are available for the nonlinear Dirac equation (1.1) compared with the nonlinear Schrödinger equation. A major difference between nonlinear Schrödinger and Dirac equations is that the Dirac operator is strongly indefinite in the sense that both the negative and positive parts of the spectrum are unbounded and consist of essential spectrum. It follows that the quadratic part of the energy functional associated to (1.1) has no longer a positive sign, moreover, the Morse index and co-index at any critical point of the energy functional are infinite.

In order to compare our result with the existing literature we first present in short the state of the art. The first result for concentration behavior of the nonlinear Dirac equation (1.1) is due to Ding [13], who considered the case $V \equiv 0$ and $f(x,|u|) = P(x)|u|^{p-2}$ with $p \in (2,3)$ subcritical, $\inf P > 0$, and $\limsup_{|x|\to\infty} P(x) < \max P$. He obtained a least energy solution u_ε for $\varepsilon > 0$ small that concentrates around a global maximum of P as $\varepsilon \to 0$. A similar result has been obtained in [14] where $f = f(|u|)$ is subcritical and V satisfies $a < \min V < \liminf_{|x|\to\infty} V(x) \le |V|_\infty < a$. Here the solutions u_ε concentrate at a global minimum of V. In both papers [13, 14] the solutions are obtained via a mountain pass argument applied to a reduced functional. In [18, 44] the authors considered the case of a local minimum of V using a penalization approach analogous to the one in [10, 11].

All papers mentioned so far consider a subcritical nonlinearity f. The only papers dealing with a critical nonlinearity, i.e. where $f(t)$ grows as t for $t \to \infty$, are [15, 16]. Both papers assume, in addition to various technical conditions, that V has a global minimum. The least energy solution is obtained again via a mountain pass argument applied to a reduced functional. It is essential that the mountain pass level is below the threshold level where the Palais-Smale condition fails. In [15] the authors were also able to obtain solutions with energy above the mountain pass level using the oddness of the equation and Lusternik-Schnirelmann type arguments, but again the energy levels of the solutions are below the level where the Palais-Smale condition fails.

The distinct new feature of our result is that we find solutions of

$$-i\varepsilon \sum_{k=1}^{3} \alpha_k \partial_k u + a\beta u + V(x)u = f(|u|)u \tag{1.3}$$

localized near a critical point of V that is not necessarily a (local or global) minimum of V. The model nonlinearity we consider is $f(t) = \kappa t + \lambda t^{p-2}$ with $\kappa, \lambda > 0$ and $p \in (2,3)$. We can deal with local minima, local maxima, or saddle points of V, both in the critical ($\kappa > 0$) and subcritical ($\kappa = 0$) case. As a consequence, a least energy solution may not exist, and in the variational setting there is no threshold value below which the Palais-Smale condition holds, so that the methods from [15, 16] do not apply. We have to work at energy levels where the Palais-Smale condition fails which, in the critical case $\kappa > 0$, leads to a subtle interplay between κ, V, λ, p. Our results are new even in the subcritical case where so far only local minima of V have been treated. They are of course new in the critical case where only global minima of V have been considered.

The paper is organized as follows. In the next section we state and discuss our main theorem. After collecting some basic results on the Dirac operator in Section 3 we investigate the family of equations

$$-i\sum_{k=1}^{3}\alpha_k\partial_k u + a\beta u + V(\xi)u = f(|u|)u \tag{1.4}$$

parametrized by $\xi \in \mathbb{R}^3$ which appear as limit equations. This will be done in Section 4. In Section 5 we introduce a truncated problem, set up the variational structure, and prove the Palais-Smale condition for the truncated functional in a certain parameter range. Then in Section 6 we develop a min-max scheme that can be applied to the truncated problem. The proof of a key result, Proposition 6.4, that is needed for the passage to the limit $\varepsilon \to 0$ will be presented in Section 7. The delicate analysis in Section 7 is not needed in the case of a local minimum of V because in that case the lower bound estimate of Proposition 6.4 is automatically satisfied. In the final Section 8 we show that the solutions of the truncated problem are actually solutions of (1.1) for $\varepsilon > 0$ small enough, thus finishing the proof of the main theorem. The proof of a technical lemma will be presented in the Appendix.

2 The main result

We set $\alpha \cdot \nabla := \sum_{k=1}^{3}\alpha_k\partial_k$ so that equation (1.3) reads as

$$-i\varepsilon\alpha\cdot\nabla u + a\beta u + V(x)u = f(|u|)u, \quad u \in H^1(\mathbb{R}^3, \mathbb{C}^4).$$

Throughout the paper, we fix the constant $a > 0$ and assume that the potential V satisfies

$(V0)$ $V \in \mathcal{C}^{0,1}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and $|V|_\infty < a$.

Here we use the notation $|\cdot|_p$ for the various L^p-norms. We also require one of the following hypotheses:

$(V1)$ V is $\mathcal{C}^1$ in a neighborhood of 0, and 0 is an isolated local maximum or minimum of V.

$(V2)$ V is $\mathcal{C}^2$ in a neighborhood of 0, 0 is an isolated critical point, and there exists a vector space $X \subset \mathbb{R}^3$ such that:

(a) $V|_X$ has a strict local maximum at 0;

(b) $V|_{X^\perp}$ has a strict local minimum at 0.

In the case of $(V2)$ we may assume that $\{0\} \neq X \neq \mathbb{R}^3$ so that 0 is a possibly degenerate saddle point of V.

The domain of the quadratic form associated to the Dirac operator is $H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{C}^4)$. This space embeds into the corresponding L^q-spaces for $2 \le q \le 3$, and the embedding is locally compact precisely if $q < 3$. Therefore the nonlinearity $f(|u|)u$ has subcritical growth if $f(s)s \sim s^{p-1}$ with $2 < p < 3$, and it has critical growth if $p = 3$. In (3.8) below we define for $\lambda > 0$, $p \in (2,3)$ a constant $\bar{\kappa}(V, \lambda, p) > 0$ that appears in the following assumptions when the nonlinearity is critical. Here $F(s) := \int_0^s f(t)t\,dt$ is the primitive of $f(s)s$.

$(f1)$ $f \in \mathcal{C}^0[0,\infty) \cap \mathcal{C}^1(0,\infty)$ satisfies $f(0) = 0$ and $f'(s) > 0$ for $s > 0$.

$(f2)$ There exist $\lambda > 0$, $p \in (2,3)$, $\kappa \in [0, \bar{\kappa})$ with $\bar{\kappa} = \bar{\kappa}(V, \lambda, p)$ defined in (3.8) such that $f(s) \ge \kappa s + \lambda s^{p-2}$ for $s > 0$, and $f'(s) \to \kappa$ as $s \to \infty$.

$(f3)$ There exists $\theta > 2$ such that $0 < \theta F(s) \le f(s)s^2 + \frac{\theta-2}{3}\kappa s^3$ for $s > 0$.

These conditions imply that $s \mapsto f(s)s$ is strictly increasing and superlinear. Condition $(f3)$ is a weakened Ambrosetti-Rabinowitz condition. If $\kappa > 0$ then the nonlinearity has critical growth.

Theorem 2.1. *Assume that V satisfies $(V0)$ and one of $(V1)$ or $(V2)$. Suppose that f satisfies $(f1)$, $(f2)$ and $(f3)$. Then (1.1) has a solution u_ε for $\varepsilon > 0$ small. These solutions have the following properties.*

(i) $|u_\varepsilon|$ possesses a global maximum point $x_\varepsilon \in \mathbb{R}^3$ such that $x_\varepsilon \to 0$ as $\varepsilon \to 0$, and

$$|u_\varepsilon(x)| \le C \exp\Big(-\frac{c}{\varepsilon}|x - x_\varepsilon|\Big)$$

with $C, c > 0$ independent of ε.

(ii) The rescaled function $U_\varepsilon(x) = u_\varepsilon(\varepsilon x + x_\varepsilon)$ converges as $\varepsilon \to 0$ uniformly to a least energy solution $U : \mathbb{R}^3 \to \mathbb{C}^4$ of

$$-i\alpha \cdot \nabla U + a\beta U + V(0)U = f(|U|)U.$$

Remark 2.2. Thus in the subcritical case $\kappa = 0$ equation (1.1) always has solutions with shape as in (i) and (ii). We do allow critical growth but the factor κ cannot be too large. The constant $\bar{\kappa}$ depends on $|V|_\infty$, $\sup V$, λ and p. It is bounded away from 0 by a positive number provided V is bounded away from $-a$ and $\sup V \le 0$. Moreover $\bar{\kappa} \to 0$ as $|V|_\infty \to a$. It is an interesting open problem whether the restriction on κ can be removed.

3 Preliminaries

We write $L^q = L^q(\mathbb{R}^3, \mathbb{C}^4)$ for $q \geq 1$ and $H^s = H^s(\mathbb{R}^3, \mathbb{C}^4)$ for $s > 0$. Let $D_a = -i\alpha \cdot \nabla + a\beta$ denote the self-adjoint operator on L^2 with domain $\mathcal{D}(D_a) = H^1$. It is well known that the spectrum of D_a is purely continuous and $\sigma(D_a) = \sigma_c(D_a) = \mathbb{R} \setminus (-a, a)$. Therefore L^2 possesses the orthogonal decomposition

$$L^2 = L^+ \oplus L^-, \quad u = u^+ + u^-, \tag{3.1}$$

so that D_a is positive definite (resp. negative definite) in L^+ (resp. L^-). Now let $E := \mathcal{D}(|D_a|^{1/2})$ be the form domain of D_a endowed with the inner product

$$\langle u, v \rangle = \mathrm{Re}\big(|D_a|^{1/2}u, |D_a|^{1/2}v\big)_2$$

and induced norm $\|\cdot\|$; here $(\cdot,\cdot)_2$ denotes the L^2-inner product. This norm is equivalent to the usual $H^{1/2}$-norm, hence E embeds continuously into L^q for all $q \in [2, 3]$ and compactly into L^q_{loc} for all $q \in [2, 3)$. Clearly E possesses the decomposition

$$E = E^+ \oplus E^- \quad \text{with } \; E^\pm = E \cap L^\pm, \tag{3.2}$$

orthogonal with respect to both $(\cdot,\cdot)_2$ and $\langle\cdot,\cdot\rangle$. Since $\sigma(D_a) = \mathbb{R} \setminus (-a, a)$, one has

$$a|u|_2^2 \leq \|u\|^2 \quad \text{for all } u \in E. \tag{3.3}$$

The decomposition of E induces also a natural decomposition of L^q for every $q \in (1, \infty)$ as proved in [18].

Proposition 3.1. *Setting $E_q^\pm := E^\pm \cap L^q$ for $q \in (1, \infty)$ there holds*

$$L^q = \mathrm{cl}_q\, E_q^+ \oplus \mathrm{cl}_q\, E_q^-$$

with cl_q denoting the closure in L^q. More precisely, for every $q \in (1, \infty)$ there exists $d_q > 0$ such that

$$d_q|u^\pm|_q \leq |u|_q \quad \text{for all } u \in E \cap L^q.$$

Moreover, the decomposition is invariant when taking derivatives.

Proposition 3.2. *For $u \in H^1$ we have $\partial_k u^\pm = (\partial_k u)^\pm$.*

Proof. The Fourier transformation of D_a is given by

$$(D_a u)\hat{}(\xi) = \begin{pmatrix} 0 & \sum_{k=1}^3 \xi_k\sigma_k \\ \sum_{k=1}^3 \xi_k\sigma_k & 0 \end{pmatrix} \hat{u} + \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \hat{u},$$

where $\hat{u}$, a $\mathbb{C}^4$-valued function, denotes the Fourier transform of $u \in L^2$. It has been proved in [18] that the Fourier transforms of the orthogonal projections $P^\pm : L^2 \to L^\pm$ are given by

$$(P^+u)\hat{}(\xi) = \Big(\frac{1}{2} + \frac{a}{2\sqrt{a^2+|\xi|^2}}\Big)\begin{pmatrix} I & \Sigma(\xi) \\ \Sigma(\xi) & A(\xi) \end{pmatrix}\hat{u}$$

and

$$(P^-u)\hat{}(\xi) = \Big(\frac{1}{2} + \frac{a}{2\sqrt{a^2+|\xi|^2}}\Big)\begin{pmatrix} A(\xi) & -\Sigma(\xi) \\ -\Sigma(\xi) & I \end{pmatrix}\hat{u}$$

with I being the 2×2 identity matrix and

$$A(\xi) = \frac{\sqrt{a^2+|\xi|^2}-a}{a+\sqrt{a^2+|\xi|^2}}\cdot I, \quad \Sigma(\xi) = \sum_{k=1}^{3}\frac{\xi_k\sigma_k}{a+\sqrt{a^2+|\xi|^2}}.$$

The proposition follows from the fact that these matrix operations commute with the multiplication by $i\xi_k$ for $k = 1,2,3$. $\square$

The proof of our main results will be achieved by variational methods applied to functionals $J : E \to \mathbb{R}$ of the form

$$J(u) = \frac{1}{2}\big(\|u^+\|^2 - \|u^-\|^2\big) + \frac{1}{2}\int_{\mathbb{R}^3} W(x)|u|^2\,dx - \int_{\mathbb{R}^3} G(x,|u|)\,dx. \tag{3.4}$$

The following reduction process will be very useful.

Theorem 3.3. *Let $W \in L^\infty$ satisfy $|W|_\infty < a$ and suppose $G : \mathbb{R}^3 \times \mathbb{R}_0^+ \to \mathbb{R}$ has the form $G(x,s) = \int_0^s g(x,t)tdt$ where g is measurable in $x \in \mathbb{R}^3$, of class $\mathcal{C}^1$ in $s \in \mathbb{R}_0^+$ and satisfies*

(i) $0 \le g(x,s)s$ *for all* $x \in \mathbb{R}^3$*;*

(ii) $g(x,s)s = o(s)$ *as* $s \to 0$ *uniformly in* $x \in \mathbb{R}^3$*;*

(iii) $0 \le \partial_s\big(g(x,s)s\big) \le Cs$ *for all* $x \in \mathbb{R}^3$*,* $s > 0$*, some* $C > 0$.

Then the following hold for J as in (3.4).

a) There exists a $\mathcal{C}^1$-map $h_J : E^+ \to E^-$ such that for $v \in E^+$ and $w \in E^-$

$$DJ(v+w)[\phi] = 0 \quad \text{for all } \phi \in E^- \qquad \Longleftrightarrow \qquad w = h_J(v)$$

and

$$\|h_J(v)\|^2 \le \frac{2|W|_\infty}{a-|W|_\infty}\|v\|^2 + \frac{2a}{a-|W|_\infty}\int_{\mathbb{R}^3} G(x,|v|)dx.$$

b) *Setting* $J^{red} : E^+ \to \mathbb{R}$, $J^{red}(v) := J(v + h_J(v))$, *the sets*

$$\mathcal{M}^+(J) := \{v \in E^+ \setminus \{0\} : DJ^{red}(v)[v] = 0\}$$

and

$$\mathcal{M}(J) := \{v+h_J(v) \in E\setminus\{0\} : v \in \mathcal{M}^+(J)\} = \{u \in E\setminus\{0\} : DJ(u)|_{\mathbb{R}u\oplus E^-} = 0\}$$

are $\mathcal{C}^1$*-submanifolds of* E*, diffeomorphic to an open subset of the unit sphere* $SE^+ = \{v \in E^+ : \|v\| = 1\}$.

c) *If* $(v_n)_n$ *is a Palais-Smale sequence for* J^{red} *then* $\{v_n + h_J(v_n)\}_n$ *is a Palais-Smale sequence for* J.

d) *If* $|g(x,s)| = O\left(|s|^{p-2}\right)$ *as* $|s| \to \infty$ *for some* $p \in (2,3)$ *then* h_J *is weakly sequentially continuous.*

The proof of Theorem 3.3 is standard. We refer the reader to [1, 18, 42] for this kind of results. The diffeomorphisms to an open subset of SE^+ are simply given by $u \mapsto \frac{u^+}{\|u^+\|}$. In the case $W \equiv \nu \in (-a,a)$ the manifold $\mathcal{M}(J)$ is the Nehari-Pankov manifold associated to J. It will be useful that the decomposition $E = E^+ \oplus E^-$ is independent of W and does not necessarily correspond to the decomposition of E into the positive and negative eigenspaces associated to $D^2J(0) = P^+ - P^- + W(x)$. We call J^{red} the reduced functional, h_J the reduction map, and (J^{red}, h_J) the *reduction couple* of J.

Remark 3.4. In the setting of Theorem 3.3, for each $v \in SE^+$ the map $\varphi_v(t) = J^{red}(tv)$ is $\mathcal{C}^2$ and has at most one critical point $t_v > 0$, which is a nondegenerate maximum. There holds:

$$\mathcal{M}^+(J) = \{t_v v : v \in SE^+,\ \varphi_v'(t_v) = 0\}.$$

If G grows super-quadratically in t as $t \to \infty$ then $J(tu) \to -\infty$ as $t \to \infty$ and $\varphi_v(t)$ has a unique maximum for each $v \in SE^+$. Then $\mathcal{M}(J)$ and $\mathcal{M}^+(J)$ are diffeomorphic to SE^+. It is clear that $\mathcal{M}(J)$ contains all nontrivial critical points of J, and that for $u \in E \setminus \{0\}$ there holds:

$$DJ(u) = 0 \qquad \Longleftrightarrow \qquad u^- = h_J(u^+) \text{ and } DJ^{red}(u^+) = 0$$

Finally, the infimum of J on $\mathcal{M}(J)$ can be described as follows:

$$\begin{aligned} \gamma(J) := \inf_{u\in\mathcal{M}(J)} J(u) &= \inf_{v\in E^+\setminus\{0\}} \sup_{u\in\mathbb{R}v\oplus E^-} J(u) \\ &= \inf_{v\in E^+\setminus\{0\}} \max_{t>0} J^{red}(tv) = \inf_{v\in\mathcal{M}^+(J)} J^{red}(v) \end{aligned} \tag{3.5}$$

If $\gamma(J)$ is achieved then it is the ground state energy.

Theorem 3.3 applies in particular to the following functional which depends on the parameters $\vec{\mu} = (\kappa, \lambda, \nu, p)$ with $\kappa, \lambda \geq 0$, $|\nu| < a$ and $p \in (2,3)$:

$$J_{\vec{\mu}}(u) = \frac{1}{2}\big(\|u^+\|^2 - \|u^-\|^2\big) + \frac{\nu}{2}|u|_2^2 - \frac{\lambda}{p}|u|_p^p - \frac{\kappa}{3}|u|_3^3. \tag{3.6}$$

In order to define the constant $\bar{\kappa}$ from Theorem 2.1 let

$$S := \inf_{0 \neq u \in H^1} \frac{|\nabla u|_2^2}{|u|_6^2} \tag{3.7}$$

be the best constant for the embedding $H^1(\mathbb{R}^3, \mathbb{C}^4) \hookrightarrow L^6(\mathbb{R}^3, \mathbb{C}^4)$. Then we define for V as in $(V0)$, $\lambda > 0$, $p \in (2,3)$ as in $(f2)$, $(f3)$:

$$\bar{\kappa} := \left(\frac{a^2 - |V|_\infty^2}{a^2}\right)^{\frac{3}{4}} S^{\frac{3}{4}} \big(6\gamma(J_{\vec{\mu}_V})\big)^{-\frac{1}{2}} \qquad \text{with } \vec{\mu}_V := (0, \lambda, \sup V, p). \tag{3.8}$$

The following technical result will be needed later.

Lemma 3.5. *For $v \in E^+ \setminus \{0\}$ the function $H(t) = I(tv) - \frac{t}{2} I'(tv)[v]$ is of class $\mathcal{C}^1$ and satisfies $H'(t) > 0$ for all $t > 0$.*

Proof. We set $\varphi_v(t) = I(tv)$ so that $H(t) = \varphi_v(t) - \frac{t}{2}\varphi_v'(t)$. Since

$$H'(t) = \frac{1}{2}\varphi_v'(t) - \frac{t}{2}\varphi_v''(t) = \frac{1}{2t}\big[\varphi_{tv}'(1) - \varphi_{tv}''(1)\big],$$

it is sufficient to check that $\varphi_v'(1) - \varphi_v''(1) > 0$ for all $v \in E^+ \setminus \{0\}$. Setting $K(u) = \int_{\mathbb{R}^3} G(x, |u|)\, dx$, we have by the definition of h_J

$$-\langle h_J(v), \phi \rangle + \operatorname{Re} \int_{\mathbb{R}^3} W(x)(v + h_J(v)) \cdot \overline{\phi}\, dx - K'(v + h_J(v))[\phi] = 0 \tag{3.9}$$

for all $\phi \in E^-$. It follows for $z_v = v + h_J(v)$ and $w_v = h_J'(v)[v] - h_J(v)$ that

$$\varphi_v'(1) = \|v\|^2 + \operatorname{Re} \int_{\mathbb{R}^3} W(x) z_v \cdot \overline{v}\, dx - K'(z_v)[v] = J'(z_v)[z_v + w_v]. \tag{3.10}$$

Since (3.9) is valid for all $v \in E^+$, differentiating yields for all $\phi \in E^-$:

$$0 = -\langle h_J'(v)[v], \phi \rangle + \operatorname{Re} \int_{\mathbb{R}^3} W(x)(v + h_J'(v)[v]) \cdot \overline{\phi}\, dx - K''(v + h_J(v))[v + h_J'(v)[v], \phi]\,.$$

Choosing $\phi = h_J'(v)[v]$ in the above identity, so that $z_v + w_v = v + \phi$, we get

$$\begin{aligned}
\varphi_v''(1) &= \|v\|^2 + \mathrm{Re}\int_{\mathbb{R}^3} W(x)(v + h_J'(v)[v]) \cdot \overline{v}\, dx - K''(z_v)[z_v + w_v, v] \\
&= \|v\|^2 - \|\phi\|^2 + \int_{\mathbb{R}^3} W(x)|v+\phi|^2\, dx - K''(z_v)[z_v + w_v, v + \phi] \\
&= J''(z_v)[z_v + w_v, z_v + w_v] \\
&= \|v\|^2 - \|h_J(v)\|^2 + \int_{\mathbb{R}^3} W(x)|z_v|^2\, dx - K''(z_v)[z_v, z_v] \\
&\quad + 2\left(- \langle h_J(v), w_v\rangle + \mathrm{Re}\int_{\mathbb{R}^3} W(x) z_v \cdot \overline{w_v}\, dx - K''(z_v)[z_v, w_v]\right) \\
&\quad + \left(-\|w_v\|^2 + \int_{\mathbb{R}^3} W(x)|w_v|^2\, dx - K''(z_v)[w_v, w_v]\right) \\
&= \varphi_v'(1) + \big(K'(z_v)[z_v] - K''(z_v)[z_v, z_v]\big) + 2\big(K'(z_v)[w_v] - K''(z_v)[z_v, w_v]\big) \\
&\quad - K''(z_v)[w_v, w_v] - \|w_v\|^2 + \int_{\mathbb{R}^3} W(x)|w_v|^2\, dx.
\end{aligned}$$

Finally we obtain:

$$\varphi_v'(1) - \varphi_v''(1) \geq \int_{\mathbb{R}^3} G'(x, |z_v|)|w_v|^2 + G''(x, |z_v|)|z_v| \Big(|z_v| + \frac{\mathrm{Re}\, z_v \cdot \overline{w_v}}{|z_v|}\Big)^2 dx > 0$$

$\square$

4 The limit problem

For $|\nu| < a$ the problem

$$-i\alpha \cdot \nabla u + a\beta u + \nu u = f(|u|)u, \quad u \in E, \tag{4.1}$$

appears as limit equation of (1.1). We begin with the model case

$$-i\alpha \cdot \nabla u + a\beta u + \nu u = \lambda |u|^{p-2} u + \kappa |u| u \quad u \in E.$$

and recall the associated energy functional $J_{\vec{\mu}}$ from (3.6) with $\vec{\mu} = (\kappa, \lambda, \nu, p)$ and κ, λ, p from $(f2)$, $(f3)$.

Proposition 4.1. *The infimum $\gamma(J_{\vec{\mu}})$ is attained provided ν satisfies*

$$\Big(\frac{a^2}{a^2 - \nu_-^2}\Big)^{\frac{3}{2}} \cdot \kappa^2 \cdot \gamma(J_{\vec{\mu}}) < \frac{S^{\frac{3}{2}}}{6}, \tag{4.2}$$

where $\nu_- = \min\{0, \nu\}$.

Proof. We only give the proof for $\kappa > 0$ since the subcritical case $\kappa = 0$ is much easier. Let $(J_{\vec{\mu}}^{red}, h_{\vec{\mu}})$ denote the reduction couple of $J_{\vec{\mu}}$ and let $(v_n)_n$ be a minimizing sequence for $J_{\vec{\mu}}^{red}$ in $\mathcal{M}^+(J_{\vec{\mu}})$. Setting $u_n = v_n + h_{\vec{\mu}}(v_n)$ it is not difficult to check that $(u_n)_n$ is bounded in E, hence it is either vanishing or non-vanishing up to a subsequence (see [34]).

If $(u_n)_n$ has a non-vanishing subsequence then we are done, so let us assume to the contrary that $(u_n)_n$ is vanishing, hence $|u_n|_p \to 0$. We first show that this implies

$$\gamma(J_{\vec{\mu}}) \geq \gamma(J_{\vec{\mu}_0}) \qquad \text{where } \vec{\mu}_0 = (\kappa, 0, \nu, p). \tag{4.3}$$

In order to see this let $t_n > 0$ be defined by $t_n v_n \in \mathcal{M}^+(J_{\vec{\mu}_0})$. Observe that $\|v_n\|$ is bounded away from 0 and the nonlinearity in $J_{\vec{\mu}_0}$ is super-quadratic, so that $(t_n)_n$ is bounded. Theorem 3.3 d) now implies $|h_{\vec{\mu}_0}(t_n v_n)|_p \to 0$ where $h_{\vec{\mu}_0}$ is the reduction map for $J_{\vec{\mu}_0}$. Now (4.3) follows from

$$\begin{aligned}\gamma(J_{\vec{\mu}_0}) &\leq J_{\vec{\mu}_0}(t_n v_n + h_{\vec{\mu}_0}(t_n v_n)) \\ &= J_{\vec{\mu}}(t_n v_n + h_{\vec{\mu}_0}(t_n v_n)) + o_n(1) \leq J_{\vec{\mu}}^{red}(v_n) + o_n(1) = \gamma(J_{\vec{\mu}}) + o_n(1).\end{aligned}$$

Next we show that

$$J_{\vec{\mu}_0}^{red}(v) \geq \frac{1}{6\kappa^2}\left(\frac{\|v\|^2 + \nu|v|_2^2}{|v|_3^2}\right)^3 \qquad \text{for all } v \in \mathcal{M}^+(J_{\vec{\mu}_0})\,. \tag{4.4}$$

For this we consider the functional

$$I : E \setminus \{0\} \to \mathbb{R}, \quad u \mapsto \frac{\|u^+\|^2 - \|u^-\|^2 + \nu|u|_2^2}{|u|_3^2}.$$

For any $v \in E^+$ it is easy to see by a direct argument that $\sup_{w \in E^-} I(v + w) > 0$ is achieved by some $w_v \in E^-$. Moreover, for any $c > 0$ the set $\{w \in E^- : I(v + w) \geq c\}$ is strictly convex because

$$w \mapsto \|v\|^2 - \|w\|^2 + \nu|v + w|_2^2 - c|v + w|_3^2$$

is strictly concave on E^-. This also uses $|\nu| < a$. Hence w_v is the unique critical point of $w \mapsto I(v + w)$. On the other hand, for $v \in \mathcal{M}^+(J_{\vec{\mu}_0})$, we have

$$0 = DJ_{\vec{\mu}_0}^{red}(v)[v] = \|v\|^2 - \|h_{\vec{\mu}_0}(v)\|^2 + \nu|v + h_{\vec{\mu}_0}(v)|_2^2 - \kappa|v + h_{\vec{\mu}_0}(v)|_3^3, \tag{4.5}$$

hence

$$J_{\vec{\mu}_0}^{red}(v) = J_{\vec{\mu}_0}^{red}(v) - \frac{1}{2}DJ_{\vec{\mu}_0}^{red}(v)[v] = \frac{\kappa}{6}|v + h_{\vec{\mu}_0}(v)|_3^3.$$

A direct calculation gives

$$DI\big(v + h_{\vec{\mu}_0}(v)\big)\big|_{E^-} = 0 \quad \text{and} \quad I\big(v + h_{\vec{\mu}_0}(v)\big) > 0$$

which implies $h_{\vec{\mu}_0}(v) = w_v$. Now (4.4) follows, using (4.5) once more:

$$J^{red}_{\vec{\mu}_0}(v) = \frac{\kappa}{6}|v + h_{\vec{\mu}_0}(v)|_3^3 = \frac{1}{6\kappa^2} I^3(v + h_{\vec{\mu}_0}(v)) \geq \frac{1}{6\kappa^2} I^3(v)$$

Finally, the proposition follows from (4.3), (4.4) and

$$\frac{\|v\|^2 + \nu|v|_2^2}{|v|_3^2} \geq \left(\frac{a^2 - \nu_-^2}{a^2}\right)^{\frac{1}{2}} S^{\frac{1}{2}} \qquad \text{for all } v \in \mathcal{M}^+(J_{\vec{\mu}_0})\,. \tag{4.6}$$

For the proof of (4.6) we pass to the Fourier domain and recall from [18] that

$$\|u\|^2 = \int_{\mathbb{R}^3} (a^2 + |\xi|^2)^{\frac{1}{2}} |\hat{u}|^2 \, d\xi \qquad \text{for all } u \in E.$$

Since $|\nu| < a$ we have

$$(a^2 + t^2)^{\frac{1}{2}} + \nu \geq \left(\frac{a^2 - \nu_-^2}{a^2}\right)^{\frac{1}{2}} |t| \quad \text{for all } t \in \mathbb{R}$$

which implies for $v \in E^+ \setminus \{0\}$:

$$\begin{aligned} \frac{\|v\|^2 + \nu|v|_2^2}{|v|_3^2} &= \frac{\int_{\mathbb{R}^3} [(a^2 + |\xi|^2)^{\frac{1}{2}} + \nu] \cdot |\hat{v}|^2 \, d\xi}{|v|_3^2} \geq \left(\frac{a^2 - \nu_-^2}{a^2}\right)^{\frac{1}{2}} \frac{\int_{\mathbb{R}^3} |\xi||\hat{u}|^2 \, d\xi}{|u|_3^2} \\ &\geq \left(\frac{a^2 - \nu_-^2}{a^2}\right)^{\frac{1}{2}} S^{\frac{1}{2}} \end{aligned}$$

Here the last inequality follows from

$$\frac{\int_{\mathbb{R}^3} |\xi|^2 |\hat{u}|^2 d\xi}{|u|_6^2} = \frac{|\widehat{\nabla u}|_2^2}{|u|_6^2} = \frac{|\nabla u|_2^2}{|u|_6^2} \geq S \quad \text{for all } u \in H^1(\mathbb{R}^3, \mathbb{C}^4)$$

and the Calderón-Lions interpolation theorem (see [41]). □

Now we consider the energy functional $I_\nu : E \to \mathbb{R}$ associated to (4.1) given by

$$I_\nu(u) = \frac{1}{2}\big(\|u^+\|^2 - \|u^-\|^2\big) + \frac{\nu}{2}|u|_2^2 - \int_{\mathbb{R}^3} F(|u|)dx. \tag{4.7}$$

The hypotheses $(f1) - (f3)$ imply that I_ν satisfies the assumptions of Theorem 3.3 for $|\nu| < a$.

Lemma 4.2. *If $\nu_0 \in (-a, a)$ satisfies* (4.2) *then $\gamma(I_\nu)$ is achieved for all $\nu \in (-a, \nu_0]$. Moreover, the map $\nu \mapsto \gamma(I_\nu)$ is continuous and strictly increasing.*

Proof. For $\nu \in (-a, \nu_0] \subset (-a, a)$ assumption $(f3)$ implies $I_\nu \leq J_{\vec{\mu}_1} \leq J_{\vec{\mu}_2}$, where $\vec{\mu}_1 = (\kappa, \lambda, \nu_0, p)$ and $\vec{\mu}_2 = (0, \lambda, \nu_0, p)$. It follows that $\gamma(I_\nu) \leq \gamma(J_{\vec{\mu}_1}) \leq \gamma(J_{\vec{\mu}_2})$. A similar argument as in the proof of Proposition 4.1 implies the existence of a nontrivial critical point u_ν for I_ν such that u_ν^+ is the minimizer for I_ν^{red} on $\mathcal{M}^+(I_\nu)$.

In order to prove the monotonicity of $\gamma(\nu)$ we consider $-a < \nu_1 < \nu_2 \le \nu_0$. Let $u \in \mathcal{M}(I_{\nu_2})$ be a minimizer for $\gamma(I_{\nu_2})$ and define $s > 0$ by $su^+ \in \mathcal{M}^+(I_{\nu_1})$. Then we have, with $(I_{\nu_1}^{red}, h_{\nu_1})$ denoting the reduction couple for I_{ν_1} and $u_1 := t_1u^+ + h_{\nu_1}(su^+) \in \mathcal{M}(I_{\nu_1})$:

$$\begin{aligned}\gamma(I_{\nu_1}) &\le I_{\nu_1}^{red}(su^+) = I_{\nu_1}(u_1) = I_{\nu_2}(u_1) - \frac{\nu_2-\nu_1}{2}|u_1|_2^2 \le I_{\nu_2}^{red}(t_1u^+) - \frac{\nu_2-\nu_1}{2}|u_1|_2^2 \\ &\le \max_{t>0} I_{\nu_2}^{red}(tu^+) - \frac{\nu_2-\nu_1}{2}|u_1|_2^2 = \gamma(I_{\nu_2}) - \frac{\nu_2-\nu_1}{2}|u_1|_2^2.\end{aligned}$$

Choosing a minimizer $v \in \mathcal{M}(I_{\nu_1})$ for $\gamma(I_{\nu_1})$, defining $t > 0$ by $tv^+ \in \mathcal{M}^+(I_{\nu_2})$, and setting $u_2 := tv^+ + h_{\nu_1}(tv^+) \in \mathcal{M}(I_{\nu_2})$, an analogous argument shows that

$$\gamma(I_{\nu_2}) \le \gamma(I_{\nu_1}) + \frac{\nu_2-\nu_1}{2}|u_2|_2^2.$$

For the continuity of $\gamma(\nu)$ it remains to prove that s, t are bounded for ν_1, ν_2 in a compact subset of $(-a, \nu_0]$ because then $|\gamma(I_{\nu_2}) - \gamma(I_{\nu_1})| = O(\nu_2 - \nu_1)$. This follows for s from

$$0 < I_{\nu_1}^{red}(su^+) \le \frac{s^2}{2}\big(\|u^+\|^2 + \nu_1|u^+|_2^2\big) - \frac{d_p\lambda}{p}s^p|u^+|_p^p.$$

where $d_p > 0$ is from Proposition 3.1. The bound for t is proved analogously. $\square$

5 The truncated problem

For a subset $\Lambda \subset \mathbb{R}^3$, let Λ^c denote its complement, and $\Lambda^\varepsilon := \{x \in \mathbb{R}^3 : \varepsilon x \in \Lambda\}$, $\varepsilon > 0$. By the change of variables $x \mapsto \varepsilon x$ and setting $V_\varepsilon(x) = V(\varepsilon x)$, the singularly perturbed problem (1.1) is equivalent to

$$-i\alpha\cdot\nabla u + a\beta u + V_\varepsilon(x)u = f(|u|)u. \tag{5.1}$$

In the sequel, we will modify the function f similar to [9, 10]. For

$$0 < \delta_0 \le \frac{a - |V|_\infty}{4}, \tag{5.2}$$

we define $\tilde f = \tilde f_{\delta_0} \in C^1(\mathbb{R}_0^+)$ by $\tilde f(0) = 0$ and

$$\frac{d}{ds}\big(\tilde f(s)s\big) = \min\{f'(s)s + f(s),\ \delta_0\}.$$

In the subcritical case $\kappa = 0$ of Theorem 2.1 the choice $\delta_0 = \frac{a-|V|_\infty}{4}$ will be fine. For the critical case $\kappa > 0$ we need to make δ_0 smaller in the course of the proof. Let $\tilde F(s) = \int_0^s \tilde f(t)t\,dt$ be the primitive of $\tilde f(s)s$. By our assumptions on V there exists $R_1 > 0$ so that

$$\nabla V(x) \notin \mathbb{R}x \quad \text{for all } x \in \mathbb{R}^3 \text{ with } |x| = R_1 \text{ and } V(x) = V(0), \tag{5.3}$$

see [8]. We define the cut-off function $\chi : \mathbb{R}^3 \to [0,1]$ by

$$\chi(x) = \begin{cases} 1, & \text{if } |x| < R_1 \\ \frac{2R_1 - |x|}{R_1}, & \text{if } R_1 \le |x| < 2R_1 \\ 0, & \text{if } |x| \ge 2R_1. \end{cases} \tag{5.4}$$

and consider

$$g(x,s) = \chi(x) f(s) + \big(1 - \chi(x)\big)\tilde{f}(s)$$

as well as

$$G(x,s) = \int_0^s g(x,t)t dt = \chi(x) F(s) + \big(1 - \chi(x)\big)\tilde{F}(s).$$

For later use, associated to the above notations, we denote $B_1 = B(0, R_1)$ and $B_2 = B(0, 2R_1)$ the open balls in $\mathbb{R}^3$ of radius R_1 and $2R_1$. The following lemma is easy to prove.

Lemma 5.1. *The function $G(x,s)$ satisfies the conditions $(i) - (iii)$ from Theorem* 3.3.

We will consider the truncated problem

$$-i\alpha \cdot \nabla u + a\beta u + V_\varepsilon(x) u = g_\varepsilon(x, |u|)u, \quad u \in E \tag{5.5}$$

where we write $g_\varepsilon(x,s) = g(\varepsilon x, s)$; we also use the notations χ_ε and G_ε for the dilations of χ and G, respectively. The corresponding energy functional is

$$\Phi_\varepsilon(u) = \frac{1}{2}\big(\|u^+\|^2 - \|u^-\|^2\big) + \frac{1}{2}\int_{\mathbb{R}^3} V_\varepsilon(x)|u|^2\, dx - \int_{\mathbb{R}^3} G_\varepsilon(x, |u|)\, dx.$$

As a direct consequence of Lemma 5.1, we can introduce $(\Phi_\varepsilon^{red}, h_\varepsilon)$ as the reduction couple of Φ_ε.

In order to establish a compactness result for Φ_ε, we first prove a bound for Palais-Smale sequences of Φ_ε that is uniform in ε.

Lemma 5.2. *For $c \in \mathbb{R}$ fixed, $(PS)_c$-sequences of Φ_ε are bounded uniformly in ε.*

Proof. Given a $(PS)_c$-sequence $(u_n)_n$ for Φ_ε we have by our conditions on f:

$$\begin{aligned}
&\int_{\mathbb{R}^3} \chi_\varepsilon(x) f(|u_n|)|u_n| \cdot |u_n^+ - u_n^-| dx \\
&\quad \le \Big(\int_{\mathbb{R}^3} \chi_\varepsilon(x)\big(f(|u_n|)|u_n|\big)^{\frac{3}{2}} dx\Big)^{\frac{2}{3}} \cdot \big|u_n^+ - u_n^-\big|_3 + \delta_0 \int_{\mathbb{R}^3} \chi_\varepsilon(x)|u_n| \cdot |u_n^+ - u_n^-| dx \\
&\quad \le C_\theta\Big(\int_{\mathbb{R}^3} \chi_\varepsilon(x)\big(f(|u_n|)|u_n|^2 - 2F(|u_n|)\big) dx\Big)^{\frac{2}{3}} \|u_n\| + \delta_0 |u_n|_2^2,
\end{aligned}$$

where $C_\theta > 0$ only depends on the constant $\theta > 2$ in $(f2)$. It follows from (5.2) that

$$\Big(1 - \frac{|V|_\infty}{a}\Big)\|u_n\|^2 \le \Phi_\varepsilon'(u_n)[u_n^+ - u_n^-] + \int_{\mathbb{R}^3} g_\varepsilon(x,|u_n|)|u_n| \cdot |u_n^+ - u_n^-| dx$$
$$\le C_\theta\Big(2\Phi_\varepsilon(u_n) - \Phi_\varepsilon'(u_n)[u_n]\Big)^{\frac{2}{3}}\|u_n\| + 2\delta_0|u_n|_2^2 + o(\|u_n\|).$$

Now the lemma follows using (3.3):

$$\Big(1 - \frac{|V|_\infty + 2\delta_0}{a}\Big)\|u_n\|^2 \le C_\theta\big(2(c+o(1)) + o(\|u_n\|)\big)^{\frac{2}{3}}\|u_n\| + o(\|u_n\|). \tag{5.6}$$

□

Now we can prove the Palais-Smale condition for Φ_ε. Recall that the nonlinearity G in Φ_ε depends on a constant δ_0; see (5.2).

Proposition 5.3. *If*

$$\kappa^2 \cdot c_0 < \Big(\frac{a^2 - |V|_\infty^2}{a^2}\Big)^{\frac{3}{2}} \cdot \frac{S^{\frac{3}{2}}}{6},$$

then there exists $\delta_0 > 0$ such that the truncated functional Φ_ε satisfies the $(PS)_c$-condition for all $c \le c_0$, all $\varepsilon > 0$.

Proof. We choose $\delta_0 > 0$ so that

$$\Big(\frac{a^2 - |V|_\infty^2}{a^2}\Big)^{\frac{3}{2}}\frac{S^{\frac{3}{2}}}{6} > \Big(\frac{a^2 - (|V|_\infty + \delta_0)^2}{a^2}\Big)^{\frac{3}{2}}\frac{S^{\frac{3}{2}}}{6} > \kappa^2 \cdot c_0.$$

Let $(u_n)_n$ be a $(PS)_c$-sequence for Φ_ε with $c \le c_0$, any $\varepsilon > 0$. By Lemma 5.2 there exists $u \in E$ such that, along a subsequence, $u_n \rightharpoonup u$ in E and $u_n \to u$ strongly in L^q_{loc} for $q \in [2,3)$. We want to show that $u_n \to u$ strongly in E.

Set $z_n = u_n - u$ so that $z_n \rightharpoonup 0$ in E and $\|u_n^\pm\|^2 = \|u^\pm\|^2 + \|z_n^\pm\|^2 + o_n(1)$. Note that

$$\lim_{s\to 0}\tilde f(s) = \lim_{s\to\infty}\frac{\tilde f(s)}{s} = 0 \qquad \text{and} \qquad \lim_{s\to 0} f(s) = \lim_{s\to\infty}\frac{f(s)}{s} - \kappa = 0.$$

By the Brezis-Lieb lemma (see for instance [45, Lemma 1.32]) there holds

$$\int_{\mathbb{R}^3} G_\varepsilon(x,|u_n|) = \int_{\mathbb{R}^3} G_\varepsilon(x,|u|) + \int_{\mathbb{R}^3}\big(1-\chi_\varepsilon(x)\big)\tilde F(|z_n|) + \frac{\kappa}{3}\int_{\mathbb{R}^3}\chi_\varepsilon(x)|z_n|^3 + o_n(1),$$

and

$$\int_{\mathbb{R}^3} g_\varepsilon(x,|u_n|)|u_n|^2 = \int_{\mathbb{R}^3} g_\varepsilon(x,|u|)|u|^2 + \int_{\mathbb{R}^3}\big(1-\chi_\varepsilon(x)\big)\tilde f(|z_n|)|z_n|^2 + \kappa\int_{\mathbb{R}^3}\chi_\varepsilon(x)|z_n|^3 + o_n(1).$$

Therefore

$$\Phi_\varepsilon(u_n) = \Phi_\varepsilon(u) + \Phi_\varepsilon(z_n) + o_n(1),$$

and

$$D\Phi_\varepsilon(u_n)[u_n] = D\Phi_\varepsilon(u)[u] + D\Phi_\varepsilon(z_n)[z_n] + o_n(1).$$

Obviously, $D\Phi_\varepsilon(u) = 0$, hence $D\Phi_\varepsilon(z_n)[z_n] = o_n(1)$. We claim that

$$D\Phi_\varepsilon(z_n) \to 0 \text{ as } n \to \infty. \tag{5.7}$$

In fact, consider $\varphi \in E$ with $\|\varphi\| \le 1$ and set $g^1(x,s) = g(x,s) - \kappa\chi(x)s$. We have

$$\begin{aligned} D\Phi_\varepsilon(u_n)[\varphi] &= \langle u_n^+ - u_n^-, \varphi\rangle + \mathrm{Re}\int_{\mathbb{R}^3} V_\varepsilon(x)u_n\cdot\bar{\varphi} - \mathrm{Re}\int_{\mathbb{R}^3} g_\varepsilon(x,|u_n|)u_n\cdot\bar{\varphi} \\ &= \langle z_n^+, \varphi^+\rangle - \langle z_n^-, \varphi^-\rangle + \langle u^+, \varphi^+\rangle - \langle u^-, \varphi^-\rangle \\ &\quad + \mathrm{Re}\int_{\mathbb{R}^3} V_\varepsilon(x)z_n\cdot\bar{\varphi} + \mathrm{Re}\int_{\mathbb{R}^3} V_\varepsilon(x)u\cdot\bar{\varphi} \\ &\quad - \mathrm{Re}\int_{\mathbb{R}^3} g_\varepsilon^1(x,|z_n|)z_n\cdot\bar{\varphi} - \mathrm{Re}\int_{\mathbb{R}^3} g_\varepsilon^1(x,|u|)u\cdot\bar{\varphi} \\ &\quad - \kappa\cdot\mathrm{Re}\int_{\mathbb{R}^3} \chi_\varepsilon(x)|z_n+u|(z_n+u)\cdot\bar{\varphi} + o_n(\|\varphi\|) \end{aligned} \tag{5.8}$$

where we used $u_n = z_n + u$ and $D\Phi_\varepsilon(u) = 0$. The estimate for the subcritical part

$$\mathrm{Re}\int_{\mathbb{R}^3} g_\varepsilon^1(x,|u_n|)u_n\cdot\bar{\varphi} - \mathrm{Re}\int_{\mathbb{R}^3} g_\varepsilon^1(x,|z_n|)z_n\cdot\bar{\varphi} - \mathrm{Re}\int_{\mathbb{R}^3} g_\varepsilon^1(x,|u|)u\cdot\bar{\varphi} = o_n(\|\varphi\|)$$

follows from a standard argument in [12, Lemma 7.10]. To estimate the last integral in (5.8), we set $\psi_n := |z_n+u|(z_n+u) - |z_n|z_n - |u|u$ and observe $|\psi_n| \le 2|z_n|\cdot|u|$. By the Egorov theorem there exists $\Theta_\sigma \subset B_2^\varepsilon$ such that *meas*$(B_2^\varepsilon \setminus \Theta_\sigma) < \sigma$ and $z_n \to 0$ uniformly on Θ_σ as $n \to \infty$. Thus, by the Hölder inequality, we have

$$\begin{aligned} \int_{\mathbb{R}^3} \chi_\varepsilon(x)|\psi_n|\cdot|\varphi| &\le \int_{\Theta_\sigma} |\psi_n|\cdot|\varphi| + \int_{B_2^\varepsilon\setminus\Theta_\sigma} |\psi_n|\cdot|\varphi| \\ &\le \int_{\Theta_\sigma} |\psi_n|\cdot|\varphi| + 2\left(\int_{B_2^\varepsilon\setminus\Theta_\sigma} |z_n|^3\right)^{\frac13}\cdot\left(\int_{B_2^\varepsilon\setminus\Theta_\sigma} |u|^3\right)^{\frac13}\cdot\left(\int_{B_2^\varepsilon\setminus\Theta_\sigma} |\varphi|^3\right)^{\frac13}. \end{aligned}$$

The first integral in the last line converges to 0 as $n \to \infty$ and the remaining integrals go to 0 uniformly in n as $\sigma \to 0$. This shows

$$\int_{\mathbb{R}^3} \chi_\varepsilon(x)|\psi_n|\cdot|\varphi| = o_n(\|\varphi\|) \quad \text{as } n \to \infty$$

and consequently, using again $D\Phi_\varepsilon(u) = 0$,

$$\begin{aligned}
D\Phi_\varepsilon(u_n)[\varphi] = \big\langle z_n^+, \varphi^+\big\rangle - \big\langle z_n^-, \varphi^-\big\rangle + \big\langle u^+, \varphi^+\big\rangle - \big\langle u^-, \varphi^-\big\rangle& \\
+ \mathrm{Re}\int_{\mathbb{R}^3} V_\varepsilon(x) z_n \cdot \bar{\varphi} + \mathrm{Re}\int_{\mathbb{R}^3} V_\varepsilon(x) u \cdot \bar{\varphi}& \\
- \mathrm{Re}\int_{\mathbb{R}^3} g_\varepsilon^1(x, |z_n|) z_n \cdot \bar{\varphi} - \mathrm{Re}\int_{\mathbb{R}^3} g_\varepsilon^1(x, |u|) u \cdot \bar{\varphi}& \\
- \kappa \cdot \mathrm{Re}\int_{\mathbb{R}^3} \chi_\varepsilon(x)|z_n| z_n \cdot \bar{\varphi} - \kappa \cdot \mathrm{Re}\int_{\mathbb{R}^3} \chi_\varepsilon(x)|u| u \cdot \bar{\varphi} + o(\|\varphi\|)& \\
= D\Phi_\varepsilon(z_n)[\varphi] + D\Phi_\varepsilon(u)[\varphi] + o_n(\|\varphi\|)& \\
= D\Phi_\varepsilon(z_n)[\varphi] + o_n(\|\varphi\|)&
\end{aligned}$$

It follows that $D\Phi_\varepsilon(z_n) \to 0$ as $n \to \infty$ as claimed in (5.7). Now $D\Phi_\varepsilon(z_n)[z_n^+ - z_n^-] = o_n(1)$ reads as

$$\begin{aligned}
&\|z_n\|^2 + \mathrm{Re}\int_{\mathbb{R}^3} V_\varepsilon(x) z_n \cdot \overline{(z_n^+ - z_n^-)} \\
&\quad = \int_{\mathbb{R}^3} \big(1 - \chi_\varepsilon(x)\big) \tilde{f}(|z_n|) z_n \cdot \overline{(z_n^+ - z_n^-)} + \kappa \cdot \mathrm{Re}\int_{\mathbb{R}^3} \chi_\varepsilon(x)|z_n| z_n \cdot \overline{(z_n^+ - z_n^-)} + o_n(1).
\end{aligned}$$

Then, by using the fact $\tilde{f}(s) \le \delta_0$ and (4.6), we obtain

$$\left(\frac{a^2 - (|V|_\infty + \delta_0)^2}{a^2}\right)^{\frac{1}{2}} S^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} \chi_\varepsilon(x)|z_n|^3 dx\right)^{\frac{2}{3}} \le \kappa \cdot \int_{\mathbb{R}^3} \chi_\varepsilon(x)|z_n|^3 dx + o_n(1).$$

If $b := \lim_{n\to\infty} \int_{\mathbb{R}^3} \chi_\varepsilon(x)|z_n|^3 dx = 0$ then $\|z_n\| = o_n(1)$ and $u_n \to u$ strongly in E, as claimed. Suppose to the contrary that $b > 0$ so that

$$\left(\frac{a^2 - (|V|_\infty + \delta_0)^2}{a^2}\right)^{\frac{3}{2}} S^{\frac{3}{2}} \le \kappa^3 \cdot \int_{\mathbb{R}^3} \chi_\varepsilon(x)|z_n|^3 dx = \kappa^3 \cdot b + o_n(1).$$

In case $\kappa = 0$, this is a contradiction. In case $\kappa > 0$, using

$$\Phi_\varepsilon(u) = \int_{\mathbb{R}^3} \frac{1}{2} g_\varepsilon(x, |u|)|u|^2 - G_\varepsilon(x, |u|) \ge 0,$$

as well as $\Phi_\varepsilon(u_n) \ge \Phi_\varepsilon(z_n) + o_n(1)$ and $D\Phi_\varepsilon(z_n)[z_n] = o_n(1)$, we obtain the contradiction

$$\kappa^2 \cdot c + o_n(1) \ge \frac{\kappa^3}{6} \cdot b + o_n(1) \ge \left(\frac{a^2 - (|V|_\infty + \delta_0)^2}{a^2}\right)^{\frac{3}{2}} \frac{S^{\frac{3}{2}}}{6} + o_n(1).$$

□

We finish this section with a couple of notations that will be of use later. For simplicity, when ν belongs to the range of $V(x)$ that is $\nu \in \{V(x) : x \in \mathbb{R}^3\}$, we denote $\nu_0 = V(0)$ and correspondingly

$$I_{\nu_0} = I_{V(0)}, \quad I_{\nu_0}^{red} = I_{V(0)}^{red}, \quad \gamma(I_{\nu_0}) = \gamma(I_{V(0)}). \tag{5.9}$$

Moreover, given arbitrarily $y \in \mathbb{R}^3$, we can define the functional $\Phi_y : E \to \mathbb{R}$,

$$\Phi_y(u) = \frac{1}{2}\big(\|u^+\|^2 - \|u^-\|^2\big) + \frac{V(y)}{2}|u|_2^2 - \int_{\mathbb{R}^3} G(y, |u|)dx,$$

and (Φ_y^{red}, h_y) the reduction couple associated to Φ_y. Plainly, the critical point of Φ_y are solutions of the problem

$$-i\alpha \cdot \nabla u + a\beta u + V(y)u = g(y, |u|)u.$$

When $y \in B_1$, we have $\Phi_y = I_{V(y)}$ and $h_y = h_{V(y)}$. Let us point out that, by virtue of [19, Lemma 4.3], we can conclude the following splitting type result, whose proof is postponed to the appendix.

Proposition 5.4. *For $y \in \mathbb{R}^3$, let us define the functional $\Phi_{\varepsilon,y} : E \to \mathbb{R}$,*

$$\Phi_{\varepsilon,y}(u) = \frac{1}{2}\big(\|u^+\|^2 - \|u^-\|^2\big) + \frac{1}{2}\int_{\mathbb{R}^3} V(\varepsilon x + y)|u|^2 dx - \int_{\mathbb{R}^3} G(\varepsilon x + y, |u|)dx,$$

and $(\Phi_{\varepsilon,y}^{red}, h_{\varepsilon,y})$ the associated reduction couple, we have that

(1) *let $\{y_\varepsilon\} \subset \mathbb{R}^3$ be such that $y_\varepsilon \to y$ for some $y \in \mathbb{R}^3$ then, up to a subsequence, $h_{\varepsilon,y_\varepsilon}(w) \to h_y(w)$ as $\varepsilon \to 0$ for each $w \in E^+$;*

(2) *let $\{y_\varepsilon\} \subset \mathbb{R}^3$ be such that $y_\varepsilon \to y$ for some $y \in \mathbb{R}^3$ and let $\{w_\varepsilon\} \subset E^+$ be such that $w_\varepsilon \rightharpoonup w$ for some $w \in E^+$ then, up to a subsequence,*

$$\|h_{\varepsilon,y_\varepsilon}(w_\varepsilon) - h_{\varepsilon,y_\varepsilon}(w_\varepsilon - w) - h_y(w)\| = o_\varepsilon(1)$$

as $\varepsilon \to 0$;

(3) *let $\{y_\varepsilon\} \subset \mathbb{R}^3$ be such that $y_\varepsilon \to y$ for some $y \in \mathbb{R}^3$ and let $\{w_\varepsilon\} \subset E^+$ be such that $w_\varepsilon \rightharpoonup w$ for some $w \in E^+$ then, up to a subsequence,*

$$\Phi_{\varepsilon,y_\varepsilon}^{red}(w_\varepsilon) - \Phi_{\varepsilon,y_\varepsilon}^{red}(w_\varepsilon - w) - \Phi_y^{red}(w) = o_\varepsilon(1)$$

and

$$D\Phi_{\varepsilon,y_\varepsilon}^{red}(w_\varepsilon)[\varphi] - D\Phi_{\varepsilon,y_\varepsilon}^{red}(w_\varepsilon - w)[\varphi] - D\Phi_y(w)[\varphi] = o_\varepsilon(1)\|\varphi\|$$

uniformly for $\varphi \in E^+$ as $\varepsilon \to 0$.

6 The min-max scheme

In this section, we will prove the existence of solutions to the truncated problem (5.5) and, by virtue of Lemma 4.2, we will restrict ourselves in the barrier $0 \le \kappa < \bar{\kappa}$ where $\bar{\kappa}$ is define in (3.8). We would like to emphasize that such a choice of $\bar{\kappa}$ can be interpreted as we choose $c_0 = \gamma(J_{\vec{\mu}_V})$ in Proposition 5.3. With all these notations, for such choice of κ, we can fix the constant $\delta_0 > 0$ properly small so that the Palais-Smale condition holds automatically in the energy range $\Phi_\varepsilon \le \gamma(J_{\vec{\mu}_V})$.

To begin with, let us mention that, under our hypotheses on V, there always exists a vector space $X \subset \mathbb{R}^3$ such that:

(a) $V|_X$ has a strict local maximum at 0;

(b) $V|_{X^\perp}$ has a strict local minimum at 0.

In fact, in case $(V1)$, $X = \mathbb{R}^3$ if 0 is local maximum or $X = \{0\}$ if 0 is local minimum, whereas, in case $(V2)$, X is the space spanned by eigenvectors associated to negative eigenvalues of $D^2V(0)$. Let $P_X : \mathbb{R}^3 \to X$ be the orthogonal projection (in the case $X = \{0\}$, P_X is simply the trivial projection).

In the next, solutions of (5.5) will be obtained as critical points of Φ_ε, and a key ingredient for the construction of a min-max scheme is using the reduction couple $(\Phi_\varepsilon^{red}, h_\varepsilon)$. However, due to the lack of information on the exact behavior of the reduction map $h_\varepsilon : E^+ \to E^-$, it seems hopeless to make a "path of least energy spikes" by proper scaling as was employed in [8, 32].

Recalling $\nu_0 = V(0)$, let us focus on functions in the subspace E^+. Denoted by $B_0 := B(0, R_0)$ for some $R_0 < R_1$. Let us choose a minimizer $U \in \mathcal{M}(I_{\nu_0})$ for $\gamma(I_{\nu_0})$ and consider the path $p_\varepsilon : B_0^\varepsilon \to \mathcal{M}^+(\Phi_\varepsilon)$ defined as

$$p_\varepsilon(\xi)(x) = t_{\xi,\varepsilon} U^+(x - \xi), \quad x \in \mathbb{R}^3,$$

where $\mathcal{M}^+(\Phi_\varepsilon) = \big\{ w \in E^+ \setminus \{0\} : \ D\Phi_\varepsilon^{red}(w)[w] = 0 \big\}$ and $t_{\xi,\varepsilon}$ is the unique $t > 0$ such that

$$t_{\xi,\varepsilon} U^+(\cdot - \xi) \in \mathcal{M}^+(\Phi_\varepsilon).$$

We also define a family of deformations on $\mathcal{M}^+(\Phi_\varepsilon)$

$$\Gamma_\varepsilon \equiv \big\{ \varphi : \mathcal{M}^+(\Phi_\varepsilon) \to \mathcal{M}^+(\Phi_\varepsilon) \text{ homeomorphism} : \ \varphi(p_\varepsilon(\xi)) = p_\varepsilon(\xi) \text{ if } \xi \in \partial B_0^\varepsilon \cap X \big\}.$$

Then we define the min-max level

$$\gamma_\varepsilon := \inf_{\varphi \in \Gamma_\varepsilon} \max_{\xi \in \overline{B_0^\varepsilon} \cap X} \Phi_\varepsilon^{red}(\varphi(p_\varepsilon(\xi))). \tag{6.1}$$

We point out here that, in the case $X = \{0\}$, $\gamma_\varepsilon = \gamma(\Phi_\varepsilon) = \inf_{\mathcal{M}^+(\Phi_\varepsilon)} \Phi_\varepsilon^{red}$. A technical point we would like to emphasize, which constitutes a crucial difference with min-max quantity defined in [8], is the fact that the elements $p_\varepsilon(\xi) + h_\varepsilon(p_\varepsilon(\xi))$ do not resemble a least energy solution of I_ν since not much is known about the map $h_\varepsilon : E^+ \to E^-$.

Proposition 6.1. *There exist ε_0, $\delta > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$*

$$\Phi_\varepsilon^{red}(p_\varepsilon(\cdot))\big|_{\partial B_0^\varepsilon \cap X} \le \gamma(J_{\nu_0}) - \delta.$$

Proof. To simplify notation, we use subscript "ξ" to indicate the coordinate translation of a function $u \in E$, that is, $u_\xi(x) = u(x - \xi)$. Then, on a fixed bounded interval $t \in [0, T_0]$ with some T_0 large, we have

$$\Phi_\varepsilon^{red}(tW_\xi) \le \frac{1}{2}\big(\|tW_\xi\|^2 - \|h_\varepsilon(tW_\xi)\|^2\big) + \frac{1}{2}\int_{\mathbb{R}^3} V_\varepsilon(x)|tW_\xi + h_\varepsilon(tW_\xi)|^2 dx$$
$$- \int_{B_1^\varepsilon} F(|tW_\xi + h_\varepsilon(tW_\xi)|)dx.$$

Let us first remark that there exists $\sigma > 0$ such that $V(\xi) \le \nu_0 - \sigma$ for all $\xi \in \partial B_0^\varepsilon \cap X$. Since $t \in [0, T_0]$ is bounded and $R_0 < R_1$, by (1) in Proposition 5.4, $h_\varepsilon(tW_\xi) = h_\varepsilon(tW)_\xi \to \mathscr{J}_{V(\xi)}(tW)$ uniformly in t as $\varepsilon \to 0$. Thus, we deduce

$$\Phi_\varepsilon^{red}(tW_\xi) \le J_{\nu_0-\sigma}^{red}(tW) + o_\varepsilon(1) \quad \forall \xi \in \partial B_0^\varepsilon \cap X.$$

Finally, since $J_{\nu_0-\sigma} < J_{\nu_0}$ strictly on compact subsets, we have that

$$\begin{aligned}
\max_{t>0} J_{\nu_0-\sigma}^{red}(tW) &= \max_{t>0} J_{\nu_0-\sigma}(tW + \mathscr{J}_{\nu_0-\sigma}(tW)) \\
&< \max_{t>0} J_{\nu_0}(tW + \mathscr{J}_{\nu_0-\sigma}(tW)) \\
&\le \max_{t>0} J_{\nu_0}^{red}(tW) = \gamma(J_{\nu_0}),
\end{aligned}$$

which completes the proof. □

Proposition 6.2. *We have that*

$$\limsup_{\varepsilon \to 0} \gamma_\varepsilon \le \gamma(J_{\nu_0}).$$

Proof. It suffices to show that

$$\limsup_{\varepsilon \to 0} \max_{\xi \in \overline{B_0^\varepsilon} \cap X} \Phi_\varepsilon^{red}(p_\varepsilon(\xi)) \le \gamma(J_{\nu_0}). \tag{6.2}$$

In the following we take a sequence $\varepsilon = \varepsilon_n \to 0$, but we drop the sub-index n for the sake of clarity. For every ε, there exists a maximum point $\xi_\varepsilon \in B_0^\varepsilon \cap X$ such that

$$\max_{\xi_\varepsilon \in \overline{B_0^\varepsilon} \cap X} \Phi_\varepsilon^{red}(p_\varepsilon(\xi)) = \Phi_\varepsilon^{red}(p_\varepsilon(\xi_\varepsilon)).$$

And we see that

$$\Phi_\varepsilon^{red}(p_\varepsilon(\xi_\varepsilon)) \le \frac{1}{2}\big(\|t_\varepsilon W_{\xi_\varepsilon}\|^2 - \|h_\varepsilon(t_\varepsilon W_{\xi_\varepsilon})\|^2\big) + \frac{1}{2}\int_{\mathbb{R}^3} V_\varepsilon(x)|t_\varepsilon W_{\xi_\varepsilon} + h_\varepsilon(t_\varepsilon W_{\xi_\varepsilon})|^2 dx$$
$$- \int_{B_1^\varepsilon} F(|t_\varepsilon W_{\xi_\varepsilon} + h_\varepsilon(t_\varepsilon W_{\xi_\varepsilon})|)dx,$$

where $t_\varepsilon = t_{\xi_\varepsilon,\varepsilon}$. Since we have $\{t_\varepsilon\}$ is bounded (up to a subsequence), we can assume that $t_\varepsilon \to t_0$ and $\varepsilon\xi_\varepsilon \to \xi_0 \in \overline{B_0} \cap X$. Then we can conclude that

$$\begin{aligned}\Phi_\varepsilon^{red}(p_\varepsilon(\xi_\varepsilon)) &\le \frac{1}{2}\big(\|t_0W\|^2 - \|\mathscr{J}_{V(\xi_0)}(t_0W)\|^2\big) + \frac{V(\xi_0)}{2}\int_{\mathbb{R}^3} |t_0W + \mathscr{J}_{V(\xi_0)}(t_0W)|^2 dx \\ &\quad - \int_{\mathbb{R}^3} F(|t_0W + \mathscr{J}_{V(\xi_0)}(t_0W)|)dx + o_\varepsilon(1) \\ &= J^{red}_{V(\xi_0)}(t_0W) + o_\varepsilon(1).\end{aligned}$$

Notice that $V(\xi_0) \le \nu_0$, then

$$J^{red}_{V(\xi_0)}(t_0W) \le \max_{t>0} J^{red}_{\nu_0}(tW) = \gamma(J_{\nu_0}),$$

and hence (6.2) holds. □

In the next, we will show that γ_ε is a critical value of Φ_ε. Motivated by [8, 11], we are going to give an estimate from below on γ_ε and show that $\gamma_\varepsilon \ge \gamma(J_{\nu_0}) + o_\varepsilon(1)$. And in order to do so, we need to compare γ_ε with another auxiliary minimization value. Firstly, set $B_3 = B(0, 3R_1)$ the open ball of radius $3R_1$ and $\zeta : \mathbb{R}^3 \to \mathbb{R}^3$ be a cut-off function

$$\zeta(x) = \begin{cases} x & \text{if } |x| < 3R_1, \\ 3R_1x/|x| & \text{if } |x| \ge 3R_1, \end{cases} \tag{6.3}$$

and let $Q_\varepsilon : \mathbb{R}^3 \to X$ be defined as $Q_\varepsilon(x) = P_X(\zeta(\varepsilon x))$. Then, let us define the barycenter type functional $\mathcal{B}_\varepsilon : E \setminus \{0\} \to \mathbb{R}$,

$$\mathcal{B}_\varepsilon(u) = \frac{\int_{\mathbb{R}^3} Q_\varepsilon(x)|u|^\theta dx}{\int_{\mathbb{R}^3} |u|^\theta dx}, \quad \text{for } u \in E \setminus \{0\}$$

where $\theta \in (2,3)$ is the constant in $(f2)$. Recall that $(\Phi_\varepsilon^{red}, h_\varepsilon)$ is the reduction couple for Φ_ε and $\mathcal{M}^+(\Phi_\varepsilon) = \big\{w \in E^+ \setminus \{0\} : \ D\Phi_\varepsilon^{red}(w)[w] = 0\big\}$, let us consider the following subset of functions in $\mathcal{M}^+(\Phi_\varepsilon)$:

$$\widetilde{\mathcal{M}^+}(\Phi_\varepsilon) = \big\{w \in \mathcal{M}^+(\Phi_\varepsilon) : \ \mathcal{B}_\varepsilon(w) = 0\big\}.$$

We also define the corresponding auxiliary minimization

$$b_\varepsilon \equiv \inf_{w \in \widetilde{\mathcal{M}^+}(\Phi_\varepsilon)} \Phi_\varepsilon^{red}(w). \tag{6.4}$$

When X is trivial, i.e. $X = \{0\}$, we have $\widetilde{\mathcal{M}^+}(\Phi_\varepsilon) = \mathcal{M}^+(\Phi_\varepsilon)$ and then $b_\varepsilon = \gamma_\varepsilon$.

The next lemma shows that b_ε is well-defined in general.

Lemma 6.3. *There exists ε_0, $\varrho > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$,*

$$\gamma_\varepsilon \ge b_\varepsilon \ge \varrho.$$

Technically, the crucial difference with the barycenter quantity defined in [8, 11] is that the integrations in $\mathcal{B}_\varepsilon$ are taken over the whole space $\mathbb{R}^3$. The reason is twofold: firstly, the orthogonal projections associated to the decomposition $E = E^+ \oplus E^-$ are of convolution type with some tempered distributions $\rho^\pm$ (see an abstract result in [27] for operators that commutes with translations), and thus, making the choice of compact-supported functions in $E^\pm$ by simply multiplying smooth cut-off functions would be in our situation hopeless since the convolution with $\rho^\pm$ do not commute with the multiplication in general. Secondly, the barycenter of an element $w \in E^+$ does not exhibit the location of the mass of those $u \in E$ with $u^+ = w$. Therefore, it is not enough if we only consider the barycenter integrations over a bounded domain as was introduced in [8, 11].

Proof of Lemma 6.3. Since $b_\varepsilon \geq \varrho$ follows directly from $(f1) - (f3)$ for some $\varrho > 0$, we only need to prove that $\gamma_\varepsilon \geq b_\varepsilon$ for all small ε.

Motivated by [8], let us take an arbitrary $\varphi \in \Gamma_\varepsilon$. We define $\psi_\varepsilon : \overline{B_0} \cap X \to X$ as

$$\psi_\varepsilon(\xi) = \mathcal{B}_\varepsilon\big(\varphi(p_\varepsilon(\xi/\varepsilon))\big).$$

We point out here that, by the definition of Γ_ε, $\varphi(p_\varepsilon(\xi/\varepsilon)) \neq 0$ for all $\xi \in \overline{B_0} \cap X$, and so ψ_ε is well defined.

For $\xi \in \partial B_0 \cap X$, it can be seen from the definition of $\mathcal{B}_\varepsilon$ that

$$\psi_\varepsilon(\xi) = \xi + o_\varepsilon(1) \text{ uniformly in } \xi \in \partial B_0 \cap X, \text{ as } \varepsilon \to 0.$$

Therefore we can choose ε_0 small enough (independent of φ) so that, for all $\varepsilon \in (0, \varepsilon_0)$,

$$\deg(\psi_\varepsilon,\, B_0 \cap X,\, 0) = \deg(id,\, B_0 \cap X,\, 0) = 1.$$

Then we can conclude that for every ε, there exists $\xi_\varepsilon \in B_0 \cap X$ such that $\psi_\varepsilon(\xi_\varepsilon) = 0$.

Therefore, since $\xi_\varepsilon/\varepsilon \in \overline{B_0^\varepsilon} \cap X$, there follows

$$\max_{\xi \in \overline{B_0^\varepsilon} \cap X} \Phi_\varepsilon^{red}(\varphi(p_\varepsilon(\xi))) \geq \Phi_\varepsilon^{red}(\varphi(p_\varepsilon(\xi_\varepsilon/\varepsilon))) \geq b_\varepsilon,$$

which concludes the proof. □

Proposition 6.4. *We have that*

$$\liminf_{\varepsilon \to 0} b_\varepsilon \geq \gamma(J_{\nu_0}).$$

The proof of this proposition contains the main difficulties of the paper. It will be presented in the next section. Assuming the conclusion for the moment, jointly with Proposition 6.2, we can obtain the following

Proposition 6.5. *We have that*

$$\lim_{\varepsilon\to 0}\gamma_\varepsilon = \gamma(J_{\nu_0}).$$

From Proposition 6.1 and 6.5, we can get $\gamma_\varepsilon > \Phi_\varepsilon^{red}(p_\varepsilon(\cdot))\big|_{\partial B_0^\varepsilon\cap X}$ for all small $\varepsilon > 0$. Recall that we have restricted $\kappa \in [0,\bar\kappa)$, it follows that $\kappa^2\cdot\gamma(J_{\nu_0}) < \left(\frac{a^2-|V|_\infty^2}{a^2}\right)^{\frac{3}{2}}\frac{S^{\frac{3}{2}}}{6}$ which guarantees the compactness. Thus, by Proposition 5.3, we easily obtain

Theorem 6.6. *There exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0,\varepsilon_0)$ there exists a solution z_ε of the problem* (5.5)*. Moreover, $\Phi_\varepsilon^{red}(z_\varepsilon^+) = \Phi_\varepsilon(z_\varepsilon) = \gamma_\varepsilon$.*

7 Proof of Proposition 6.4

The proof will be divided into several parts. As a first step, we prove the existence of a minimizer u_ε to be auxiliary problem (6.4).

Lemma 7.1. *There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0,\varepsilon_0)$, there exist $u_\varepsilon \in E\setminus\{0\}$ with $\mathcal{B}_\varepsilon(u_\varepsilon^+) = 0$ and $\lambda_\varepsilon \in X$ such that*

$$-i\alpha\cdot\nabla u_\varepsilon + a\beta u_\varepsilon + V_\varepsilon(x)u_\varepsilon = g_\varepsilon(x,|u_\varepsilon|)u_\varepsilon + \big(\lambda_\varepsilon\cdot Q_\varepsilon(x)|u_\varepsilon^+|^{\theta-2}u_\varepsilon^+\big)^+ \tag{7.1}$$

and

$$\Phi_\varepsilon(u_\varepsilon) = b_\varepsilon.$$

Moreover, the sequence $\{u_\varepsilon\}$ is bounded in E.

Proof. We sketch the proof as follows. For $\varepsilon > 0$ fixed, by the Ekeland variational principle, there exists a sequence $\{w_n\} \subset \widetilde{\mathcal{M}^+}(\Phi_\varepsilon)$ which is a constrained (PS)-sequence for Φ_ε^{red} at level b_ε, moreover, it can be deduced that there exists $\{\lambda_n\} \subset X$ such that

$$\Phi_\varepsilon^{red}(w_n) \to b_\varepsilon, \quad \text{as } n\to\infty, \tag{7.2}$$

$$D\Phi_\varepsilon^{red}(w_n) - \frac{(\lambda_n\cdot Q_\varepsilon(x)|w_n|^{\theta-2}w_n)^+}{|w_n|_\theta^\theta} \to 0, \quad \text{as } n\to\infty. \tag{7.3}$$

Now, let us set $u_n = w_n + h_\varepsilon(w_n)$. Since $\mathcal{B}_\varepsilon(u_n^+) = \mathcal{B}_\varepsilon(w_n) = 0$, by (7.2) and (7.3), repeating the arguments of Lemma 5.2, we get that $\{u_n\}$ is bounded in E (uniformly with respect to ε) and, therefore, up to a subsequence, it converges weakly to some $u_\varepsilon \in E$. Since we have assumed $0 \le \kappa < \bar\kappa$, it follows that $b_\varepsilon \le \gamma_\varepsilon \le \gamma(J_{\nu_0}) + o_\varepsilon(1) \le \gamma(J_{\vec\mu_V})$ for small ε. By Proposition 5.3, $\{u_n\}$ converges strongly in E, i.e. $u_n \to u_\varepsilon$ as $n\to\infty$. Note that $u_\varepsilon \ne 0$, $\liminf_{\varepsilon\to 0} b_\varepsilon > 0$, also the sequence λ_n is bounded, we have u_ε is the desired minimizer and this concludes the proof. □

Lemma 7.2. *We have that $u_\varepsilon^+\chi_{B_2^\varepsilon}$ is non-vanishing.*

Proof. We only consider the case $\kappa > 0$ since it is much easier when $\kappa = 0$. To the contrary, we assume that $u_\varepsilon^+ \chi_{B_2^\varepsilon}$ vanishes. Then we have $u_\varepsilon^+ \chi_{B_2^\varepsilon} \to 0$ in L^q for all $q \in (2,3)$. At this point we first claim that

$$u_\varepsilon^+ \chi_{B_2^\varepsilon} \not\to 0 \ \text{ in } L^3. \tag{7.4}$$

Accepting this fact for the moment, let us consider the function

$$t \mapsto \Phi_\varepsilon(tu_\varepsilon^+)$$

and denote $t_\varepsilon > 0$ the unique maximum point which realizes its maximum. Then $\{t_\varepsilon\}$ is bounded. Set $z_\varepsilon = t_\varepsilon u_\varepsilon^+ \in E^+$, we have that $D\Phi_\varepsilon(z_\varepsilon)[z_\varepsilon] = 0$ and hence

$$\|z_\varepsilon\|^2 + \int_{\mathbb{R}^3} V_\varepsilon(x)|z_\varepsilon|^2 dx = \int_{\mathbb{R}^3} (1-\chi_\varepsilon(x))\tilde{f}(|z_\varepsilon|)|z_\varepsilon|^2 dx + \kappa \int_{\mathbb{R}^3} \chi_\varepsilon(x)|z_\varepsilon|^3 dx + o_\varepsilon(1).$$

Since $u_\varepsilon^+ \chi_{B_2^\varepsilon} \not\to 0$ in L^3, similarly as that was argued in Proposition 5.3, we soon have that

$$\kappa^3 \int_{\mathbb{R}^3} \chi_\varepsilon(x)|z_\varepsilon|^3 dx + o_\varepsilon(1) \geq \Big(\frac{a^2 - (|V|_\infty + \delta_0)^2}{a^2}\Big)^{\frac{3}{2}} S^{\frac{3}{2}}.$$

And hence, thanks to our choice of $\kappa \in (0, \bar{\kappa})$, we get

$$\begin{aligned} \kappa^2 \Phi_\varepsilon(z_\varepsilon) &= \kappa^2 \Big(\Phi_\varepsilon(z_\varepsilon) - \frac{1}{2}\Phi_\varepsilon'(z_\varepsilon)[z_\varepsilon]\Big) \\ &\geq \Big(\frac{a^2 - (|V|_\infty + \delta_0)^2}{a^2}\Big)^{\frac{3}{2}} \frac{S^{\frac{3}{2}}}{6} + o_\varepsilon(1) \\ &> \kappa^2 \gamma(J_{\vec{\mu}_V}). \end{aligned}$$

Therefore, we have that

$$\gamma(J_{\vec{\mu}_V}) < \Phi_\varepsilon(z_\varepsilon) \leq \max_{t>0} \Phi_\varepsilon^{red}(tu_\varepsilon^+) = b_\varepsilon \leq \gamma(J_{\nu_0}) \quad \text{as } \varepsilon \to 0$$

which is impossible due to Lemma 4.2.

Now, it remains to show (7.4) is valid. Indeed, it follows from Lemma 7.1 that, for some $C > 0$,

$$\begin{aligned} b_\varepsilon = \Phi_\varepsilon(u_\varepsilon) &= \max_{t>0} \Phi_\varepsilon^{red}(tu_\varepsilon^+) \geq \max_{t>0} \Phi_\varepsilon(tu_\varepsilon^+) \\ &\geq \max_{t>0} \Big[\frac{t^2}{2}\Big(1 - \frac{|V|_\infty + \delta_0}{a}\Big)\|u_\varepsilon^+\|^2 - C\kappa t^3 \int_{B_2^\varepsilon} |u_\varepsilon^+|^3 dx \Big]. \end{aligned}$$

Then, if $u_\varepsilon^+ \chi_{B_2^\varepsilon} \to 0$ in L^3 as $\varepsilon \to 0$, we can choose $T_0 > 0$ (independent of ε) large enough such that $\Phi_\varepsilon(T_0 u_\varepsilon^+) > 2\gamma(J_{\vec{\mu}_V})$ for all small $\varepsilon > 0$, and we soon have that

$$\liminf_{\varepsilon \to 0} b_\varepsilon \geq \liminf_{\varepsilon \to 0} \Phi_\varepsilon(T_0 u_\varepsilon^+) > \gamma(J_{\vec{\mu}_V})$$

which is absurd. □

Lemma 7.3. *We have that* $\{\lambda_\varepsilon\} \subset X$ *is bounded.*

Proof. Let us assume that $\lambda_\varepsilon \neq 0$, otherwise we are done. In the sequel, let us set $\tilde{\lambda}_\varepsilon = \lambda_\varepsilon/|\lambda_\varepsilon|$. By elliptic regularity arguments we have that $u_\varepsilon \in \cap_{q\geq 2} W^{1,q}(\mathbb{R}^3, \mathbb{C}^4)$ and then, jointly with Proposition 3.2, we are allowed to multiply (7.1) by $\partial_{\tilde{\lambda}_\varepsilon} u_\varepsilon$. Then, we have

$$\begin{aligned} &\mathrm{Re}\int_{\mathbb{R}^3} \Big(- i\alpha\cdot\nabla u_\varepsilon + a\beta u_\varepsilon + V_\varepsilon(x)u_\varepsilon - g_\varepsilon(x,|u_\varepsilon|)u_\varepsilon\Big)\cdot \overline{\partial_{\tilde{\lambda}_\varepsilon} u_\varepsilon}\, dx \\ &= \mathrm{Re}\int_{\mathbb{R}^3} \lambda_\varepsilon \cdot Q_\varepsilon(x)|u_\varepsilon^+|^{\theta-2}u_\varepsilon^+ \cdot \overline{\partial_{\tilde{\lambda}_\varepsilon} u_\varepsilon^+}\, dx. \end{aligned} \tag{7.5}$$

Now, let us evaluate each term of the previous equality. We get

$$0 = \mathrm{Re}\int_{\mathbb{R}^3} \partial_{\tilde{\lambda}_\varepsilon}\big[(-i\alpha\cdot\nabla u_\varepsilon)\cdot\overline{u_\varepsilon}\big]dx = 2\mathrm{Re}\int_{\mathbb{R}^3}(-i\alpha\cdot\nabla u_\varepsilon)\cdot\overline{\partial_{\tilde{\lambda}_\varepsilon} u_\varepsilon}dx$$

and so

$$\mathrm{Re}\int_{\mathbb{R}^3}(-i\alpha\cdot\nabla u_\varepsilon)\cdot\overline{\partial_{\tilde{\lambda}_\varepsilon} u_\varepsilon}dx = 0 \tag{7.6}$$

Analogously, we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}^3} \partial_{\tilde{\lambda}_\varepsilon}\big[V_\varepsilon(x)|u_\varepsilon|^2\big]dx \\ &= \varepsilon\int_{\mathbb{R}^3} \partial_{\tilde{\lambda}_\varepsilon} V(\varepsilon x)|u_\varepsilon|^2 dx + 2\mathrm{Re}\int_{\mathbb{R}^3} V_\varepsilon(x)u_\varepsilon\cdot\overline{\partial_{\tilde{\lambda}_\varepsilon} u_\varepsilon}dx \end{aligned}$$

and so

$$\mathrm{Re}\int_{\mathbb{R}^3} V_\varepsilon(x)u_\varepsilon\cdot\overline{\partial_{\tilde{\lambda}_\varepsilon} u_\varepsilon}dx = -\frac{\varepsilon}{2}\int_{\mathbb{R}^3}\partial_{\tilde{\lambda}_\varepsilon} V(\varepsilon x)|u_\varepsilon|^2dx = O(\varepsilon). \tag{7.7}$$

It also follows that

$$\mathrm{Re}\int_{\mathbb{R}^3} a\beta u_\varepsilon\cdot\overline{\partial_{\tilde{\lambda}_\varepsilon} u_\varepsilon}dx = 0. \tag{7.8}$$

For the nonlinear part, let us recall the definition of G_ε,

$$\partial_{\tilde{\lambda}_\varepsilon} G_\varepsilon(x,|u_\varepsilon|) = \varepsilon\partial_{\tilde{\lambda}_\varepsilon}\chi(\varepsilon x)\big(F(|u_\varepsilon|) - \tilde{F}(|u_\varepsilon|)\big) + \mathrm{Re}\, g_\varepsilon(x,|u_\varepsilon|)u_\varepsilon\cdot\overline{\partial_{\tilde{\lambda}_\varepsilon} u_\varepsilon},$$

then we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}^3}\partial_{\tilde{\lambda}_\varepsilon}\big[G_\varepsilon(x,|u_\varepsilon|)\big]dx \\ &= \varepsilon\int_{\mathbb{R}^3}\big(F(|u_\varepsilon|) - \tilde{F}(|u_\varepsilon|)\big)\big(\partial_{\tilde{\lambda}_\varepsilon}\chi(\varepsilon x)\big)dx + \mathrm{Re}\int_{\mathbb{R}^3} g_\varepsilon(x,|u_\varepsilon|)u_\varepsilon\cdot\overline{\partial_{\tilde{\lambda}_\varepsilon} u_\varepsilon}dx \end{aligned}$$

and it follows that

$$\mathrm{Re}\int_{\mathbb{R}^3} g_\varepsilon(x,|u_\varepsilon|)u_\varepsilon\cdot\overline{\partial_{\tilde{\lambda}_\varepsilon} u_\varepsilon}dx = O(\varepsilon). \tag{7.9}$$

Finally

$$
\begin{aligned}
0 &= \int_{\mathbb{R}^3} \partial_{\tilde{\lambda}_\varepsilon}\big[\lambda_\varepsilon \cdot Q_\varepsilon(x)|u_\varepsilon^+|^\theta\big]dx \\
&= \varepsilon|\lambda_\varepsilon| \int_{B_3^\varepsilon} |u_\varepsilon^+|^\theta dx + \varepsilon|\lambda_\varepsilon| \int_{\mathbb{R}^3\setminus B_3^\varepsilon} \frac{R_3}{\varepsilon|x|}\Big[1-\frac{(\lambda_\varepsilon\cdot x)^2}{|\lambda_\varepsilon|^2|x|^2}\Big]|u_\varepsilon^+|^\theta dx \\
&\quad + \theta \mathrm{Re}\int_{\mathbb{R}^3} \lambda_\varepsilon \cdot Q_\varepsilon(x)|u_\varepsilon^+|^{\theta-2}u_\varepsilon^+\cdot\overline{\partial_{\tilde{\lambda}_\varepsilon} u_\varepsilon^+}dx
\end{aligned}
$$

Observe that $0 \le \partial_{\tilde{\lambda}_\varepsilon}\lambda_\varepsilon \cdot Q_\varepsilon(x) \le \varepsilon|\lambda_\varepsilon|$ for all $x\in\mathbb{R}^3\setminus B_3^\varepsilon$; this is the key point of our estimates. And hence

$$
\begin{aligned}
\mathrm{Re}\int_{\mathbb{R}^3} \lambda_\varepsilon \cdot Q_\varepsilon(x)|u_\varepsilon^+|^{\theta-2}u_\varepsilon^+\cdot\overline{\partial_{\tilde{\lambda}_\varepsilon} u_\varepsilon^+}dx &= -\frac{\varepsilon|\lambda_\varepsilon|}{\theta}\int_{B_3^\varepsilon}|u_\varepsilon^+|^2dx \\
&\quad -\frac{\varepsilon|\lambda_\varepsilon|}{\theta}\int_{\mathbb{R}^3\setminus B_3^\varepsilon} \frac{R_3}{\varepsilon|x|}\Big[1-\frac{(\lambda_\varepsilon\cdot x)^2}{|\lambda_\varepsilon|^2|x|^2}\Big]|u_\varepsilon^+|^2 dx.
\end{aligned} \tag{7.10}
$$

By (7.5)-(7.10) and Lemma 7.2, we conclude the boundedness of $\lambda_\varepsilon \in X$. □

In what follows, we consider a sequence $\varepsilon_k \to 0$ and assume that $\lambda_{\varepsilon_k} \to \bar{\lambda} \in X$. For simplicity, we still denote ε_k by ε. For a small $\delta > 0$, let us define

$$
H_\varepsilon = \big\{x\in\mathbb{R}^3 :\ \bar{\lambda}\cdot Q_\varepsilon(x) \le \delta\big\}.
$$

The next proposition gives a complete description of u_ε as $\varepsilon \to 0$. We recall the notations $B_2 = B(0, 2R_1)$ and $B_3 = B(0, 3R_1)$.

Proposition 7.4. *Passing to a subsequence if necessary, there exist $y_\varepsilon^1 \in H_\varepsilon$, $y_1 \in B_2$ and $u_1 \in E\setminus\{0\}$ with*

$$
-i\alpha\cdot\nabla u_1 + a\beta u_1 + V(y_1)u_1 = g(y_1, |u_1|)u_1,
$$

such that $\bar{\lambda}\cdot y_1 = 0$ and

$$
\varepsilon y_\varepsilon^1 \to y_1, \quad \|u_\varepsilon - u_1(\cdot - y_\varepsilon^1)\| \to 0 \quad \textit{as } \varepsilon\to 0.
$$

Proof. We divide the proof into different steps:

Step 1. $u_\varepsilon^+|_{H_\varepsilon} \not\to 0$ in the L^2-norm and L^θ-norm.

Let us first show that $u_\varepsilon^+ \not\to 0$ in $L^\theta(H_\varepsilon)$. Suppose contrarily that

$$
\int_{H_\varepsilon} |u_\varepsilon^+|^\theta dx \to 0, \quad \text{as } \varepsilon \to 0.
$$

Since $\mathcal{B}_\varepsilon(u_\varepsilon^+) = 0$ and $\bar\lambda \in X$, we have

$$\begin{aligned} 0 = \int_{\mathbb{R}^3} \bar\lambda \cdot Q_\varepsilon(x)|u_\varepsilon^+|^\theta dx &= \int_{H_\varepsilon} \bar\lambda \cdot Q_\varepsilon(x)|u_\varepsilon^+|^\theta dx + \int_{H_\varepsilon^c} \bar\lambda \cdot Q_\varepsilon(x)|u_\varepsilon^+|^\theta dx \\ &\geq \delta \int_{H_\varepsilon^c} |u_\varepsilon^+|^\theta dx + \int_{H_\varepsilon} \bar\lambda \cdot Q_\varepsilon(x)|u_\varepsilon^+|^\theta dx. \end{aligned}$$

Therefore

$$\delta \int_{H_\varepsilon^c} |u_\varepsilon^+|^\theta dx \leq \Big| \int_{H_\varepsilon} \bar\lambda \cdot Q_\varepsilon(x)|u_\varepsilon^+|^\theta dx \Big| \leq |\bar\lambda| R_3 \int_{H_\varepsilon} |u_\varepsilon^+|^\theta dx$$

and so

$$\int_{H_\varepsilon^c} |u_\varepsilon^+|^\theta dx \to 0, \quad \text{as } \varepsilon \to 0.$$

Then we get $u_\varepsilon^+ \to 0$ in L^θ which is a contradiction with Lemma 7.2. Now, by the boundedness of $\{u_\varepsilon\}$ in E and so in L^3, we can conclude by interpolation: for a suitable $\mu \in (0,1)$

$$0 < c \leq \|u_\varepsilon^+\|_{L^\theta(H_\varepsilon)} \leq \|u_\varepsilon^+\|^\mu_{L^2(H_\varepsilon)} \|u_\varepsilon^+\|^{1-\mu}_{L^3(H_\varepsilon)} \leq C\|u_\varepsilon^+\|^\mu_{L^2(H_\varepsilon)}.$$

Step 2. Passing to be limit by concentration-compactness.

By Step 1, we can conclude that $\{u_\varepsilon^+|_{H_\varepsilon}\}$ is non-vanishing. And hence, by concentration-compactness arguments (see [34]), there exist $y_\varepsilon^1 \in H_\varepsilon$ and $r > 0$ such that

$$\int_{B(y_\varepsilon^1, r)\cap H_\varepsilon} |u_\varepsilon^+|^2 \geq c > 0.$$

Therefore there exits $u_1 \in E \setminus \{0\}$ such that $v_\varepsilon^1 = u_\varepsilon(\cdot + y_\varepsilon^1) \rightharpoonup u_1$ in E.

Claim 7.1. *$\{\varepsilon y_\varepsilon^1\}$ is bounded and, up to a subsequence, $\varepsilon y_\varepsilon^1 \to y_1 \in B_2$ as $\varepsilon \to 0$.*

To see this, let us assume that $\varepsilon y_\varepsilon^1 \notin B_2$ and $\mathrm{dist}(\varepsilon y_\varepsilon^1, \partial B_2)/\varepsilon \to \infty$. Observe that v_ε^1 solves the equation

$$-i\alpha\cdot\nabla v_\varepsilon^1 + a\beta v_\varepsilon^1 + V(\varepsilon x + \varepsilon y_\varepsilon^1)v_\varepsilon^1 = g(\varepsilon x + \varepsilon y_\varepsilon^1, |v_\varepsilon^1|)v_\varepsilon^1 + \big(\lambda_\varepsilon \cdot Q_\varepsilon(x + y_\varepsilon^1)|v_\varepsilon^{1+}|^{\theta-2}v_\varepsilon^{1+}\big)^+,$$

and if we assume that $V(\varepsilon y_\varepsilon^1) \to \nu_1$ as $\varepsilon \to 0$ (passing to a subsequence), we have that u_1 is a weak solution of

$$-i\alpha\cdot\nabla u + a\beta u + \nu_1 u = \tilde f(|u|)u + \big(\bar\lambda \cdot \tilde y_1 |u^+|^{\theta-2}u^+\big)^+ \tag{7.11}$$

where $\tilde y_1 \in B_3$ is given by

$$\tilde y_1 = \begin{cases} \lim\limits_{\varepsilon\to 0} \varepsilon y_\varepsilon^1 & \text{if } \varepsilon y_\varepsilon^1 \in B_3, \\ \lim\limits_{\varepsilon\to 0} \dfrac{3R_1 y_\varepsilon^1}{|y_\varepsilon^1|} & \text{if } \varepsilon y_\varepsilon^1 \in B_3^c. \end{cases}$$

Since $y_\varepsilon^1 \in H_\varepsilon$, we have that $\bar\lambda \cdot \tilde y_1 \le \delta$ and, by the definition of $\tilde f$, we easily get that $\bar\lambda \cdot \tilde y_1 > 0$ (otherwise u_1^+ should be 0). Now we let $\widetilde\Phi_1 : E \to \mathbb{R}$ denote the associated energy functional for (7.11), that is

$$\widetilde\Phi_1(u) = \frac{1}{2}\big(\|u^+\|^2 - \|u^-\|^2\big) + \frac{\nu_1}{2}|u|_2^2 - \int_{\mathbb{R}^3} \tilde F(|u|)dx - \frac{\bar\lambda \cdot \tilde y_1}{\theta}\int_{\mathbb{R}^3} |u^+|^\theta dx.$$

Remark that, for any $u \in E$ with $u^+ \ne 0$ and arbitrary $v \in E$, there holds that

$$\bar\lambda \cdot \tilde y_1 \int_{\mathbb{R}^3} |u^+|^{\theta-2}|v^+|^2 dx + (\theta-2)\bar\lambda \cdot \tilde y_1 \int_{\mathbb{R}^3} |u^+|^{\theta-2}\Big(|u^+| + \frac{\operatorname{Re} u^+ \cdot \overline{v^+}}{|u^+|}\Big)^2 dx > 0.$$

As a consequence of [1, Theorem 5.1] (see also [19, Lemma 4.6]), we have that Theorem 3.3 applies to the situation here. So, we can take $(\widetilde\Phi_1^{red}, \tilde h_1)$ to be the reduction couple for $\widetilde\Phi_1$ and let $\tilde\gamma_1$ stand for the critical level realized by u_1, we then have

$$\begin{aligned}
\tilde\gamma_1 &= \widetilde\Phi_1^{red}(u_1^+) = \max_{t>0} \widetilde\Phi_1^{red}(tu_1^+) \ge \max_{t>0} \widetilde\Phi_1(tu_1^+) \\
&\ge \max_{t>0} \frac{t^2}{2}\big(\|u_1^+\|^2 - (|V|_\infty + \delta_0)|u_1|_2^2\big) - \frac{\bar\lambda \cdot \tilde y_1}{\theta} t^\theta \int_{\mathbb{R}^3} |u_1^+|^\theta dx \\
&\ge \max_{t>0} \frac{t^2}{2}\big(\|u_1^+\|^2 - (|V|_\infty + \delta_0)|u_1|_2^2\big) - \frac{\delta}{\theta} t^\theta \int_{\mathbb{R}^3} |u_1^+|^\theta dx.
\end{aligned}$$

Since $\|u_1\| \le \|v_\varepsilon^1\| = \|u_\varepsilon\| < \infty$, we can conclude that $\tilde\gamma_1 > 2\gamma(J_{\nu_0})$ provided that δ is fixed small enough. However, by Fatou's lemma, we get

$$\begin{aligned}
\tilde\gamma_1 &= \widetilde\Phi_1(u_1) - \frac{1}{2}D\widetilde\Phi_1(u_1)[u_1] = \int_{\mathbb{R}^3} \frac{1}{2}\tilde f(|u_1|)|u_1|^2 - \tilde F(|u_1|)dx + \big(\frac{1}{2} - \frac{1}{\theta}\big)\bar\lambda \cdot \tilde y_1 |u_1^+|_\theta^\theta \\
&\le \int_{\mathbb{R}^3} \frac{1}{2}\tilde f(|u_1|)|u_1|^2 - \tilde F(|u_1|)dx + O(\delta) \\
&\le O(\delta) + \liminf_{\varepsilon\to 0} \int_{\mathbb{R}^3} \frac{1}{2} g(\varepsilon x + \varepsilon y_\varepsilon^1, |v_\varepsilon^1|)|v_\varepsilon^1|^2 - G(\varepsilon x + \varepsilon y_\varepsilon^1, |v_\varepsilon^1|)dx \\
&= O(\delta) + \liminf_{\varepsilon\to 0} \Phi_\varepsilon(u_\varepsilon) \le 2\gamma(J_{\nu_0})
\end{aligned}$$

which is impossible. This proves the claim.

Now by Claim 7.1, passing to the limit, we have u_1 is a weak solution of

$$-i\alpha \cdot \nabla u_1 + a\beta u_1 + V(y_1)u_1 = g(y_1, |u_1|)u_1 + \big(\bar\lambda \cdot y_1 |u_1^+|^{\theta-2}u_1^+\big)^+,$$

with $\varepsilon y_\varepsilon^1 \to y_1 \in B_2$ such that $\bar\lambda \cdot y_1 \le \delta$ and there exits $\bar c > 0$ such that

$$\|u_\varepsilon\| \ge \|u_1\| \ge \bar c > 0.$$

Let us define $z_{1,\varepsilon} = u_\varepsilon - u_1(\cdot - y_\varepsilon^1)$. We consider two possibilities: either $\|z_{1,\varepsilon}^+\| \to 0$ or not. In the first case the proposition should be proved. In the second case, there are two sub-cases: either $z_{1,\varepsilon}^+|_{H_\varepsilon} \to 0$ in the L^θ-norm or not.

Step 3. Assume that $z_{1,\varepsilon}^+|_{H_\varepsilon} \not\to 0$ in the L^θ-norm.

In this case, we can repeat the previous argument to the sequence $\{z_{1,\varepsilon}\}$ to obtain $y_\varepsilon^2 \in H_\varepsilon$ such that

$$\int_{B(y_\varepsilon^2, r)\cap H_\varepsilon} |z_{1,\varepsilon}^+|^2 \ge c > 0.$$

Therefore there exists $u_2 \in E \setminus \{0\}$ such that $v_\varepsilon^2 = z_{1,\varepsilon}(\cdot + y_\varepsilon^2) \rightharpoonup u_2$ in E. Moreover, $|y_\varepsilon^1 - y_\varepsilon^2| \to \infty$, $\varepsilon y_\varepsilon^2 \to y_2 \in B_2$, $\bar\lambda \cdot y_2 \le \delta$ and

$$-i\alpha \cdot \nabla u_2 + a\beta u_2 + V(y_2)u_2 = g(y_2, |u_2|)u_2 + \big(\bar\lambda \cdot y_2 |u_2^+|^{\theta-2}u_2^+\big)^+,$$

and $\|u_2\| \ge \bar c > 0$. Also, it follows from the weak convergence,

$$\|u_\varepsilon\|^2 \ge \|u_1\|^2 + \|u_2\|^2.$$

Let us set $z_{2,\varepsilon} = u_\varepsilon - u_1(\cdot - y_\varepsilon^1) - u_2(\cdot - y_\varepsilon^2)$. Suppose that $\|z_{2,\varepsilon}^+\| \not\to 0$ and $z_{2,\varepsilon}^+|_{H_\varepsilon} \not\to 0$ in L^θ, then we can argue again as above. And it is all clear that there exists $l \in \mathbb{N}$ such that, after repeating the above argument for l times, we can get that $z_{l,\varepsilon}^+|_{H_\varepsilon} \to 0$ in the L^θ-norm.

Step 4. $\|z_{l,\varepsilon}^+\| \to 0$ as $\varepsilon \to 0$.

To the contrary let us assume that $\|z_{l,\varepsilon}^+\| \not\to 0$. Since $Q_\varepsilon(\cdot)$ is bounded, it follows from a standard argument that

$$\begin{aligned}
\mathrm{Re}\int_{\mathbb{R}^3} &\lambda_\varepsilon \cdot Q_\varepsilon(x)|u_\varepsilon^+|^{\theta-2}u_\varepsilon^+ \cdot \overline{\varphi^+}dx \\
&= \sum_{j=1}^{l} \bar\lambda \cdot y_j \mathrm{Re}\int_{\mathbb{R}^3} |u_j^+(\cdot - y_\varepsilon^j)|^{\theta-2}u_j^+(\cdot - y_\varepsilon^j)\cdot \overline{\varphi^+}dx \\
&\qquad + \mathrm{Re}\int_{\mathbb{R}^3} \lambda_\varepsilon \cdot Q_\varepsilon(x)|z_{l,\varepsilon}^+|^{\theta-2}z_{l,\varepsilon}^+ \cdot \overline{\varphi^+}dx + o_\varepsilon(1)\|\varphi\|,
\end{aligned}$$

uniformly for $\varphi \in E$ as $\varepsilon \to 0$ and, particularly,

$$\int_{\mathbb{R}^3} \lambda_\varepsilon \cdot Q_\varepsilon(x)|u_\varepsilon^+|^\theta dx = \sum_{j=1}^{l} \bar\lambda \cdot y_j \int_{\mathbb{R}^3} |u_j^+|^\theta dx + \int_{\mathbb{R}^3} \lambda_\varepsilon \cdot Q_\varepsilon(x)|z_{l,\varepsilon}^+|^\theta dx + o_\varepsilon(1). \tag{7.12}$$

Since $\mathcal{B}_\varepsilon(u_\varepsilon^+) = 0$, together with Proposition 5.4, we can deduce from (7.12) that

$$\begin{aligned}
o_\varepsilon(1) = \|z_{l,\varepsilon}^+ + h_\varepsilon(z_{l,\varepsilon}^+)\|^2 &+ \mathrm{Re}\int_{\mathbb{R}^3} V_\varepsilon(x)\big(z_{l,\varepsilon}^+ + h_\varepsilon(z_{l,\varepsilon}^+)\big)\cdot \overline{\big(z_{l,\varepsilon}^+ - h_\varepsilon(z_{l,\varepsilon}^+)\big)}dx \\
&- \mathrm{Re}\int_{\mathbb{R}^3} g_\varepsilon\big(x, |z_{l,\varepsilon}^+ + h_\varepsilon(z_{l,\varepsilon}^+)|\big)\big(z_{l,\varepsilon}^+ + h_\varepsilon(z_{l,\varepsilon}^+)\big)\cdot \overline{\big(z_{l,\varepsilon}^+ - h_\varepsilon(z_{l,\varepsilon}^+)\big)}dx \\
&- \int_{\mathbb{R}^3} \lambda_\varepsilon \cdot Q_\varepsilon(x)|z_{l,\varepsilon}^+|^\theta dx.
\end{aligned} \tag{7.13}$$

Therefore, by $(f2)$ and Proposition 3.1, we obtain

$$\|z_{l,\varepsilon}^+ + h_\varepsilon(z_{l,\varepsilon}^+)\|^2 \le C|z_{l,\varepsilon}^+ + h_\varepsilon(z_{l,\varepsilon}^+)|_3^3 \le C'\|z_{l,\varepsilon}^+ + h_\varepsilon(z_{l,\varepsilon}^+)\|^3,$$

for some $C, C' > 0$ which implies there exists $c > 0$ such that $\|z_{l,\varepsilon}^+ + h_\varepsilon(z_{l,\varepsilon}^+)\| \ge c$. In what follows, for simplicity of notation, we denote $\bar z_{l,\varepsilon} = z_{l,\varepsilon}^+ + h_\varepsilon(z_{l,\varepsilon}^+)$. By (7.13) again, and a similar argument as in the proof of Lemma 5.2, we get that

$$\begin{aligned}\|\bar z_{l,\varepsilon}\|^2 &\le C_\theta\Big(\int_{\mathbb{R}^3}\chi_\varepsilon(x)\big(f(|\bar z_{l,\varepsilon}|)|\bar z_{l,\varepsilon}|^2 - 2F(|\bar z_{l,\varepsilon}|)\big)dx\Big)^{\frac{2}{3}}|\bar z_{l,\varepsilon}^+ - \bar z_{l,\varepsilon}^-|_3 \\ &\qquad + C\int_{\mathbb{R}^3}\lambda_\varepsilon\cdot Q_\varepsilon(x)|\bar z_{l,\varepsilon}^+|^\theta dx + o_\varepsilon(1) \\ &\le C_\theta'\Big(2\Phi_\varepsilon^{red}(z_{l,\varepsilon}^+) - D\Phi_\varepsilon^{red}(z_{l,\varepsilon}^+)[z_{l,\varepsilon}^+]\Big)^{\frac{2}{3}}\|\bar z_{l,\varepsilon}\| + C\int_{\mathbb{R}^3}\lambda_\varepsilon\cdot Q_\varepsilon(x)|\bar z_{l,\varepsilon}^+|^\theta dx + o_\varepsilon(1)\end{aligned}$$

for some $C, C_\theta, C_\theta' > 0$. Remark that $\bar z_{l,\varepsilon}^+ = z_{l,\varepsilon}^+ \to 0$ in $L^\theta(H_\varepsilon)$. Then, it follows from $\|\bar z_{l,\varepsilon}\| \ge c$ and $(f2)$ that there exists constant $c' > 0$ (independent of R_1) such that

$$\liminf_{\varepsilon\to 0}\Big(\Phi_\varepsilon^{red}(z_{l,\varepsilon}^+) - \frac{1}{2}D\Phi_\varepsilon^{red}(z_{l,\varepsilon}^+)[z_{l,\varepsilon}^+]\Big) \ge c'. \tag{7.14}$$

Next, let us distinguish two possible situations.

• *Case 1.* $\bar\lambda\cdot y_j \ge 0$ for all $j = 1,\dots,l$.
Since $\mathcal{B}_\varepsilon(u_\varepsilon^+) = 0$, we have that

$$0 = \int_{H_\varepsilon}\lambda_\varepsilon\cdot Q_\varepsilon(x)|u_\varepsilon^+|^\theta dx + \int_{H_\varepsilon^c}\lambda_\varepsilon\cdot Q_\varepsilon(x)|u_\varepsilon^+|^\theta dx.$$

By virtue of $z_{l,\varepsilon}^+|_{H_\varepsilon} \to 0$ in the L^θ-norm and $\bar\lambda\cdot y_j \ge 0$ for all $j = 1,\dots,l$, we know that

$$\int_{H_\varepsilon}\lambda_\varepsilon\cdot Q_\varepsilon(x)|u_\varepsilon^+|^\theta dx \to \sum_{j=1}^l \bar\lambda\cdot y_j\int_{\mathbb{R}^3}|u_j^+|^\theta dx \ge 0,$$

whereas $\lambda_\varepsilon\cdot Q_\varepsilon(x) \ge \frac{1}{2}\delta > 0$ in H_ε^c. Thus we have

$$\bar\lambda\cdot y_j = 0, \quad \text{for all } j = 1,\dots,l,$$

and so

$$\frac{\delta}{2}\int_{H_\varepsilon^c}|u_\varepsilon^+|^\theta dx \le \int_{H_\varepsilon^c}\lambda_\varepsilon\cdot Q_\varepsilon(x)|u_\varepsilon^+|^\theta dx \to 0, \quad \text{as } \varepsilon\to 0.$$

We also deduce from (7.12) that

$$\int_{H_\varepsilon^c}\lambda_\varepsilon\cdot Q_\varepsilon(x)|z_{l,\varepsilon}^+|^\theta dx \to 0, \quad \text{as } \varepsilon\to 0.$$

With all those information in hand, by Proposition 5.4, we can estimate the energy $\Phi_\varepsilon^{red}(u_\varepsilon^+)$ as

$$\Phi_\varepsilon^{red}(u_\varepsilon^+) = \Phi_\varepsilon^{red}(z_{l,\varepsilon}^+) + \sum_{j=1}^{l} \mathscr{T}_{y_j}^{red}(u_j^+) + o_\varepsilon(1).$$

Moreover, we have that

$$D\Phi_\varepsilon^{red}(u_\varepsilon^+)[u_\varepsilon^+] = D\Phi_\varepsilon^{red}(z_{l,\varepsilon}^+)[z_{l,\varepsilon}^+] + \sum_{j=1}^{l} D\mathscr{T}_{y_j}^{red}(u_j^+)[u_j^+] + o_\varepsilon(1).$$

Since $\bar\lambda \cdot y_j = 0$ for all $j = 1, \ldots, l$, we have u_j^+'s are critical points of $\mathscr{T}_{y_j}^{red}$. And so, we get the estimate

$$\liminf_{\varepsilon\to 0} b_\varepsilon = \liminf_{\varepsilon\to 0} \Phi_\varepsilon^{red}(u_\varepsilon^+) = \liminf_{\varepsilon\to 0} \Big(\Phi_\varepsilon^{red}(z_{l,\varepsilon}^+) - \frac{1}{2} D\Phi_\varepsilon^{red}(z_{l,\varepsilon}^+)[z_{l,\varepsilon}^+]\Big) + \sum_{j=1}^{l} \mathscr{T}_{y_j}^{red}(u_j^+).$$

Recall that we have denoted $\bar z_{l,\varepsilon} = z_{l,\varepsilon}^+ + h_\varepsilon(z_{l,\varepsilon}^+)$, hence, by (7.14), we have

$$\liminf_{\varepsilon\to 0} b_\varepsilon \geq c' + \sum_{j=1}^{l} \mathscr{T}_{y_j}^{red}(u_j^+).$$

Since, by Lemma 5.1, we have that $\mathscr{T}_{y_j}^{red}(w) \geq J_{V(y_j)}^{red}(w)$, $\forall w \in E^+$, for all $j = 1, \ldots, l$, we can infer that

$$\mathscr{T}_{y_j}^{red}(u_j^+) \geq \gamma(J_{V(y_j)}), \quad j = 1, \ldots, l.$$

And therefore

$$\liminf_{\varepsilon\to 0} b_\varepsilon \geq l \cdot \min_{j=1,\ldots,l} \gamma(J_{V(y_j)}) + c'.$$

Remark that $y_j \in B_2 = B(0, 2R_1)$, by shrinking R_1 if necessary, we can conclude from the continuity of the map $\nu \mapsto \gamma(J_\nu)$ that

$$\big|\gamma(J_{V(y_j)}) - \gamma(J_{\nu_0})\big| < \frac{1}{2}c' \quad \text{for all } j = 1, \ldots, l,$$

and then we obtain

$$\liminf_{\varepsilon\to 0} b_\varepsilon \geq \gamma(J_{\nu_0}) + \frac{1}{2}c' > \gamma(J_{\nu_0})$$

which contradicts to Proposition 6.2 and Lemma 6.3.

• *Case 2.* There exists $\{j_1, \ldots, j_k\} \subset \{1, \ldots, l\}$ such that $\bar\lambda \cdot y_{j_m} < 0$ for $m = 1, \ldots, k$.

In this case, similar as that in Case 1, we can apply Proposition 5.4 to obtain

$$\Phi_\varepsilon^{red}(u_\varepsilon^+) = \Big(\Phi_\varepsilon^{red}(z_{l,\varepsilon}^+) - \frac{1}{2} D\Phi_\varepsilon^{red}(z_{l,\varepsilon}^+)[z_{l,\varepsilon}^+]\Big) + \sum_{j=1}^{l} \Big(\mathscr{T}_{y_j}^{red}(u_j^+) - \frac{1}{2} D\mathscr{T}_{y_j}^{red}(u_j^+)[u_j^+]\Big) + o_\varepsilon(1).$$

By the definition of $G(x,s)$, we have $\mathscr{T}_{y_j}^{red}(u_j^+) - \frac{1}{2}D\mathscr{T}_{y_j}^{red}(u_j^+)[u_j^+] \geq 0$ for all $j = 1,\dots,l$. Then, we conclude that

$$\begin{aligned}\Phi_\varepsilon^{red}(u_\varepsilon^+) &\geq \Big(\Phi_\varepsilon^{red}(z_{l,\varepsilon}^+) - \frac{1}{2}D\Phi_\varepsilon^{red}(z_{l,\varepsilon}^+)[z_{l,\varepsilon}^+]\Big) \\ &\quad + \sum_{m=1}^{k}\Big(\mathscr{T}_{y_{j_m}}^{red}(u_{j_m}^+) - \frac{1}{2}D\mathscr{T}_{y_{j_m}}^{red}(u_{j_m}^+)[u_{j_m}^+]\Big) + o_\varepsilon(1). \qquad (7.15)\end{aligned}$$

To evaluate the above inequality, let us denote $\mathcal{M}^+(\mathscr{T}_{y_{j_m}}) = \big\{w \in E^+ \setminus \{0\} : D\mathscr{T}_{y_{j_m}}^{red}(w)[w] = 0\big\}$, for $m = 1,\dots,k$, and $t_m > 0$ be the unique point such that $t_m u_{j_m}^+ \in \mathcal{M}^+(\mathscr{T}_{y_{j_m}})$. Observe that $\bar\lambda \cdot y_{j_m} < 0$, by Step 2 and Step 3, we get that

$$D\mathscr{T}_{y_{j_m}}^{red}(u_{j_m}^+)[u_{j_m}^+] - \bar\lambda \cdot y_{j_m}\int_{\mathbb{R}^3}|u_{j_m}^+|^\theta dx = 0,$$

and hence we have $t_m < 1$. Observe that, by applying Lemma 3.5, we have

$$\mathscr{T}_{y_{j_m}}^{red}(u_{j_m}^+) - \frac{1}{2}D\mathscr{T}_{y_{j_m}}^{red}(u_{j_m}^+)[u_{j_m}^+] > \mathscr{T}_{y_{j_m}}^{red}(t_m u_{j_m}^+) - \frac{1}{2}D\mathscr{T}_{y_{j_m}}^{red}(t_m u_{j_m}^+)[t_m u_{j_m}^+].$$

Then, it follows from $t_m u_{j_m}^+ \in \mathcal{M}^+(\mathscr{T}_{y_{j_m}})$ that

$$\mathscr{T}_{y_{j_m}}^{red}(u_{j_m}^+) - \frac{1}{2}D\mathscr{T}_{y_{j_m}}^{red}(u_{j_m}^+)[u_{j_m}^+] > \gamma(J_{V(y_{j_m})}), \quad \text{for all } m = 1,\dots,k.$$

Finally, by (7.14) and (7.15), we obtain the inequality

$$\liminf_{\varepsilon\to 0} b_\varepsilon = \liminf_{\varepsilon\to 0}\Phi_\varepsilon^{red}(u_\varepsilon^+) \geq k\cdot \min_{m=1,\dots,k}\gamma(J_{V(y_{j_m})}) + c'.$$

And therefore, as in Case 1, we conclude easily a contradiction.

Step 5. Complete description of u_ε as $\varepsilon \to 0$.

As was argued in the previous steps, we know that there exists $l \in \mathbb{N}$ and, for any $j = 1,\dots,l$, $y_\varepsilon^j \in H_\varepsilon$, $y_j \in B_2$ and $u_j \in E\setminus\{0\}$ such that

$$\begin{aligned}&|y_\varepsilon^j - y_\varepsilon^{j'}| \to \infty, \quad \text{if } j \neq j', \\ &\varepsilon y_\varepsilon^j \to y_j, \\ &\Big\|u_\varepsilon^+ - \sum_{j=1}^{l} u_j^+(\cdot - y_\varepsilon^j)\Big\| \to 0, \\ &D\mathscr{T}_{y_j}^{red}(u_j^+) - \bar\lambda\cdot y_j\big(|u_j^+|^{\theta-2}u_j^+\big)^+ = 0.\end{aligned}$$

Observe that there strictly holds

$$\mathscr{T}_{y_j}^{red}(u_j^+) - \frac{1}{2}D\mathscr{T}_{y_j}^{red}(u_j^+)[u_j^+] > \gamma(J_{V(y_j)})$$

provided that $\bar\lambda\cdot y_j < 0$. Moreover, Lemma 4.2 implies that $\gamma(J_{V(y_j)}) \geq \gamma(J_{\nu_0}) - \sigma$ for any $y_j \in B_2$, where $\sigma > 0$ can be taken arbitrary small by appropriately shrinking R_1. Therefore, by Proposition 6.2 and Lemma 6.3, we conclude that $l = 1$ and $\bar\lambda\cdot y_1 = 0$. And thus we have $\|u_\varepsilon - u_1(\cdot - y_\varepsilon^1)\| \to 0$ as $\varepsilon \to 0$ which complete the proof. $\square$

Corollary 7.5. *$y_1 \in X^{\perp}$ and $\liminf_{\varepsilon\to 0} b_\varepsilon \geq \gamma(J_{\nu_0})$.*

Proof. Since $\mathcal{B}_\varepsilon(u_\varepsilon^+) = 0$, by Proposition 7.4, we get

$$\begin{aligned} 0 &= \int_{\mathbb{R}^3} Q_\varepsilon(x)|u_\varepsilon^+(x)|^\theta dx \\ &= \int_{\mathbb{R}^3} P_X(\zeta(\varepsilon x + \varepsilon y_\varepsilon^1))|u_\varepsilon^+(x + y_\varepsilon^1)|^\theta dx \to P_X(y_1) \int_{\mathbb{R}^3} |u_1^+|^\theta dx. \end{aligned}$$

Then $y_1 \in X^{\perp}$, and we soon conclude

$$\liminf_{\varepsilon\to 0} b_\varepsilon \geq \gamma(J_{V(y_1)}) \geq \gamma(J_{\nu_0}).$$

□

This finishes the proof of Proposition 6.4.

8 Profile of the solutions

In this section, let us study the asymptotic behavior of the solution z_ε obtained in Theorem 6.6. We will show that z_ε is actually a solution of the original problem (5.1), and consequently, we can complete the proof of Theorem 2.1.

Let us recall that z_ε is the critical point of Φ_ε at level γ_ε, that is,

$$-i\alpha \cdot \nabla z_\varepsilon + a\beta z_\varepsilon + V_\varepsilon(x) z_\varepsilon = g_\varepsilon(x, |z_\varepsilon|) z_\varepsilon. \tag{8.1}$$

Moreover, Proposition 6.5 implies that $\Phi_\varepsilon(z_\varepsilon) \to \gamma(J_{\nu_0})$ as $\varepsilon \to 0$.

In what follows, we will give the asymptotic behavior of z_ε as $\varepsilon \to 0$.

Proposition 8.1. *Given a sequence $\varepsilon_j \to 0$, up to a subsequence, there exists $\{y_{\varepsilon_j}\} \subset \mathbb{R}^3$ such that*

$$\varepsilon_j y_{\varepsilon_j} \to 0, \quad \|z_{\varepsilon_j} - Z(\cdot - y_{\varepsilon_j})\| \to 0,$$

where $Z \in \mathscr{L}_{\nu_0}$ (see (5.9)).

Proof. For the sake of clarity, let us write $\varepsilon = \varepsilon_j$. Our argument here has been used already in the previous section, so we will be sketchy. First of all, analogous to Proposition 7.4, we can conclude that: there exist $\bar{y}_\varepsilon^1 \in \mathbb{R}^3$, $\bar{y}_1 \in B_2$ and $z_1 \in E \setminus \{0\}$ with

$$-i\alpha \cdot \nabla z_1 + a\beta z_1 + V(\bar{y}_1) z_1 = g(\bar{y}_1, |z_1|) z_1,$$

such that

$$\varepsilon \bar{y}_\varepsilon^1 \to \bar{y}_1, \quad \|z_\varepsilon - z_1(\cdot - y_\varepsilon^1)\| \to 0 \quad \text{as } \varepsilon \to 0.$$

So, the only thing that need to be proved is that $\bar{y}_1 = 0$.

By regularity arguments, $\{z_\varepsilon\} \subset \cap_{q\geq 2} W^{1,q}(\mathbb{R}^3, \mathbb{C}^4)$. For arbitrary $\xi \in \mathbb{R}^3$, multiplying (8.1) by $\partial_\xi z_\varepsilon$ and integrating, we get

$$-\frac{\varepsilon}{2}\int_{\mathbb{R}^3} \partial_\xi V(\varepsilon x)|z_\varepsilon|^2 dx + \varepsilon \int_{\mathbb{R}^3} \big(F(|z_\varepsilon|) - \tilde{F}(|z_\varepsilon|)\big)\partial_\xi \chi(\varepsilon x) dx = 0. \tag{8.2}$$

And if χ is C^1 around $\bar{y}_1$, we shall divide by ε and pass to the limit to obtain

$$-\frac{\partial_\xi V(\bar{y}_1)}{2}\int_{\mathbb{R}^3} |z_1|^2 dx + \partial_\xi \chi(\bar{y}_1) \int_{\mathbb{R}^3} \big(F(|z_1|) - \tilde{F}(|z_1|)\big) dx = 0. \tag{8.3}$$

At this point, similar as that in [8], we consider three different cases.

• *Case 1.* $\bar{y}_1 \in B_1$.

By (8.3), we get that $\partial_\xi V(\bar{y}_1) = 0$. Since $\xi \in \mathbb{R}^3$ is arbitrary, $\bar{y}_1$ is a critical point of V in B_1, and therefore $\bar{y}_1 = 0$.

• *Case 2.* $\bar{y}_1 \in B_2 \setminus \overline{B_1}$.

In this case, let us first fix $\xi = \frac{1}{|\bar{y}_1|}\bar{y}_1$. By the definition of χ (see (5.4)), we have that $\partial_\xi \chi(\bar{y}_1) = -1/R_1$.

Now, using $(f3)$ and the fact $\tilde{F}(s) \leq \frac{\delta_0}{2}s^2$, it follow easily that there exists a constant $c > 0$ (which is independent of the choice of δ_0) such that

$$\int_{\mathbb{R}^3} F(|z_1|) dx \geq c,$$

and so by the boundedness of $z_1 \in E$ (see an argument of Lemma 5.2) we get

$$c' \int_{\mathbb{R}^3} |z_1|^2 dx \leq \int_{\mathbb{R}^3} \big(F(|z_1|) - \tilde{F}(|z_1|)\big) dx.$$

Thus, it suffices to take R_1 smaller, if necessary, to get a contradiction with (8.3).

• *Case 3.* $\bar{y}_1 \in \partial B_2$.

In this case, observe that $\chi(\bar{y}_1) = 1$, and so z_1 is a solution of

$$-i\alpha \cdot \nabla z_1 + a\beta z_1 + V(\bar{y}_1) z_1 = f(|z_1|) z_1.$$

Since $J^{red}_{V(\bar{y}_1)}(z_1^+) = J_{V(\bar{y}_1)}(z_1) = \gamma(J_{\nu_0})$, Lemma 4.2 implies that $V(\bar{y}_1) = \nu_0$. Then, by (5.3), there exists $\tau \in \mathbb{R}^3$ tangent to ∂B_1 at $\bar{y}_1$ such that $\partial_\tau V(\bar{y}_1) \neq 0$.

Remark that χ is not C^1 on ∂B_1, let us go back to consider (8.2). Take $\xi = \tau$ and $r < R_1$, we can estimate by the dominated convergence theorem and the strong convergence of $z_\varepsilon(\cdot + \bar{y}^1_\varepsilon)$ that

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^3} \partial_\tau \chi(\varepsilon x)\big[F(|z_\varepsilon|) - \tilde{F}(|z_\varepsilon|)\big] dx \Big| \\
&\leq \frac{1}{R_1}\int_{B(0,r/\sqrt{\varepsilon})} \Big[\frac{|x\cdot\tau|}{|x+\bar{y}^1_\varepsilon|} + \frac{|\bar{y}^1_\varepsilon\cdot\tau|}{|x+\bar{y}^1_\varepsilon|}\Big]\big[F(|z_\varepsilon(x+\bar{y}^1_\varepsilon)|) - \tilde{F}(|z_\varepsilon(x+\bar{y}^1_\varepsilon)|)\big] dx \\
&\quad + \frac{1}{R_1}\int_{\mathbb{R}^3\setminus B(0,r/\sqrt{\varepsilon})} \frac{|(x+\bar{y}^1_\varepsilon)\cdot\tau|}{|x+\bar{y}^1_\varepsilon|}\big[F(|z_\varepsilon(x+\bar{y}^1_\varepsilon)|) - \tilde{F}(|z_\varepsilon(x+\bar{y}^1_\varepsilon)|)\big] dx \to 0.
\end{aligned}$$

Dividing by ε and passing to the limit in (8.2), we can conclude

$$\frac{1}{2}\partial_\tau V(\bar{y}_1)\int_{\mathbb{R}^3}|z_1|^2dx=0,$$

a contradiction. □

Complete proof of Theorem 2.1. It suffices to show that $|z_\varepsilon(x)|\to 0$ uniformly in $\mathbb{R}^3\setminus B_1^\varepsilon$ as $\varepsilon\to 0$. In fact, from the regularity argument in [17, Lemma 3.19], we have that there exists $C>0$ (independent of ε) such that $|z_\varepsilon|_\infty\le C$. Then we can use elliptic esitmate to get

$$|z_\varepsilon(x)|\le C_0\int_{B(x,1)}|z_\varepsilon(y)|dy$$

with $C_0>0$ independent of both ε and $x\in\mathbb{R}^3$. And thus, by Proposition 8.1, we have that for any $x\in\mathbb{R}^3\setminus B_1^\varepsilon$,

$$\begin{aligned}|z_\varepsilon(x)|&\le C_0\Big(\int_{B(x,1)}|z_\varepsilon|^2\Big)^{1/2}\\&\le C_0\Big(\int_{\mathbb{R}^3}\big|z_\varepsilon-Z(\cdot-y_\varepsilon)\big|^2\Big)^{1/2}+C_0\Big(\int_{B(x,1)}\big|Z(\cdot-y_\varepsilon)\big|^2\Big)^{1/2}\to 0,\end{aligned}$$

as $\varepsilon\to 0$. Finally, by the decay estimates obtained in [18, Lemma 4.2], it is standard to prove that there exists $C,c>0$ independent of ε such that

$$|z_\varepsilon(x)|\le C\exp\big(-c|x-y_\varepsilon|\big).$$

This concludes the whole proof. □

A Appendix

Here we sketch the proof of Proposition 5.4. Firstly for later use let us point out that, under the assumptions of Proposition 5.4, $V(\varepsilon\cdot+y_\varepsilon)\to V(y)$ in $L^\infty_{loc}(\mathbb{R}^N)$ as $\varepsilon\to 0$. Now, denote $V_\varepsilon^0(x)=V(\varepsilon x+y_\varepsilon)-V(y)$, we soon have

$$\Phi_{\varepsilon,y_\varepsilon}(u)=\mathscr{T}_y(u)+\frac{1}{2}\int_{\mathbb{R}^3}V_\varepsilon^0(x)|u|^2dx-\int_{\mathbb{R}^3}\big(G(\varepsilon x+y_\varepsilon,|u|)-G(y,|u|)\big)dx \quad (A.1)$$

for all $u\in E$. We also remark that, for arbitrary $w\in E^+$ and $v\in E^-$, by setting $\tilde{v}=v-h_{\varepsilon,y_\varepsilon}(w)$ and $\ell(t)=\Phi_{\varepsilon,y_\varepsilon}\big(w+h_{\varepsilon,y_\varepsilon}(w)+t\tilde{v}\big)$, one has $\ell(1)=\Phi_{\varepsilon,y_\varepsilon}(w+v)$, $\ell(0)=\Phi_{\varepsilon,y_\varepsilon}\big(w+h_{\varepsilon,y_\varepsilon}(w)\big)$ and $\ell'(0)=0$. Hence we deduce $\ell(1)-\ell(0)=\int_0^1(1-s)\ell''(s)ds$. And consequently, we have

$$\begin{aligned}&\int_0^1(1-s)\Psi''_{\varepsilon,y_\varepsilon}\big(w+h_{\varepsilon,y_\varepsilon}(w)+s\tilde{v}\big)[\tilde{v},\tilde{v}]\,ds\\&+\frac{1}{2}\|\tilde{v}\|^2+\frac{1}{2}\int_{\mathbb{R}^N}V(\varepsilon x+y_\varepsilon)|\tilde{v}|^2dx=\Phi_{\varepsilon,y_\varepsilon}\big(w+h_{\varepsilon,y_\varepsilon}(w)\big)-\Phi_{\varepsilon,y_\varepsilon}(z+v),\end{aligned} \quad (A.2)$$

where, for notation convenience, we denote $\Psi_{\varepsilon,y}(u) \equiv \int_{\mathbb{R}^3} G(\varepsilon x + y, |u|)dx$ for $u \in E$ and $y \in \mathbb{R}^3$.

Observe that assertion (1) follows directly from [19, Lemma 4.3] and that assertion (3) can be viewed as an immediate corollary of assertion (2). Hence, to complete the proof, it suffices to show that, as $\varepsilon \to 0$,

$$\begin{cases} y_\varepsilon \to y \text{ in } \mathbb{R}^3 \\ w_\varepsilon \rightharpoonup w \text{ in } E^+ \end{cases} \implies \|h_{\varepsilon,y_\varepsilon}(w_\varepsilon) - h_{\varepsilon,y_\varepsilon}(w_\varepsilon - w) - h_y(w)\| = o_\varepsilon(1). \tag{A.3}$$

To this end, we first claim that

$$\begin{aligned} &y_\varepsilon \to y \text{ in } \mathbb{R}^3 \text{ and } u_\varepsilon \rightharpoonup u \text{ in } E \text{ as } \varepsilon \to 0 \\ &\qquad \implies \Phi_{\varepsilon,y_\varepsilon}(u_\varepsilon) - \Phi_{\varepsilon,y_\varepsilon}(u_\varepsilon - u) - \Phi_{\varepsilon,y_\varepsilon}(u) = o_\varepsilon(1) \quad \text{as } \varepsilon \to 0. \end{aligned} \tag{A.4}$$

This can be proved similarly as (5.8) in Proposition 5.3, therefore we omit the details. We only point out here that, for the nonlinear part, it suffices to check

$$\int_{\mathbb{R}^3} \big(G^1(\varepsilon x + y_\varepsilon, |u_\varepsilon|) - G^1(\varepsilon x + y_\varepsilon, |u_\varepsilon - u|) - G^1(\varepsilon x + y_\varepsilon, |u|)\big)dx = o_\varepsilon(1)$$

where $G^1(x,s) = G(x,s) - \frac{\kappa}{3}\chi(x)s^3$. Since G^1 is subcritical, the proof follows from a standard argument in [12, Lemma 7.10].

As a direct consequence of (A.4), we soon conclude that

$$\text{For any sequence } w_\varepsilon \rightharpoonup 0 \text{ in } E^+, \text{ we have that } h_{\varepsilon,y_\varepsilon}(w_\varepsilon) \rightharpoonup 0 \text{ in } E^-. \tag{A.5}$$

Indeed, notice that $h_{\varepsilon,y_\varepsilon}(w_\varepsilon)$ is bounded (see Theorem 3.3), we may assume up to a subsequence that $h_{\varepsilon,y_\varepsilon}(w_\varepsilon) \rightharpoonup u_0 \in E^-$. Then $u_\varepsilon \equiv w_\varepsilon + h_{\varepsilon,y_\varepsilon}(w_\varepsilon) \rightharpoonup u_0$. Now, remark that $\Psi_{\varepsilon,y_\varepsilon} \geq 0$, we conclude from (A.4) that

$$\frac{a - |V|_\infty}{2a}\|u_0\|^2 \leq -\Phi_{\varepsilon,y_\varepsilon}(u_0) = \Phi_{\varepsilon,y_\varepsilon}(u_\varepsilon - u_0) - \Phi_{\varepsilon,y_\varepsilon}(u_\varepsilon) + o_\varepsilon(1) \leq o_\varepsilon(1)$$

as $\varepsilon \to 0$. And hence $u_0 = 0$.

Now we are ready to show (A.3). Let $w_\varepsilon \rightharpoonup w$ in E^+. We may assume $h_{\varepsilon,y_\varepsilon}(w_\varepsilon) \rightharpoonup v$ in E^-. By (A.5), there holds $h_{\varepsilon,y_\varepsilon}(w_\varepsilon - w) \rightharpoonup 0$. Using (A.4) and assertion (1) (i.e. the fact that $h_{\varepsilon,y_\varepsilon}(w) \to h_y(w)$ as $\varepsilon \to 0$), we conclude that

$$\begin{aligned} \Phi_{\varepsilon,y_\varepsilon}\big(w_\varepsilon + h_{\varepsilon,y_\varepsilon}(w_\varepsilon)\big) &= \Phi_{\varepsilon,y_\varepsilon}(w + v) + \Phi_{\varepsilon,y_\varepsilon}\big(w_\varepsilon - w + h_{\varepsilon,y_\varepsilon}(w_\varepsilon) - v\big) + o_\varepsilon(1) \\ &\leq \Phi_{\varepsilon,y_\varepsilon}\big(w + h_{\varepsilon,y_\varepsilon}(w)\big) + \Phi_{\varepsilon,y_\varepsilon}\big(w_\varepsilon - w + h_{\varepsilon,y_\varepsilon}(w_\varepsilon - w)\big) + o_\varepsilon(1) \\ &= \Phi_{\varepsilon,y_\varepsilon}\big(w + h_y(w)\big) + \Phi_{\varepsilon,y_\varepsilon}\big(w_\varepsilon - w + h_{\varepsilon,y_\varepsilon}(w_\varepsilon - w)\big) + o_\varepsilon(1) \\ &= \Phi_{\varepsilon,y_\varepsilon}\big(w_\varepsilon + h_{\varepsilon,y_\varepsilon}(w_\varepsilon - w) + h_y(w)\big) + o_\varepsilon(1) \end{aligned}$$

as $\varepsilon \to 0$. Now use (A.2), we can deduce that

$$\frac{a - |V|_\infty}{2a}\|h_{\varepsilon,y_\varepsilon}(w_\varepsilon) - h_{\varepsilon,y_\varepsilon}(w_\varepsilon - w) - h_y(w)\|^2 \leq o_\varepsilon(1)$$

and hence (A.3) is proved.

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