Solutions Concentrating on Higher Dimensional Subsets for singularly Perturbed Elliptic Equations I

Thomas Bartsch
Mathematisches Institut
Universität Giessen
Arndtstr. 2, 35392 Giessen
Germany

Shuangjie Peng*
School of Mathematics and Statistics
Central China Normal University
Wuhan, 430079
P. R. China

Abstract

We consider the singularly perturbed equation $-\varepsilon^2\Delta u + V(x)u = K(x)u^{p-1}$ on a domain $\Omega \subset \mathbb{R}^N$ which may be bounded or unbounded. Under suitable hypotheses on V, K we construct layered solutions $u \in H_0^1(\Omega)$ which concentrate on certain high-dimensional subsets of Ω. This gives a positive answer to a problem proposed by Ambrosetti, Malchiodi and Ni in [1].

Keywords: singularly perturbed elliptic equation; variational method, critical point, concentrating solutions, higher dimensional subsets.
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1 Introduction and main results

In this paper, we are concerned with the existence of solutions which concentrate on some higher dimensional subsets of $\mathbb{R}^N$ for the following singularly perturbed elliptic equation

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u = K(x)u^{p-1}, \quad u > 0, \qquad x \in \Omega, \\ u \in H_0^1(\Omega) \end{cases} \tag{1.1}$$

on a domain $\Omega \subset \mathbb{R}^N$ which may be bounded or unbounded.

A basic motivation for the study of (1.1) comes from looking for standing-wave solutions

$$\psi(x,t) = \exp(-iEt/\varepsilon)u(x)$$

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of the nonlinear Schrödinger equation

$$i\varepsilon\frac{\partial\psi}{\partial t} = -\frac{\varepsilon^2}{2m}\Delta_x\psi + (V(x)+E)\psi - K(x)|\psi|^{p-2}\psi \quad \text{for } (t,x)\in\mathbb{R}\times\Omega, \tag{1.2}$$

where ε is the Planck constant. Plugging the standing-wave ansatz into (1.2) one is lead to equation (1.1) for u. Equation (1.2) arises in many applications, for instance in nonlinear optics, plasma physics, and in condensed matter physics. The presence of many particles leads one to consider nonlinear terms which model the interaction effect among them. We shall find solutions for $\varepsilon > 0$ small, i. e. in the semiclassical case which describes the transition from quantum to classical mechanics. Another motivation for studying equation (1.1) are models for pattern formation in mathematical biology or reaction-diffusion equations with small diffusion coefficients; see [21].

Equation (1.1) has been in the focus of research in nonlinear analysis in the last two decades since the pioneering work [17] of Floer and Weinstein. This is of course due to its importance in applications but also to the fascinating complexity and richness of the structure of the solution set of (1.1) and the necessity to develop new techniques to investigate this. Most papers deal with single- or multi-peak spike-layer solutions, i. e. solutions u_ε which develop as $\varepsilon \to 0$ one or several spikes whose peaks are located at critical points of the potential function V. We refer to the recent papers [3, 6, 7, 9, 10, 11, 12, 15, 16, 18, 19, 23, 24, 30, 31] and the references therein. In all these papers, the authors used the least energy solution of the related limiting equation in $\mathbb{R}^N$ to construct spike-layer solutions for problem (1.1). Furthermore, all the solutions concentrate near one or more isolated points.

Malchiodi and Montenegro [20] seem to be the first to construct solutions of (1.1) which concentrate on higher-dimensional subsets of $\mathbb{R}^N$. They considered (1.1) with Neumann boundary conditions on smooth bounded domains in $\mathbb{R}^2$ and for $V, K \equiv 1$. The new type of solutions found in [20] concentrate on the boundary of the domain. Recently, Ambrosetti, Malchiodi and Ni [1] extended [20] to higher-dimensional layers for problem (1.1) on $\Omega = \mathbb{R}^N$ with $K \equiv 1$ and $V(x) = V(|x|)$ being radially symmetric. Under certain conditions on V they found radial solutions which concentrate near an $(N-1)$-dimensional sphere $\{|x| = \rho\}$. In [1] they considered the case of a ball or an annulus with Dirichlet or Neumann boundary conditions. Del Pino, Kowalczyk and Wei [13] considered (1.1) on $\mathbb{R}^2$ with $K \equiv 1$ and without any symmetry conditions on V. For certain values $0 < \varepsilon \ll 1$ they obtained solutions of (1.1) concentrating on a prescribed curve $\Gamma \subset \mathbb{R}^2$ which is stationary and nondegenerate for the weighted area functional $\int_\Gamma V^\sigma$, $\sigma = \frac{p}{p-2} - \frac{1}{2}$.

In [4], we considered problem (1.1) in the non-autonomous case with V and K being radially symmetric. We constructed constructed radially symmetric solutions which concentrate simultaneously on several spheres. Recently, Dancer and Yan [14] studied (1.1) with $V = K \equiv 1$ on certain domains Ω and obtained solutions which concentrate near $(m-1)$ dimensional spheres, $1 < m \leq N$. In the terminology of [21], these solutions are $(m-1)$-dimensional layer solutions. In this paper, we extend the work of [14] to the non-autonomous case. Whereas in [14] the location of the spheres was determined by the geometry of the domain, in our case the potential functions V and K are essential. Our techniques also allow to find solutions concentrating on other types of manifolds like tori.

Now we describe the class of domains $\Omega \subset \mathbb{R}^N$ and potentials we consider.

(Ω) There is an integer m, $1 < m \le N$, and a relatively open subset $\Omega_0 \subset \mathbb{R}_0^+ \times \mathbb{R}^{N-m}$ such that $\Omega = \{x = (x', x'') \in \mathbb{R}^m \times \mathbb{R}^{N-m} = \mathbb{R}^N : \ (|x'|, x'') \in \Omega_0\}$

For $x = (x', x'') \in \mathbb{R}^m \times \mathbb{R}^{N-m}$ we write $\tilde{x} = (|x'|, x'') \in \mathbb{R}_0^+ \times \mathbb{R}^{N-m}$.

$(VK)_1$ There exist functions $V_0, K_0 \in C^1(\Omega_0, \mathbb{R})$ such that $V(x) = V_0(\tilde{x})$ and $K(x) = K_0(\tilde{x})$ for $x \in \Omega$.

$(VK)_2$ V, K are bounded and $\inf V > 0$, $\inf K > 0$.

The solutions we obtain will have the same symmetry, there is a limiting equation on $\mathbb{R}^{N-m+1}$, so the critical exponent is

$$p_c := \begin{cases} 2(N-m+1)/(N-m-1) & \text{if } m < N-1; \\ \infty & \text{if } m \ge N-1. \end{cases}$$

For $2 < p < p_c$ we define

$$\theta = \frac{p}{p-2} - \frac{N-m+1}{2}$$

and the function $\Gamma : \Omega_0 \to \mathbb{R}$ by:

$$\Gamma(z_0, \cdots, z_{N-m}) = z_0^{m-1} \big(V_0(z_0, \dots, z_{N-m})\big)^{\theta} \big(K_0(z_0, \dots, z_{N-m})\big)^{-2/(p-2)}. \tag{1.3}$$

Our last assumption concerns the localization of the spheres where the solutions concentrate.

$(VK)_3$ Γ has k critical points $Z_j = (Z_{j,0}, \dots, Z_{j,N-m}) \in \Omega_0$ such that $Z_{j,0} > 0$, $D^2\Gamma(Z_j)$ exists and is nondegenerate, $j = 1, \dots, k$

In order to formulate our results let $U \in H^1(\mathbb{R}^{N-m+1})$ be the unique solution of the problem

$$\begin{cases} -\Delta v + v = v^{p-1}, \quad v > 0, \\ v(0) = \max v, \\ v \in H^1(\mathbb{R}^{N-m+1}). \end{cases} \tag{1.4}$$

For given $z \in \mathbb{R}^{N-m+1}$ we set

$$\alpha(z) = \left(\frac{V_0(z)}{K_0(z)}\right)^{1/(p-2)} \quad \text{and} \quad \beta(z) = \sqrt{V_0(z)}.$$

Finally we define

$$(u,v)_\varepsilon = \int_\Omega (\varepsilon^2 \nabla u \nabla v + V(x)uv)dx \quad \text{and} \quad \|u\|_\varepsilon^2 = (u,u)_\varepsilon.$$

The main result of this paper is the following:

Theorem 1.1. *Let* (Ω) *and* $(VK)_1 - (VK)_3$ *hold,* $p \in (2, p_c)$*. Then for* $\varepsilon > 0$ *sufficiently small,* (1.1) *has a solution* $u_\varepsilon \in H_0^1(\Omega)$ *of the form*

$$u_\varepsilon(x) = \sum_{j=1}^k \alpha\left(Z_{\varepsilon,j}\right) U\left(\beta\left(Z_{\varepsilon,j}\right)\left(\frac{\tilde{x} - Z_{\varepsilon,j}}{\varepsilon}\right)\right) + w_\varepsilon(\tilde{x}),$$

with $\tilde{x} = (|x'|, x'')$ *as above and*

$$Z_{\varepsilon,j} \in \Omega_0, \quad |Z_{\varepsilon,j} - Z_j| = O\left(\varepsilon^{\min\{1,p-2\}}\right), \quad \|w_\varepsilon\|_\varepsilon^2 = O\left(\varepsilon^{N-m+3}\right), \quad |w_\varepsilon|_\infty = O(\varepsilon).$$

This result can be extended in various directions. We state one such variation dealing with the case where the critical points are allowed to be degenerate.

$(VK)_4$ Γ has k isolated critical points $Z_1, \dots, Z_k \in \Omega_0$ with nontrivial local degree: $\deg(\nabla\Gamma, B_\delta(Z_j), 0) \neq 0$ for $\delta > 0$ small, $j = 1, \dots, k$.

Theorem 1.2. *Suppose* (Ω)*,* $(VK)_1$*,* $(VK)_2$*, and* $(VK)_4$ *hold,* $p \in (2, p_c)$*. Then for* $\varepsilon > 0$ *sufficiently small,* (1.1) *has a solution* $u_\varepsilon \in H_0^1(\Omega)$ *of the form*

$$u_\varepsilon(x) = \sum_{j=1}^k \alpha\left(Z_{\varepsilon,j}\right) U\left(\beta\left(Z_{\varepsilon,j}\right)\left(\frac{\tilde{x} - Z_{\varepsilon,j}}{\varepsilon}\right)\right) + w_\varepsilon(\tilde{x}),$$

with

$$Z_{\varepsilon,j} \in \Omega_0, \quad |Z_{\varepsilon,j} - Z_j| = o(1), \quad \|w_\varepsilon\|_\varepsilon^2 = O\left(\varepsilon^{N-m+3}\right), \quad |w_\varepsilon|_\infty = O(\varepsilon).$$

Theorem 1.1 and Theorem 1.2 continue to hold with Neumann boundary conditions if the boundary is non-empty.

Our arguments can also be used to construct other types of solutions. Fix integers $N_1, \dots, N_h \in \mathbb{N}$ with $N_1 + \ldots + N_h = N$ and write

$$x = (x_1, \dots, x_h) \in \mathbb{R}^N = \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_h}$$

accordingly. Setting

$$z = z(x) = (z_1, \dots, z_h) \quad \text{with} \quad z_i = \begin{cases} x_i & \text{if } N_i = 1, \\ |x_i| & \text{if } N_i \geq 2, \end{cases}$$

we require $\Omega = \{x \in \mathbb{R}^N : z(x) \in \Omega_0\}$, $V(x) = V_0(z)$ and $K(x) = K_0(z)$ with C^1-functions $V_0, K_0 : \Omega_0 \subset \mathbb{R}^h \to \mathbb{R}$. Thus Ω, V, K are radially symmetric in x_i for all i with $N_i \geq 2$. In Theorem 1.1 we have $h = N - m + 1$, $N_1 = m$, $N_i = 1$ for $i = 2, \dots, h$. Here we consider the function

$$\Gamma(z_1, \dots, z_h) = z_1^{N_1 - 1} \cdot \ldots \cdot z_h^{N_h - 1} V_0(z)^{\frac{p}{p-2} - \frac{h}{2}} K_0(z)^{-\frac{2}{p-2}},$$

and assume that Γ has k nondegenerate critical points $Z_1, \dots, Z_k$ with $Z_{ji} > 0$, if $N_i \geq 2$, $j = 1, \dots, k$, $i = 1, \dots, h$. The critical exponent here is $p_c = 2h/(h-2)$ if $h > 2$ since the limiting equation lives on $\mathbb{R}^h$.

Theorem 1.3. *Suppose $p \in (2, p_c)$. Then for $\varepsilon > 0$ sufficiently small,* (1.1) *has a solution $u_\varepsilon \in H_0^1(\Omega)$ of the form*

$$u_\varepsilon(x) = \sum_{j=1}^{k} \alpha(Z_{\varepsilon,j}) U \left(\beta(Z_{\varepsilon,j}) \left(\frac{z - Z_{\varepsilon,j}}{\varepsilon} \right) \right) + w_\varepsilon(z),$$

with $z = z(x)$ as above and

$$Z_{\varepsilon,j} \in \Omega_0, \quad |Z_{\varepsilon,j} - Z_j| = O(\varepsilon^{\min\{1, p-2\}}), \quad \|w_\varepsilon\|_\varepsilon^2 = O(\varepsilon^{h+2}), \quad |w_\varepsilon|_\infty = O(\varepsilon).$$

where U is the unique solution of equation

$$\begin{cases} -\Delta v + v = v^{p-1}, \ v > 0, \\ v(0) = \max v, \\ v \in H^1(\mathbb{R}^h). \end{cases} \tag{1.5}$$

The solutions of Theorem 1.3 concentrate near the k manifolds $M_j = \{x \in \mathbb{R}^N : z(x) = Z_j\}$, $j = 1, \dots, k$. Observe that each M_j is diffeomorphic to the product of spheres $S^{N_1 - 1} \times \ldots \times S^{N_h - 1}$.

In the subsequent paper [5] we construct yet another type of solutions of (1.1) which concentrate simultaneously on a prescribed number of $(m-1)$-dimensional spheres and such that the spheres converge towards the same sphere as $\varepsilon \to 0$. Thus these solutions have the form as in Theorem 1.1 with $Z_{\varepsilon,j} \to Z$ as $\varepsilon \to 0$ for all j.

In the radially symmetric case $m = N$, our result with $k = 1$ is the same as that of [1]. We believe that our arguments are simpler and can be applied to more general cases. The case $k > 1$ is new even if $m = N$.

Our arguments are based on variational methods. The basic idea is to use the least energy solution of the related limiting equation in $\mathbb{R}^{N-m+1}$ as a building block to construct solutions for (1.1). We first reduce the problem we are dealing with to a finite-dimensional one by a kind of Lyapunov-Schmidt reduction (see [8] or [28]). For this reduction it is essential to work in the subspace

$$H_s := \{u \in H_0^1(\Omega) : u(x) = u_0(|x'|, x'')\}.$$

of $H_0^1(\Omega)$ consisting of functions having the same symmetry as the problem. The reason is the lack of control in some directions for the corresponding linearized operator $Lv(x) := -\Delta v(x) + v(x) - (p-1)\big(U(|x'|,x'')\big)^{p-2}v(x)$ in $H^1(\mathbb{R}^N)$. This leads to spectral problems for L which disappear in $H_s^1(\mathbb{R}^N)$. After making the reduction we use the Brouwer degree and apply energy comparison techniques. In order to obtain the existence result in the supercritical range $p \in (2N/(N-2), p_c)$, we also employ a penalty function argument which needs some truncation. Consequently we use a local approach in the finite dimensional reduction. This is essential for finding a fixed-point in a subspace where the functions are L^∞ uniformly bounded. In [1] where $m = N$, Strauss's inequality[29] and Green's function were used in the procedure of reduction. But it seems that the argument of [1] fails to work in our case since we do not have Strauss's inequality if $1 < m < N$. We also improve the techniques developed in [14]. We believe that our arguments can work well to generalize most of the results obtained in the case $m = 1$ to the case $1 < m \le N$.

The paper is organized as follows: in Section 2 we first introduce some notation and explain the framework of the proof. Then we prove some preliminary estimates which play a key role in the rest of the proof. In Section 3 we reduce the problem to the study of a finite dimensional variational problem. The proofs of the main results are given in Section 4. Finally, in the Appendix we prove a technical result.

Throughout this paper, we will use C, c and C_j, $j \in \mathbb{N}$, to denote various positive constants. $O(t)$, $o(t)$ means $|O(t)| \le C|t|$ and $o(t)/t \to 0$ respectively as $t \to 0$. Given $D_0 \subset \mathbb{R}^{N-m+1}$ such that $f(|x'|,x'')$ is integrable over $D := \{x \in \mathbb{R}^N : (|x'|,x'') \in D_0\}$ we write $\int_D f(|x'|,x'')dx = \int_{D_0} z_0^{m-1} f(z)dz$. So $z = (z_0,\dots,z_{N-m})$ and $\int dz$ includes the factor ω_{m-1}, the $(m-1)$-dimensional volume of the unit sphere in $\mathbb{R}^m$.

2 Preliminaries

Recall that the unique solution $U \in H^1(\mathbb{R}^{N-m+1})$ of (1.4) is radially symmetric and satisfies

$$\lim_{|z|\to\infty} |z|^{(N-m)/2} e^{|z|} U(z) = \alpha_{N,m,p} > 0 \quad \text{and} \quad \lim_{|z|\to\infty} \frac{U'(z)}{U(z)} = -1,$$

where $\alpha_{N,m,p}$ is a constant depending only on N, m and the exponent p. Moreover, U is non-degenerate, that is, the kernel of the operator $w \mapsto -\Delta w + w - (p-1)U^{p-2}w$ in $H^1(\mathbb{R}^{N-m+1})$ is spanned by $\{\partial U/\partial z_l : l = 0,\dots,N-m\}$; see [6] or [8] for instance.

For fixed $\varepsilon > 0$ and $y \in \Omega_0 \subset \mathbb{R}^{N-m+1}$ we define

$$U_{\varepsilon,y}(z) = \alpha_y U\left(\frac{\beta_y(z-y)}{\varepsilon}\right).$$

where

$$\alpha_y := \left(\frac{V_0(y)}{K_0(y)}\right)^{1/(p-2)} \quad \text{and} \quad \beta_y := V_0(y)^{1/2}$$

It is easy to check that $U_{\varepsilon,y}$ satisfies

$$-\varepsilon^2\Delta v(z)+V_0(y)v(z)=K_0(y)v(z)^{p-1}\quad \text{in } \mathbb{R}^{N-m+1}. \tag{2.1}$$

Moreover, there exist constants $c, C, \lambda>0$ such that

$$\begin{aligned} U_{\varepsilon,y}(z) &\le Ce^{-\lambda(z-y)/\varepsilon},\\ U'_{\varepsilon,y}(z) &\le C\varepsilon^{-1}e^{-\lambda(z-y)/\varepsilon}+e^{-c/\varepsilon},\\ U''_{\varepsilon,y}(z) &\le C\varepsilon^{-2}e^{-\lambda(z-y)/\varepsilon}+e^{-c/\varepsilon}. \end{aligned} \tag{2.2}$$

The function $\widetilde{U}_{\varepsilon,y}(x)=U_{\varepsilon,y}(\tilde{x})$ satisfies

$$\begin{aligned} &-\varepsilon^2\Delta\widetilde{U}_{\varepsilon,y}+V_0(y)\widetilde{U}_{\varepsilon,y}\\ &\qquad=K_0(y)\widetilde{U}^{p-1}_{\varepsilon,y}-\varepsilon\beta_j\alpha_y\frac{m-1}{|x'|}\frac{|x'|-y_0}{|\tilde{x}-y|}U'\left(\frac{\beta_y(\tilde{x}-y)}{\varepsilon}\right) \end{aligned} \tag{2.3}$$

with $\tilde{x}=(|x'|,x'')$.

Set $\kappa=\min\{\operatorname{dist}(Z_j,\partial\Omega_0):\ j=1,\dots,k\}$ and let $\eta\in C^\infty(\mathbb{R}^{N-m+1},[0,1])$ be such that

$$\begin{cases}\eta(z)=1, & \text{if } z\in\Omega_0,\ \operatorname{dist}(z,\partial\Omega_0)\ge\kappa/4,\\ \eta(z)=0, & \text{if } z\notin\Omega_0 \text{ or } \operatorname{dist}(z,\partial\Omega_0)\le\kappa/8.\end{cases}$$

The function

$$W_{\varepsilon,y}(x)=\eta(\tilde{x})\widetilde{U}_{\varepsilon,y}(x),$$

satisfies

$$\begin{cases}-\varepsilon^2\Delta W_{\varepsilon,y}+V_0(y)W_{\varepsilon,y}=\eta K_0(y)\widetilde{U}^{p-1}_{\varepsilon,y}+f_{\varepsilon,y}(x) & \text{in } \Omega,\\ W_{\varepsilon,y}=0, & \text{on } \partial\Omega,\end{cases} \tag{2.4}$$

where

$$\begin{aligned} f_{\varepsilon,y}(x)=&-\eta(\tilde{x})\varepsilon\beta_y\alpha_y\frac{m-1}{|x'|}\frac{|x'|-y_0}{|\tilde{x}-y|}U'\left(\frac{\beta_y(\tilde{x}-y)}{\varepsilon}\right)\\ &-2\varepsilon\alpha_y\beta_y\nabla\eta(\tilde{x})\nabla U\left(\frac{\beta_y(\tilde{x}-y)}{\varepsilon}\right)-\varepsilon^2\alpha_y U\left(\frac{\beta_y(\tilde{x}-y)}{\varepsilon}\right)\Delta\eta(\tilde{x}). \end{aligned}$$

Hence we can easily check that $f_{\varepsilon,y}$ depends smoothly on x and y. Moreover,

$$|f_{\varepsilon,y}|\le C\varepsilon U\left(\frac{\beta_y(\tilde{x}-y)}{\varepsilon}\right). \tag{2.5}$$

Fix $\delta>0$ small so that $\overline{B}_{4\delta}(Z_j)\subset\operatorname{int}\Omega_0$ and $\overline{B}_{4\delta}(Z_j)\cap\overline{B}_{4\delta}(Z_i)=\emptyset$ for $i\ne j$, $i,j=1,\dots,k$. Set

$$D_\delta=B_\delta(Z_1)\times\ldots\times B_\delta(Z_k)\subset\mathbb{R}^{k(N-m+1)} \tag{2.6}$$

and $B_j = \{x \in \Omega : \tilde{x} \in B_\delta(Z_j)\}$. We replace the nonlinearity u^{p-1} by

$$g(x,t) = \sum_{j=1}^{k} \chi_{B_j} t_+^{p-1} + \left(1 - \sum_{j=1}^{k} \chi_{B_j}\right) g_0(t), \tag{2.7}$$

where χ_{B_j} is the characteristic function of B_j and

$$g_0(t) = \begin{cases} t_+^{p-1} & \text{for } t \le a, \\ a^{p-2} t & \text{for } t > a, \end{cases}$$

with $a := kU(0) + 1$.

Now we consider the following new problem

$$\begin{cases} -\varepsilon^2 \Delta u + V(x) u = K(x) g(x,u), \quad u > 0, & \text{in } \Omega, \\ u \in H_0^1(\Omega) \end{cases} \tag{2.8}$$

The functional associated to problem (2.8) is

$$I_\varepsilon(u) = \frac{1}{2} \int_\Omega \left(\varepsilon^2 |\nabla u|^2 + V(x) u^2 \right) dx - \int_\Omega K(x) G(x,u) dx,$$

where $G(x,t) = \int_0^t g(x,s) ds$. For any $x \in B_j$, we have $|x'| \ge c > 0$, so I_ε is well defined in H_s for $2 < p < p_c$. It is easy to check that $I_\varepsilon \in C^1(H_s)$, hence its positive critical points are solutions of problem (2.8).

For $Y = (Y_1, \dots, Y_k) \in D_\delta$ we define the subspace

$$E_{\varepsilon,Y} = \left\{ v \in H_s : \left(v, \partial W_{\varepsilon,Y_j} / \partial Y_{j,l} \right)_\varepsilon = 0, \ j = 1, \dots, k, \ l = 0, \dots, N - m \right\}$$

of H_s of codimension $k(N - m + 1)$. We restrict our arguments to the existence of critical points of I_ε of the form

$$u = \sum_{j=1}^{k} W_{\varepsilon,Y_j} + w_\varepsilon,$$

where Y_j is close to Z_j, $w_\varepsilon \in E_{\varepsilon,Y}$ and $\|w_\varepsilon\|_\varepsilon^2 = o(\varepsilon^{N-m+1})$. In order to do this we consider the functional

$$J_\varepsilon(Y, w) := I_\varepsilon \left(\sum_{j=1}^{k} W_{\varepsilon,Y_j} + w \right)$$

defined for $Y \in \mathbb{R}^{k(N-m+1)}$ and $w \in H_s$. Clearly J_ε is of class C^1. We need to constrain J_ε to the $k(N - m + 1)$-codimensional submanifold

$$M_{\varepsilon,\delta} := \{(Y, w) : Y \in D_\delta, \ w \in E_{\varepsilon,Y}\}$$

of $\mathbb{R}^{k(N-m+1)} \times H_s$.

Lemma 2.1. *There exist $\delta_0 > 0$ and $\varepsilon_0 > 0$ such that if $\varepsilon \in (0, \varepsilon_0]$ and $\delta \in (0, \delta_0]$, then (Y, w) is a critical point of J_ε constrained to $M_{\varepsilon,\delta}$ if and only if*

$$u = \sum_{j=1}^{k} W_{\varepsilon,Y_j} + w_\varepsilon$$

is a critical point of I_ε in H_s.

Proof. The proof of Lemma 2.1 proceeds analogous to the proof of [4, Lemma 2.3]. We therefore just give a sketch. We define

$$\varphi(Y) = \sum_{j=1}^{k} W_{\varepsilon,Y_j}$$

and, for $\delta, \varepsilon > 0$,

$$W(\delta, \varepsilon) = \{u \in H_s : \|u - \varphi(Y)\|_\varepsilon < \delta\varepsilon^{(N-m+1)/2} \text{ for some } Y \in D_\delta\}.$$

Then one shows that there exist $\delta_0, \varepsilon_0 > 0$ such that if $\delta \in (0, \delta_0]$ and $\varepsilon \in (0, \varepsilon_0]$, then given $u \in W(\delta, \varepsilon)$ the minimization problem

$$\inf\{\|u - \varphi(Y)\|_\varepsilon : Y \in D_\delta\} \tag{2.9}$$

is achieved in $D_{2\delta}$ and not in $D_{4\delta} \setminus \overline{D_{2\delta}}$. If $Y \in D_{2\delta}$ is a minimizer of (2.9) then $w := u - \varphi(Y)$ satisfies

$$\left(w, \frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}}\right)_\varepsilon = 0 \quad \text{for } j = 1, \dots, k, \ l = 1, \dots, N - m + 1.$$

Finally one shows that (2.9) admits a unique solution provided δ_0, ε_0 are small enough. □

We notice that, according to the Lagrange multiplier rule, (Y, w) is a critical point of J_ε constrained to $M_{\varepsilon,\delta}$ if and only if there are scalars $A_{j,l} \in \mathbb{R}$, $j = 1, \dots, k$, $l = 0, \dots, N - m$, such that

$$\frac{\partial J_\varepsilon}{\partial Y_{j,l}}(Y, w) = \sum_{n=0}^{N-m} A_{j,n} \left(\frac{\partial^2 W_{\varepsilon,Y_j}}{\partial Y_{j,n} \partial Y_{j,l}}, w\right)_\varepsilon, \tag{2.10}$$

and

$$\frac{\partial J_\varepsilon}{\partial w}(Y, w) = \sum_{j=1}^{k} \sum_{l=0}^{N-m} A_{j,l} \frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}}. \tag{2.11}$$

In order to prove Theorem 1.1, we show first that for given Y and ε small enough, there exist $w_{\varepsilon,Y} \in E_{\varepsilon,Y}$ and scalars $A_{j,l}$, $j = 1, \dots, k$, $l = 0, \dots, N - m$, such that

(2.11) is satisfied and the mapping $Y \mapsto w_{\varepsilon,Y}$ is C^1. We then show that for sufficiently small ε, there exists a point $Y \in D_\delta$, such that $(Y, w_{\varepsilon,Y}) \in M_{\varepsilon,\delta}$ solves (2.10).

Expand $J_\varepsilon(Y,w)$ near $w=0$ as follows

$$J_\varepsilon(Y,w) = J_\varepsilon(Y,0) + h_{\varepsilon,Y}(w) + \frac{1}{2}Q_{\varepsilon,Y}(w) - R_{\varepsilon,Y}(w),$$

where

$$\begin{aligned}
h_{\varepsilon,Y}(w) &= \sum_{j=1}^{k}\int_\Omega \left(\varepsilon^2\nabla W_{\varepsilon,Y_j}\nabla w + V(x)W_{\varepsilon,Y_j}w\right) - \int_\Omega K(x)\left(\sum_{j=1}^{k} W_{\varepsilon,Y_j}\right)^{p-1} w,\\
Q_{\varepsilon,Y}(w) &= \int_\Omega (\varepsilon^2|\nabla w|^2 + V(x)w^2) - (p-1)\int_\Omega K(x)\left(\sum_{j=1}^{k} W_{\varepsilon,Y_j}\right)^{p-2} w^2,\\
R_{\varepsilon,Y}(w) &= \int_\Omega K(x)G\left(x, \sum_{j=1}^{k} W_{\varepsilon,Y_j} + w\right) - \frac{1}{p}\int_\Omega K(x)\left(\sum_{j=1}^{k} W_{\varepsilon,Y_j}\right)^{p}\\
&\quad - \int_\Omega K(x)\left(\sum_{j=1}^{k} W_{\varepsilon,Y_j}\right)^{p-1} w\\
&\quad - \frac{p-1}{2}\int_\Omega K(x)\left(\sum_{j=1}^{k} W_{\varepsilon,Y_j}\right)^{p-2} w^2.
\end{aligned}$$

Lemma 2.2. *$h_{\varepsilon,Y} : E_{\varepsilon,Y} \to \mathbb{R}$ is a bounded linear map satisfying*

$$|h_{\varepsilon,Y}(w)| \le C\varepsilon^{(N-m+1)/2}(\varepsilon + e^{-c/\varepsilon})\|w\|_\varepsilon$$

for some constants $c, C > 0$.

Proof. By (2.4), we have

$$\begin{aligned}
h_{\varepsilon,Y}(w) = &\sum_{j=1}^{k}\int_\Omega \left(\eta K_0(Y_j)\widetilde{U}^{p-1}_{\varepsilon,Y_j} + f_{\varepsilon,j}\right) w - \sum_{j=1}^{k}\int_\Omega K_0(Y_j)\left(W_{\varepsilon,Y_j}\right)^{p-1} w\\
&+ \sum_{j=1}^{k}\int_\Omega V(x)W_{\varepsilon,Y_j}w - \sum_{j=1}^{k}\int_\Omega V_0(Y_j)W_{\varepsilon,Y_j}w\\
&+ \sum_{j=1}^{k}\int_\Omega K_0(Y_j)\left(W_{\varepsilon,Y_j}\right)^{p-1} w - \int_\Omega K(x)\left(\sum_{j=1}^{k} W_{\varepsilon,Y_j}\right)^{p-1} w.
\end{aligned}$$

On the other hand, by (2.2),

$$\left|\sum_{j=1}^{k}\int_\Omega \left(\eta K_0(Y_j)\widetilde{U}^{p-1}_{\varepsilon,Y_j}\right) w - \sum_{j=1}^{k}\int_\Omega K_0(Y_j)\left(W_{\varepsilon,Y_j}\right)^{p-1} w\right|$$

$$
\begin{aligned}
&\le \sum_{j=1}^{k} \int_\Omega K_0(Y_j) \left| \eta \widetilde{U}_{\varepsilon,Y_j}^{p-1} - W_{\varepsilon,Y_j}^{p-1} \right| |w| \\
&= O(\varepsilon^{\frac{N-m+1}{2}} e^{-\frac{c}{\varepsilon}}) \|w\|_\varepsilon,
\end{aligned}
$$

$$
\begin{aligned}
&\left| \sum_{j=1}^{k} \int_\Omega V(x) W_{\varepsilon,Y_j} w - \sum_{j=1}^{k} \int_\Omega V_0(Y_j) W_{\varepsilon,Y_j} w \right| \\
&\qquad \le \sum_{j=1}^{k} \int_\Omega |V(x) - V_0(Y_j)| W_{\varepsilon,Y_j} |w| \\
&\qquad \le \left(\int_\Omega |V(x) - V_0(Y_j)|^2 W_{\varepsilon,Y_j}^2 \right)^{\frac{1}{2}} \|w\|_\varepsilon \\
&\qquad = O(\varepsilon \varepsilon^{\frac{N-m+1}{2}}) \|w\|_\varepsilon,
\end{aligned}
$$

$$
\begin{aligned}
&\left| \sum_{j=1}^{k} \int_\Omega K_0(Y_j) \left(W_{\varepsilon,Y_j} \right)^{p-1} w - \int_\Omega K(x) \left(\sum_{j=1}^{k} W_{\varepsilon,Y_j} \right)^{p-1} w \right| \\
&\qquad = \sum_{j=1}^{k} \int_\Omega |K_0(Y_j) - K(x)| W_{\varepsilon,y}^{p-1} |w| \\
&\qquad\qquad + \begin{cases} O\left(\sum_{i \neq j} \int_\Omega W_{\varepsilon,Y_j}^{\frac{p-1}{2}} W_{\varepsilon,Y_i}^{\frac{p-1}{2}} |w| \right) & (2 < p < 3) \\ O\left(\sum_{i \neq j} \int_\Omega W_{\varepsilon,Y_j}^{p-2} W_{\varepsilon,Y_i} |w| \right) & (p \ge 3) \end{cases} \\
&\qquad \le \sum_{j=1}^{k} \left(\int_\Omega |K_0(Y_j) - K(x)|^2 W_{\varepsilon,Y_j}^{2(p-1)} \right)^{\frac{1}{2}} \|w\|_\varepsilon \\
&\qquad\qquad + \begin{cases} O\left(\sum_{i \neq j} \left(\int_\Omega W_{\varepsilon,Y_j}^{p-1} W_{\varepsilon,Y_i}^{p-1} \right)^{\frac{1}{2}} \right) \|w\|_\varepsilon & (2 < p < 3) \\ O\left(\sum_{i \neq j} \left(\int_\Omega W_{\varepsilon,Y_j}^{2(p-2)} W_{\varepsilon,Y_i}^{2} \right)^{\frac{1}{2}} \right) \|w\|_\varepsilon & (p \ge 3) \end{cases} \\
&\qquad = O(\varepsilon \varepsilon^{\frac{N-m+1}{2}}) \|w\|_\varepsilon + O(\varepsilon^{\frac{N-m+1}{2}} e^{-\frac{c}{\varepsilon}}) \|w\|_\varepsilon,
\end{aligned}
$$

$$
\left| \int_\Omega f_{\varepsilon,j}(x) w \right| \le \left(\int_\Omega |f_{\varepsilon,j}|^2 \right)^{\frac{1}{2}} \|w\|_\varepsilon = O(\varepsilon \varepsilon^{\frac{N-m+1}{2}}) \|w\|_\varepsilon.
$$

Adding the above four inequalities we obtain the desired estimate. $\square$

3 The finite-dimensional reduction

In this section, we solve equation (2.11) for any given $Y = (Y_1, \dots, Y_k) \in D_\delta$. Associated to the quadratic form $Q_{\varepsilon,Y} : H_s \to \mathbb{R}$ is the bounded linear map $L_{\varepsilon,Y} : H_s \to H_s$, defined by

$$(L_{\varepsilon,Y} w_1, w_2)_\varepsilon = \int_\Omega (\varepsilon^2 \nabla w_1 \nabla w_2 + V(x) w_1 w_2) - (p-1) \int_\Omega K(x) \left(\sum_{j=1}^k W_{\varepsilon,Y_j} \right)^{p-2} w_1 w_2$$

so that $Q_{\varepsilon,Y}(w) = (L_{\varepsilon,Y} w, w)$. Constraining $Q_{\varepsilon,Y}$ yields the quadratic form $Q^E_{\varepsilon,Y} : E_{\varepsilon,Y} \to \mathbb{R}$ which induces a bounded linear map $L^E_{\varepsilon,Y} : E_{\varepsilon,Y} \to E_{\varepsilon,Y}$ given by $L^E_{\varepsilon,Y} w = P L_{\varepsilon,Y} w$ with $P : H_s \to E_{\varepsilon,Y}$ the orthogonal projection (with respect to the scalar product $(\,\cdot\,,\,\cdot\,)_\varepsilon$).

Proposition 3.1. *For ε small enough and $Y \in D_\delta$, the operator $L^E_{\varepsilon,Y}$ is invertible with uniformly bounded inverse. In other words, there exist constants $\varepsilon_0 > 0$ and $\tau > 0$, such that for each $\varepsilon \in (0, \varepsilon_0]$ and $Y \in D_\delta$*

$$\|L^E_{\varepsilon,Y} w\|_\varepsilon \ge \tau \|w\|_\varepsilon, \quad \textit{for all } w \in E_{\varepsilon,Y}.$$

The proof of Proposition 3.1 will be postponed to the Appendix.

Let

$$\mathcal{F}_{\varepsilon,Y} = \left\{ w \in H_s : |w(x)| \le \sum_{j=1}^k \eta e^{-\nu|\tilde{x} - Y_j| / \varepsilon} + C e^{-\frac{\sigma}{\varepsilon}} \right\},$$

where $\nu, \sigma > 0$ are small constants to be determined later. The following estimates were essentially observed by Dancer and Yan in [14].

Lemma 3.2. *For any $w \in \mathcal{F}_\varepsilon$ we have, provided $\varepsilon > 0$ is small:*

(3.1) $\quad |R_{\varepsilon,Y}(w)| \le C\varepsilon^{N-m+1}(\varepsilon^{-(p^*-2)(N-m+1)/2}\|w\|_\varepsilon^{p^*-2}) + Ce^{-c/\varepsilon}\|w\|_\varepsilon^2$

(3.2) $\quad |R'_{\varepsilon,Y}(w)v| \le C\varepsilon^{(N-m+1)/2}\varepsilon^{-(p^*+1)(N-m+1)/2}\|w\|_\varepsilon^{p^*+1}\|v\|_\varepsilon + Ce^{-c/\varepsilon}\|w\|_\varepsilon\|v\|_\varepsilon$

(3.3) $\quad |R''_{\varepsilon,Y}(w)[v_1, v_2]| \le C(\varepsilon^{-p^*(N-m+1)/2}\|w\|_\varepsilon^{p^*} + e^{-c/\varepsilon})\|v_1\|_\varepsilon\|v_2\|_\varepsilon$

where $p^* = \min\{1, p-2\}$.

Proof. For any $w \in \mathcal{F}_\varepsilon$ and small ε, we have $|w| \le \frac{1}{2}$ and $\sum\limits_{j=1}^k W_{\varepsilon,Y_j} \le \frac{1}{2}$ in $\Omega \setminus \bigcup_{j=1}^k B_j$. This implies

$$|R_{\varepsilon,Y}(w)| \;=\; \left| \frac{1}{p} \int_\Omega K(x) \left(\sum_{j=1}^k W_{\varepsilon,y} + w \right)_+^p - \frac{1}{p} \int_\Omega K(x) \left(\sum_{j=1}^k W_{\varepsilon,Y_j} \right)^p \right.$$

$$
\begin{aligned}
& -\int_\Omega K(x)\left(\sum_{j=1}^k W_{\varepsilon,Y_j}\right)^{p-1} w - \frac{p-1}{2}\int_\Omega K(x)\left(\sum_{j=1}^k W_{\varepsilon,Y_j}\right)^{p-2} w^2\Bigg| \\
\leq\ & C\int_{\bigcup_{j=1}^k B_j} |w|^{p^*-2} + C\int_{\Omega\setminus\bigcup_{j=1}^k B_j} |w|^{p^*-2} \\
\leq\ & C\int_{\bigcup_{j=1}^k B_j} |w|^{p^*-2} + C|w|^{p^*}_{L^\infty(\Omega\setminus\bigcup_{j=1}^k B_j)}\int_{\Omega\setminus\bigcup_{j=1}^k B_j} w^2 \\
\leq\ & C\int_{\bigcup_{j=1}^k B_j} |w|^{p^*-2} + Ce^{-c/\varepsilon}\|w\|_\varepsilon^2.
\end{aligned}
$$

Fix some $j = 1,\dots,k$ and set $\tilde{w}(z) = w(\varepsilon z + Y_j)$ and $B_{\varepsilon,j} = \{z : \varepsilon z + Y_j \in B_\delta(Z_j)\}$. By the fact that $|x'| \geq c > 0$ for $x \in B_j$, we see

$$
\begin{aligned}
\int_{B_j} |w|^{p^*-2}dx &= \int_{B_\delta(Z_j)} z_0^{m-1}|w|^{p^*-2}dz \leq C\int_{B_\delta(Z_j)} |w|^{p^*-2}dz \\
&= C\varepsilon^{N-m+1}\int_{B_{\varepsilon,j}} |\tilde{w}|^{p^*-2}dz \leq C\varepsilon^{N-m+1}\left(\int_{B_{\varepsilon,j}} (|\nabla\tilde{w}|^2 + \tilde{w}^2)dz\right)^{(p^*-2)/2} \\
&= C\varepsilon^{N-m+1}\left(\varepsilon^{-(N-m+1)}\int_{B_\delta(Z_j)} (\varepsilon^2|\nabla w|^2 + w^2)dz\right)^{(p^*-2)/2} \\
&\leq C\varepsilon^{N-m+1}\left(\varepsilon^{-(p^*-2)(N-m+1)/2}\|w\|_\varepsilon^{p^*-2}\right).
\end{aligned}
$$

Combining the last two estimates, we obtain (3.1). (3.2) and (3.3) can be verified similarly. □

Now we are in the position to state the main result of this section.

Proposition 3.3. *For ε sufficiently small, there exists a C^1-map $D_\delta \to H_s$, $Y \mapsto w_{\varepsilon,Y}$, such that $w_{\varepsilon,Y} \in E_{\varepsilon,Y}$ and $(Y, w_{\varepsilon,Y})$ satisfies (2.11) for some $A_{j,l} \in \mathbb{R}$, $j = 1,\dots,k$, $l = 0,\dots,N-m$. Moreover,*

$$
\|w_{\varepsilon,Y}\|_\varepsilon^2 \leq C\varepsilon^{N-m+3}. \tag{3.4}
$$

Proof. Lemma 2.2 implies that the map $h_{\varepsilon,Y}|_{E_{\varepsilon,Y}}$ is represented by an element of $E_{\varepsilon,Y}$ which we denote by $h^E_{\varepsilon,Y}$. So $h^E_{\varepsilon,Y} \in E_{\varepsilon,Y}$ satisfies

$$
(h^E_{\varepsilon,Y}, w)_\varepsilon = h_{\varepsilon,Y}(w), \quad \text{for all } w \in E_{\varepsilon,Y}.
$$

Thus, solving (2.11) is equivalent to solving

$$
h^E_{\varepsilon,Y} + L^E_{\varepsilon,Y}w + (R^E_{\varepsilon,Y})'(w) = 0, \quad w \in E_{\varepsilon,Y} \tag{3.5}
$$

where $(R^E_{\varepsilon,Y})'(w) \in E_{\varepsilon,Y}$ represents $R'_{\varepsilon,Y}(w)|_{E_{\varepsilon,Y}}$. As a consequence of Proposition 3.1, $Q^E_{\varepsilon,Y}$ is invertible. So we can rewrite (3.5) as

$$
w = -(Q^E_{\varepsilon,Y})^{-1}(h^E_{\varepsilon,Y} + (R^E_{\varepsilon,Y})'(w)) =: A_{\varepsilon,Y}(w).
$$

For $\nu > 0$, $0 < \sigma \ll \nu$, and $\gamma > 0$ to be determined later, define

$$\mathcal{C}_\varepsilon := \left\{ w \in E_{\varepsilon,Y} : |w(x)| \le \gamma\varepsilon \sum_{j=1}^{k} \eta e^{-\nu|\tilde{x}-Y_j|/\varepsilon} + \gamma e^{-\sigma/\varepsilon},\ \|w\|_\varepsilon \le \gamma\varepsilon^{(N-m+3)/2} \right\}.$$

Now we prove that for a suitable choice of γ, the map $A_{\varepsilon,Y}$ is a contraction on the set $\mathcal{C}_\varepsilon$ endowed with the norm $\|\,.\,\|_\varepsilon$. For any $w_1, w_2 \in \mathcal{C}_\varepsilon$, we have by (3.3),

$$\begin{aligned} &\|A_{\varepsilon,Y}(w_1) - A_{\varepsilon,Y}(w_2)\|_\varepsilon \\ &\qquad \le C\|(R^E_{\varepsilon,Y})'(w_1) - (R^E_{\varepsilon,Y})'(w_2)\|_\varepsilon \\ &\qquad \le C(\varepsilon^{-p^*(N-m+1)/2}\|w_1 + (1-\varrho)w_2\|_\varepsilon^{p^*} + e^{-c/\varepsilon})\|w_1 - w_2\|_\varepsilon \\ &\qquad \le C\gamma^{p^*}\varepsilon^{p^*}\|w_1 - w_2\|_\varepsilon, \end{aligned}$$

where $\varrho \in [0,1]$. Thus $A_{\varepsilon,Y}$ is a contraction for ε small enough.

For $w \in \mathcal{C}_\varepsilon$ we have

$$\begin{aligned} \|A_{\varepsilon,Y}(w)\|_\varepsilon &\le C\|h_{\varepsilon,Y}\|_\varepsilon + C\|(R^E_{\varepsilon,Y})'(w)\|_\varepsilon \\ &\le C(\varepsilon\varepsilon^{(N-m+1)/2} + e^{-c/\varepsilon}) \\ &\qquad + C\varepsilon^{(N-m+1)/2}(\varepsilon^{-(p^*+1)(N-m+1)/2}\|w\|_\varepsilon^{p^*+1}) + Ce^{-c/\varepsilon}\|w\|_\varepsilon \\ &\le C_0\varepsilon^{(N-m+3)/2}(1 + \gamma^{p^*+1}\varepsilon^{p^*}). \end{aligned} \tag{3.6}$$

In order to see that $A_{\varepsilon,Y}(w) \in \mathcal{C}_\varepsilon$, it suffices to prove that for a suitable $\gamma > 0$,

$$|A_\varepsilon(w)(x)| \le \gamma\varepsilon \sum_{j=1}^{k} \eta e^{-\nu|\tilde{x}-Y_j|/\varepsilon} + \gamma e^{-\sigma/\varepsilon}.$$

Setting $w_1 = A_{\varepsilon,Y}(w)$ we obtain

$$L^E_{\varepsilon,Y} w_1 = -h^E_{\varepsilon,Y} - (R^E_{\varepsilon,Y})'(w)$$

that is,

$$L_{\varepsilon,Y} w_1 + h_{\varepsilon,Y} + R'_{\varepsilon,Y}(w) = \sum_{j=1}^{k}\sum_{l=0}^{N-m} A_{j,l}\frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}}, \tag{3.7}$$

for some $A_{j,l} \in \mathbb{R}$, $j = 1, \dots, k$, $l = 0, \dots, N-m$; here we identify the bounded linear maps $h_{\varepsilon,Y}, R'_{\varepsilon,Y}(w) : H_s \to \mathbb{R}$ with elements of H_s using the scalar product $(\,\cdot\,,\,\cdot\,)_\varepsilon$.

We claim that

$$|A_{j,l}| \le C_1\varepsilon^2(1 + \gamma^{p^*+1}\varepsilon^{p^*}), \quad j = 1, \dots, k,\ l = 0, \dots, N-m. \tag{3.8}$$

In fact, first, we can easily check that

$$\left(\frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}}, \frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}}\right)_\varepsilon = C\varepsilon^{N-m-1} + O(\varepsilon^{N-m}e^{-\frac{c}{\varepsilon}}),$$

$$\left(\frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}}, \frac{\partial W_{\varepsilon,Y_i}}{\partial Y_{i,n}}\right)_\varepsilon = O(\varepsilon^{N-m} + \varepsilon^{N-m} e^{-\frac{c}{\varepsilon}}), \quad i \neq j \text{ or } l \neq n.$$

Thus, taking the scalar product in H_s of (3.7) with $\frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}}$ for $j = 1, \dots, k$, $l = 0, \dots, N-m$, respectively, we get a quasi-diagonal linear system with $A_{j,l}$ as unknowns, which yields,

$$\begin{aligned}
|A_{j,l}| &\leq C\varepsilon^{-\frac{N-m-1}{2}} (\|w_1\|_\varepsilon + \|h_{\varepsilon,Y}\|_\varepsilon + \|R'_{\varepsilon,Y}(w)\|_\varepsilon) \\
&\leq C\varepsilon^{-(N-m-1)/2} \Big(C_0 \varepsilon^{(N-m+3)/2} (1 + \gamma^{p^*+1} \varepsilon^{p^*}) + C\varepsilon^{(N-m+3)/2} \\
&\qquad + C\gamma^{p^*-1} \varepsilon^{(N-m+3)/2} \varepsilon^{p^*} \Big) \\
&\leq C_1 \varepsilon^2 (1 + \gamma^{p^*+1} \varepsilon^{p^*}).
\end{aligned}$$

By duality, (3.7) can be written as

$$\begin{aligned}
&-\varepsilon^2 \Delta w_1 + V(x) w_1 - (p-1) K(x) \Big(\sum_{j=1}^k W_{\varepsilon,Y_j} \Big)^{p-2} w_1 \\
&= -\sum_{j=1}^k \Big(\eta K_0(Y_j) \widetilde{U}_{\varepsilon,Y_j}^{p-1} + f_{\varepsilon,j} \Big) + K(x) \Big(\sum_{j=1}^k W_{\varepsilon,Y_j} \Big)^{p-1} + \sum_{j=1}^k (V(x) - V_0(Y_j)) W_{\varepsilon,Y_j} \\
&\qquad - K(x) \Big\{ g\Big(x, \sum_{j=1}^k W_{\varepsilon,Y_j} + w\Big) - \Big(\sum_{j=1}^k W_{\varepsilon,Y_j} \Big)^{p-1} - (p-1) \Big(\sum_{j=1}^k W_{\varepsilon,Y_j} \Big)^{p-2} w \Big\} \\
&\qquad + \sum_{j=1}^k \sum_{l=0}^{N-m} A_{j,l} \Big(\frac{\partial q_{\varepsilon,j}}{\partial Y_{j,l}} - \frac{\partial V_0(Y_j)}{\partial Y_{j,l}} W_{\varepsilon,y} \Big) \\
&=: G_{\varepsilon,Y}(x),
\end{aligned}$$

where

$$q_{\varepsilon,j} = \eta K_0(Y_j) \widetilde{U}_{\varepsilon,Y_j}^{p-1} + f_{\varepsilon,j}.$$

Since $w \in \mathcal{C}_\varepsilon$, we have $|w| \leq 1/2$ in $\Omega \setminus \bigcup_{j=1}^k B_j$. Therefore,

(3.9)

$$\begin{aligned}
&\Big| K(x) g\Big(x, \sum_{j=1}^k W_{\varepsilon,Y_j} + w\Big) - K(x) \Big(\sum_{j=1}^k W_{\varepsilon,Y_j} \Big)^{p-1} - (p-1) K(x) \Big(\sum_{j=1}^k W_{\varepsilon,y} \Big)^{p-2} w \Big| \\
&\qquad \leq C|w|^{p^*+1}.
\end{aligned}$$

Now direct calculations yield:

(3.10)

$$
\begin{aligned}
&\Big| -\sum_{j=1}^{k}\Big(\eta K_0(Y_j)\widetilde{U}^{p-1}_{\varepsilon,Y_j}\Big) + K(x)\Big(\sum_{j=1}^{k} W_{\varepsilon,Y_j}\Big)^{p-1}\Big| \\
&\qquad \le \Big| -\sum_{j=1}^{k}\Big(K_0(Y_j)W^{p-1}_{\varepsilon,Y_j} - K(x)W^{p-1}_{\varepsilon,y}\Big)\Big| \\
&\qquad\qquad + K(x)\Big|\Big(\sum_{j=1}^{k} W_{\varepsilon,Y_j}\Big)^{p-1} - \sum_{j=1}^{k} W^{p-1}_{\varepsilon,Y_j}\Big| + O(e^{-c/\varepsilon}) \\
&\qquad \le C\sum_{j=1}^{k}|\tilde{x}-Y_j|W^{p-1}_{\varepsilon,Y_j} + O(e^{-c/\varepsilon}) + \begin{cases} C\sum_{i\neq j} W^{(p-1)/2}_{\varepsilon,Y_j}W^{(p-1)/2}_{\varepsilon,Y_i} & 2<p<3 \\ C\sum_{i\neq j} W^{p-2}_{\varepsilon,Y_j}W_{\varepsilon,Y_i} & p\ge 3 \end{cases} \\
&\qquad \le C\sum_{j=1}^{k}|\tilde{x}-Y_j|W^{p-1}_{\varepsilon,Y_j} + O(e^{-c/\varepsilon}),
\end{aligned}
$$

and

$$
\sum_{j=1}^{k}|V(x)-V_0(Y_j)|W_{\varepsilon,Y_j} \le C\sum_{j=1}^{k}|\tilde{x}-Y_j|W_{\varepsilon,Y_j} + O(e^{-c/\varepsilon}). \tag{3.11}
$$

On the other hand, it follows from the definitions of W_{ε,Y_j} and $f_{\varepsilon,j}$ that for $j=1,\dots,k$

$$
\Big|\frac{\partial q_{\varepsilon,j}}{\partial Y_{j,l}} - \frac{\partial V_0(Y_j)}{\partial Y_{j,l}}W_{\varepsilon,y}\Big| \le C\varepsilon^{-1}U^{p-1}\left(\frac{\beta_j(\tilde{x}-Y_j)}{\varepsilon}\right) + CU\left(\frac{\beta_j(\tilde{x}-Y_j)}{\varepsilon}\right). \tag{3.12}
$$

Hence combining (2.2), (2.5) and (3.8)-(3.12), we obtain

$$
\begin{aligned}
|G_{\varepsilon,Y}(x)| &\le C\sum_{j=1}^{k}|\tilde{x}-Y_j|U^{p-1}\left(\frac{\beta_j(\tilde{x}-Y_j)}{\varepsilon}\right) + C\sum_{j=1}^{k}|\tilde{x}-Y_j|U\left(\frac{\beta_j(\tilde{x}-Y_j)}{\varepsilon}\right) \\
&\qquad + C_1\sum_{j=1}^{k}\left(\varepsilon(1+\gamma^{p^*+1}\varepsilon^{p^*})U\left(\beta_j\frac{(\tilde{x}-Y_j)}{\varepsilon}\right)\right) + Ce^{-c/\varepsilon} + C|w|^{p^*+1} \\
&\le C\sum_{j=1}^{k}|\tilde{x}-Y_j|e^{-(p-1)\lambda|\tilde{x}-Y_j|/\varepsilon} + C\sum_{j=1}^{k}|\tilde{x}-Y_j|e^{-\lambda|\tilde{x}-Y_j|/\varepsilon} \\
&\qquad + C_1\sum_{j=1}^{k}\left(\varepsilon(1+\gamma^{p^*+1}\varepsilon^{p^*})e^{-\lambda|\tilde{x}-Y_j|/\varepsilon}\right) + Ce^{-c/\varepsilon} \\
&\qquad + C\gamma^{p^*+1}\varepsilon^{p^*+1}\sum_{j=1}^{k}\eta e^{-(p^*+1)\nu|\tilde{x}-Y_j|/\varepsilon} + C\gamma^{p^*+1}e^{-(p-1)\sigma/\varepsilon}.
\end{aligned}
$$

It is easy to check that for $0 < \nu < \lambda$ and $r \geq 0$

$$\frac{re^{-\lambda r/\varepsilon}}{\varepsilon e^{-\nu r/\varepsilon}} \leq \frac{e^{-1}}{\lambda - \nu}.$$

Consequently,

$$|G_{\varepsilon,Y}(x)| \leq \frac{C_2\varepsilon(1+\gamma^{p^*+1}\varepsilon^{p^*})}{\lambda-\nu}\sum_{j=1}^{k}\eta e^{-\nu|\tilde{x}-Y_j|/\varepsilon} + C_2 e^{-c/\varepsilon} + C_2\gamma^{p^*+1}e^{-(p^*+1)\sigma/\varepsilon}, \tag{3.13}$$

where C_2, c, ν and σ are independent of γ, $0 < \nu < \lambda$.

We claim that there exists $C_3 > 0$ independent of γ, such that

$$|w_1| \leq \frac{C_3\varepsilon(1+\gamma^{p^*+1}\varepsilon^{p^*})}{\lambda-\nu} \qquad \text{in } \bigcup_{j=1}^{k} B_j. \tag{3.14}$$

Indeed, setting

$$g_{\varepsilon,Y}(x,w_1) = -V(x)w_1 + (p-1)K(x)\left(\sum_{j=1}^{k} W_{\varepsilon,Y_j}\right)^{p-2} w_1,$$

and $\tilde{w_1}(z) = w_1(\varepsilon z + Y_j)$, $z = \tilde{x} = (|x'|, x'')$, then

$$-\Delta\tilde{w}_1(z) = G_{\varepsilon,Y}(\varepsilon z + Y_j) + g_\varepsilon(\varepsilon z + Y_j, \tilde{w}_1) \quad \text{in } B_{\varepsilon,j},$$

where $B_{\varepsilon,j} = \{z : \varepsilon z + Y_j \in B_\delta(Z_j)\}$.

For any $\bar{z} \in B_{\varepsilon,j}$, since $|x'| > c > 0$ in B_j, we have

$$\begin{aligned}\int_{B_1(\bar{z})}|\tilde{w_1}|^2 dz &\leq \varepsilon^{-(N-m+1)}\int_{B_j}|w_1|^2 dz \leq C\varepsilon^{-(N-m+1)}\|w_1\|_\varepsilon^2 \\ &= C\varepsilon^{-(N-m+1)}\|A_{\varepsilon,Y}w\|_\varepsilon^2 \leq CC_0\varepsilon^2(1+\gamma^{p^*+1}\varepsilon^{p^*}).\end{aligned}$$

So, we deduce from (3.13) that

$$\begin{aligned}|\tilde{w}_1|_{L^\infty(B_1(\bar{z}))} &\leq C\|g_{\varepsilon,Y}(\varepsilon z + Y_j, \tilde{w}_1)\|_{L^2(B_1(\bar{z}))} + C|G_{\varepsilon,Y}(\varepsilon z + Y_j)|_{L^\infty B_1(\bar{z})} \\ &\leq C\|\tilde{w}_1\|_{L^2(B_1(\bar{z}))} + C|G_{\varepsilon,Y}|_{L^\infty(\Omega)} \\ &\leq \frac{C_3\varepsilon(1+\gamma^{p^*+1}\varepsilon^{p^*})}{\lambda-\nu}.\end{aligned}$$

Thus, our claim follows.

For a smooth function ψ satisfying $\psi = 0$ in B_j, define

$$a_\varepsilon(x) = K(x)\left(\sum_{j=1}^{k} W_{\varepsilon,Y_j}(x)\right)^{p-2}\psi(x).$$

It is easy to see that $a_\varepsilon \to 0$ uniformly in Ω as $\varepsilon \to 0$. It follows from (3.14) that w_1 satisfies

$$\begin{aligned} & -\varepsilon^2 \Delta w_1 + (V(x) - (p-1)a_\varepsilon(x))w_1 \\ & \qquad \le G_{\varepsilon,Y}(x) + \frac{C_3\varepsilon(1+\gamma^{p^*+1}\varepsilon^{p^*})}{\lambda-\nu}\left(\sum_{j=1}^k W_{\varepsilon,Y_j}\right)^{p-2}. \end{aligned} \tag{3.15}$$

Setting

$$v = \gamma\varepsilon \sum_{j=1}^k \eta e^{-\nu|\tilde{x}-Y_j|/\varepsilon} + \gamma e^{-\sigma/\varepsilon}.$$

direct computations yield that for ε sufficiently small, there exists $C_4 > 0$ independent of ε and γ such that

$$\begin{aligned} & -\varepsilon^2 \Delta v + (V(x) - (p-1)a_\varepsilon(x))v \\ & \ge (V(x) - (p-1)a_\varepsilon(x) - \nu^2)\left(\gamma\varepsilon\sum_{j=1}^k \eta e^{-\nu|\tilde{x}-Y_j|/\varepsilon} + \gamma e^{-\sigma/\varepsilon}\right) \\ & \qquad + O\left(\varepsilon|\nabla\eta| + \varepsilon^2\sum_{l,n=0}^{N-m}\left|\frac{\partial^2\eta}{\partial z_l \partial z_n}\right|\right)\left(\gamma\varepsilon\sum_{j=1}^k e^{-\nu|\tilde{x}-Y_j|/\varepsilon} + \gamma e^{-\sigma/\varepsilon}\right) \\ & \ge \frac{(\lambda-\nu^2)\gamma}{2}\left(\varepsilon\sum_{j=1}^k \eta e^{-\frac{\nu|\tilde{x}-Y_j|}{\varepsilon}} + e^{-\sigma/\varepsilon}\right) \\ & \ge \frac{C_4\varepsilon(1+\gamma^{p^*+1}\varepsilon^{p^*})}{\lambda-\nu}\sum_{j=1}^k \eta e^{-\nu|\tilde{x}-Y_j|/\varepsilon} + C_4 e^{-c/\varepsilon} + C_4\gamma^{p^*+1}e^{-(p^*+1)\sigma/\varepsilon} \\ & \ge G_{\varepsilon,Y}(x) + \frac{C_3\varepsilon(1+\gamma^{p^*+1}\varepsilon^{p^*})}{\lambda-\nu}\left(\sum_{j=1}^k W_{\varepsilon,Y_j}\right)^{p-2}, \end{aligned}$$

provided $\sigma > 0$ is small, $0 < \nu < \lambda$, $0 < \nu^2 < \lambda$ and $(\lambda - \nu^2)\gamma/4 \ge C_4/\lambda - \nu)$.

Using the comparison principle, we obtain

$$w_1 \le v = \gamma\varepsilon \sum_{j=1}^k \eta e^{-\nu|\tilde{x}-Y_j|/\varepsilon} + \gamma e^{-\sigma/\varepsilon}.$$

Therefore, choosing $\gamma > \max\{2C_0, 4C_4/(\lambda-\nu)(\lambda-\nu^2)\}$ and $0 < \sigma \ll \nu \ll \lambda$, where C_0 is from (3.6), we see that $w_1 \in \mathcal{C}_\varepsilon$. Thus, $A_{\varepsilon,Y}$ is a contraction from $\mathcal{C}_\varepsilon$ into itself. As a consequence, there exists $w_{\varepsilon,Y} \in \mathcal{C}_\varepsilon$ satisfying $w_{\varepsilon,Y} = A_{\varepsilon,Y}(w_{\varepsilon,Y})$, that is, $w_{\varepsilon,Y}$ satisfies (2.11) for some scalars $A_{j,l}$, $j = 1, \dots, k$, $l = 0, \dots, N-m$. Moreover,

$$\|w_{\varepsilon,Y}\|_\varepsilon^2 \le C\varepsilon^{N-m+3}. \tag{3.16}$$

Finally we claim that $w_{\varepsilon,Y}$ is C^1-smooth with respect to Y. Using similar arguments as in [8], we can deduce that there exists a unique C^1-map $\overline{w}_{\varepsilon,Y} : D_\delta \to E_{\varepsilon,Y}$ which satisfies (2.11). As a consequence of the uniqueness, $w_{\varepsilon,Y} = \overline{w}_{\varepsilon,Y}$ and the claim follows. $\square$

4 Proofs of the main results

In this section, we prove the main theorems stated in section 1.

Lemma 4.1. *Let $w_{\varepsilon,Y}$ and $A_{j,l}$, $j = 1, \dots, k$, $l = 0, \dots, N - m$ be as in Proposition 3.3. Then for each $j = 1, \dots, k$ and $l = 0, \dots, N - m$,*

$$\frac{\partial J_\varepsilon}{\partial Y_{j,l}}(Y, w_{\varepsilon,Y}) = \left(\frac{1}{2} - \frac{1}{p}\right) B \frac{\partial \Gamma(Y_j)}{\partial Y_{j,l}} \varepsilon^{N-m+1} + O(\varepsilon^{N-m+1+\min\{1,p-2\}})$$

where $B = \displaystyle\int_{\mathbb{R}^{N-m+1}} U^p dz$.

Proof. For $w := w_{\varepsilon,Y} \in \mathcal{C}_\varepsilon$, $j = 1, \dots, k$, $l = 0, \dots, N - m$, we have:

$$\begin{aligned}
\frac{\partial J_\varepsilon}{\partial Y_{j,l}} &= \int_\Omega \left(\varepsilon^2 \nabla \left(\sum_{i=1}^k W_{\varepsilon,Y_i} + w \right) \nabla \frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}} + V(x) \left(\sum_{i=1}^k W_{\varepsilon,Y_i} + w \right) \frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}} \right) dx \\
&\quad - \int_\Omega K(x) \left(\sum_{i=1}^k W_{\varepsilon,Y_i} + w \right)^{p-1} \frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}} dx \\
&= \int_\Omega \left(\varepsilon^2 \nabla \left(\sum_{i=1}^k W_{\varepsilon,Y_i} \right) \nabla \frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}} + V(x) \left(\sum_{i=1}^k W_{\varepsilon,Y_i} \right) \frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}} \right) dx \\
&\quad - \int_\Omega K(x) \left(\sum_{i=1}^k W_{\varepsilon,Y_i} + w \right)^{p-1} \frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}} dx.
\end{aligned}$$

For $i \neq j$, $i, j = 1, \dots, k$, the exponential decay of W_{ε,Y_i} and $\frac{\partial W_{\varepsilon,Y_i}}{\partial Y_{i,l}}$ implies:

$$\begin{aligned}
&\int_\Omega \left(va^2 \nabla W_{\varepsilon,Y_i} \nabla \frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}} + V(x) W_{\varepsilon,Y_i} \frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}} \right) dx \\
&\qquad = \int_\Omega \left(\eta K_0(Y_i) \widetilde{U}^{p-1}_{\varepsilon,Y_i} \frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}} + f_{\varepsilon,j} \frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}} \right) dx \\
&\qquad\quad + \int_\Omega (V(x) - V_0(Y_i)) W_{\varepsilon,Y_i} \frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}} dx \\
&\qquad = O(\varepsilon^{N-m} e^{-c/\varepsilon})
\end{aligned} \tag{4.1}$$

for some $c > 0$. Hence

$$
\begin{aligned}
\frac{\partial J_\varepsilon}{\partial Y_{j,l}} &= \int_\Omega \left(\varepsilon^2 \nabla W_{\varepsilon,Y_j} \nabla \frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}} + V(x) W_{\varepsilon,Y_j} \frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}} \right) dx \\
&\qquad - \int_\Omega K(x) \left(\sum_{i=1}^k W_{\varepsilon,Y_i} + w \right)^{p-1} \frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}} dx + O(\varepsilon^{N-m} e^{-c/\varepsilon}) \\
&=: I_1 + I_2 + O(\varepsilon^{N-m} e^{-c/\varepsilon}).
\end{aligned} \tag{4.2}
$$

A direct calculation gives

$$
\begin{aligned}
2I_1 &= \frac{\partial}{\partial Y_{j,l}} \int_\Omega \left(\varepsilon^2 |\nabla W_{\varepsilon,Y_j}|^2 + V(x) |W_{\varepsilon,Y_j}|^2 \right) dx \\
&= \frac{\partial}{\partial Y_{j,l}} \left(\int_\Omega \eta K_0(Y_j) \widetilde{U}^{p-1}_{\varepsilon,Y_i} W_{\varepsilon,Y_j} dx + \int_\Omega f_{\varepsilon,j} W_{\varepsilon,Y_j} dx + \int_\Omega (V(x) - V_0(Y_j)) W^2_{\varepsilon,Y_j} dx \right) \\
&= \frac{\partial}{\partial Y_{j,l}} \int_\Omega \eta K_0(Y_j) \widetilde{U}^{p-1}_{\varepsilon,Y_i} W_{\varepsilon,Y_j} dx + \frac{\partial}{\partial Y_{j,l}} \int_\Omega f_{\varepsilon,j} W_{\varepsilon,Y_j} dx \\
&\qquad + \frac{\partial}{\partial Y_{j,l}} \int_\Omega (V(x) - V_0(Y_j)) W^2_{\varepsilon,Y_j} dx \\
&=: I_3 + I_4 + I_5.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
I_3 &= \frac{\partial}{\partial Y_{j,l}} \int_\Omega \eta^2 K_0(Y_j) \widetilde{U}^p_{\varepsilon,Y_j} dx \\
&= \frac{\partial}{\partial Y_{j,l}} \int_\Omega K_0(Y_j) \widetilde{U}^p_{\varepsilon,Y_i} dx + \frac{\partial}{\partial Y_{j,l}} \int_\Omega (\eta^2 - 1) K_0(Y_j) \widetilde{U}^p_{\varepsilon,Y_j} dx \\
&= \frac{\partial}{\partial Y_{j,l}} \int_{\mathbb{R}^N} K_0(Y_j) \widetilde{U}^p_{\varepsilon,Y_j} dx + O(\varepsilon^{N-m} e^{-c/\varepsilon}) \\
&= \frac{\partial}{\partial Y_{j,l}} \int_0^\infty \int_{\mathbb{R}^{N-m}} z_0^{m-1} K_0(Y_j) \alpha_i^p U^p \left(\frac{(z - Y_j)\beta_j}{\varepsilon} \right) dz + O(\varepsilon^{N-m} e^{-c/\varepsilon}) \\
&= \varepsilon^{N-m+1} \frac{\partial}{\partial Y_{j,l}} \int_{-\frac{\beta_j}{\varepsilon} Y_{j,0}}^{+\infty} \int_{\mathbb{R}^{N-m}} \left| Y_{j,0} + \frac{\varepsilon}{\beta_j} z_0 \right|^{m-1} V_0(Y_j)^\theta K_0(Y_j)^{-\frac{2}{p-2}} U^p(z) dz \\
&\qquad + O(\varepsilon^{N-m} e^{-c/\varepsilon}) \\
&= \varepsilon^{N-m+1} \Bigg\{ \frac{\partial}{\partial Y_{j,l}} \int_{-\frac{\beta_j}{\varepsilon} Y_{j,0}}^{+\infty} \int_{\mathbb{R}^{N-m}} Y_{j,0}^{m-1} V_0(Y_j)^\theta K_0(Y_j)^{-2/(p-2)} U^p(z) dz \\
&\qquad + \frac{\partial}{\partial Y_{j,l}} \int_{-\frac{\beta_j}{\varepsilon} Y_{j,0}}^{+\infty} \int_{\mathbb{R}^{N-m}} \left(\left| Y_{j,0} + \frac{\varepsilon}{\beta_i} z_1 \right|^{m-1} - Y_{j,0}^{m-1} \right) V_0(Y_j)^\theta K_0(Y_j)^{-\frac{2}{p-2}} U^p(z) \Bigg\} \\
&\qquad + O(\varepsilon^{N-m} e^{-c/\varepsilon}) \\
&= \varepsilon^{N-m+1} \frac{\partial}{\partial Y_{i,l}} \Gamma(Y_i) \int_{\mathbb{R}^{N-m+1}} U^p dz + O(\varepsilon^{N-m+2}) + O(\varepsilon^{N-m} e^{-c/\varepsilon}).
\end{aligned}
$$

Similarly we obtain

$$\begin{aligned}
I_4 &= \frac{\partial}{\partial Y_{j,l}} \int_{\mathbb{R}^N} f_{\varepsilon,j} W_{\varepsilon,Y_j} dx + O(\varepsilon^{N-m} e^{-c/\varepsilon}) \\
&= \frac{\partial}{\partial Y_{j,l}} \int_0^\infty \int_{\mathbb{R}^{N-m}} z_0^{m-1} f_{\varepsilon,j}(z) \alpha_j U\left(\frac{(z-Y_i)\beta_i}{\varepsilon}\right) dz + O(\varepsilon^{N-m} e^{-c/\varepsilon}) \\
&= \varepsilon^{N-m+1} \frac{\partial}{\partial Y_{j,l}} \int_{\mathbb{R}^{N-m+1}} \alpha_i \beta_i^{-N+m-1} \left| Y_{j,0} + \frac{\varepsilon}{\beta_j} z_0 \right|^{m-1} f_{\varepsilon,j}(Y_j + \frac{\varepsilon}{\beta_j} z) U(z) dz \\
&\qquad + O(\varepsilon^{N-m} e^{-c/\varepsilon}) \\
&= O(\varepsilon^{N-m+2}) + O(\varepsilon^{N-m} e^{-c/\varepsilon})
\end{aligned}$$

and,

$$\begin{aligned}
I_5 &= \frac{\partial}{\partial Y_{j,l}} \int_0^\infty \int_{\mathbb{R}^{N-m}} z_0^{m-1} (V(z) - V_0(Y_j)) \alpha_j^2 U^2 \left(\frac{(z-Y_i)\beta_i}{\varepsilon}\right) dz + O(\varepsilon^{N-m} e^{-c/\varepsilon}) \\
&= \varepsilon^{N-m+1} \frac{\partial}{\partial Y_{j,l}} \int_{\mathbb{R}^{N-m+1}} \alpha_j^2 \beta_j^{-N+m-1} \left| Y_{j,0} + \frac{\varepsilon}{\beta_i} z_1 \right|^{m-1} \left(V_0\left(Y_j + \frac{\varepsilon}{\beta_j} z\right) - V_0(Y_j)\right) U^2(z) dz \\
&\qquad + O(\varepsilon^{N-m} e^{-c/\varepsilon}) \\
&= O(\varepsilon^{N-m+2}) + O(\varepsilon^{N-m} e^{-c/\varepsilon}).
\end{aligned}$$

Therefore

$$2I_1 = \varepsilon^{N-m+1} \frac{\partial}{\partial Y_{j,l}} \Gamma(Y_j) \int_{\mathbb{R}^{N-m+1}} U^p dz + O(\varepsilon^{N-m+2}) + O(\varepsilon^{N-m} e^{-c/\varepsilon}). \tag{4.3}$$

Since $w \in \mathcal{C}_\varepsilon$ we deduce from the exponential decay of W_{ε,Y_j} and $\left|\frac{\partial}{\partial Y_{j,l}} W_{\varepsilon,Y_j}\right|$ that

$$\begin{aligned}
I_2 &= \int_\Omega K(x) W_{\varepsilon,Y_j}^{p-1} \frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}} dx + \int_\Omega K(x) \sum_{i \neq j}^k W_{\varepsilon,Y_i}^{p-1} \frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}} dx \\
&\quad + (p-1) \int_\Omega K(x) W_{\varepsilon,Y_j}^{p-2} w \frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}} dx + (p-1) \int_\Omega K(x) \sum_{i \neq j}^k W_{\varepsilon,Y_i}^{p-2} w \frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}} dx \\
&\quad + \int_\Omega |w|^{p^*+1} \left| \frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}} \right| dx \\
&= \int_\Omega K(x) W_{\varepsilon,Y_j}^{p-1} \frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}} dx + (p-1) \int_\Omega K(x) W_{\varepsilon,Y_j}^{p-2} w \frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}} dx \\
&\quad + \int_\Omega |w|^{p^*+1} \left| \frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}} \right| dx + O(\varepsilon^{N-m} e^{-c/\varepsilon}).
\end{aligned}$$

On the other hand, similar computations as for I_1 lead to

$$\begin{aligned}\int_\Omega K(x)W_{\varepsilon,Y_j}^{p-1}\frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}}dx &= \frac{1}{p}\frac{\partial}{\partial Y_{j,l}}\left(\int_\Omega K_0(Y_j)W_{\varepsilon,Y_j}^{p}dx - \int_\Omega (K(x)-K_0(Y_j))W_{\varepsilon,Y_j}^{p}dx\right)\\ &= \varepsilon^{N-m+1}\frac{1}{p}\frac{\partial}{\partial Y_{j,l}}\Gamma(Y_j)\int_{\mathbb{R}^{N-m+1}}U^p dz + O(\varepsilon^{N-m+2})\\ &\quad + O(\varepsilon^{N-m}e^{-c/\varepsilon}),\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega K(x)W_{\varepsilon,Y_j}^{p-2}w\frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}}dx\\ &= \int_\Omega K_0(Y_j)W_{\varepsilon,Y_j}^{p-2}w\frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}}dx + \int_\Omega (K(x)-K_0(Y_j))W_{\varepsilon,Y_j}^{p-2}w\frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}}dx\\ &= \int_\Omega K_0(Y_j)\eta\widetilde{U}_{\varepsilon,Y_j}^{p-2}w\frac{\partial \widetilde{U}_{\varepsilon,Y_j}}{\partial Y_{j,l}}dx + O\left(|w|_{L^\infty(\Omega)}\int_\Omega |K(x)-K_0(Y_j)|W_{\varepsilon,Y_j}^{p-2}\left|\frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}}\right|dx\right)\\ &\quad + O(\varepsilon^{N-m+1}e^{-c/\varepsilon})\\ &= \frac{1}{p-1}\left(\frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}},\, w\right)_\varepsilon\\ &\quad + \frac{1}{p-1}\int_\Omega\left(\frac{\partial V_0(Y_j)}{\partial Y_{j,l}}W_{\varepsilon,Y_j}w - \eta\frac{\partial K_0(Y_j)}{\partial Y_{j,l}}\widetilde{U}_{\varepsilon,Y_j}^{p-1}w - \frac{\partial f_{\varepsilon,j}}{Y_{j,l}}w\right)dx\\ &+ \int_\Omega (V_0(Y_j)-V(x))w\frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}}dx + O(\varepsilon^{2+N-m}) + O(\varepsilon^{N-m+1}e^{-c/\varepsilon})\\ &= O(\varepsilon^{2+N-m}) + O(\varepsilon^{N-m+1}e^{-c/\varepsilon}),\end{aligned}$$

and

$$\begin{aligned}\int_\Omega |w|^{p^*+1}\left|\frac{\partial W_{\varepsilon,Y_i}}{\partial Y_{i,l}}\right|dx &= O(|w|_{L^\infty(\Omega)}^{p^*+1})\left(\int_\Omega\left|\frac{\partial W_{\varepsilon,Y_j}}{\partial Y_{j,l}}\right|dx\right)\\ &\le C\varepsilon^{p^*+1}\left(\varepsilon^{N-m}+\varepsilon^{N-m}e^{-c/\varepsilon}\right).\end{aligned}$$

It follows that

$$I_2 = \varepsilon^{N-m+1}\frac{1}{p}\frac{\partial}{\partial Y_{j,l}}\Gamma(Y_j)\int_{\mathbb{R}^{N-m+1}}U^p dz + O(\varepsilon^{N-m+2}) + O(\varepsilon^{N-m+1+p^*}). \tag{4.4}$$

Now plugging the estimates for I_1 and I_2 into (4.2), we obtain for ε sufficiently small:

$$\frac{\partial J_\varepsilon}{\partial Y_{j,l}}(Y, w_{\varepsilon,Y}) = \left(\frac{1}{2}-\frac{1}{p}\right)B\frac{\partial\Gamma(Y_j)}{\partial Y_{j,l}}\varepsilon^{N-m+1} + O(\varepsilon^{N-m+1+p^*}), \tag{4.5}$$

as required. □

Lemma 4.2. *For $j = 0, \dots, k$ there holds:*

$$\sum_{n=0}^{N-m} A_{j,n} \left(\frac{\partial^2 W_{\varepsilon,Y_j}}{\partial Y_{j,l} \partial Y_{j,n}}, w \right) = O(\varepsilon^{2+N-m\}}).$$

Proof. By direct computation, we have

$$\left\| \frac{\partial^2 \widetilde{U}_{\varepsilon,Y_j}}{\partial Y_{j,l} \partial Y_{j,n}} \right\|_\varepsilon^2 = O(\varepsilon^{N-m-3}).$$

Hence, (3.8) and (3.16) yield

$$\begin{aligned} \left| A_{j,n} \left(\frac{\partial^2 W_{\varepsilon,Y_j}}{\partial Y_{j,l} \partial Y_{j,n}}, w \right) \right| &\le |A_{j,n}| \left\| \frac{\partial^2 \widetilde{U}_{\varepsilon,Y_j}}{\partial Y_{j,l} \partial Y_{j,n}} \right\|_\varepsilon \|w\|_\varepsilon + O(\varepsilon^{N-m+1} e^{-c/\varepsilon}) \\ &= O(\varepsilon^{2+N-m}) \end{aligned}$$

and the lemma follows. □

Proof of Theorem 1.1. We have to show that for $\varepsilon > 0$ small, there exists $Z_\varepsilon \in D_\delta$ solving (2.10), that is

$$\frac{\partial J_\varepsilon}{\partial Y_{j,l}}(Y, w_{\varepsilon,Y}) = \sum_{n=0}^{N-m} A_{j,n} \left(\frac{\partial^2 W_{\varepsilon,Y_j}}{\partial Y_{j,n} \partial Y_{j,l}}, w \right)_\varepsilon \tag{4.6}$$

holds at $Y = Z_\varepsilon$ for $j = 1, \dots, k$, $l = 0, \dots, N-m$. By Lemma 4.1 and Lemma 4.2, equation (4.6) is equivalent to

$$\begin{aligned} &\left(\frac{1}{2} - \frac{1}{p} \right) B \frac{\partial \Gamma}{\partial Y_{j,l}}(Y_j) + O\left(\varepsilon^{p^*}\right) \\ &\qquad = \varepsilon^{-1-N+m} \sum_{n=0}^{N-m} A_{j,n} \left(\frac{\partial^2 W_{\varepsilon,Y_j}}{\partial Y_{j,n} \partial Y_{j,l}}, w \right)_\varepsilon = O(\varepsilon) \end{aligned} \tag{4.7}$$

for $j = 1, \dots, k$, $l = 0, \dots, N-m$. We use a degree argument to prove the existence of a solution. Equation (4.7) has the form $\Phi(Y) = \Psi_\varepsilon(Y)$ where $\Phi, \Psi : D_\delta \subset \mathbb{R}^{k(N-m+1)} \to \mathbb{R}^{k(N-m+1)}$ are continuous, $D\Phi(Z)$ exists and is an isomorphism by assumption $(VK)_3$, and $|\Psi_\varepsilon|_\infty \le C\varepsilon^{p^*}$. It follows that there exist constants $C_0, C_\rho > 0$ with $C_\rho \to 0$ as $\rho \to 0$ such that $|\Phi(Y)| > (C_0 - C_\rho)\rho$ for $|Y| = \rho$, hence $|\Phi(Y)| > |\Psi_\varepsilon(Y)|$ for $|Y| = \rho = C\varepsilon^{p^*}/(C_0 - C_\rho)$ and ε small. Now the Brouwer degree yields a solution Z_ε of (4.7) with $|Z_\varepsilon| < C\varepsilon^{p^*}/(C_0 - C_\rho)$.

Now $u_\varepsilon = \sum_{j=1}^{k} W_{\varepsilon,Z_{\varepsilon,j}} + w_{\varepsilon,Z_\varepsilon}$ is a critical point of I_ε by Lemma 2.1. Since $g(x,t) = 0$ for $t \le 0$, we see that u_ε is non-negative. Hence the maximum principle yields that

u_ε satisfies (2.8). Moreover, $w_{\varepsilon,Z_\varepsilon} \in \mathcal{C}_\varepsilon$, and thus $|w_{\varepsilon,Z_\varepsilon}| \le \frac{1}{2}$ for ε small enough. This implies that for ε sufficiently small

$$g\left(x, \sum_{j=1}^{k} W_{\varepsilon,Z_{\varepsilon,j}} + w_{\varepsilon,Z_\varepsilon}\right) = \left(W_{\varepsilon,Z_{\varepsilon,j}} + w_{\varepsilon,Z_\varepsilon}\right)_+^{p-1}.$$

Consequently, $u_\varepsilon = \sum_{j=1}^{k} W_{\varepsilon,Z_{\varepsilon,j}} + w_{\varepsilon,Z_\varepsilon}$ is a solution of the original problem (1.1). Recalling the definition of W_{ε,Y_j}, the proof of Theorem 1.1 can be finished easily. □

Proof of Theorem 1.2. The proof proceeds as the one of Theorem 1.1 except that in the application of the Brouwer degree we only obtain a solution Z_ε with $|Z_{\varepsilon,j} - Z_j| \to 0$ as $\varepsilon \to 0$. □

We leave the details f the proof of Theorem 1.3 to the interested reader.

5 Appendix

Proof of Proposition 3.1. The proof can be carried out in a similar way to [4, Proposition 2.1]. However, here we give a different proof which essentially goes back to [14].

Suppose to the contrary that Proposition 3.1 does not hold. Then there exist $\varepsilon_n \to 0$, $Y^{(n)} = (Y_1^{(n)}, \ldots, Y_k^{(n)}) \in D_\delta$ and $w_n \in E_{\varepsilon_n, Y^{(n)}}$, such that

$$\|L_{\varepsilon_n,Y^{(n)}} w_n\|_{\varepsilon_n} = o_n(1)\|w_n\|_{\varepsilon_n}. \tag{5.1}$$

So we have for $\varphi \in E_{\varepsilon_n,Y^{(n)}}$:

$$\int_\Omega (\varepsilon^2 \nabla w_n \nabla \varphi + V(x) w_n \varphi) - (p-1)\int_\Omega K(x)\left(\sum_{j=1}^{k} W_{\varepsilon_n, Y_j^{(n)}}\right)^{p-2} w_n \varphi \tag{5.2}$$
$$= o_n(1)\|w_n\|_{\varepsilon_n}\|\varphi\|_{\varepsilon_n}.$$

We may assume without loss of generality that

$$\|w_n\|_{\varepsilon_n} = \varepsilon_n^{(N-m+1)/2}. \tag{5.3}$$

For each fixed $j = 1, \ldots, k$ let

$$\tilde{w}_{n,j}(\tilde{x}) := w_n\left(\frac{\varepsilon_n}{\beta_{n,j}}\tilde{x} + Y_j^{(n)}\right),$$

where $\beta_{n,j} = \left(V_0(Y_j^{(n)})\right)^{1/2}$. Since $Y_{j,0}^{(n)} > c > 0$, by (5.3),

$$\int_{B_R(0)} |\nabla \tilde{w}_{n,j}|^2 + \frac{V_0\left(\frac{\varepsilon_n}{\beta_{n,j}}\tilde{x} + Y_j^{(n)}\right)}{\beta_{n,j}^2}|\tilde{w}_{n,j}|^2 \le C, \tag{5.4}$$

for any $R>0$ large; here $C>0$ is independent of R and $B_R(0)$ is the ball in $\mathbb{R}^{N-m+1}$ with radius R and centered at the origin. After passing to a subsequence we have for any $R>0$,

$$\tilde{w}_{n,j} \to \tilde{w}_j \quad \text{as } n\to\infty \text{ weakly in } H^1(B_R(0)), \tag{5.5}$$

and

$$\tilde{w}_{n,j} \to \tilde{w}_j \quad \text{as } n\to\infty \text{ strongly in } L^2(B_R(0)). \tag{5.6}$$

We claim that $\tilde{w}_j \equiv 0$.

In fact, as a consequence of (5.2), $\tilde{w}_{n,j}$ satisfies for $\varphi \in \tilde{E}_n$

$$\begin{aligned}\int_{D_n} \left|\frac{\varepsilon_n}{\beta_{n,j}} z_0 + Y_{j,0}^{(n)}\right|^{m-1} &\left(\nabla \tilde{w}_{n,j} \nabla \varphi + \frac{V_0\left(\frac{\varepsilon_n}{\beta_{n,j}} z + Y_j^{(n)}\right)}{\beta_{n,j}^2} \tilde{w}_{n,j}\varphi\right) dz \\ &- (p-1)\int_{\Omega_n} \left|\frac{\varepsilon_n}{\beta_{n,j}} z_0 + Y_{j,0}^{(n)}\right|^{m-1} \frac{K_0\left(\frac{\varepsilon_n}{\beta_{n,j}} z + Y_j^{(n)}\right)}{\beta_{n,j}^2} \left(\sum_{i=1}^k H_{n,i}\right)^{p-2} \tilde{w}_{n,j}\varphi dz \\ = o_n(1)\|\varphi\|_{\varepsilon_n},\end{aligned} \tag{5.7}$$

where $\Omega_{n,j} = \{z \in \mathbb{R}^{N-m+1} : \frac{\varepsilon_n}{\beta_{n,j}} z + Y_j^{(n)} \in \Omega_0\}$, $H_{n,j}(z) = \left(W_{\varepsilon_n, Y_j^{(n)}}\right)\left(\frac{\varepsilon_n}{\beta_{n,j}} z + Y_j^{(n)}\right)$, and

$$\begin{aligned}\tilde{E}_{n,j} = \Bigg\{\varphi : \varphi\left(\frac{\beta_{n,j}(\tilde{x} - Y_j^{(n)})}{\varepsilon_n}\right) \in H_s,\ \int_{\Omega_n} \left|\frac{\varepsilon_n}{\beta_{n,j}} z_0 + Y_{j,0}^{(n)}\right|^{m-1} &\left(\nabla \frac{\partial H_{n,i}}{\partial Y_{i,l}^{(n)}} \nabla\varphi\right. \\ + \frac{V_0\left(\frac{\varepsilon_n}{\beta_{n,j}} z + Y_j^{(n)}\right)}{\beta_{n,j}^2} \frac{\partial H_{n,i}}{\partial Y_{i,l}^{(n)}}\varphi\Bigg) dz = 0,\ i = 1,\dots,k,\ l = 1,\dots,N-m+1\Bigg\}.\end{aligned}$$

Now decompose $\varphi \in C_0^\infty(\mathbb{R}^{N-m+1})$ as follows:

$$\varphi = \varphi_{n,j} + \sum_{i=1}^k \sum_{l=0}^{N-m} a_{n,i,l} \frac{\partial H_{n,i}}{\partial Y_{i,l}^{(n)}},$$

where $\varphi_{n,j} \in \tilde{E}_{n,j}$, $a_{n,i,l} \in \mathbb{R}$ for $i = 1,\dots,k$, $l = 0,\dots,N-m$. Then due to the exponential decay of $\partial H_{n,i}/\partial Y_{i,l}^{(n)}$, we obtain for $i = 1,\dots,k$, $h = 1,\dots,k$, $i \neq j$ and $i \neq h$:

$$\int_{\Omega_n} \left|\frac{\varepsilon_n}{\beta_{n,j}} z_0 + Y_{j,0}^{(n)}\right|^{m-1} \left(\nabla \frac{\partial H_{n,i}}{\partial Y_{i,l}^{(n)}} \nabla\varphi + \frac{V_0\left(\frac{\varepsilon_n}{\beta_{n,j}} z + Y_j^{n}\right)}{\beta_{n,j}^2} \frac{\partial H_{n,i}}{\partial Y_{i,l}^{(n)}}\varphi\right) dz = o_n(1),$$

and

$$\int_{\Omega_n}\left|\frac{\varepsilon_n}{\beta_{n,j}}z_0+Y_{j,0}^{(n)}\right|^{m-1}\left(\nabla\frac{\partial H_{n,i}}{\partial Y_{i,l}^{(n)}}\nabla\frac{\partial H_{n,h}}{\partial Y_{h,l}^{(n)}}+\frac{V_0\left(\frac{\varepsilon_n}{\beta_{n,j}}z+Y_j^{(n)}\right)}{\beta_{n,j}^2}\frac{\partial H_{n,i}}{\partial Y_{i,l}^{(n)}}\frac{\partial H_{n,h}}{\partial Y_{h,l}^{(n)}}\right)dz=o_n(1).$$

On the other hand,

$$\int_{\Omega_n}\left|\frac{\varepsilon_n}{\beta_{n,j}}z_0+Y_{j,0}^{(n)}\right|^{m-1}\left(\left|\nabla\frac{\partial H_{n,i}}{\partial Y_{i,l}^{(n)}}\right|^2+\frac{V_0\left(\frac{\varepsilon_n}{\beta_{n,j}}z+Y_j^{(n)}\right)}{\beta_{n,j}^2}\left|\frac{\partial H_{n,i}}{\partial Y_{i,l}^{(n)}}\right|^2\right)dz\geq C>0.$$

It follows that $a_{n,i,l}\to 0$ as $n\to\infty$ for $i\neq j$, while $a_{n,j,l}\to a_{j,l}$ up to a subsequence.

It is easy to verify that for $l=0,\dots,N-m$,

$$\frac{\partial W_{\varepsilon_n,Y_j^{(n)}}}{\partial Y_{j,l}^{(n)}}=-\frac{\partial W_{\varepsilon_n,Y_j^{(n)}}}{\partial z_l}+W_{\varepsilon_n,Y_j^{(n)}}\frac{\partial\alpha_{n,j}}{\partial Y_j^{(n)}}+\frac{\tilde{x}-Y_j^{(n)}}{\varepsilon_n}W'_{\varepsilon_n,Y_j^{(n)}}\frac{\partial\beta_{n,j}}{\partial Y_j^{(n)}}. \tag{5.8}$$

Hence, plugging $\varphi_{n,j}$ into (5.7) and letting $n\to\infty$, we deduce from the fact $Y_{j,0}^{(n)}>c>0$ that

$$\int_{\mathbb{R}^{N-m+1}}(\nabla\tilde{w}_j\nabla\varphi+\tilde{w}_j\varphi)dz-(p-1)\int_{\mathbb{R}^{N-m+1}}U^{p-2}\tilde{w}_j\varphi\,dz$$
$$+\sum_{l=0}^{N-m}a_{j,l}\left(\int_{\mathbb{R}^{N-m+1}}\left(\nabla\tilde{w}_j\nabla\frac{\partial U}{\partial z_l}+\tilde{w}_j\frac{\partial U}{\partial z_l}\right)dz-(p-1)\int_{\mathbb{R}^{N-m+1}}U^{p-2}\tilde{w}_j\frac{\partial U}{\partial z_l}dz\right)=0.$$

From the fact that U solves (1.4), we deduce for $l=0,\dots,N-m$ that

$$\int_{\mathbb{R}^{N-m+1}}\left(\nabla\tilde{w}_j\nabla\frac{\partial U}{\partial z_l}+\tilde{w}_j\frac{\partial U}{\partial z_l}\right)dz-(p-1)\int_{\mathbb{R}^{N-m+1}}U^{p-2}\tilde{w}_j\frac{\partial U}{\partial z_l}dz=0.$$

Therefore

$$\int_{\mathbb{R}^{N-m+1}}(\nabla\tilde{w}_j\nabla\varphi+\tilde{w}_j\varphi)dz-(p-1)\int_{\mathbb{R}^{N-m+1}}U^{p-2}\tilde{w}_j\varphi\,dz=0. \tag{5.9}$$

Since $\varphi\in C_0^\infty(\mathbb{R}^{N-m+1})$ is arbitrary in (5.9), the non-degeneracy of U yields that

$$\tilde{w}_j\in\operatorname{span}\{\partial U/\partial z_l: l=0,\dots,N-m\} \tag{5.10}$$

But (5.8) and $w_n\in E_{\varepsilon_n,k}$ imply

$$\int_{\mathbb{R}^{N-m+1}}\left(\nabla\tilde{w}_j\nabla\frac{\partial U}{\partial z_l}+\tilde{w}_j\frac{\partial U}{\partial z_l}\right)dz=0$$

for $l=0,\dots,N-m$. Therefore, $\tilde{w}_j\equiv 0$, which is exactly our claim.

Now, for $j = 1, \ldots, k$ let $B_{n,j,R} = \{x \in \Omega : \tilde{x} \in B_{\frac{\varepsilon}{\beta_{n,j}}R}(Y_j)\}$. Then, using (5.6) we deduce

$$
\begin{aligned}
\int_\Omega K(x)\left(\sum_{j=1}^k W_{\varepsilon_n,Y_j^{(n)}}\right)^{p-2} w_n^2 dx &= \int_{\bigcup_{j=1}^k B_{n,j,R}} K(x)\left(\sum_{j=1}^k W_{\varepsilon_n,Y_j^{(n)}}\right)^{p-2} w_n^2 dx \\
&\quad + \int_{\Omega\setminus\bigcup_{j=1}^k B_{n,j,R}} K(x)\left(\sum_{j=1}^k W_{\varepsilon_n,Y_j^{(n)}}\right)^{p-2} w_n^2 dx \\
&\le C\int_{\bigcup_{j=1}^k B_{n,j,R}} w_n^2 dx + o_R(1)\|w_n\|_{\varepsilon_n}^2 \\
&= o(\varepsilon_n^{N-m+1}) + o_R(1)\varepsilon_n^{N-m+1},
\end{aligned}
$$

where $o_R(1) \to 0$ as $R \to \infty$. Hence from (5.2), we have

$$
o(\varepsilon_n^{N-m+1}) = \|w_n\|_{\varepsilon_n}^2 + o(\varepsilon_n^{N-m+1}) + o_R(1)\varepsilon_n^{N-m+1},
$$

which is impossible. This completes the proof. $\square$

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