

# Liouville-type theorems and asymptotic behavior of nodal radial solutions of semilinear heat equations

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**Abstract.** We prove a Liouville type theorem for sign-changing radial solutions of a subcritical semilinear heat equation  $u_t = \Delta u + |u|^{p-1}u$ . We use this theorem to derive a priori bounds, decay estimates, and initial and final blow-up rates for radial solutions of rather general semilinear parabolic equations whose nonlinearities have a subcritical polynomial growth. Further consequences on the existence of steady states and time-periodic solutions are also shown.

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# 1 Introduction

In this paper we study the asymptotic behavior of classical solutions of problems of the form

$$\left. \begin{aligned} u_t - \Delta u &= f(|x|, t, u, \nabla u), & x \in \Omega, \quad t > 0, \\ u &= 0, & x \in \partial\Omega, \quad t > 0, \\ u(x, 0) &= u_0(x), & x \in \Omega, \end{aligned} \right\} \quad (1.1)$$

where  $\Omega$  is a radial domain in  $\mathbb{R}^N$  (that is a ball, an annulus, an exterior of a ball or the whole of  $\mathbb{R}^N$ ),  $u_0$  is radially symmetric and  $f$  behaves like the power nonlinearity  $|u|^{p-1}u$ ,  $p > 1$ , for large values of  $u$ . (If  $N = 1$  then the radial symmetry of  $\Omega$  and  $u_0$  is not needed.) Solutions of (1.1) are radially symmetric in the  $x$ -variable and we will often consider such functions as functions of the radial variable  $r = |x|$  and  $t$ . Hence, without fearing confusion we use both the notation  $u(x, t)$  and  $u(r, t)$ . The key ingredient of our study is a Liouville-type theorem for radial solutions of the corresponding limiting problem

$$u_t - \Delta u = |u|^{p-1}u, \quad x \in \mathbb{R}^N, \quad t \in \mathbb{R}, \quad (1.2)$$

which enables us to derive universal estimates on solutions of (1.1).

We start with a short description of analogous results for elliptic problems which play an important role also in the parabolic case. It is well known (see [19, 9] or [35] and the references therein) that the problem

$$-\Delta u = |u|^{p-1}u, \quad x \in \mathbb{R}^N, \quad (1.3)$$

possesses positive classical solutions if and only if  $p \geq p_S$ , where  $p_S$  is the critical Sobolev exponent:

$$p_S := \begin{cases} \frac{N+2}{N-2}, & \text{if } N \geq 3, \\ \infty, & \text{if } N \in \{1, 2\}. \end{cases}$$

The theorem on nonexistence of positive solutions of (1.3) in the subcritical case, often referred to as a Liouville theorem for (1.3), is very useful in the study of nonnegative solutions of problems of the form

$$-\Delta u = f(x, u, \nabla u), \quad x \in \Omega, \quad (1.4)$$

where  $\Omega$  is an arbitrary domain in  $\mathbb{R}^N$  and  $f$  behaves like the power  $|u|^{p-1}u$  for  $u$  large. In combination with scaling arguments, the Liouville theorem is very effective for derivation of a priori estimates on positive solutions (see [30] for recent theorems on universal a priori estimates and a discussion of earlier results).

It is well known that the Liouville theorem for (1.3) is not valid for nodal solutions, that is, solutions which may change sign. However, one can hope that it does hold (and, as a consequence, a priori estimates for (1.4) can be established), if the class of solutions considered is restricted by an additional structure. An example is the class of solutions with finite Morse index. The nonexistence of nodal solutions of (1.3) with finite Morse index (or, more generally, solutions which are stable outside a compact set) for all subcritical and even some supercritical values of  $p$  was proved in [4, 17].

Another interesting and natural class is that of radial solutions with a finite number of zeros. As we explain below, in the context of parabolic equations this class is more relevant, thus we discuss it in more detail. Given an open interval  $I \subset \mathbb{R}$  and  $v \in C(I)$ , we define

$$z_I(v) := \sup\{j : \exists x_1, \dots, x_{j+1} \in I, x_1 < x_2 < \dots < x_{j+1}, \\ v(x_i) \cdot v(x_{i+1}) < 0 \text{ for } i = 1, 2, \dots, j\},$$

where  $\sup(\emptyset) := 0$ . We usually refer to  $z_I(v)$  as the *zero number* of  $v$  in  $I$ . Note that  $z_I(v)$  is actually the number of sign changes of  $v$ ; it coincides with the number of zeros of  $v$  if  $v \in C^1(I)$  and all its zeros are simple.

The following result is an elliptic Liouville-type theorem for radial solutions with finite zero number. It is a direct consequence of [28, Theorem 2.5].

**Theorem 1.1.** *Let  $1 < p < p_S$  and let  $u = u(r)$  be a classical radial solution of (1.3) with  $z_{(0,\infty)}(u) < \infty$ . Then  $u \equiv 0$ .*

*The same result remains true for non-radial solutions  $u = u(x)$  if  $N = 1$  and  $z_{\mathbb{R}}(u) < \infty$ .*

We remark that the result in Theorem 1.1 is not true without the assumption  $z_{(0,\infty)}(u) < \infty$  (or  $z_{\mathbb{R}}(u) < \infty$ ).

Finally, we mention a yet another class of solutions admissible for the Liouville theorem, namely solutions (not necessarily radially symmetric) lying in the energy space

$$\left. \begin{aligned} \mathcal{E} &:= \{v \in L^{p+1}(\mathbb{R}^N) : \nabla v \in L^2(\mathbb{R}^N)\}, \\ \|v\|_{\mathcal{E}} &:= \|v\|_{L^{p+1}(\mathbb{R}^N)} + \|\nabla v\|_{L^2(\mathbb{R}^N)}. \end{aligned} \right\} \quad (1.5)$$

In fact, if  $u \in \mathcal{E}$  is a solution of (1.3) then the Pohozaev identity and the equality  $\int_{\mathbb{R}^N} |\nabla u|^2 dx = \int_{\mathbb{R}^N} |u|^{p+1} dx$  (which can be proved by using the same cut-off function as in the proof of the Pohozaev identity [37, Theorem B.3]) guarantee  $u \equiv 0$ .

Let us now turn to parabolic problems, first considering positive solutions. A natural extension of the elliptic Liouville theorem for positive solutions would state that for any subcritical  $p$  there are no positive classical solutions of (1.2) (note that in (1.2) we are dealing with *entire solutions*, that is, solutions defined for all times  $t \in \mathbb{R}$ ). So far such an extension has been proved only for exponents  $p \in (1, p_B)$ , where  $p_B := N(N+2)/(N-1)^2$ , see [6] or [35, Theorem 21.2]. In particular, if  $N = 1$  then the following theorem is true.

**Theorem 1.2.** *Let  $N = 1$  and  $p > 1$ . Then equation (1.2) does not possess positive classical solutions.*

The restriction  $p < p_B$  in the result mentioned above is hardly optimal. On the other hand, if we restrict ourselves to the radial solutions then the following result due to [29, 31] is valid for the optimal range of exponents.

**Theorem 1.3.** *Let  $1 < p < p_S$ . Then the equation (1.2) does not possess positive classical radial solutions.*

Similarly as in the elliptic case, Theorems 1.2 and 1.3 can be used for proving optimal estimates of positive radial solutions of problems of the form (1.1), see [31]. In particular, one can deduce initial and final blow up rates of local solutions, as well as decay rates for global solutions, all with universal constants.

The previous discussion raises a natural question whether as in the elliptic case a Liouville-type theorem and a priori estimates are valid for a suitable class of nodal solutions. Unlike in elliptic equations, the class of solutions with finite Morse index does not seem to be appropriate. While one can make sense of a Morse index along a solution  $u(\cdot, t)$ , defining it for each fixed  $t$  using the “elliptic part” of the equation, a discrete quantity defined this way is not a Lyapunov functional for the parabolic semiflow (cf. [18]), hence it can increase along a solution of (1.1) and possibly be unbounded. For this reason, extensions of the elliptic Liouville theorem for solutions with finite Morse index to parabolic equations are probably not very meaningful.

On the other hand, it is well known (see [3, 12]) that the zero number is a discrete-valued Lyapunov functional for radial solutions of many problems

of the form (1.1). Therefore, we focus our attention on radial nodal solutions with finite zero number. Our main aim is to extend Theorems 1.2, 1.3 and their applications in [31] to such nodal solutions. It is also well known that for many parabolic problems, the usual energy functional serves as a real-valued Lyapunov functional, at least for solutions in a suitable energy space. Hence, the class of solutions with finite energy can also be considered and we prove several results for such solutions, as well. In fact, we use energy estimates at several places to complement zero number arguments.

Our first result is the following Liouville-type theorem.

**Theorem 1.4.** *Let  $1 < p < p_S$  and let  $u = u(r, t)$  be a classical radial solution of (1.2). Assume that there exists  $Z \in \mathbb{N}$  such that*

$$z_{(0,\infty)}(u(\cdot, t)) \leq Z \quad (t \in \mathbb{R}). \quad (1.6)$$

*Then  $u \equiv 0$ .*

Similarly as in the elliptic case, the finiteness of  $z_{(0,\infty)}(u(\cdot, t))$  for some  $t \in \mathbb{R}$  is necessary for the nonexistence result in Theorem 1.4, in general. On the other hand, if we restrict ourselves to bounded radial solutions with suitable spatial decay then the assumption (1.6) is not needed, see Corollary 2.8.

In dimension one, we can treat nonsymmetric solutions as well. The following two theorems are results of independent interest, but also they will be needed, together with Theorem 1.4, for the derivation of a priori estimates on radial solutions of (1.1).

**Theorem 1.5.** *Let  $N = 1$ ,  $p > 1$  and let  $u = u(x, t)$  be a classical solution of (1.2). Assume that there exists  $Z \in \mathbb{N}$  such that*

$$z_{\mathbb{R}}(u(\cdot, t)) \leq Z \quad (t \in \mathbb{R}). \quad (1.7)$$

*Then  $u \equiv 0$ .*

**Theorem 1.6.** *Let  $p > 1$  and let  $u = u(x, t)$  be a classical solution of the problem*

$$\begin{aligned} u_t - u_{xx} &= |u|^{p-1}u, & x \in (0, \infty), \quad t \in \mathbb{R}, \\ u(0, t) &= 0, & t \in \mathbb{R}. \end{aligned} \quad (1.8)$$

*Assume that there exists  $Z \in \mathbb{N}$  such that*

$$z_{(0,\infty)}(u(\cdot, t)) \leq Z \quad (t \in \mathbb{R}). \quad (1.9)$$

*Then  $u \equiv 0$ .*

The proof of Theorem 1.5 for bounded solutions is based on zero number arguments only, see Proposition 2.1. On the other hand, in the proof of Theorem 1.4 for  $N > 1$  we employ zero number arguments as well as energy arguments. First, the bound on the zero number of the solution  $u(\cdot, t)$  is used in order to get a uniform bound on  $u(\cdot, t)$  in the energy space  $\mathcal{E}$ . This bound is very useful since the Cauchy problem

$$\left. \begin{aligned} u_t - \Delta u &= |u|^{p-1}u, & x \in \mathbb{R}^N, t > 0, \\ u(x, 0) &= u_0(x), & x \in \mathbb{R}^N, \end{aligned} \right\} \quad (1.10)$$

is well posed in  $\mathcal{E}$  (see the case  $\lambda = 0$  in [35, Example 51.28]) and the corresponding solution satisfies the energy identity

$$E(u(\cdot, t_2)) - E(u(\cdot, t_1)) = - \int_{t_1}^{t_2} \int_{\mathbb{R}^N} u_t^2(x, t) dx dt, \quad (1.11)$$

where

$$E(v) := \int_{\mathbb{R}^N} \left( \frac{1}{2} |\nabla v|^2 - \frac{1}{p+1} |v|^{p+1} \right) dx$$

is the energy functional. Using the energy estimates, we reduce the proof to the problem of nonexistence of nontrivial equilibria which is guaranteed by Theorem 1.1.

As a by-product of various energy estimates derived in this paper, we obtain two alternative proofs of Theorem 1.3. These proofs are completely different from the proof of the main result in [29], hence they might be of interest to some readers. The first one (see Remark 2.6) makes use of the nonexistence result in [6] in the one-dimensional case. The second alternative proof is essentially self-contained but requires the additional assumption  $p < 3$  (see Section 3).

Let us now discuss some consequences of our nonexistence results. Using Theorems 1.4, 1.5 and 1.6 we obtain the following universal estimates.

**Theorem 1.7.** *Let  $0 \leq R_1 < R_2 \leq \infty$  and  $I := (R_1, R_2)$  if  $R_1 > 0$ ,  $I := [0, R_2)$  if  $R_1 = 0$ . Consider the problem*

$$\left. \begin{aligned} u_t - \Delta u &= f(|x|, t, u, \nabla u), & x \in \Omega, t \in (0, T) \\ u &= 0, & x \in \partial\Omega, t \in (0, T) \end{aligned} \right\} \quad (1.12)$$

with  $\Omega := \{x \in \mathbb{R}^N : |x| \in I\}$  and  $T \in (0, \infty]$ . Let  $Z \in \mathbb{N}$ ,  $p \in (1, p_S)$ ,  $q \in (0, 2p/(p+1))$  and let  $f$  be a Carathéodory function satisfying

$$|f(r, t, s, \xi)| \leq C_1(1 + |s|^p + |\xi|^q) \quad (r \in I, t \in (0, T), s \in \mathbb{R}, \xi \in \mathbb{R}^N),$$

and, for all  $(r, t) \in [R_1, R_2] \times [0, T]$ ,

$$\lim_{|s| \rightarrow \infty, I \times (0, T) \ni (\rho, \tau) \rightarrow (r, t)} \frac{f(\rho, \tau, s, |s|^{(p+1)/2} \xi)}{|s|^{p-1} s} = \ell(r, t) \in (0, \infty),$$

uniformly for  $\xi$  bounded. Then there exists a positive constant  $C = C(f, \Omega, Z)$  such that any radially symmetric solution  $u$  of (1.12) satisfying

$$z_{(R_1, R_2)}(u(\cdot, t)) \leq Z \quad (t \in (0, T)) \quad (1.13)$$

fulfills the following estimates.

(i) If  $T < \infty$  then

$$|u(x, t)| \leq C(1 + t^{-1/(p-1)} + (T - t)^{-1/(p-1)}) \quad (x \in \Omega, t \in (0, T)).$$

(ii) If  $T = \infty$  then

$$|u(x, t)| \leq C(1 + t^{-1/(p-1)}) \quad (x \in \Omega, t > 0).$$

(iii) If  $T = \infty$ ,  $\Omega = \mathbb{R}^N$  and  $f(r, t, s, \xi) = |s|^{p-1} s$  then

$$|u(x, t)| \leq Ct^{-1/(p-1)} \quad (x \in \mathbb{R}^N, t > 0).$$

Of course, analogous statements are true for non-radial solutions if  $N = 1$ . The universal a priori estimates stated in Theorem 1.7 extend those proved earlier for positive solutions (i.e.  $Z = 0$  in (1.13)), see [31, Theorem 4.1, Corollary 3.2 and Section 6].

We now present consequences of the above a priori estimates.

An application of Theorem 1.7 shows that global solutions of the model Cauchy problem (1.10) with  $1 < p < p_S$  satisfy the decay estimate

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq Ct^{-1/(p-1)} \quad (1.14)$$

provided  $u_0 \in L^\infty$  is continuous, radially symmetric and  $z_{(0, \infty)}(u_0) < \infty$  (and we consider classical solutions satisfying  $u(\cdot, t) \in L^\infty(\mathbb{R}^N)$  for all  $t > 0$ ). The assumption  $z_{(0, \infty)}(u_0) < \infty$  is also necessary since there exist radial stationary

solutions  $v$  of (1.2) with  $z_{(0,\infty)}(v) = \infty$ . Estimate (1.14) has interesting consequences for global solutions of (1.10) with initial data in  $H^1(\mathbb{R}^N)$  and  $p < p_S$ . In this case it is known (see [36, the proof of Theorem 2]) that

$$\|u(\cdot, t)\|_{L^2(\mathbb{R}^N)} = o(t), \quad t \rightarrow \infty. \quad (1.15)$$

Interpolation between (1.14) and (1.15) yields

$$\|u(\cdot, t)\|_{L^{p+1}(\mathbb{R}^N)} \rightarrow 0, \quad t \rightarrow \infty. \quad (1.16)$$

We will prove that the same estimate is true without the assumption  $u_0 \in H^1(\mathbb{R}^N)$  provided  $N = 1$ ,  $u_0 \in C^1$  and the zero numbers of  $u_0$  and  $u'_0$  are finite. In addition, in this case Proposition 5.4 below also shows

$$\|u(\cdot, t)\|_{\mathcal{E}} \rightarrow 0, \quad t \rightarrow \infty. \quad (1.17)$$

Let us mention that (1.17) is well known for initial data with exponential decay (see [24], [25], [35, Proposition 20.13 and Example 51.24]). On the other hand, it seems to be an interesting open problem whether (1.17) remains true for all global solutions of (1.10) with  $p < p_S$  and  $u_0 \in \mathcal{E}$ .

Another result, in which Theorem 1.7 is a key ingredient, concerns non-trivial periodic solutions for a class of periodic-parabolic problems. In Section 6 we consider the model problem

$$\left. \begin{aligned} u_t - \Delta u &= m(t)f(u), & x \in \Omega, \quad t \in (0, T), \\ u &= 0, & x \in \partial\Omega, \quad t \in (0, T), \\ u(x, 0) &= u(x, T), & x \in \Omega, \end{aligned} \right\} \quad (1.18)$$

where

$$m \in W^{1,\infty}([0, T]) \text{ is positive, } m(0) = m(T), \quad (1.19)$$

$$\left. \begin{aligned} f &\in C^1(\mathbb{R}), \quad f(0) = 0, \quad f'(0) \leq 0, \\ |f'(u)| &\leq C(1 + |u|^{r-1}), \quad r < p_S, \end{aligned} \right\} \quad (1.20)$$

and

$$\lim_{|u| \rightarrow \infty} \frac{f(u)}{|u|^{p-1}u} = 1 \quad \text{for some } p \in (1, p_S). \quad (1.21)$$

Existence of positive solutions of (1.18) in a bounded domain  $\Omega$  was proved in [14, 15, 21, 22, 34] under various additional assumptions on  $m, f$  and  $p$ . Here we assume that  $\Omega$  is a ball or the whole of  $\mathbb{R}^N$ . If  $\Omega = B_R := \{x \in \mathbb{R}^N : |x| < R\}$  then we find infinitely many radial solutions of (1.18). More precisely, we prove the following theorem.

**Theorem 1.8.** *Consider problem (1.18) with  $\Omega = B_R$  and assume (1.19), (1.20), (1.21). Fix  $Z \in \{0, 1, 2, \dots\}$ . Then there exists a radial solution  $u = u(r, t)$  of (1.18) satisfying  $z_{(0,R)}(u(\cdot, t)) = Z$  for all  $t$ .*

We also consider equations on  $\Omega = \mathbb{R}^N$ , in which case we fix  $f(u) = |u|^{p-1}u - u$  for simplicity. We prove the existence of a positive radial solution of (1.18) (with the boundary condition  $u = 0$  on  $\partial\Omega \times (0, T)$  replaced by the condition  $u(x, t) \rightarrow 0$  as  $|x| \rightarrow \infty$ ,  $t \in (0, T)$ ), see Theorem 6.1 below.

It is clear that if  $m$  is independent of  $t$ , then Theorem 1.8 guarantees the existence of infinitely many radial equilibria of the autonomous problem (1.18). Applications of the uniform a priori estimates of global solutions go even farther; one can use them to show additional properties of equilibria and establish the existence of connecting orbits between equilibria (see [1, 2, 32, 34] and the references therein).

An interesting aspect of the construction by way of Theorem 1.7 is that the resulting equilibria or periodic orbits belong to the boundary of the domain of attraction of the zero solution. This has a curious consequence for the problem

$$\left. \begin{aligned} u_t - \Delta u &= |u|^{p-1}u, & x \in \Omega, \quad t > 0, \\ u &= 0, & x \in \partial\Omega, \quad t > 0, \\ u(x, 0) &= u_0(x), & x \in \Omega, \end{aligned} \right\} \quad (1.22)$$

where  $\Omega = B_R$  and  $1 < p < p_S$ . Using scaling invariance one easily proves that given  $Z \in \{0, 1, \dots\}$ , there exists a *unique* radial equilibrium  $u$  satisfying  $z_{(0,R)}(u) = Z$  and  $u(0) > 0$  (see [7]). On the other hand the proof of Theorem 1.8 shows that there exists a radial equilibrium  $u$  satisfying  $z_{(0,R)}(u) = Z$  on the boundary of the domain of attraction of the zero solution. Using these observations and the symmetry of the problem, we conclude that in fact all radial equilibria of (1.22) belong to the boundary of the domain of attraction of zero. This result is well known for positive solutions, but seems to be new for nodal solutions (cf. [8]).

## 2 Liouville-type theorems for nodal solutions

We first consider problem (1.2) with  $N = 1$ .

**Proposition 2.1.** *Let  $u$  be a bounded classical solution of (1.2) with  $N = 1$  and  $p > 1$ . Assume that there exists  $Z \in \mathbb{N}$  such that  $z_{\mathbb{R}}(u(\cdot, t)) \leq Z$  for all  $t \in \mathbb{R}$ . Then  $u \equiv 0$ .*

A crucial ingredient of the proof of this result is the following lemma on the zero number of solutions of a linear equation

$$v_t = v_{xx} + a(x, t)v, \quad (x, t) \in \mathbb{R} \times (s, T), \quad (2.1)$$

where  $-\infty \leq s < T \leq \infty$  and  $a$  is continuous on  $\mathbb{R} \times (s, T)$ . Given a solution  $v$  of (2.1), we examine the function  $t \mapsto z_{(\theta_1(t), \theta_2(t))}(v(\cdot, t))$ , where either  $(\theta_1(t), \theta_2(t)) = (-\infty, \infty)$  for all  $t \in (s, T)$  or  $\theta_1 < \theta_2$  are continuous functions on  $(s, T)$ . In the latter case, we shall also assume that

$$v(\theta_1(t), t) \neq 0 \neq v(\theta_2(t), t) \quad (s < t < T). \quad (2.2)$$

**Lemma 2.2.** *Let  $v$  be a nontrivial solution of (2.1). Then for each  $t \in (s, T)$  the function  $v(\cdot, t)$  has only isolated zeros; in particular,  $z_{(R_1, R_2)}(v(\cdot, t)) < \infty$  for each bounded interval  $(R_1, R_2)$ . Further, with  $\theta_1, \theta_2$  as above, assume that (2.2) holds if  $\theta_1, \theta_2$  are finite; in case they are infinite, assume that*

$$z_{(\theta_1(t), \theta_2(t))}(v(\cdot, t)) < \infty \quad (t \in (s, T)). \quad (2.3)$$

Then the following assertions hold true.

- (i)  $t \mapsto z_{(\theta_1(t), \theta_2(t))}(v(\cdot, t))$  is a monotone nonincreasing function on  $(s, T)$ ;
- (ii) if for some  $t_1 \in (s, T)$  the function  $v(\cdot, t_1)$  has a multiple zero in  $(\theta_1(t_1), \theta_2(t_1))$ , then  $t \mapsto z_{(\theta_1(t), \theta_2(t))}(v(\cdot, t))$  is discontinuous at  $t = t_1$  (hence, by (i), it drops at  $t = t_1$ ).

These are standard results in case  $\theta_1, \theta_2$  are independent of  $t$  (see [3, 10]). It is straightforward to extend them to the case of variable  $\theta_1, \theta_2$ . Indeed, the monotonicity property, which is a consequence of the maximum principle and Jordan curve theorem (see for example [26]), is proved in the same way as for constant  $\theta_1, \theta_2$ . The finiteness and dropping properties are derived from the local structure of the nodal set of  $v$  near points  $(x_0, t_0)$ , where  $x_0$  is a multiple zero of  $v(\cdot, t_0)$  [3, 10]. These apply in our setting in the same way. An interested reader can also verify that the results for variable  $\theta_1, \theta_2$  can

be derived from those for constant  $\theta_1, \theta_2$  via an approximation argument. For this one first chooses neighborhoods of the graphs of  $\theta_1, \theta_2$  on which  $v$  does not vanish. Approximating the functions  $\theta_1, \theta_2$  by suitable piecewise constant functions  $\tilde{\theta}_1, \tilde{\theta}_2$ , with graphs in the nonvanishing neighborhoods, one can prove the desired conclusion by repeatedly using Lemma 2.2 on time-independent intervals.

Observe that if (2.3) holds (in particular, if  $\theta_1, \theta_2$  are finite) then statements (i), (ii) of the lemma imply that  $v(\cdot, t)$  can have a multiple zero in  $(\theta_1(t), \theta_2(t))$  only for isolated values of  $t$ . Regardless of (2.3), we can say that given any bounded interval  $(R_1, R_2)$ , the function  $v(\cdot, t)$  can have a multiple zero in  $(R_1, R_2)$  only for isolated values of  $t$ . To show this, fix any  $t_0$  and enlarge the interval  $(R_1, R_2)$  slightly, if necessary, so that  $v(R_1, t) \neq 0 \neq v(R_2, t)$  for  $t \approx t_0$  (this is possible by the first statement of the lemma). Then we can use statements (i), (ii) on the interval  $(\theta_1(t), \theta_2(t)) = (R_1, R_2)$  for  $t \approx t_0$ .

In applications of Lemma 2.2 below, we set  $v = u_1 - u_2$ , where  $u_1, u_2$  are solutions of (1.2) with  $N = 1$  and  $p > 1$ . Such  $v$  is a solution of (2.1) with  $a(x, t) = p|\zeta|^{p-1}(x, t)$ , where  $\zeta(x, t)$  is between  $u_1(x, t)$  and  $u_2(x, t)$ .

*Proof of Proposition 2.1.* The proof is by contradiction: we assume  $u \not\equiv 0$ . By Theorem 1.2 and Lemma 2.2,  $u(\cdot, t)$  must have at least one zero for each sufficiently large negative  $t$ . Lemma 2.2 further implies that there exists  $\tau_0 \in \mathbb{R}$  such that  $z_{\mathbb{R}}(u(\cdot, t)) \geq 1$  is constant on  $(-\infty, \tau_0]$  and therefore all zeros of  $u(\cdot, t)$  are simple if  $t \in (-\infty, \tau_0]$ . For  $t \leq \tau_0$  we let  $\eta(t) \leq \xi(t)$  denote the smallest and largest zero of  $u(\cdot, t)$ , respectively. By the implicit function theorem,  $\xi$  and  $\eta$  are  $C^1$ -functions on  $(-\infty, \tau_0]$ . Clearly,  $u$  is of one sign in each of the sets  $\{(x, t) \in \mathbb{R}^2 : x > \xi(t), t \leq \tau_0\}$  and  $\{(x, t) \in \mathbb{R}^2 : x < \eta(t), t \leq \tau_0\}$ . Replacing  $u$  with  $-u$ , if necessary, we shall assume below that  $u(x, t) > 0$  for  $x > \xi(t), t \leq \tau_0$ .

We distinguish the following two possibilities:

- (a) Either  $\xi(t)$  is not bounded from above or  $\eta(t)$  is not bounded from below.
- (b) There exists  $R_0$  such that  $-R_0 < \eta(t) \leq \xi(t) < R_0$  for all  $t \leq \tau_0$ .

Consider case (a) and for definiteness assume that  $\xi(t)$  is unbounded from above (the case of  $\eta(t)$  being unbounded from below is analogous). For any  $\tau \in (-\infty, \tau_0]$  and  $\lambda > \xi(\tau)$  set

$$Q_\tau^\lambda := \{(x, t) : \xi(t) < x < \lambda, t \in (s_\lambda(\tau), \tau]\}, \quad (2.4)$$

where

$$s_\lambda(\tau) := \sup\{s < \tau : \xi(s) = \lambda\}. \quad (2.5)$$

Since  $\lambda > \xi(\tau)$  and  $\xi$  is unbounded from above,  $s_\lambda(\tau) \in (-\infty, \tau)$ . Define further

$$v^\lambda(x, t) = u(2\lambda - x, t) - u(x, t) \quad ((x, t) \in \mathbb{R}^2). \quad (2.6)$$

Then  $v^\lambda$  solves a linear parabolic equation (2.1) on  $Q = \mathbb{R}^2$  and  $v^\lambda \geq 0$  on the parabolic boundary of  $Q_\tau^\lambda$ ; more specifically,  $v^\lambda(\lambda, t) = 0$  and  $v^\lambda(\xi(t), t) > 0$  for  $t \in (s_\lambda(\tau), \tau)$  (the latter follows from the positivity of  $u(x, t)$  for  $x > \xi(t)$ ). Therefore, by the maximum principle,  $v^\lambda > 0$  in  $Q_\tau^\lambda$  and then, by the Hopf boundary principle,

$$0 > v_x^\lambda(\lambda, t) = -2u_x(\lambda, t) \quad (t \in (s_\lambda(\tau), \tau]). \quad (2.7)$$

In particular,  $u_x(\lambda, \tau) > 0$ . Since  $\lambda > \xi(\tau)$  was arbitrary, we conclude that for each  $\tau \leq \tau_0$

$$u_x(x, \tau) > 0 \quad (x > \xi(\tau)). \quad (2.8)$$

Now choose any sequence  $y_k \rightarrow \infty$  and consider the function

$$u_k(x, t) := u(x + y_k, t) \quad ((x, t) \in \mathbb{R}^2).$$

Since  $u$  is bounded, parabolic estimates imply that  $u_k$ , replaced by a subsequence if necessary, converges locally uniformly on  $\mathbb{R}^2$  to a solution  $\bar{u}$  of (1.2). Now, (2.8) implies that for each fixed  $t \leq \tau_0$  the limit  $\bar{u}(x, t)$  is positive and independent of  $x$  (it is equal to  $\lim_{y \rightarrow \infty} u(y, t)$ ). Consequently,  $\bar{u}$  is an entire positive  $x$ -independent solution of (1.2), which is absurd.

Next we consider case (b), dividing it further into the following two possibilities.

(b1)  $\|u(\cdot, t)\|_{L^\infty(-R, R)} \rightarrow 0$  as  $t \rightarrow -\infty$  for each  $R > 0$ .

(b2) There exist a sequence  $t_k \rightarrow -\infty$  and positive constants  $R_1, \beta_0$  such that  $\|u(\cdot, t_k)\|_{L^\infty(-R_1, R_1)} \geq \beta_0$  for each  $k = 0, 1, \dots$

For  $\tau \leq \tau_0$  and  $\lambda > \xi(\tau)$  we define  $v^\lambda, Q_\tau^\lambda$  as in (2.6), (2.4). The value  $s_\lambda(\tau)$  is as in (2.5) with the understanding that  $\sup \emptyset = -\infty$ .

Assume (b1). We show that in this case, too, (2.8) holds for all sufficiently large negative  $\tau$  and this leads to a contradiction as above. As in case (a), (2.8) follows from the Hopf boundary principle if we prove that for each

$\lambda > \xi(\tau)$  one has  $v^\lambda > 0$  in  $Q_\tau^\lambda$ . This holds, by the same arguments as in case (a), if  $s_\lambda(\tau) > -\infty$ . We thus only need to consider the possibility  $s_\lambda(\tau) = -\infty$ , that is,  $\xi(t) < \lambda$  for all  $t \leq \tau$ .

Let  $\mu_1, \varphi_1$  be, respectively, the principal eigenvalue and a positive eigenfunction of the eigenvalue problem

$$\begin{aligned} \varphi_{xx} + \mu\varphi &= 0, & x \in (-R_0 - 1, \lambda + 1), \\ \varphi &= 0 & x \in \{-R_0 - 1, \lambda + 1\}. \end{aligned}$$

By (b1), for each sufficiently large negative  $\tau \leq \tau_0$

$$\max\{|u|^{p-1}(2\lambda - x, t), |u|^{p-1}(x, t)\} \leq \frac{\mu_1}{2p} \quad (x \in [-R_0, \lambda], t \leq \tau). \quad (2.9)$$

Fix any such  $\tau$ . Observe that in the equation (2.1) satisfied by  $v^\lambda$ , we have  $a(x, t) = p|\zeta|^{p-1}(x, t)$ , where  $\zeta(x, t)$  is between  $u(x, t)$  and  $u(2\lambda - x, t)$ . By (2.9),

$$a(x, t) \leq \mu_1/2 \quad ((x, t) \in [-R_0, \lambda] \times (-\infty, \tau)).$$

Also note that for each  $t \leq \tau$

$$v^\lambda(\lambda, t) = 0 > -\varphi_1(\lambda) \text{ and } v^\lambda(\xi(t), t) > 0 > -\varphi_1(\xi(t)).$$

Therefore, using a comparison argument on the set  $\tilde{Q}_\tau^\lambda(s) := Q_\tau^\lambda \cap (\mathbb{R} \times (s, \tau])$ , with an arbitrary  $s \in (-\infty, \tau)$ , we obtain the following conclusion. If  $\epsilon > 0$  is a constant and  $v^\lambda(\cdot, s) \geq -\epsilon\varphi_1$  on  $[\xi(s), \lambda]$  then  $v^\lambda \geq -\epsilon\varphi_1$  on  $\tilde{Q}_\tau^\lambda(s)$ . Now, in view of (b1), we can take  $\epsilon > 0$  arbitrarily small, upon taking  $s < 0$  large enough. Sending  $\epsilon \rightarrow 0$  (and  $s \rightarrow -\infty$ ) we obtain  $v^\lambda \geq 0$  on  $Q_\tau^\lambda$ . The strong maximum principle then gives  $v^\lambda > 0$  on  $Q_\tau^\lambda$ , as desired. We have thus completed our argument in case (b1).

Finally, assume that (b2) holds. Set

$$u_k(x, t) = u(x, t_k + t) \quad (x \in \mathbb{R}, t \in (-\infty, \tau_0 - t_k)).$$

A suitable subsequence of these functions converges locally uniformly on  $\mathbb{R}^2$  to a bounded solution  $\tilde{u}$  of (1.2) such that  $\|\tilde{u}(\cdot, 0)\|_{L^\infty(-R_1, R_1)} \geq \beta_0$  and  $\tilde{u} \geq 0$  in  $(R_0, \infty) \times \mathbb{R}$ . By Lemma 2.2 and the strong maximum principle,  $\tilde{u} > 0$  in  $(R_0, \infty) \times \mathbb{R}$ . Similarly,  $\tilde{u}$  does not vanish in  $(-\infty, -R_0) \times \mathbb{R}$ , hence

$$\text{for each } t \text{ all zeros of } \tilde{u}(\cdot, t) \text{ are contained in } [-R_0, R_0]. \quad (2.10)$$

Moreover,

$$z_{(-R_0-1, R_0+1)}(\tilde{u}(\cdot, t)) = m \quad (t \in \mathbb{R}), \quad (2.11)$$

where

$$m := z_{\mathbb{R}}(u(\cdot, s)) = z_{(-R_0, R_0)}(u(\cdot, s)) \quad (s \leq \tau_0).$$

Indeed, if  $z_{(-R_0-1, R_0+1)}(\tilde{u}(\cdot, t)) < m$  for some  $t$ , then the same is true for any larger  $t$  and hence we can assume in addition that all zeros of  $\tilde{u}(\cdot, t)$  are simple. Then, since  $u(\cdot, t_k + t) \rightarrow \tilde{u}(\cdot, t)$  in  $C^1[-R_0 - 1, R_0 + 1]$ , we have  $z_{(-R_0, R_0)}(u(\cdot, t_k + t)) < m$  for each sufficiently large  $k$ , a contradiction. The inequality  $z_{(-R_0-1, R_0+1)}(\tilde{u}(\cdot, t)) > m$  is ruled out by a similar argument.

By (2.11), all zeros of  $\tilde{u}(\cdot, t)$  are simple and hence the largest zero, which we denote by  $\tilde{\xi}(t)$ , is a  $C^1$  function of  $t$ . Clearly,

$$\tilde{\xi}(t) = \lim_{k \rightarrow \infty} \xi(t_k + t) \leq \sigma := \limsup_{s \rightarrow -\infty} \xi(s).$$

Our goal is to prove that the points of local maxima and minima of  $\tilde{u}(\cdot, t)$  in  $(\tilde{\xi}(t), \infty)$ , if there are any, are independent of  $t$ . We then show that that leads to a contradiction.

First we show that for each  $\tau \in \mathbb{R}$  one has

$$\tilde{u}_x(\lambda, \tau) \geq 0 \quad (2.12)$$

whenever  $\tilde{\xi}(\tau) \leq \lambda \leq \sigma$ . Obviously, (2.12) holds (with the strict inequality) at the simple zero  $\lambda = \tilde{\xi}(\tau)$  of  $\tilde{u}(\cdot, \tau)$ . Hence, it is sufficient to prove (2.12) for any  $\lambda$  satisfying  $\tilde{\xi}(\tau) < \lambda < \sigma$  (if there is no such  $\lambda$ , i.e. if  $\tilde{\xi}(\tau) = \sigma$ , the previous argument alone gives the desired conclusion).

Fix any  $\tau \in \mathbb{R}$  and  $\tilde{\xi}(\tau) < \lambda < \sigma$ . For all sufficiently large  $k$  we have  $\xi(\tau + t_k) < \lambda$ . Also,  $\lambda < \sigma$  implies that  $s_\lambda(\tau + t_k) > -\infty$ . Therefore, as in case (a),  $u_x(\lambda, \tau + t_k) > 0$ . Taking the limit as  $k \rightarrow \infty$ , we obtain (2.12).

Now let  $\lambda > \sigma$ . For a sufficiently large  $\tau_1 < \tau_0$  we have  $\xi(t) < \lambda$  for all  $t \leq \tau_1$ . Let  $\tilde{v}^\lambda$  be defined as  $v^\lambda$  with  $u$  in (2.6) replaced with  $\tilde{u}$ . Since  $\tilde{v}^\lambda \not\equiv 0$  (we have  $\tilde{v}^\lambda(\tilde{\xi}(t), t) > 0$ ), we can fix  $\hat{t} < \tau_1$  such that  $v^\lambda(\cdot, \hat{t})$  has only simple zeros in  $[-R_0 - 1, 2\lambda + R_0 + 1]$ . It then follows from the convergence

$$v^\lambda(\cdot, t_k + \hat{t}) \rightarrow \tilde{v}^\lambda(\cdot, \hat{t})$$

in  $C^1[-R_0 - 1, 2\lambda + R_0 + 1]$  that there is  $M$  such that

$$z_{(\xi(t_k + \hat{t}), 2\lambda - \xi(t_k + \hat{t}))}(v^\lambda(\cdot, t_k + \hat{t})) \leq M \quad (2.13)$$

for all sufficiently large  $k$ . Note that  $0 < v^\lambda(\xi(t), t) = -v^\lambda(2\lambda - \xi(t), t)$  for each  $t \leq \tau_1$ . Hence, Lemma 2.2 and (2.13) imply that  $z_{(\xi(t), 2\lambda - \xi(t))}(v^\lambda(\cdot, t))$  is independent of  $t$  for large negative  $t$ , say for  $t \leq \tau_2$  ( $\tau_2$  may depend on  $\lambda$ ). Using Lemma 2.2 again we infer that  $v^\lambda(\cdot, t)$  has only simple zeros in  $(\xi(t), 2\lambda - \xi(t))$  for  $t \leq \tau_2$ . In particular, since  $v^\lambda(\lambda, t) = 0$ , we have  $-2u_x(\lambda, t) = v_x^\lambda(\lambda, t) \neq 0$  for all  $t \leq \tau_2$ .

The above considerations show that for each  $\lambda > \sigma$ , there exists

$$\lim_{t \rightarrow -\infty} \text{sign } u_x(\lambda, t) \in \{-1, 1\} \quad (2.14)$$

(this argument was inspired by [11]). This has the following consequence on  $\tilde{u}$ : if  $\lambda > \sigma$  then either  $\tilde{u}_x(\lambda, \cdot) \geq 0$  on  $\mathbb{R}$  or  $\tilde{u}_x(\lambda, \cdot) \leq 0$  on  $\mathbb{R}$ . Combining this information with the fact that (2.12) holds whenever  $\tilde{\xi}(\tau) \leq \lambda \leq \sigma$ , we conclude that either  $\tilde{u}(\cdot, t)$  is monotone nondecreasing on  $(\tilde{\xi}(t), \infty)$  for each  $t \in \mathbb{R}$  or else there is  $\lambda \geq \sigma$  such that  $\tilde{u}_x(\lambda, \cdot) \equiv 0$  (namely,  $\lambda = \inf\{\mu \geq \sigma : \tilde{u}_x(\mu, \cdot) \leq 0 \text{ on } \mathbb{R}\}$ ). The former leads to a contradiction, as we have seen above. In the latter case, the function  $\tilde{v}^\lambda(\cdot, t)$  has a multiple zero at  $x = \lambda$  for each  $t \in \mathbb{R}$ , which is possible only if  $\tilde{v}^\lambda \equiv 0$ . In case  $\lambda > \tilde{\xi}(t)$  for some  $t$  this contradicts the positivity of  $\tilde{u}(\cdot, t)$  in  $(\tilde{\xi}(t), \infty)$ . If  $\tilde{\xi} \equiv \lambda$  then  $-2\tilde{u}_x(\xi(t), t) = \tilde{v}_x^\lambda(\cdot, t) = 0$  contradicts the simplicity of  $\tilde{\xi}(t)$  as a zero of  $\tilde{u}(\cdot, t)$ .

We have thus finished the arguments in case (b2) and thereby completed the proof.  $\square$

**Remark 2.3.** In order to make the proof of Proposition 2.1 self-contained we have based it entirely on the maximum and intersection-comparison principles. At some steps we could have used alternative arguments and perhaps it is worthwhile to mention the following ones. Inequality (2.8) could also be ruled out using results of [27] (see (5.1) below). Case (b2) in the proof of Proposition 2.1 could also be resolved by energy arguments similar to those used in Proposition 2.4 below. In fact, the boundedness of  $\tilde{u}$ , (2.10) and [31, Theorem 3.1(ii) and Remark 3.4(e)] used with  $\Omega = \{x : |x| > R_0\}$  guarantee estimate (2.15) with  $u$  replaced by  $\tilde{u}$ .

Next we consider bounded radial solutions of (1.2).

**Proposition 2.4.** *Let  $1 < p < p_s$  and let  $u$  be a classical bounded radial solution of (1.2). Assume that there exists  $Z \in \mathbb{N}$  such that (1.6) is true. Then  $u \equiv 0$ .*

The main ingredients of the proof of this result are the Liouville-type results of Proposition 2.1 and Theorem 1.1, energy estimates, and the following Doubling Lemma of [30].

**Lemma 2.5.** *Let  $(X, d)$  be a complete metric space and let  $\emptyset \neq D \subset \Sigma \subset X$ , with  $\Sigma$  closed. Set  $\Gamma = \Sigma \setminus D$ . Finally let  $M : D \rightarrow (0, \infty)$  be bounded on compact subsets of  $D$  and fix a real  $k > 0$ . If  $y \in D$  is such that*

$$M(y) \operatorname{dist}(y, \Gamma) > 2k,$$

then there exists  $x \in D$  such that

$$M(x) \operatorname{dist}(x, \Gamma) > 2k, \quad M(x) \geq M(y),$$

and

$$M(z) \leq 2M(x) \quad \text{for all } z \in D \cap \overline{B}_X(x, kM^{-1}(x)).$$

*Proof of Proposition 2.4.* The proof is by contradiction: we assume  $u \not\equiv 0$ . First we prove that there is a constant  $C > 0$  such that

$$|u(r, t)|r^{2/(p-1)} + |u_r(r, t)|r^{(p+1)/(p-1)} \leq C \quad (r > 0, t \in \mathbb{R}). \quad (2.15)$$

Assume on the contrary that there exist  $r_k > 0$  and  $t_k \in \mathbb{R}$  such that

$$|u(r_k, t_k)|r_k^{2/(p-1)} + |u_r(r_k, t_k)|r_k^{(p+1)/(p-1)} \rightarrow \infty.$$

Set

$$M(r, t) := |u(r, t)|^{(p-1)/2} + |u_r(r, t)|^{(p-1)/(p+1)} \quad (r > 0, t \in \mathbb{R}).$$

Passing to a subsequence we may assume  $M(r_k, t_k) > 2k/r_k$ . Notice that  $r_k = \operatorname{dist}_P((r_k, t_k), \partial Q)$ , where  $\operatorname{dist}_P((r_1, t_1), (r_2, t_2)) := |r_1 - r_2| + \sqrt{|t_1 - t_2|}$  denotes the parabolic distance and  $Q := (0, \infty) \times \mathbb{R}$ . Now the Doubling Lemma (Lemma 2.5 with  $X = \mathbb{R}^2$ ,  $\operatorname{dist} = \operatorname{dist}_P$ ,  $D = Q$ ,  $\Gamma = \partial Q$ ) guarantees the existence of  $(\tilde{r}_k, \tilde{t}_k) \in Q$  such that  $M_k := M(\tilde{r}_k, \tilde{t}_k) > 2k/\tilde{r}_k$  and

$$M(r, t) \leq 2M_k \quad \text{whenever} \quad |r - \tilde{r}_k| + \sqrt{|t - \tilde{t}_k|} < \frac{k}{M_k}.$$

Set  $\lambda_k := 1/M_k$  and

$$v_k(\rho, s) := \lambda_k^{2/(p-1)} u(\tilde{r}_k + \lambda_k \rho, \tilde{t}_k + \lambda_k^2 s).$$

Then

$$\begin{aligned} |v_k(0, 0)|^{(p-1)/2} + |\partial_\rho v_k(0, 0)|^{(p-1)/(p+1)} &= 1, \\ |v_k(\rho, s)|^{(p-1)/2} + |\partial_\rho v_k(\rho, s)|^{(p-1)/(p+1)} &\leq 2 \quad (|\rho| + \sqrt{|s|} < k), \\ z_{(0, \infty)}(v_k(\cdot, s)) &\leq Z \quad (s \in \mathbb{R}), \end{aligned}$$

and  $v_k$  solves the equation

$$\partial_t v_k - \partial_{\rho\rho} v_k = \frac{N-1}{\tilde{r}_k/\lambda_k + \rho} \partial_\rho v_k + |v_k|^{p-1} v_k.$$

Since  $\tilde{r}_k/\lambda_k = \tilde{r}_k M_k \rightarrow \infty$ , it is easy to pass to the limit to get a nontrivial bounded solution  $v$  of (1.2) with  $N = 1$  satisfying (1.7). However, this contradicts Proposition 2.1. Consequently, (2.15) is true.

We now complete the proof using energy arguments. Estimate (2.15) guarantees  $\|u(\cdot, t)\|_{\mathcal{E}} \leq C$  and  $|E(u(\cdot, t))| \leq C$  with  $C$  independent of  $t$  (recall that the space  $\mathcal{E}$  is defined in (1.5)). This also implies (cf. (1.11))

$$\int_{\mathbb{R}} \int_{\mathbb{R}^N} u_t^2 dx dt < \infty.$$

Choose  $t_k \rightarrow -\infty$  such that  $\int_{\mathbb{R}} u_t^2(x, t_k) dx \rightarrow 0$ . Then

$$\|u(\cdot, t_k)\|_{L^\infty(\mathbb{R}^N)} \rightarrow 0. \quad (2.16)$$

Indeed, if not then we may assume  $\|u(\cdot, t_k)\|_{L^\infty(\mathbb{R}^N)} \geq c$  for some  $c > 0$ . Choose  $r_k > 0$  such that  $|u(r_k, t_k)| \geq \frac{1}{2} \|u(\cdot, t_k)\|_{L^\infty(\mathbb{R}^N)}$ . We may assume that either  $r_k \rightarrow r_\infty \in [0, \infty)$  or  $r_k \rightarrow \infty$ . In the former case a subsequence of  $v_k(r) := u_k(r, t_k)$  converges in  $C_{loc}([0, \infty))$  to some function  $v$  which is a radial solution of  $-\Delta v = |v|^{p-1} v$  in  $\mathbb{R}^N$ , the zero number of  $v$  is finite and  $v(r_\infty) \geq c/2$ , which contradicts Theorem 1.1. In the latter case we set  $v_k(r) := u(r_k + r, t_k)$ . Then a subsequence of  $v_k$  converges in  $C_{loc}(\mathbb{R})$  to a nontrivial solution  $v$  of the limiting problem  $-v_{rr} = |v|^{p-1} v$ ,  $r \in \mathbb{R}$ , and  $z_{\mathbb{R}}(v) < \infty$ , which contradicts Theorem 1.1 again. Hence indeed (2.16) is true and parabolic regularity estimates guarantee

$$\|u(\cdot, t_k + 1)\|_{L^\infty(\mathbb{R}^N)} + \|\nabla u(\cdot, t_k + 1)\|_{L^\infty(\mathbb{R}^N)} \rightarrow 0.$$

Analogous arguments show the existence of  $\tilde{t}_k \rightarrow \infty$  such that

$$\|u(\cdot, \tilde{t}_k + 1)\|_{L^\infty(\mathbb{R}^N)} + \|\nabla u(\cdot, \tilde{t}_k + 1)\|_{L^\infty(\mathbb{R}^N)} \rightarrow 0.$$

Now estimate (2.15) enables us to show  $E(u(\cdot, t_k + 1)) \rightarrow 0$  and  $E(u(\cdot, \tilde{t}_k + 1)) \rightarrow 0$  which implies  $E(u(\cdot, t)) \equiv 0$  and  $u_t \equiv 0$ . However, this contradicts Theorem 1.1.  $\square$

*Proof of Theorems 1.4 and 1.5.* Propositions 2.4 and 2.1 prove the Liouville theorems under the extra assumption of boundedness of  $u$ . To remove this assumption, thus showing that Theorems 1.4 and 1.5 follow from Propositions 2.4 and 2.1, respectively, one uses scaling and the Doubling Lemma in the same way as in the proof of [31, Theorem 3.1]. We omit the details.  $\square$

*Proof of Theorem 1.6.* Set  $\tilde{u}(x, t) := u(x, t)$  for  $x \geq 0$  and  $t \in \mathbb{R}$ ,  $\tilde{u}(x, t) := -u(-x, t)$  for  $x < 0$  and  $t \in \mathbb{R}$ . Then  $\tilde{u}$  is a solution of (1.2) satisfying the assumptions of Theorem 1.5 (with  $Z$  replaced by  $2Z + 1$ ), hence  $\tilde{u} \equiv 0$ .  $\square$

**Remark 2.6.** In the proof of Proposition 2.4 we showed how energy and scaling arguments can be used to derive the Liouville result for radial solutions from the Liouville theorem for nonsymmetric solutions of the one-dimensional problem. This sort of reasoning can of course be applied to positive solutions as well. Specifically, if  $u$  is a positive bounded radial solution of (1.2) (with  $N \geq 1$  and  $p > 1$ ) then one can use Theorem 1.2 instead of Proposition 2.1 in order to prove estimate (2.15). Consequently, the proof of Proposition 2.4 (and the doubling arguments used in the proof of Theorem 1.4) give one of the alternative proofs of Theorem 1.3 mentioned in the introduction.

Estimate (2.15) was used in the second part of the proof of Proposition 2.4 in order to prove a uniform bound for  $u(\cdot, t)$  in the energy space  $\mathcal{E}$ . It is clear that the same bound can be obtained if we replace estimate (2.15) by the estimate

$$|u(r, t)|r^\alpha + |u_r(r, t)|r^\beta \leq C \quad (r > 0, t \in \mathbb{R}), \quad (2.17)$$

for some  $\alpha > N/(p + 1)$  and  $\beta > N/2$ . In what follows we find a sufficient condition guaranteeing (2.17). Our estimates will also be needed in the subsequent section.

**Proposition 2.7.** *Let  $p > 1$  and let  $u$  be a radial solution of (1.2) satisfying*

$$|u(r, t)| \leq C_1 \min(1, r^{-\alpha}) \quad (r > 0, t \in \mathbb{R}), \quad (2.18)$$

where  $0 < \alpha < 2/(p - 1)$ . Fix  $\beta < \alpha(p + 1)/2$ . Then there exists a constant  $C = C(C_1, N, p, \alpha, \beta)$  such that

$$|u_r(r, t)| \leq C \min(1, r^{-\beta}) \quad (r > 0, t \in \mathbb{R}). \quad (2.19)$$

In particular, if  $p < p_S$  and (2.18) is true with some  $\alpha > N/(p+1)$  then (2.17) is true with some  $\alpha > N/(p+1)$  and  $\beta > N/2$ .

**Corollary 2.8.** *Let  $1 < p < p_S$  and let  $u = u(r, t)$  be a bounded radial solution of (1.2) satisfying (2.18) with some  $\alpha > N/(p+1)$ . Then  $u \equiv 0$ .*

*Proof.* The result follows from the remarks preceding Proposition 2.7, the proof of Proposition 2.4 and the nonexistence of non-trivial stationary solutions in the energy space  $\mathcal{E}$ .  $\square$

In the proof of Proposition 2.7 we will need the following lemma.

**Lemma 2.9.** *Let  $p > 1$  and let  $u$  be a radial solution of (1.2) satisfying (2.18) with some  $\alpha \in (0, 2/(p-1)]$ . Let  $R \geq 1$ ,  $T \in \mathbb{R}$  and  $q \in (1, \infty)$ . Then there exists  $C_2 = C_2(C_1, N, q) > 0$  such that*

$$\left( \int_{T-2R^2}^T \int_{3R}^{7R} |u_{rr}(r, t)|^q dr dt \right)^{1/q} \leq C_2 R^{3/q-p\alpha}. \quad (2.20)$$

*Proof.* By  $C$  we denote various positive constants which depend only on  $C_1, N, q$ . Given  $r \in (0, R]$ , denote

$$Q_r(R, T) := \{(x, t) \in \mathbb{R}^{N+1} : 5R - 4r < |x| < 5R + 4r, T - 8r^2 < t < T\}$$

and  $Q_r := Q_r(1, 0)$ . Set also  $v(y, s) := u(Ry, T + R^2s)$ ,  $(y, s) \in Q_1$ . Then  $v_s - \Delta_y v = R^2|v|^{p-1}v$  and  $|v| \leq C_1 R^{-\alpha}$  in  $Q_1$ . This fact and interior parabolic  $L^p$ -estimates guarantee

$$\begin{aligned} \left( \int_{Q_{1/2}} |D_y^2 v|^q \right)^{1/q} &\leq C \left( \left( \int_{Q_1} |v|^q \right)^{1/q} + R^2 \left( \int_{Q_1} |v|^{pq} \right)^{1/q} \right) \\ &\leq C(R^{-\alpha} + R^{2-p\alpha}) \leq CR^{2-p\alpha}. \end{aligned}$$

Consequently,

$$\begin{aligned} CR^{(N-1)/q} \left( \int_{T-2R^2}^T \int_{3R}^{7R} |u_{rr}(r, t)|^q dr dt \right)^{1/q} \\ \leq \left( \int_{Q_{R/2}(R, T)} |D_x^2 u|^q dx dt \right)^{1/q} \\ = R^{-2} \left( \int_{Q_{1/2}} |D_y^2 v|^q R^{N+2} dy ds \right)^{1/q} \leq CR^{(N+2)/q-p\alpha}, \end{aligned}$$

which concludes the proof.  $\square$

*Proof of Proposition 2.7.* Obviously it is sufficient to prove the result for  $\beta$  sufficiently close to  $\alpha(p+1)/2$ . We may thus assume that

$$\alpha(p+1)/2 > \beta > \max(\alpha, p\alpha - 1).$$

By  $C$  we denote various positive constants depending only on  $C_1, N, p, \alpha, \beta$ .

Since  $|u| \leq C_1$ , standard parabolic regularity estimates show  $|u_r| \leq C$ , hence (2.19) is true for the restricted range  $r \leq 5$  and  $t \in \mathbb{R}$ . Now, if (2.19) is not true in the whole range  $r > 0$  then, given  $C_M = C_M(C_1, N, p, \alpha, \beta) > 0$ , there exists  $R \geq 1$  and  $T \in \mathbb{R}$  such that

$$|u_r(5R, T)| > C_M R^{-\beta}. \quad (2.21)$$

We show that if  $C_M = C_M(C_1, N, p, \alpha, \beta)$  is sufficiently large, as specified below, then (2.21) leads to a contradiction.

First we prove that for  $C_M$  large enough (2.21) implies

$$(\forall t \in [T-2, T-1]) (\exists r_t \in [4R, 6R]) \quad |u_r(r_t, t)| > \sqrt{C_M} R^{-\beta}. \quad (2.22)$$

Assume on the contrary,

$$(\exists t_0 \in [T-2, T-1]) (\forall r \in [4R, 6R]) \quad |u_r(r, t_0)| \leq \sqrt{C_M} R^{-\beta}. \quad (2.23)$$

We have  $(u_r)_t - \Delta u_r = a u_r$ , where  $a := -(N-1)/r^2 + p|u|^{p-1}$ , hence  $a \leq c_a := pC_1^{p-1}$ . The comparison principle implies  $|u_r| \leq v$  for  $t \geq t_0$ , where  $v$  is the solution of the linear Cauchy problem

$$v_t - \Delta v = c_a v, \quad t > t_0, \quad v(\cdot, t_0) = |u_r|(\cdot, t_0).$$

Denoting  $w := e^{-c_a(t-t_0)}v$  and  $c_w := e^{2c_a}$  we have

$$w_t - \Delta w = 0, \quad t > t_0, \quad w(\cdot, t_0) = |u_r|(\cdot, t_0),$$

and  $|u_r| \leq c_w w$  for  $t \in [t_0, t_0 + 2]$ . Fix  $x_0 \in \mathbb{R}^N$  with  $|x_0| = 5R$ . Then estimates (2.21), (2.23) and the boundedness of  $u_r$  guarantee

$$\begin{aligned} C_M R^{-\beta} &< |u_r(5R, T)| \leq c_w w(x_0, T) \leq c_w \int_{\mathbb{R}^N} e^{-|x_0-y|^2/8} w(y, t_0) dy \\ &\leq c_w \sqrt{C_M} R^{-\beta} \int_{|y-x_0| < R} e^{-|x_0-y|^2/8} dy \\ &\quad + C \int_{|y-x_0| > R} e^{-|x_0-y|^2/8} dy \\ &\leq C \sqrt{C_M} R^{-\beta}, \end{aligned}$$

so that  $C_M < C^2$ . This shows that if  $C_M$  in (2.21) is sufficiently large then (2.21) implies (2.22).

Fix  $q > 1$  such that  $q((p+1)\alpha - \beta) \geq 3 + \alpha - \beta$ . Notice that this choice of  $q$  and the inequality  $\beta > p\alpha - 1$  imply

$$\theta := q'(p\alpha - \beta - 3/q) \in [\beta - \alpha, 1],$$

where  $q' = q/(q-1)$ . Lemma 2.9 guarantees the existence of  $t_0 \in [T-2, T-1]$  such that

$$\left( \int_{3R}^{7R} |u_{rr}(r, t_0)|^q dr \right)^{1/q} \leq C_2 R^{3/q - p\alpha}. \quad (2.24)$$

Due to (2.22) there exists  $r_0 \in [4R, 6R]$  such that

$$|u_r(r_0, t_0)| > \sqrt{C_M} R^{-\beta}. \quad (2.25)$$

Using the Mean Value Theorem, Hölder's inequality and estimate (2.24) we obtain for any  $r \in (0, R^\theta]$

$$|u_r(r_0 + r, t_0) - u_r(r_0, t_0)| \leq C_2 R^{3/q - p\alpha + \theta/q'} = C_2 R^{-\beta} \leq \frac{1}{2} \sqrt{C_M} R^{-\beta},$$

provided  $C_M \geq 4C_2^2$ . This inequality and (2.25) imply

$$|u_r(r_0 + r, t_0)| \geq \frac{1}{2} \sqrt{C_M} R^{-\beta} \quad (r \in (0, R^\theta)),$$

so that

$$|u(r_0 + R^\theta, t_0) - u(r_0, t_0)| \geq \frac{1}{2} \sqrt{C_M} R^{-\beta + \theta} \geq \frac{1}{2} \sqrt{C_M} R^{-\alpha},$$

which contradicts (2.18) for  $C_M$  large enough.  $\square$

### 3 Alternative proof of Theorem 1.3

This section is a slight detour from the main course of the paper. We utilize here the energy estimates derived above in order to give an alternative proof of Theorem 1.3 under the additional assumption  $p < 3$ . As mentioned in the introduction, although the result is weaker than that in [29], new ideas of this proof might be of interest to some readers.

In the following three lemmas we assume that  $1 < p < p_S$ ,  $p < 3$  and  $u$  is a positive bounded radial solution of (1.2). We set  $B_r := \{x \in \mathbb{R}^N : |x| < r\}$ . By  $C, c$  we will denote various positive constants which may vary from step to step but which are independent of  $r$  and  $t$ .

**Lemma 3.1.** *There exists a positive constant  $C_0$  independent of  $r > 0$  and  $t \in \mathbb{R}$  such that*

$$\int_{B_r} u(x, t) dx \leq C_0 r^{N-2/(p-1)}. \quad (3.1)$$

*Proof.* Set  $\varphi_1(x) := \pi^{-N/2} e^{-|x|^2}$  and  $y(t) := \int_{\mathbb{R}^N} u(x, t) \varphi_1(x) dx$ . Then [35, (17.3)] implies  $y(t) \leq (2N)^{1/(p-1)}$ , hence

$$\pi^{-N/2} e^{-1} \int_{B_1} u(x, t) dx \leq y(t) \leq (2N)^{1/(p-1)},$$

which proves (3.1) for  $r = 1$ .

If  $r > 0$  is general, we set  $v(y, s) := r^{2/(p-1)} u(ry, r^2 s)$ . Then  $v$  is a positive radial solution of (1.2), hence the above estimate shows

$$C_0 \geq \int_{B_1} v(y, s) dy = r^{-N+2/(p-1)} \int_{B_r} u(x, t) dx.$$

□

**Lemma 3.2.** *There exist positive constants  $C_1, \alpha_1$  such that  $u(r, t) \leq C_1 r^{-\alpha_1}$  for all  $r > 0$  and  $t \in \mathbb{R}$ .*

*Proof.* Parabolic regularity estimates imply  $|u_r| \leq C_2$  for some  $C_2 > 0$ . Since  $p < 3$  we can choose  $\alpha \in (0, 1/(p-1) - 1/2)$ . Assume  $u(r, t) \geq 2C_2 r^{-\alpha}$  for some  $t$  and  $r \geq R \geq 2$ . Then  $r^{-\alpha} \leq r/2$  and  $u(\rho, t) \geq C_2 r^{-\alpha}$  for  $|\rho - r| < r^{-\alpha}$  due to  $|u_r| \leq C_2$ , hence

$$\int_{B_{2r}} u(x, t) dx \geq C \int_{|\rho-r| < r^{-\alpha}} u(\rho, t) \rho^{N-1} d\rho \geq \tilde{C} r^{N-1-2\alpha},$$

which contradicts Lemma 3.1 for  $R = R(C_0, \tilde{C})$  large enough. □

Set

$$\begin{aligned} \alpha^* &:= \sup\{\alpha > 0 : (\exists C > 0) u(r, t) \leq Cr^{-\alpha} \text{ for all } r \geq C, t \in \mathbb{R}\} \\ &= \sup\{\alpha > 0 : (\exists D > 0) u(r, t) \leq Dr^{-\alpha} \text{ for all } r > 0, t \in \mathbb{R}\}. \end{aligned}$$

**Lemma 3.3.**  $\alpha^* \geq 2/(p-1)$ .

*Proof.* Lemma 3.2 guarantees  $\alpha^* \geq \alpha_1 > 0$ . Assume on the contrary  $\alpha^* < 2/(p-1)$ . Fix  $\delta > 0$  small (to be specified later) and set  $\alpha := \alpha^* - \delta$ ,  $\beta := \alpha^* + \delta$ . We can assume  $\alpha > 0$ . By the definition of  $\alpha^*$  we have

$$u(r, t) \leq Dr^{-\alpha} \quad (r > 0, t \in \mathbb{R}), \quad (3.2)$$

and there exist  $r_k \rightarrow \infty$  and  $t_k \in \mathbb{R}$  such that

$$u(r_k, t_k) > kr_k^{-\beta}. \quad (3.3)$$

Lemma 2.9 (used with  $R = r_k/5$  and  $T = t_k + 2$ ) guarantees the existence of  $\tau_k \in [t_k + 1, t_k + 2]$  such that

$$\left( \int_{3r_k/5}^{7r_k/5} |u_{rr}(r, \tau_k)|^q dr \right)^{1/q} \leq Cr_k^{3/q-p\alpha}. \quad (3.4)$$

Here  $q > 1$  can be taken arbitrarily large (and  $C$  depends on  $q$ ). Since  $u \geq v$  for  $t \geq t_k$ , where  $v$  is the solution of the Cauchy problem

$$v_t - \Delta v = 0, \quad t > t_k, \quad v(\cdot, t_k) = u(\cdot, t_k),$$

estimate (3.3) and the boundedness of  $u_r$  imply

$$u(r_k, \tau_k) \geq \min_{\tau \in [t_k+1, t_k+2]} v(r_k, \tau) \geq 2c^*kr_k^{-\beta} \quad (3.5)$$

for some (small) constant  $c^* > 0$ . Estimate (3.4), the Mean Value Theorem and Hölder's inequality imply

$$\left. \begin{aligned} |u_r(\rho_1, \tau_k) - u_r(\rho_2, \tau_k)| &\leq Cr_k^{3/q-p\alpha} |\rho_1 - \rho_2|^{1/q'}, \\ (\rho_1, \rho_2 \in (3r_k/5, 7r_k/5)) & \end{aligned} \right\} \quad (3.6)$$

(as usual,  $q' = q/(q-1)$ ).

Set  $\theta := (p-1)\alpha/2 - 2\delta$ . Choosing  $\delta$  small enough we may assume  $\theta \in (0, 1)$ , hence  $r_k^\theta < 2r_k/5$  for  $k$  large. Taking  $q > 1$  large enough we also have  $3/q - p\alpha + \theta(1 + 1/q') < -\beta$ . If  $u_r(r_k, \tau_k) \geq 0$  then (3.6) guarantees

$$u_r(r_k + r, \tau_k) \geq -Cr_k^{3/q-p\alpha} r^{1/q'} \quad (r \in (0, r_k^\theta)),$$

hence, given  $r \in I := (r_k, r_k + r_k^\theta)$ , we have

$$u(r, \tau_k) \geq 2c^*kr_k^{-\beta} - Cr_k^{3/q-p\alpha} (r - r_k)^{1+1/q'} \geq c^*kr_k^{-\beta}$$

provided  $k$  is large enough. Consequently,

$$\int_I u(r, \tau_k) dr \geq c^* k r_k^{-\beta+\theta}. \quad (3.7)$$

Analogously, if  $k$  is large enough and  $u_r(r_k, \tau_k) < 0$  then estimate (3.7) is true with  $I := (r_k - r^\theta, r_k)$ . Consequently, in either case

$$\int_{B_{r_k}} u(x, \tau_k) dx \geq c r_k^{N-1} \int_I u(r, \tau_k) dr \geq \tilde{c} k r_k^{N-1-\beta+\theta}$$

which contradicts Lemma 3.1 due to  $N - 1 - \beta + \theta > N - 2/(p - 1)$  for  $\delta$  small enough.  $\square$

*Proof of Theorem 1.3 for  $p < 3$ .* If  $u$  is a nonnegative bounded radial solution of (1.2), then  $u \equiv 0$  due to Lemma 3.3, Corollary 2.8 and the inequality  $2/(p - 1) > N/(p + 1)$ . Once this is proved, the nonexistence of unbounded positive radial solutions of (1.2) follows in the same way as in the proof of [31, Theorem 2.3].  $\square$

## 4 Proof of Theorem 1.7

The proof of Theorem 1.7 mimics the proof of [31, Theorem 4.1] (cf. also the proof of [30, Theorem 6.1]) so we will only sketch it.

*Sketch of the proof of Theorem 1.7.* As in the case of nonnegative solutions (see [31]) it is sufficient to prove assertion (i). Assume on the contrary that there exist  $T_k \in (0, \infty)$ ,  $\rho_k \in I$ ,  $s_k \in (0, T_k)$  and radial solutions  $u_k$  of (1.12) satisfying (1.13) (with  $T$  replaced by  $T_k$ ) such that the function

$$M_k := |u_k|^{(p-1)/2} + |\nabla u_k|^{(p-1)/(p+1)}$$

satisfies  $M_k(\rho_k, s_k) > 2k(1 + d_k^{-1}(s_k))$ , where  $d_k(t) := \min(t, (T_k - t))^{1/2}$ . Using the Doubling Lemma (see Lemma 2.5), we find  $r_k \in I$  and  $t_k \in (0, T_k)$  such that  $M_k(r_k, t_k) > 2k \max(1, d_k^{-1}(t_k))$  and

$$M_k \leq 2M_k(r_k, t_k) \quad \text{in } \{(r, t) \in I \times (0, T_k) : |r - r_k| + |t - t_k|^{1/2} \leq k\lambda_k\},$$

where  $\lambda_k := M_k(r_k, t_k)^{-1}$ . We may assume  $r_k \rightarrow r_0 \in [R_1, R_2]$  and  $t_k \rightarrow t_0 \in [0, \infty]$ . We may also assume that either (a)  $r_k/\lambda_k \rightarrow \rho_0 \geq 0$  or (b)  $r_k/\lambda_k \rightarrow \infty$ .

In case (a) we set

$$v_k(\rho, s) := \lambda_k^{2/(p-1)} u_k(\lambda_k \rho, t_k + \lambda_k^2 s).$$

The function  $v_k$  is radially symmetric, it has a bounded zero number,

$$N_k(\rho, s) := |v_k(\rho, s)|^{(p-1)/2} + |\partial_\rho v_k(\rho, s)|^{(p-1)/(p+1)} \leq 2$$

$$(\rho < \min(k/2, R_2/\lambda_k), |s| < k^2/4, k \text{ large}),$$

$N_k(r_k/\lambda_k, 0) = 1$ ,  $v_k$  solves the equation

$$\partial_s v_k - \partial_{\rho\rho}^2 v_k - \frac{N-1}{\rho} \partial_\rho v_k = f_k(\rho, s), \quad \text{where}$$

$$f_k(\rho, s) := \lambda_k^{2p/(p-1)} f(\lambda_k \rho, t_k + \lambda_k^2 s, \lambda_k^{-2/(p-1)} v_k(\rho, s),$$

$$\lambda_k^{-(p+1)/(p-1)} \partial_\rho v_k(\rho, s)),$$

together with the corresponding Dirichlet boundary condition (if  $\Omega \neq \mathbb{R}^N$ ). Due to our assumptions on  $f$  it is easy to pass to the limit to get a nontrivial radial solution  $v$  of the limit problem

$$v_s - \Delta v = \ell(0, t_0) |v|^{p-1} v, \quad \rho \in \mathbb{R}^N, s \in \mathbb{R},$$

satisfying (1.6) with  $u$  replaced by  $v$ . However, this clearly contradicts Theorem 1.4.

In case (b) we set

$$v_k(\rho, s) := \lambda_k^{2/(p-1)} u_k(r_k + \lambda_k \rho, t_k + \lambda_k^2 s)$$

and notice that  $v_k$  satisfies the equation

$$\partial_s v_k - \partial_{\rho\rho}^2 v_k - \frac{N-1}{\rho + r_k/\lambda_k} \partial_\rho v_k = f_k(\rho, s), \quad \text{where}$$

$$f_k(\rho, s) := \lambda_k^{2p/(p-1)} f(r_k + \lambda_k \rho, t_k + \lambda_k^2 s, \lambda_k^{-2/(p-1)} v_k(\rho, s),$$

$$\lambda_k^{-(p+1)/(p-1)} \partial_\rho v_k(\rho, s)).$$

Setting  $\eta_k := \min(R_2 - r_k, r_k - R_1)$  we may assume that either (i)  $\eta_k/\lambda_k \rightarrow \infty$  or (ii)  $\eta_k/\lambda_k \rightarrow c_0 \geq 0$ . Passing to the limit in case (i) or (ii) we obtain a contradiction with Theorem 1.5 or Theorem 1.6, respectively.  $\square$

## 5 Decay of global solutions in the energy norm

Our aim in this section is to give sufficient conditions for the decay of the solution  $u(\cdot, t)$  of (1.10) in the norm of  $\mathcal{E}$  (see the paragraph containing (1.17) for a discussion of this problem).

Throughout this section we assume that  $u$  is a global solution of problem (1.10) with  $N = 1$ ,  $p > 1$  and  $u_0 \in C(\mathbb{R})$ ,  $z_{\mathbb{R}}(u_0) < \infty$ . This assumption and [27, Theorem 1.3] guarantee that the limits  $\liminf_{x \rightarrow \infty} |u_0(x)| \cdot |x|^{2/(p-1)}$  and  $\liminf_{x \rightarrow -\infty} |u_0(x)| \cdot |x|^{2/(p-1)}$  are finite. Consequently, if there exists

$$u_0(\infty) := \lim_{x \rightarrow \infty} u_0(x) \quad \text{or} \quad u_0(-\infty) := \lim_{x \rightarrow -\infty} u_0(x)$$

then

$$u_0(\infty) = 0 \quad \text{or} \quad u_0(-\infty) = 0, \tag{5.1}$$

respectively. The proof of the following lemma is based on arguments used in the proof of [27, Theorem 1.3].

**Lemma 5.1.** *Let  $u$  be a global solution of (1.10) with  $N = 1$ ,  $p > 1$  and  $u_0 \in C(\mathbb{R})$ ,  $Z := z_{\mathbb{R}}(u_0) < \infty$ . Let  $-\infty \leq x_1 < x_2 \leq \infty$  be such that the restriction of  $u_0$  to  $I := (x_1, x_2)$  is a monotone function which does not change sign and which is not identically zero. Let  $i \in \{1, 2\}$  be such that  $|u_0(x_i)| = \max\{|u_0(x_1)|, |u_0(x_2)|\}$ . Then  $|x_i| < \infty$  and there exists a constant  $C^* = C^*(Z, p)$  independent of  $u_0, x_1, x_2$  such that*

$$|u_0(x)| \leq C^* |x - x_i|^{-2/(p-1)} \quad (x \in (x_1, x_2)). \tag{5.2}$$

*Proof.* Assume  $|x_i| = \infty$ . Then (5.1) guarantees  $u_0(x_i) = 0$ . Since  $|u_0(x_i)| \geq |u_0(x)| \geq |u_0(x_{3-i})|$  for  $x \in (x_1, x_2)$ , we have  $u_0 \equiv 0$  on  $(x_1, x_2)$ . However, this contradicts our assumptions. Consequently,  $|x_i| < \infty$ .

Without loss of generality we may assume  $i = 1$  (otherwise consider the function  $\tilde{u}_0(x) := u_0(-x)$ ) and  $u(x_1) > 0$  (otherwise consider  $\tilde{u}_0(x) := -u_0(x)$ ). Consequently,  $x_1 \in \mathbb{R}$ ,  $u_0 \geq 0$  on  $(x_1, x_2)$  and  $u_0$  is nonincreasing in that interval. We may also assume  $x_1 = 0$  (otherwise consider  $\tilde{u}_0(x) := u_0(x + x_1)$ ).

The proof of [27, Theorem 1.3] shows the existence of  $0 < \alpha < \beta < \infty$  and a smooth function  $u_0^\tau$  with support in  $[\alpha, \beta]$  such that the solution  $u^\tau$  of (1.10) with initial data  $u_0^\tau$  exhibits  $(Z + 1)$ -polar blow-up. More precisely, there exists  $T^\tau < \infty$  and  $-\infty < y_1 < y_2 < \dots < y_{Z+1} < \infty$  such that either

$\lim_{t \rightarrow T^\tau} (-1)^j u^\tau(y_j, t) = \infty$ ,  $j = 1, 2, \dots, Z + 1$ , or  $\lim_{t \rightarrow T^\tau} (-1)^j u^\tau(y_j, t) = -\infty$ ,  $j = 1, 2, \dots, Z + 1$ .

Set  $L := \sup_{\mathbb{R}} |u_0^\tau|$  and, given  $\lambda > 0$ , denote

$$\begin{aligned} v^\lambda(x, t) &:= \lambda^{2/(p-1)} u^\tau(\lambda x, \lambda^2 t), & x \in \mathbb{R}, t \in [0, T^\tau/\lambda^2), \\ v_0^\lambda(x) &:= \lambda^{2/(p-1)} u_0^\tau(\lambda x), & x \in \mathbb{R}. \end{aligned}$$

Then  $|v_0^\lambda(x)| \leq L\lambda^{2/(p-1)}$  if  $x \in [\alpha/\lambda, \beta/\lambda]$ ,  $v_0^\lambda(x) = 0$  otherwise.

Fix  $\lambda > \beta/x_2$  and notice that  $\beta/\lambda < x_2$ . The relation

$$\min_{[\alpha/\lambda, \beta/\lambda]} u_0 > L\lambda^{2/(p-1)}$$

leads to a contradiction as in the proof of [27, Theorem 1.3] (denoting  $z(t) := z_{\mathbb{R}}(u(\cdot, t) - v^\lambda(\cdot, t))$  we would have  $z(0) \leq Z$  and  $z(t) \geq Z + 1$  for  $t \rightarrow T^\tau$ ). Hence,

$$u_0(\beta/\lambda) = \min_{[\alpha/\lambda, \beta/\lambda]} u_0 \leq L\lambda^{2/(p-1)}.$$

Considering  $x \in (0, x_2)$  and choosing  $\lambda := \beta/x$  we obtain

$$u_0(x) \leq L\beta^{2/(p-1)} x^{-2/(p-1)} \quad (x \in (0, x_2)),$$

which proves the assertion.  $\square$

**Corollary 5.2.** *Let  $u$  be a global classical solution of (1.10) with  $N = 1$ ,  $p > 1$  and  $u_0 \in C^1(\mathbb{R})$  satisfying  $z_{\mathbb{R}}(u_0) < \infty$  and  $z_{\mathbb{R}}(u_0') < \infty$ . Then  $\|u(\cdot, t)\|_{L^q(\mathbb{R})} \rightarrow 0$  as  $t \rightarrow \infty$  for any  $q > (p-1)/2$ ,  $q \geq 1$ .*

*Proof.* Theorem 1.7(iii) guarantees the estimate  $|u(x, t)| \leq Ct^{-1/(p-1)}$ . This estimate and Lemma 5.1 guarantee both  $u(\cdot, t) \in L^q(\mathbb{R})$  and  $\|u(\cdot, t)\|_{L^q(\mathbb{R})} \rightarrow 0$  as  $t \rightarrow \infty$ .  $\square$

**Remark 5.3.** It is likely that Lemma 5.1 (hence also Corollary 5.2 with  $q > N(p-1)/2$ ) remain true in the higher dimensional case if we consider radial solutions and  $p < p_S$ . In fact, the continuity of the blow-up time needed in the corresponding proof of existence of solutions with  $(Z+1)$ -polar blow-up follows from [33]. We refrain from proving such results since the proof would be quite long.

**Proposition 5.4.** *Let  $u$  be a classical solution of (1.10) with  $N = 1$ ,  $p > 1$  and  $u_0 \in C^1(\mathbb{R})$  satisfying  $z_{\mathbb{R}}(u_0), z_{\mathbb{R}}(u_0') \leq Z < \infty$ . Then  $\|u(t)\|_{\mathcal{E}} \rightarrow 0$  as  $t \rightarrow \infty$ .*

*Proof.* The decay of the  $L^{p+1}$ -norm follows from Corollary 5.2. Statement (iii) of Theorem 1.7 gives the estimate  $|u(x, t)| \leq Ct^{-1/(p-1)}$  and parabolic regularity estimates show

$$\|u_x\|_\infty := \|u_x\|_{L^\infty(\mathbb{R}^N \times [1, \infty))} < \infty.$$

Fix  $t \geq 1$ . Let  $I = (x_1, x_2)$  be any interval where  $u(\cdot, t)$  and  $u_x(\cdot, t)$  do not change sign. Then

$$\begin{aligned} \|u(\cdot, t)\|_{L^\infty(\mathbb{R})} &\geq |u(x_2, t) - u(x_1, t)| = \left| \int_{x_1}^{x_2} u_x(x, t) dx \right| \\ &\geq \frac{1}{\|u_x\|_\infty} \int_{x_1}^{x_2} u_x^2(x, t) dx, \end{aligned}$$

hence

$$\int_{\mathbb{R}} u_x^2(x, t) dx \leq (2Z + 1) \|u(\cdot, t)\|_{L^\infty(\mathbb{R})} \|u_x\|_\infty, \quad (5.3)$$

which implies  $\|u_x(\cdot, t)\|_{L^2(\mathbb{R})} \rightarrow 0$  as  $t \rightarrow \infty$  and concludes the proof.  $\square$

## 6 Periodic solutions

*Proof of Theorem 1.8.* In the proof we will employ with solutions of the Cauchy-Dirichlet problem

$$\left. \begin{aligned} u_t - \Delta u &= m(t)f(u), & x \in \Omega, \quad t > 0, \\ u &= 0, & x \in \partial\Omega, \quad t > 0, \\ u(x, 0) &= u_0(x), & x \in \Omega, \end{aligned} \right\} \quad (6.1)$$

where  $u_0$  is radially symmetric and  $m(t + kT) := m(t)$  for  $k = 1, 2, \dots$  and  $t \in [0, T]$ . The solution of (6.1) at time  $t$  will also be denoted by  $u(t; u_0)$  or  $u(\cdot, t; u_0)$  (if we want to emphasize its dependence on  $u_0$ ). The maximal existence time of this solution will be denoted by  $T_{\max}(u_0)$ . Our aim is to prove the existence of infinitely many solutions of (1.18). We adapt the basic idea of the proof of the existence of infinitely many equilibria in [32, Example 2]: solutions of (1.18) will be found in the  $\omega$ -limit set of a suitable subset of the boundary of the domain of attraction of zero in problem (6.1). Set

$$X := \{u \in H^1(\Omega) : u \text{ is radially symmetric}\},$$

$$D_A := \{u_0 \in X : T_{\max}(u_0) = \infty \text{ and } u(t; u_0) \rightarrow 0 \text{ in } X \text{ as } t \rightarrow \infty\}.$$

Then  $D_A$  is an open set in  $X$  containing 0. Indeed, this is an immediate consequence of the fact that the Cauchy problem (6.1) is well posed on  $X$  and its trivial solution is asymptotically stable. Further, given  $u_0 \in X$  and  $t \in [0, T_{\max}(u_0))$ , let

$$V(t) := V_{u_0}(t) := \frac{1}{2} \int_{\Omega} |\nabla u(x, t; u_0)|^2 dx - m(t) \int_{\Omega} F(u(x, t; u_0)) dx,$$

where  $F(s) := \int_0^s f(\xi) d\xi$ . Estimate (5.19) in [34] guarantees the existence of a constant  $C_V > 0$  (independent of  $u_0$ ) such that

$$V_{u_0}(t) \geq -C_V \quad (t \geq 0) \tag{6.2}$$

provided  $T_{\max}(u_0) = \infty$ . Let  $Y_m$  ( $m = 0, 1, 2, \dots$ ) denote the linear hull of the set  $\{\varphi_0, \varphi_1, \dots, \varphi_m\}$ , where  $\varphi_j(x) = \varphi_j(|x|) = \cos((2j+1)\pi|x|/2R)$ ,  $j = 0, 1, \dots$ , and let  $P_m$  be the orthogonal projection in  $X$  onto  $Y_m$ . Similarly as in [32] we have  $z(\varphi) \leq m$  for  $\varphi \in Y_m$  and  $z(\varphi) > m$  for  $\varphi \perp Y_m$ ,  $\varphi \neq 0$ , where  $z := z_{(0,R)}$  denotes the zero number in the interval  $(0, R)$ . In addition,  $z(\varphi') \leq m$  for all  $\varphi \in Y_m$ .

Let  $\mathcal{P} := \{u_0 \in X : T_{\max}(u_0) > T \text{ and } u(T; u_0) = u_0\}$ , thus  $\mathcal{P}$  is the set of initial data of  $T$ -periodic solutions. Assume that there exists  $k \in \{0, 1, \dots\}$  such that  $z(u_0) \neq k$  for all  $u_0 \in \mathcal{P}$ . We show that this leads to a contradiction which proves the theorem (notice that  $z(u(t; u_0)) = z(u_0)$  for all  $u_0 \in \mathcal{P}$  and  $t > 0$  due to the periodicity of  $u$  and monotonicity of  $z$ ).

Set  $Y := Y_k$  and  $M := \partial_Y(D_A \cap Y)$ , where  $\partial_Y$  denotes the boundary in  $Y$ . Since  $\lim_{u_0 \in Y, \|u_0\|_X \rightarrow \infty} V_{u_0}(0) = -\infty$ , estimate (6.2) guarantees that the set  $D_A \cap Y$  is bounded. This fact and Theorem 1.7 imply the existence of  $C > 0$  (independent of  $u_0, x$  and  $t$ ) such that

$$|u(x, t; u_0)| \leq C \quad (x \in \Omega, t > 0, u_0 \in \overline{D_A \cap Y}). \tag{6.3}$$

Denote

$$M^\tau := \{u(\tau; u_0) : u_0 \in M\},$$

$$O(M) := \bigcup_{\tau \geq 0} M^\tau, \quad \omega(M) := \bigcap_{s > 0} \overline{\bigcup_{\tau \geq s} M^\tau}.$$

We remark that the trajectories and limit sets of points or sets considered here are those of the time-periodic dynamical process generated by (6.1).

Equivalently, we could work with the discrete-time dynamical system generated by the period map of (6.1), however, working with continuous time has some advantages in applications of degree arguments. We refer to [20] for basic properties of  $\omega$ -limit sets and related concepts.

Standard parabolic regularity estimates and (6.3) guarantee that  $\omega(M)$  is nonempty and compact. Moreover, with any  $u_1 \in \omega(M)$ ,  $\omega(M)$  contains the trajectory  $\{u(\cdot, t) : t \in \mathbb{R}\}$  of an entire solution  $u$  of (6.1) satisfying  $u(\cdot, s) = u_1$  for some  $s \in [0, T]$ . Notice also that  $0 \notin \omega(M)$  due to the stability of the zero solution.

Consider an entire solution  $u : \mathbb{R} \rightarrow \omega(M)$ . By [12, Theorem 3.2], the  $\omega$ - and  $\alpha$ -limit sets of this solution are given by  $T$ -periodic solutions. Consequently,  $\omega(M)$  consists of  $T$ -periodic solutions and their connecting orbits. Since  $z(u_0) \leq k$  for any  $u_0 \in M \subset Y_k$ , the monotonicity of the zero number [12] implies  $z(u) \leq k$  for all  $u \in O(M) \cup \omega(M)$ . In addition  $z(u) < k$  for any  $u \in \omega(M)$  since  $z(u) \neq k$  for  $u \in \mathcal{P}$ . In particular, this implies  $k > 0$ . Since  $z(u) > k$  for  $u \perp Y_k$ ,  $u \neq 0$ , we have  $0 \notin P_k(O(M))$ . Similarly,  $0 \notin P_{k-1}(\omega(M))$ , hence  $0 \notin P_{k-1}(M^\tau) = P_{k-1}P_k(M^\tau)$  for  $\tau$  sufficiently large. This guarantees that  $H_1(t, u) := (1-t)u + tP_{k-1}u \neq 0$  for any  $t \in [0, 1]$  and  $u \in P_k M^\tau$ . Since  $H_2(t, u) := (1-t)u + t\varphi_k \neq 0$  for any  $t \in [0, 1]$  and  $u \in Y_{k-1} \setminus \{0\}$ , we may use the homotopies  $H_1$  and  $H_2$  to contract the set  $P_k M^\tau$  in  $Y \setminus \{0\}$  to the single point  $\varphi_k \neq 0$ . Consequently, the Brouwer degree  $\deg(P_k u(\tau; \cdot), 0, D_A \cap Y)$  in the finite dimensional space  $Y$  equals zero. However, this gives a contradiction, since  $0 \notin P_k(O(M))$  implies

$$\begin{aligned} \deg(P_k u(\tau; \cdot), 0, D_A \cap Y) &= \deg(P_k u(t; \cdot), 0, D_A \cap Y) \\ &= \deg(P_k u(0; \cdot), 0, D_A \cap Y) = \deg(\text{Id}, 0, D_A \cap Y) = 1 \end{aligned}$$

for any  $t \in [0, \tau]$ . □

Next we consider the problem

$$\left. \begin{aligned} u_t - \Delta u &= m(t)(u^p - u), & x \in \mathbb{R}^N, t \in (0, T), \\ u(x, t) &\rightarrow 0, & |x| \rightarrow \infty, t \in (0, T), \\ u(x, 0) &= u(x, T), & x \in \mathbb{R}^N. \end{aligned} \right\} \quad (6.4)$$

**Theorem 6.1.** *Assume (1.19) and  $1 < p < p_S$ . Then there exists a positive radially symmetric solution of problem (6.4).*

In what follows we set  $\underline{m} := \inf_t m(t)$ ,  $\overline{m} := \sup_t m(t)$ ,

$$\overline{f}(u) := \begin{cases} \overline{m}(u^p - u) & \text{if } u \geq 1, \\ \underline{m}(u^p - u) & \text{if } u \leq 1, \end{cases} \quad \underline{f}(u) := \begin{cases} \underline{m}(u^p - u) & \text{if } u \geq 1, \\ \overline{m}(u^p - u) & \text{if } u \leq 1. \end{cases}$$

In the proof of Theorem 6.1 we will need a supersolution whose existence is guaranteed by the following lemma.

**Lemma 6.2.** *There exists  $\varepsilon > 0$  and a  $C^2$ -function  $g : [0, \infty) \rightarrow (0, \infty)$  with exponential decay satisfying  $g(0) = 1 + \varepsilon$ ,  $g''(r) + \overline{f}(g(r)) = 0$  for  $r > 0$ .*

*Proof.* Let  $\alpha > 0$  satisfy  $\alpha^2 < \underline{m}(2^{p-1} - 1)/2^p$  and  $r_0 := (\log 2)/\alpha$ . Then the function  $h(r) := e^{-\alpha(r+r_0)}$  is a supersolution for the problem

$$g(0) = 1/2, \quad g'' + \overline{f}(g) = 0 \quad \text{for } r > 0. \quad (6.5)$$

Since zero is a subsolution for (6.5), problem (6.5) possesses a solution  $g_1$  satisfying  $0 < g_1 \leq h$  on  $[0, \infty)$ . Obviously  $g_1'(0) \leq h'(0) < 0$ . Solving the initial value problem  $g_2(0) = 1/2$ ,  $g_2'(0) = g_1'(0)$ ,  $g_2'' + \overline{f}(g_2) = 0$  for  $r < 0$ , we obtain  $\varepsilon > 0$  and  $r_1 > 0$  such that  $g_2(-r_1) = 1 + \varepsilon$  and  $g_2' < 0$  on  $(-r_1, 0)$  (notice that  $g_2$  is convex on the set  $\{r : g_2(r) < 1\}$ ). Now it is sufficient to set  $g(r) := g_2(r - r_1)$  for  $r < r_1$  and  $g(r) := g_1(r - r_1)$  for  $r \geq r_1$ .  $\square$

*Proof of Theorem 6.1.* Consider the Cauchy problem

$$\left. \begin{aligned} u_t - \Delta u &= m(t)(u^p - u), & x \in \mathbb{R}^N, t > 0, \\ u(x, 0) &= u_0(x), & x \in \mathbb{R}^N, \end{aligned} \right\} \quad (6.6)$$

where  $u_0 \geq 0$  is radially symmetric and  $m(t + kT) := m(t)$  for  $k = 1, 2, \dots$  and  $t \in [0, T]$ . Let  $\phi \in C^\infty(\mathbb{R}^N, [0, \infty))$  be a radially symmetric and radially nonincreasing function with compact support,  $\phi \not\equiv 0$ , and consider initial data in the form  $u_0 := \alpha\phi$ , where  $\alpha > 0$ . Clearly, the trivial solution is asymptotically stable, hence if  $\alpha$  is small then the solution  $u = u_\alpha$  of (6.6) exists globally and tends to zero as  $t \rightarrow \infty$ . On the other hand, if  $\alpha$  is large enough then  $u_\alpha$  blows up in finite time. Indeed, this is shown easily by a comparison with the Dirichlet problem for the same equation on the ball  $B_R$ , where  $R$  chosen so large that  $B_R$  contains the support of  $\phi$ . For the Dirichlet problem, blow up for solutions with initial data  $\alpha\phi|_{B_R}$  with  $\alpha$  large follows by an energy argument (cp. (6.2)). We can thus conclude that the number

$\alpha^* := \sup\{\alpha > 0 : u_\alpha \text{ exists globally}\}$  is finite and positive. In what follows we fix  $u_0 := \alpha^* \phi$  and consider the threshold solution  $u = u_{\alpha^*}$  as a function depending on the radial variable  $r$  and  $t \geq 0$ .

Theorem 1.7(ii) guarantees that  $u$  exists globally and is bounded. Also, since  $\phi$  is radially nonincreasing,  $u(\cdot, t)$  is such for each  $t \geq 0$ . Let  $\varepsilon$  and  $g$  be as in Lemma 6.2. We next show that for  $R > 0$  large enough, we have

$$u(R, t) < 1 + \varepsilon \quad \text{for all } t \geq 0. \quad (6.7)$$

Assume on the contrary that there exist  $R_k \rightarrow \infty$  and  $t_k \in [0, \infty)$  such that  $u(R_k, t_k) \geq 1 + \varepsilon$ . Then  $u(r, t_k) \geq 1 + \varepsilon$  for all  $r \in [0, R_k]$  due to the radial monotonicity of  $u$ . Consequently,  $u(r, t + t_k) \geq w_k(r, t)$ , where  $w_k$  is the solution of

$$\begin{aligned} w_t - \Delta w &= \underline{f}(w), \quad x \in \mathbb{R}^N, \quad t > 0, \\ w(\cdot, 0) &= (1 + \varepsilon)\chi_{B_{R_k}}. \end{aligned}$$

Let  $\psi$  be the solution of the ODE  $\psi'(t) = \underline{f}(\psi(t))$ ,  $t \geq 0$ , with  $\psi(0) = 1 + \varepsilon$ . The function  $\psi$  blows up at some  $T < \infty$  and  $w_k(r, t) \nearrow \psi(t)$  as  $k \rightarrow \infty$  for all  $r \in [0, \infty)$  and  $t < T$ , which contradicts the boundedness of  $u$ . This proves that (6.7) is true. Enlarging  $R$  if necessary, we may assume  $u_0(r, 0) = 0$  for  $r \geq R$ . Then it follows (using the equation for  $u$  in the radial variable and  $u_r \leq 0$ ) that the function  $U(r, t) = g(r - R)$  is a supersolution to  $u$  for  $r \geq R$ . We conclude that

$$u(r, t) \leq g(r - R) \quad (r \geq R, t \geq 0). \quad (6.8)$$

We now use a standard zero-number argument (cp. [16, Proof of Lemma 3.5]) to prove that  $u(\cdot, t)$  approaches a periodic solution as  $t \rightarrow \infty$ . Set  $v(r, t) := u(r, t + T) - u(r, t)$ ,  $r, t \geq 0$ . Since  $u(\cdot, 0)$  has compact support and  $u(r, t) > 0$  for  $r \geq 0$  and  $t > 0$ , we see that  $v(r, 0) > 0$  for  $r$  large enough. Hence, by [12], the zero number  $z_v(t) := z_{(0, \infty)}v(\cdot, t)$  is finite for  $t > 0$ , it is nonincreasing in  $t$ , and, since  $v_r(0, t) = 0$  for all  $t$ ,  $z_v(t)$  drops whenever  $v(0, t) = 0$ . The latter can occur only finitely many times which implies that  $v(0, t)$  is of constant (non-zero) sign for all  $t$  sufficiently large, say for  $t \geq t_0$ . Fixing  $t \geq t_0$ , the sequence  $\{u(0, t + kT)\}_k$  is monotone and bounded. Let  $W(t)$  denote its limit.

Consider the sequence  $\{u(\cdot, \cdot + kT)\}_k$ . Parabolic regularity estimates guarantee that this sequence is relatively compact in  $C_{loc} = C_{loc}([0, \infty) \times [0, \infty))$ . Fix a subsequence  $\{u(\cdot, \cdot + k_j T)\}_j$  converging to a limit function  $w$

in  $C_{loc}$ . Then  $w$  is a global radial nonnegative solution of (6.6) with initial condition  $u_0 = w(\cdot, 0)$ . In view of (6.8),  $\{u(\cdot, k_j T)\}_j$  converges to  $w(\cdot, 0)$  in  $L^\infty([0, \infty))$ . In particular, by the asymptotic stability of the trivial solution,  $w$  is not identical to zero, hence it is positive by the maximum principle. In addition,  $w(r, t) \leq g(r - R)$  for  $r \geq R$  and  $t \geq 0$ . Let now  $k_j \rightarrow \infty$  be any other sequence such that  $\{u(\cdot, \cdot + \tilde{k}_j T)\}_j$  converges to a limit function  $\tilde{w}$  in  $C_{loc}$ . We shall prove that  $\tilde{w} = w$  for  $t \geq t_0$ , hence the whole sequence  $u(\cdot, \cdot + kT)$  converges for  $t \geq t_0$ . It is then easy to see that the limit  $w$  is  $T$ -periodic for  $t \geq t_0$  which concludes the proof.

Assume that  $\tilde{w}(r_1, t_1) \neq w(r_1, t_1)$  for some  $r_1 \geq 0$  and  $t_1 \geq t_0$ . We obviously have  $w(0, t) = \tilde{w}(0, t) = W(t)$  for all  $t \geq t_0$ , hence, in particular,  $r_1 > 0$ . Fix  $t_2 > t_1$  such that  $\tilde{w}(r_1, t) \neq w(r_1, t)$  for all  $t \in [t_1, t_2]$ . Then the zero number  $z_w(t) := z_{(0, r_1)}(\tilde{w}(\cdot, t) - w(\cdot, t))$  is finite for  $t \in (t_1, t_2]$  and, as  $\tilde{w}(0, t) = w(0, t)$  and  $\tilde{w}_r(0, t) = w_r(0, t) = 0$ , it has to drop at each  $t \in (t_1, t_2)$ , which is an obvious contradiction.  $\square$

**Remark 6.3.** If  $N = 1$  then the positive radially symmetric solution  $u$ , as constructed in Theorem 6.1, is the unique positive solution of (6.4) up to translations. Moreover, (6.4) does not possess any nodal radially symmetric solutions if  $N = 1$  (see [16, Theorem 1.2]). Thus there are exactly three solutions of (6.4) up to translations:  $u$ ,  $-u$ , and 0 (as usual,  $u^p$  is interpreted as  $|u|^{p-1}u$  here). On the other hand, if  $N > 1$  and  $m$  is independent of  $t$  then an analogue of Theorem 1.8 for problem (6.4) was shown in [23]; cf. also [5] for an earlier result and [13] and the references therein for possible generalizations. These results suggest that (6.4) might possess infinitely many solutions if  $N > 1$ .

## References

- [1] N. Ackermann and T. Bartsch, Superstable manifolds of semilinear parabolic problems, *J. Dynam. Differential Equations* **17** (2005), 115–173.
- [2] N. Ackermann, T. Bartsch, P. Kaplický and P. Quittner, A priori bounds, nodal equilibria and connecting orbits in indefinite superlinear parabolic problems, *Trans. Amer. Math. Soc.* **360** (2008), 3493–3539.
- [3] S. Angenent, The zero set of a solution of a parabolic equation, *J. reine angew. Math.* **390** (1988), 79–96.

- [4] A. Bahri and P.-L. Lions, Solutions of superlinear elliptic equations and their Morse indices, *Commun. Pure Appl. Math.* **45** (1992), 1205–1215.
- [5] H. Berestycki and P.-L. Lions, Nonlinear scalar field equations. II: Existence of infinitely many solutions, *Arch. Ration. Mech. Anal.* **82** (1983), 347–375.
- [6] M.-F. Bidaut-Véron, Initial blow-up for the solutions of a semilinear parabolic equation with source term. Equations aux dérivées partielles et applications, articles dédiés à Jacques-Louis Lions, Gauthier-Villars, Paris, 1998, pp. 189–198.
- [7] T. Cazenave, F. Dickstein and F.B. Weissler, Sign-changing stationary solutions and blow up for the nonlinear heat equation in a ball, Preprint 2008.
- [8] T. Cazenave, F. Dickstein and F.B. Weissler, On the structure of global solutions of the nonlinear heat equation in a ball, Preprint 2008.
- [9] W. Chen and C. Li, Classification of solutions of some nonlinear elliptic equations, *Duke Math. J.* **63** (1991), 615–622.
- [10] X.-Y. Chen, A strong unique continuation theorem for parabolic equations, *Math. Ann.* **311** (1998), 603–630.
- [11] X.-Y. Chen and H. Matano, Convergence, asymptotic periodicity, and finite-point blow-up in one-dimensional semilinear heat equations, *J. Differential Equations*, **78** (1989), 160–190.
- [12] X.-Y. Chen and P. Poláčik, Asymptotic periodicity of positive solutions of reaction diffusion equations on a ball, *J. Reine Angew. Math.* **472** (1996), 17–51.
- [13] M. Conti, L. Merizzi and S. Terracini, Radial solutions of superlinear equations on  $\mathbb{R}^N$ . I: A global variational approach, *Arch. Ration. Mech. Anal.* **153** (2000), 291–316.
- [14] M.J. Esteban, On periodic solutions of superlinear parabolic problems, *Trans. Amer. Math. Soc.* **293** (1986), 171–189.

- [15] M.J. Esteban, A remark on the existence of positive periodic solutions of superlinear parabolic problems, *Proc. Amer. Math. Soc.* **102** (1988), 131–136.
- [16] E. Feireisl and P. Poláčik, Structure of periodic solutions and asymptotic behavior for time-periodic reaction-diffusion equations on  $\mathbb{R}$ , *Adv. Differential Equations*, **5** (2000), 583–622.
- [17] A. Farina, On the classification of solutions of the Lane-Emden equation on unbounded domains of  $\mathbb{R}^N$ , *J. Math. Pures Appl.* **87** (2007), 537–561.
- [18] G. Fusco and S.M. Verduyn Lunel, Order structures and the heat equation, *J. Differential Equations* **139** (1997), 104–145.
- [19] B. Gidas and J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, *Comm. Pure Appl. Math.* **34** (1981), 525–598.
- [20] J. K. Hale, *Asymptotic behavior of dissipative systems*, American Mathematical Society, Providence, RI, 1988.
- [21] N. Hirano and N. Mizoguchi, Positive unstable periodic solutions for superlinear parabolic equations, *Proc. Amer. Math. Soc.* **123** (1995), 1487–1495.
- [22] J. Hůska, Periodic solutions in superlinear parabolic problems, *Acta Math. Univ. Comenian. (N.S.)* **71** (2002), 19–26.
- [23] C. Jones, T. Küpper, On the infinitely many solutions of a semilinear elliptic equation, *SIAM J. Math. Anal.* **17** (1986), 803–835.
- [24] O. Kavian, Remarks on the large time behaviour of a nonlinear diffusion equation, *Ann. Inst. H. Poincaré, Analyse Non Linéaire* **4** (1987), 423–452.
- [25] T. Kawanago, Asymptotic behavior of solutions of a semilinear heat equation with subcritical nonlinearity, *Ann. Inst. Henri Poincaré, Anal. non linéaire* **13** (1996), 1–15.
- [26] H. Matano, Nonincrease of the lap-number of a solution for a one-dimensional semilinear parabolic equation, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **29** (1982), 401–441.

- [27] N. Mizoguchi and E. Yanagida, Critical exponents for the blowup of solutions with sign changes in a semilinear parabolic equation, II, *J. Differ. Equations* **145** (1998), 295–331.
- [28] W.-M. Ni, Uniqueness of solutions of nonlinear Dirichlet problems, *J. Differ. Equations* **50** (1983), 289–304.
- [29] P. Poláčik and P. Quittner, A Liouville-type theorem and the decay of radial solutions of a semilinear heat equation, *Nonlinear Analysis, TMA* **64** (2006), 1679–1689.
- [30] P. Poláčik and P. Quittner and Ph. Souplet, Singularity and decay estimates in superlinear problems via Liouville-type theorems, I: Elliptic equations and systems, *Duke Math. J.* **139** (2007), 555–579.
- [31] P. Poláčik and P. Quittner and Ph. Souplet, Singularity and decay estimates in superlinear problems via Liouville-type theorems. Part II: Parabolic equations, *Indiana Univ. Math. J.* **56** (2007), 879–908.
- [32] P. Quittner, Boundedness of trajectories of parabolic equations and stationary solutions via dynamical methods, *Differ. Integral Equations* **7** (1994), 1547–1556.
- [33] P. Quittner, Continuity of the blow-up time and a priori bounds for solutions in superlinear parabolic problems, *Houston J. Math.* **29** (2003), 757–799.
- [34] P. Quittner, Multiple equilibria, periodic solutions and a priori bounds for solutions in superlinear parabolic problems, *NoDEA Nonlinear Differential Equations Appl.* **11** (2004), 237–258.
- [35] P. Quittner and Ph. Souplet, *Superlinear parabolic problems. Blow-up, global existence and steady states*, Birkhäuser Advanced Texts, Birkhäuser, Basel 2007.
- [36] Ph. Souplet, Sur l’asymptotique des solutions globales pour une équation de la chaleur semi-linéaire dans des domaines non bornés, *C. R. Acad. Sci. Paris Sér. I Math.* **323** (1996), 877–882.
- [37] M. Willem, *Minimax theorems*, Birkhäuser, Boston, 1996.