Fučik Spectrum for Schrödinger Equations and Applications

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Abstract

We investigate Fučik spectrum for Schrödinger equations $-\Delta u + V(x)u = \alpha u^+ + \beta u^-,\ x \in \mathbb{R}^N$. We construct the first nontrivial curve in the spectrum by minimax methods, and show some properties of the curve, for example, we show that the eigenfunctions corresponding to eigenvalues (α, β) in the first Fucik curve are foliated Schwarz symmetric if $V(x) = V(|x|), \forall x \in \mathbb{R}^N$. Finally we establish some existence results of multiple solutions for jumping nonlinearity problems.

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1 Introduction

In this paper, we are concerned with Fučik Spectrum for Schrödinger equations.

$$-\Delta u + V(x)u = \alpha u^+ + \beta u^-,\ x \in \mathbb{R}^N, \tag{1.1}$$

where $u^+ = \max\{u, 0\}, u^- = \min\{u, 0\}$, and u satisfies the boundary condition $u(x) \to 0$ as $|x| \to \infty$.

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The Fučik spectrum of $-\Delta + V$ is defined as the set Σ of those $(\alpha, \beta) \in \mathbb{R}^2$ such that (1.1) has a nontrivial solution. The generalized notion of spectrum was introduced in the 1970s by Fučik [Fu] and Dancer [Da] in connection with the study of the so-called jumping non-linearities

$$-\Delta u = \alpha u^+ + \beta u^-, \ x \in \Omega, u|_{\partial\Omega} = 0. \tag{1.2}$$

Several works have been devoted since that time to the Fučik Spectrum Σ of (1.2) for bounded domain Ω, in [Sch] the author got the existence of Fučik spectrum near the points (λ_k, λ_k), where λ_k is the eigenvalues of $-\Delta$ with $\lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$. In [CFG], the authors studied the Fučik spectrum of the p-Laplacian on a bounded domain in $\mathbb{R}^N$, and get the first curve in the Fučik spectrum, they also obtain some properties for the curve . In [ACC], the authors studied the beginning of the Fučik spectrum with weights for the right side.

In this paper, we investigate Fučik spectrum for Schrödinger equations. We construct the first nontrivial curve in the spectrum by minmax methods, and show some properties of the curve, for example, we show that the eigenfunctions corresponding to eigenvalues (α, β) in the first Fucik curve are foliated Schwarz symmetric if $V(x) = V(|x|), \forall x \in \mathbb{R}^N$. We finally establish some existence results of multiple solutions for nonlinear jumping problems in $\mathbb{R}^N$. The potential function V satisfies the following conditions:

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies $\inf V(x) = a_0 > 0$.
(V_2) $\lim\limits_{|x|\to\infty} V(x) = +\infty$.
Let $H := \{u \in H^1(\mathbb{R}^N) : \|u\| < \infty\}$ is a Hilbert space with the inner product

$$(u, v) := \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(x)uv)dx,$$

and with the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx \right)^{\frac{1}{2}}.$$

Under $(V_1), (V_2)$, $H \hookrightarrow L^2(\mathbb{R}^N)$ is compact.

If the Schrödinger operator $-\Delta + V$ on $L^2(\mathbb{R}^N)$ has essential spectrum, the problem becomes more difficult. In this paper we first investigate a situation where $-\Delta + V$ may have only isolated eigenvalues. Under assumptions $(V_1), (V_2)$, it is well known that $\sigma(-\Delta + V) = \{\lambda_1, \lambda_2, \cdots, \lambda_k \cdots\}$, and $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \lambda_k \cdots$ with eigenfunctions $e_i, i = 1, 2, \cdots$, and $\int_{\mathbb{R}^N} e_i^2 = 1$, $e_1 > 0$.

2 Construction of the curve by minimax methods

This section is devoted to the construction of a nontrivial curve in the spectrum.

Consider the functional

$$J_s(u) = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) - s \int_{\mathbb{R}^N} (u^+)^2. \tag{2.1}$$

J_s is a C^1 functional on H. The restriction $\tilde{J}_s$ of J_s to

$$S = \{u \in H : I(u) = \int_{\mathbb{R}^N} u^2 = 1\}. \tag{2.2}$$

By Lagrange multiplier rule, $u \in S$ is a critical point of $\tilde{J}_s$ if and only if there exists $t \in \mathbb{R}$ such that $J_s'(u) = tI'(u)$ i.e.,

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla v + V(x)uv - s \int_{\mathbb{R}^N} u^+ v = t \int_{\mathbb{R}^N} uv, \forall v \in H. \tag{2.3}$$

This implies that

$$-\Delta u + V(x)u = (s+t)u^+ + tu^- \quad \text{in} \quad \mathbb{R}^N \tag{2.4}$$

hold in the weak sense. i.e., $(s+t, t) \in \Sigma$. Taking $v = u$ in (2.3), one also see that the Lagrange multiplier t is equal to the corresponding critical value $\tilde{J}_s(u)$, we have thus the following

Lemma 2.1. *The points in Σ on the parallel to the diagonal passing through $(s, 0)$ are exactly of the form $(s + \tilde{J}_s(u), \tilde{J}_s(u))$ with u a critical point of $\tilde{J}_s$.*

From now on we assume $s \geq 0$, which is no restriction since Σ is clearly symmetric with respect to the diagonal.

A first critical point of $\tilde{J}_s$ comes from global minimization.

Indeed,

$$\tilde{J}_s \geq \lambda_1 \int_{\mathbb{R}^N} u^2 - s \int_{\mathbb{R}^N} (u^+)^2 \geq \lambda_1 - s, \text{ for all } u \in S,$$

and one has $\tilde{J}_s(u) = \lambda_1 - s$ for $u = e_1$. So we get the following

Proposition 2.1. *e_1 is a global minimum of $\tilde{J}_s$ with $\tilde{J}_s(e_1) = \lambda_1 - s$, The corresponding point in Σ is $(\lambda_1, \lambda_1 - s)$, which lies on the vertical line through (λ_1, λ_1).*

A second critical point of $\tilde{J}_s$ comes from the following

Proposition 2.2. *$-e_1$ is a strict local minimum of $\tilde{J}_s$ and $\tilde{J}_s(-e_1) = \lambda_1$. The corresponding point in Σ is $(\lambda_1 + s, \lambda_1)$, which lies on the horizontal line through (λ_1, λ_1).*

When $s = 0$, the two critical values $\tilde{J}_s(e_1)$ and $\tilde{J}_s(-e_1)$ coincide as well as the corresponding points in Σ.

Proof. Assume by contradiction that there exists a sequence $u_n \in S$ with $u_n \neq -e_1, u_n \to -e_1$ in H and $\tilde{J}_s(u_n) \leq \lambda_1$. We first observe that u_n changes sign for n sufficiently large. Indeed since $u_n \to -e_1$, u_n must be negative somewhere. Moreover, if $u_n \leq 0$ a.e. in $\mathbb{R}^N$, then

$$\tilde{J}_s(u_n) = \int_{\mathbb{R}^N} |\nabla u_n|^2 + V(x)u_n^2 > \lambda_1 \tag{2.5}$$

since $u_n \neq \pm e_1$, and this contradicts $\tilde{J}_s(u_n) \leq \lambda_1$. So u_n changes sign.

Let $\gamma_n = \int_{\mathbb{R}^N}[|\nabla u_n^+|^2 + V(x)(u_n^+)^2]/\int_{\mathbb{R}^N}(u_n^+)^2$, we have

(2.6)
$$\begin{aligned} \tilde{J}_s(u_n) &= \int_{\mathbb{R}^N}[|\nabla u_n^+|^2 + V(x)(u_n^+)^2] + \int_{\mathbb{R}^N}[|\nabla u_n^-|^2 + V(x)(u_n^-)^2] - s\int_{\mathbb{R}^N}(u_n^+)^2 \\ &\geq (r_n - s)\int_{\mathbb{R}^N}(u_n^+)^2 + \lambda_1\int_{\mathbb{R}^N}(u_n^-)^2. \end{aligned}$$

On the other hand,

$$\tilde{J}_s(u_n) \leq \lambda_1 = \lambda_1\int_{\mathbb{R}^N}(u_n^+)^2 + \lambda_1\int_{\mathbb{R}^N}(u_n^-)^2. \tag{2.7}$$

Therefore, by $\int_{\mathbb{R}^N}(u_n^+)^2 > 0$ we obtain

$$\gamma_n - s \leq \lambda_1. \tag{2.8}$$

Now let $A_n := \{x \in \mathbb{R}^N | u_n(x) > 0\} \cap A$, where $A := B(0,r) = \{x \in \mathbb{R}^N, |x| \leq r\}$. Since $u_n \to -e_1$ in $L^2(\mathbb{R}^N)$ by (V_2), then we have that $mes(A_n) \to 0$, otherwise, $\int_{\mathbb{R}^N}|u_n - (-e_1)|^2 \geq \int_{A_n}(e_1)^2 > \delta > 0$, which contradicts that $u_n \to -e_1$ in $L^2(\mathbb{R}^N)$.

We first prove that on A,

$$\frac{\int_A[|\nabla u_n^+|^2 + V(x)(u_n^+)^2]}{\int_A(u_n^+)^2} \to +\infty, \text{ as } n \to \infty. \tag{2.9}$$

If not, then let $w_n = u_n/(\int_A(u_n^+)^2)^{1/2}$, we have $\int_A[|\nabla w_n^+|^2 + V(x)(w_n^+)^2]$ contains a bounded subsequence. Then for a further subsequence, $w_n \to w$ in $L^2(A)$. Clearly $w \geq 0$ a.e. and $\int_A w^2 \geq 1$. Thus for some $\varepsilon > 0, \rho = mes\{x \in A : w(x) > \varepsilon\} > 0$. We deduce that $mes\{x \in A : w_n(x) > \varepsilon/2\} > \rho/2$ for n sufficiently large, which contradicts that $mesA_n \to 0$.

On the other hand, on $\mathbb{R}^N \setminus A$, we have for any n

$$\frac{\int_{\mathbb{R}^N\setminus A}[|\nabla u_n^+|^2 + V(x)(u_n^+)^2]}{\int_{\mathbb{R}^N\setminus A}(u_n^+)^2} > C_r > 0, \tag{2.10}$$

where $C_r \to +\infty$ as $r \to +\infty$ by (V_2). By (2.9)(2.10), we get that for n sufficiently large

(2.11)
$$\int_A[|\nabla u_n^+|^2 + V(x)(u_n^+)^2] + \int_{\mathbb{R}^N\setminus A}[|\nabla u_n^+|^2 + V(x)(u_n^+)^2] \geq C_r[\int_A(u_n^+)^2 + \int_{\mathbb{R}^N\setminus A}(u_n^+)^2],$$

i.e.,

$$\gamma_n \geq C_r. \tag{2.12}$$

Which is a contradiction by (2.8), since $C_r \to +\infty$ as $r \to +\infty$. $\square$

To get a third critical point, we will use a version of the mountain pass theorem on a C^1 manifold, which we now recall.

Let E be a real Banach space and let

$$M = \{u \in E : g(u) = 1\},$$

where $g \in C^1(E, \mathbb{R})$ and 1 is a regular value of g. For $f \in C^1(E, \mathbb{R})$, the norm of the derivative at u of the restriction $\tilde{f}$ of f to M is defined as

$$\|\tilde{f}'(u)\|_* = \min\{\|f'(u) - tg'(u)\|_{E^*} : t \in \mathbb{R}\},$$

where $\|\cdot\|_{E^*}$ denotes the norm on the dual space E^*. We recall that f is said to satisfy the (P.S.) condition on M, if for any sequence $u_n \in M$ such that $f(u_n)$ is bounded and $\|\tilde{f}'(u_n)\|_* \to 0$, one has that u_n admits a convergent subsequence. The proposition below follows from Theorem 3.2 in [GN].

Proposition 2.3. *Let $u_0, u_1 \in M$ and let $\varepsilon > 0$ be such that $\|u_1 - u_0\|_E > \varepsilon$ and*

$$\inf\{f(u) : u \in M \text{ and } \|u - u_0\|_E = \varepsilon\} > \max\{f(u_0), f(u_1)\}. \tag{2.13}$$

Assume that f satisfies the (P.S.) condition on M and that

$$\Gamma = \{\gamma \in C([-1, +1], M) : \gamma(-1) = u_0 \text{ and } \gamma(1) = u_1\}$$

is nonempty. Then

$$c = \inf_{\gamma \in \Gamma} \max_{u \in \gamma[-1,1]} f(u)$$

is a critical value of $\tilde{f}$ (i.e., there exists $u \in M$ with $\|\tilde{f}'(u)\|_* = 0$ and $f(u) = c$).

We will apply Proposition 2.3 with $E = H, f = J_s$ and $g = I$. First we give two preliminary results. The first one concerns the (P.S.) condition while the second one describes the geometry of $\tilde{J}_s$ near the local minimum $-e_1$.

Lemma 2.2. *$\tilde{J}_s$ satisfies the (P.S.) condition on S.*

Proof. Let $u_n \in S$ and $t_n \in \mathbb{R}$ be sequences such that for some constant C,

$$|J_s(u_n)| \leq C \tag{2.14}$$

and

$$\left| \int_{\mathbb{R}^N} [\nabla u_n \nabla v + V(x) u_n v] - s \int_{\mathbb{R}^N} u_n^+ v - t_n \int_{\mathbb{R}^N} u_n v \right| \leq \varepsilon_n \|v\| \tag{2.15}$$

for all $v \in H$ where $\varepsilon_n \to 0$. By (2.14), we know that u_n remains bounded in H, consequently, for a subsequence, $u_n \to u_0$ weakly in H and strongly in $L^2(\mathbb{R}^N)$.

Let $v = u_n$ in (2.15), we get that t_n is bounded.

Let $v = u_n - u_0$ in (2.15), we get

(2.16)
$$\begin{aligned} &\left|\int_{\mathbb{R}^N} \nabla u_n \nabla(u_n - u_0) + V(x)u_n(u_n - u_0)\right| \\ \leq\ & s\left|\int_{\mathbb{R}^N} u_n^+(u_n - u_0)\right| + \left|t_n \int_{\mathbb{R}^N} u_n(u_n - u_0)\right| + \varepsilon_n\|u_n - u_0\| \to 0, (n \to \infty) \end{aligned}$$

i.e.,

$$\|u_n\| \to \|u_0\|, (n \to \infty).$$

Then we get

$$u_n \to u_0 \text{ in } H, (n \to \infty).$$

□

Lemma 2.3. *Let $\varepsilon_0 > 0$ be such that*

$$\tilde{J}_s(u) > \tilde{J}_s(-e_1) \tag{2.17}$$

for all $u \in B(-e_1, \varepsilon_0) \cap S$ with $u \neq -e_1$, where the Ball B is taken in H. Then, for any $0 < \varepsilon < \varepsilon_0$,

$$\inf\{\tilde{J}_s(u) : u \in S, \|u - (-e_1)\| = \varepsilon\} > \tilde{J}_s(-e_1). \tag{2.18}$$

Proof. Assume by contradiction that the infimum in (2.18) is equal to $\tilde{J}_s(-e_1) = \lambda_1$ for some ε with $0 < \varepsilon < \varepsilon_0$. So there exists a sequence $u_n \in S$ with $\|u_n - (-e_1)\| = \varepsilon$ and $\tilde{J}_s(u_n) \leq \lambda_1 + 1/2n^2$. Let

$$C = \{u \in S : \varepsilon - \delta \leq \|u - (-e_1)\| \leq \varepsilon + \delta\},$$

where $\delta > 0$ is chosen such that $0 < \varepsilon - \delta$ and $\varepsilon + \delta < \varepsilon_0$. Under the contradiction hypothesis and (2.17), we have that $\inf\{\tilde{J}_s(u) : u \in C\} = \lambda_1$. We now apply for each n Ekeland's principle to the functional $\tilde{J}_s$ on C to get the existence of $v_n \in C$ such that

$$\tilde{J}_s(v_n) \leq \tilde{J}_s(u_n), \tag{2.19}$$

$$\|v_n - u_n\| \leq 1/n, \tag{2.20}$$

$$\tilde{J}_s(v_n) \leq \tilde{J}_s(u) + \frac{1}{n}\|u - v_n\|, \forall u \in C. \tag{2.21}$$

Our purpose is to show that v_n is a (P.S.) sequence for $\tilde{J}_s$ on S, i.e., that $\tilde{J}_s(v_n)$ is bounded (which is obvious by (2.19)) and that $\|\tilde{J}_s'(v_n)\|_* \to 0$. Once this is proved, we get by Lemma 2.2, that for a subsequence, $v_n \to v$ in H. Clearly $v \in S$ and satisfies $\|v - (-e_1)\| = \varepsilon$ and $\tilde{J}_s(v) = \lambda_1$, which contradicts (2.17).

To prove that $\|\tilde{J}_s'(v_n)\|_* \to 0$, we first fix $n > 1/\delta$, take $w \in H$ tangent to S at v_n i.e., such that

$$\int_{\mathbb{R}^N} v_n w = 0, \tag{2.22}$$

and consider for $t \in \mathbb{R}$

$$u_t = \frac{v_n + tw}{\|v_n + tw\|_{L^2(\mathbb{R}^N)}}. \tag{2.23}$$

We first observe that for $|t|$ sufficiently small, $u_t \in C$. Indeed,

$$\lim_{t\to 0} \|u_t - (-e_1)\| = \|v_n - (-e_1)\|,$$

and

$$\varepsilon - \delta < \varepsilon - \frac{1}{n} \le \|(-e_1) - u_n\| - \|u_n - v_n\| \le \|v_n - (-e_1)\|,$$

$$\|v_n - (-e_1)\| \le \|u_n - (-e_1)\| + \|u_n - v_n\| \le \varepsilon - \frac{1}{n} < \delta + \varepsilon.$$

Take $u = u_t$ in (2.21), let $r(t) = \|v_n + tw\|_{L^2(\mathbb{R}^N)}$, we get for $t > 0$

$$\frac{J_s(v_n) - J_s(v_n + tw)}{t} \le \frac{1}{n}\frac{1}{r(t)t}\|v_n(1 - r(t)) + tw\| + \frac{1}{t}\left(\frac{1}{r^2(t)} - 1\right)J_s(v_n + tw). \tag{2.24}$$

Since by (2.22),

$$\frac{d}{dt} r^2(t)|_{t=0} = 2\int_{\mathbb{R}^N} v_n w = 0,$$

we have that $(r^2(t) - 1)/t \to 0$ as $t \to 0$, thus the second term in the right hand side of (2.24) goes to 0 as $t \to 0$. The first term in the right-hand side of (2.24) involves $(1 - r(t))/t$, which also goes to zero as $t \to 0$ (by (2.22)). Finally, as $t \to 0$, we get $\langle J_s'(v_n), w\rangle \le 1/n\|w\|$. Consequently,

$$|\langle J_s'(v_n), w\rangle| \le \frac{1}{n}\|w\| \tag{2.25}$$

for all $w \in H$ tangent to S at v_n.

Now if w is arbitrary in H, we choose α_n so that $w - \alpha_n v_n$ satisfies (2.22), i.e., $\alpha_n = \int_{\mathbb{R}^N} v_n w$. Replacing in (2.25), we get

$$|\langle J_s'(v_n), w\rangle - \langle J_s'(v_n), v_n\rangle \int_{\mathbb{R}^N} v_n w| \le \frac{1}{n}\|w - \alpha_n v_n\|.$$

Since $\|\alpha_n v_n\| \le C\|w\|$, we get

$$|\langle J_s'(v_n), w\rangle - t_n \int_{\mathbb{R}^N} v_n w| \le \varepsilon_n \|w\|,$$

where $t_n = \langle J_s'(v_n), v_n\rangle$ and $\varepsilon_n \to 0$. Thus $\|\tilde{J}_s'(v_n)\|_* \to 0$ and v_n is a (P.S.) sequence for $\tilde{J}_s$ on S. □

Let

$$\Gamma = \{\gamma \in C([-1,1], S) : \gamma(-1) = -e_1, \gamma(1) = e_1\}$$

then $\Gamma \neq \emptyset$. Let

$$c(s) = \inf_{\gamma\in\Gamma} \max_{u\in\gamma[-1,1]} \tilde{J}_s(u), \tag{2.26}$$

then by Lemma 2.2,2.3 and proposition 2.3, $c(s)$ is a critical value of $\tilde{J}_s$. Moreover,

$$c(s) > \max\{\tilde{J}_s(-e_1), \tilde{J}_s(e_1)\} = \lambda_1. \tag{2.27}$$

A third critical point of $\tilde{J}_s$ is obtained in this way. Thus we have proved that (noticing the spectrum of $-\Delta + V$, and Lemma 2.1)

Theorem 2.1. *For each $s \geq 0$, the point $(s + c(s), c(s))$, where $c(s) > \lambda_1$ is defined by the minimax formula* (2.26)*, belongs to Σ*

This yields for $s > 0$ (resp. $s = 0$) a third (resp. second) point $(s + c(s), c(s))$ in Σ on the parallel to the diagonal passing through $(s, 0)$. So for $s > 0$ we get a nontrivial curve $s \in \mathbb{R}^+ \to (s+c(s), c(s)) \in \mathbb{R}^2$ in Σ. Of course the symmetric points with respect to the diagonal also belong to Σ. The whole curve will be denoted by Θ.

3 The first nontrivial curve

This section is devoted to the proof that the curve Θ constructed above is the first nontrivial curve in Σ, in the following sense:

Theorem 3.1. *For $s \geq 0$ the points $(s + c(s), c(s))$ is the first nontrivial point of Σ on the parallel to the diagonal through $(s, 0)$.*

Before going to the proof of Theorem 3.1, we will show that the trivial line $\lambda_1 \times \mathbb{R}$ and $\mathbb{R} \times \lambda_1$ are isolated in Σ.

Proposition 3.1. *There does not exist $(\alpha_n, \beta_n) \in \Sigma$ with $\alpha_n > \lambda_1$ and $\beta_n > \lambda_1$ such that $(\alpha_n, \beta_n) \to (\alpha, \beta)$ with α or $\beta = \lambda_1$.*

Proof. Assume by contradiction the existence of $(\alpha_n, \beta_n) \in \Sigma$ with the properties above, and let $u_n \in H$ be a solution of

$$-\Delta u_n + V(x)u_n = \alpha_n u_n^+ + \beta_n u_n^- \text{ in } \mathbb{R}^N \tag{3.1}$$

with $\|u_n\|_{L^2(\mathbb{R}^N)} = 1$. Thus by (3.1) u_n remains bounded in H and consequently, for a subsequence, $u_n \to u$ weakly in H, and strongly in $L^2(\mathbb{R}^N)$. Therefore,

$$-\Delta u + V(x)u = \lambda_1 u^+ + \beta u^- \text{ in } \mathbb{R}^N, \tag{3.2}$$

where we have considered the case $\alpha = \lambda_1$. Multiplying by u^+ and integrating, we get that

$$\int_{\mathbb{R}^N} |\nabla u^+|^2 + V(x)(u^+)^2 = \lambda_1 \int_{\mathbb{R}^N} (u^+)^2,$$

so either $u^+ = 0$ or $u = e_1$. So in any case u_n converges in $L^2(\mathbb{R}^N)$ to either e_1 or $-e_1$. So for any $r > 0$ we have as $n \to +\infty$

(3.3)

$$\text{either } mes(\{x \in B(0,r) : u_n(x) < 0\}) \to 0 \text{ or } mes(\{x \in B(0,r) : u_n(x) > 0\}) \to 0.$$

Now consider (3.1) again and observe that since (α_n, β_n) does not belong to the trivial lines of Σ, u_n changes sign. Multiplying (3.1) by u_n^+ and integrating, we get

$$\int_{\mathbb{R}^N} |\nabla u_n^+|^2 + V(x)(u_n^+)^2 = \alpha_n \int_{\mathbb{R}^N} (u_n^+)^2 \tag{3.4}$$

i.e.,

$$\begin{aligned} &\int_{B(0,r)} [|\nabla u_n^+|^2 + V(x)(u_n^+)^2] + \int_{\mathbb{R}^N \setminus B(0,r)} [|\nabla u_n^+|^2 + V(x)(u_n^+)^2] \\ = \ &\alpha_n \int_{B(0,r)} (u_n^+)^2 + \alpha_n \int_{\mathbb{R}^N \setminus B(0,r)} (u_n^+)^2 \end{aligned} \tag{3.5}$$

By (V_2) and α_n is bounded, as r sufficiently large, we get

$$\int_{\mathbb{R}^N \setminus B(0,r)} [|\nabla u_n^+|^2 + V(x)(u_n^+)^2] > \alpha_n \int_{\mathbb{R}^N \setminus B(0,r)} (u_n^+)^2, \tag{3.6}$$

thus,

$$\int_{B(0,r)} [|\nabla u_n^+|^2 + V(x)(u_n^+)^2] < \alpha_n \int_{B(0,r)} (u_n^+)^2. \tag{3.7}$$

Using Hölder's inequality and Sobolev inequality, we have

(3.8)

$$\begin{aligned} \int_{B(0,r)} (u_n^+)^2 &\leq [mes(\{x \in B(0,r) : u_n(x) > 0\})]^{1-2/q} \|u_n^+\|^2_{L^q[\{x \in B(0,r): u_n(x) > 0\}]}, \\ &\leq [mes(\{x \in B(0,r) : u_n(x) > 0\})]^{1-2/q} \cdot c(r) \|\nabla u_n^+\|^2_{L^2(B(0,r))} \end{aligned}$$

where q is chosen with $2 < q \leq 2^* = \frac{2N}{N-2}$ if $N > 2$, and $2 < q < \infty$ if $N \leq 2$ and $c(r) > 0$ is a constant depending on r. Thus, by (3.7),(3.8), we have

$$mes(\{x \in B(0,r) : u_n(x) > 0\}) \geq [c(r)]^{-q/(q-2)} \alpha_n^{-q/(q-2)}. \tag{3.9}$$

Since we can also get a similar estimate for $mes(\{x \in B(0,r) : u_n(x) < 0\})$, one reaches a contradiction with (3.3). $\square$

Similarly to Lemma 3.5 and Lemma 3.6 of [CFG], we have the following two Lemmas too.

Lemma 3.1. *(i) S is locally arcwise connected. (ii) Any connected open subset ϑ of S is arcwise connected. (iii) If ϑ' is a component(i.e., a nonempty maximal open connected subset) of an open set $\vartheta \subset S$, then $\partial\vartheta' \cap \vartheta$ is empty.*

Lemma 3.2. *Let $\vartheta = \{u \in S : \tilde{J}_s(u) < r\}$, any component of ϑ contains a critical point of $\tilde{J}_s$.*

Similarly to the proof of Theorem 3.1 of [CFG], we give the proof of Theorem 3.1 here.

Proof. Assume by contradiction the existence of a point of the form $(s+\mu,\mu)$ in Σ with $\lambda_1 < \mu < c(s)$. By Proposition 3.1 and the fact Σ is closed (see the proof from (3.1) to (3.2)), we can choose such a point with μ minimum. In other words, $\tilde{J}_s$ has a critical value μ with $\lambda_1 < \mu < c(s)$, but there is no critical value in (λ_1, μ). We will construct a path in Γ on which $\tilde{J}_s$ remains $\leq \mu$, which yields a contradiction with the definition of $c(s)$.

Let $u \in S$ be a critical point of $\tilde{J}_s$ at level μ. So u satisfies the equation

$$-\Delta u + V(x)u = (s+\mu)u^+ + \mu u^- \quad \text{in} \quad \mathbb{R}^N, \tag{3.10}$$

and we know that u changes sign in $\mathbb{R}^N$. From this equation, we also have

$$\int_{\mathbb{R}^N} |\nabla u^+|^2 + V(x)(u^+)^2 = (s+\mu)\int_{\mathbb{R}^N}(u^+)^2, \quad \int_{\mathbb{R}^N} |\nabla u^-|^2 + V(x)(u^-)^2 = \mu\int_{\mathbb{R}^N}(u^-)^2, \tag{3.11}$$

and consequently

$$\tilde{J}_s(u) = \tilde{J}_s\Big(\frac{u^+}{\|u^+\|_{L^2(\mathbb{R}^N)}}\Big) = \tilde{J}_s\Big(\frac{u^-}{\|u^-\|_{L^2(\mathbb{R}^N)}}\Big) = \mu, \tag{3.12}$$

$$\tilde{J}_s\Big(\frac{-u^-}{\|u^-\|_{L^2(\mathbb{R}^N)}}\Big) = \mu - s. \tag{3.13}$$

We will consider the following three paths in S, which go respectively from u to $u^+/\|u^+\|_{L^2(\mathbb{R}^N)}$, from $u^+/\|u^+\|_{L^2(\mathbb{R}^N)}$ to $-u^-/\|u^-\|_{L^2(\mathbb{R}^N)}$ and from $u^-/\|u^-\|_{L^2(\mathbb{R}^N)}$ to u:

$$u_1(t) = \frac{tu + (1-t)u^+}{\|tu + (1-t)u^+\|_{L^2(\mathbb{R}^N)}}, u_2(t) = \frac{tu^+ - (1-t)u^-}{\|tu^+ - (1-t)u^-\|_{L^2(\mathbb{R}^N)}}$$

$$u_3(t) = \frac{tu^- + (1-t)u}{\|tu^- + (1-t)u\|_{L^2(\mathbb{R}^N)}}.$$

By (3.11), for all $t \in [0,1]$,

$$\tilde{J}_s(u_1(t)) = \mu, \ \tilde{J}_s(u_2(t)) \leq \mu, \ \tilde{J}_s(u_3(t)) = \mu.$$

And

$$\tilde{J}_s(\frac{-u^-}{\|u^-\|_{L^2(\mathbb{R}^N)}}) = \mu - s,$$

To continue, we have to investigate the levels below $\mu - s$. Let $\vartheta = \{v \in S : \tilde{J}_s(v) < \mu - s\}$. Clearly $e_1 \in \vartheta$, while $-e_1 \in \vartheta$ if $\mu - s > \lambda_1$. Moreover, e_1 and $-e_1$ are the only possible critical points of $\tilde{J}_s$ in ϑ (because of the choice of μ).

By Lemma 3.1 (i), there exists $v \in \vartheta$ and a path from $\frac{-u^-}{\|u^-\|_{L^2(\mathbb{R}^N)}}$ to v, with exception of the point $\frac{-u^-}{\|u^-\|_{L^2(\mathbb{R}^N)}}$, lies at levels $< \mu - s$. And by Lemma 3.1, 3.2, we can continue from v to e_1 (or to $-e_1$) with a path in S at levels $< \mu - s$. Let's assume it is e_1 (the end of the argument would be similar in the other case). From $\frac{-u^-}{\|u^-\|_{L^2(\mathbb{R}^N)}}$ to e_1 is $u_4(t)$ staying at level $\leq \mu - s$. Then $-u_4(t)$ goes from $\frac{u^-}{\|u^-\|_{L^2(\mathbb{R}^N)}}$ to $-e_1$. We observe that for any $w \in S$,

$$|\tilde{J}_s(w) - \tilde{J}_s(-w)| \leq s,$$

thus

$$\tilde{J}_s(-u_4(t)) \leq \tilde{J}_s(u_4(t)) + s \leq (\mu - s) + s = \mu.$$

So we have a continuous path in S from $-e_1$ to e_1 staying at level $\leq \mu$. This concludes the proof of Theorem 3.1. □

4 Some properties of the curve and the corresponding eigenfunctions

In this section we study some monotonicity and regularity properties of the curve Θ as well as its asymptotic behavior. Moreover, the partial symmetry of the corresponding eigenfunctions are obtained.

Lemma 4.1. *Let $(\alpha, \beta) \in \Theta$, $\lambda_1 < \alpha' \leq \alpha$, $\lambda_1 < \beta' \leq \beta$, and either $\alpha' < \alpha$ or $\beta' < \beta$. Then*

$$-\Delta u + V(x)u = \alpha' u^+ + \beta' u^-, \ x \in \mathbb{R}^N \tag{4.1}$$

has only the trivial solution.

Proof. Replacing u by $-u$ if necessary in (4.1), we can assume that the point $(\alpha, \beta) \in \Theta$ is such that $\alpha \geq \beta$. Let u be a non-trivial solution of (4.1). We first show that u changes sign in $\mathbb{R}^N$. Suppose by contradiction that this is not the case, say $u \geq 0$ a.e. (a similar argument would work in the other case). So u solves

$$-\Delta u + V(x)u = \alpha' u, \ x \in \mathbb{R}^N. \tag{4.2}$$

This implies that $\alpha' = \lambda_1$, which contradicts the assumption.

We give the proof for the case $\alpha' < \alpha$, another case $\beta' < \beta$ is similar. Put $\alpha - \beta = s \geq 0$. So, with the notations of Section 2, $\beta = c(s)$ where $c(s)$ is given by (2.26). We will show the existence of a path $\gamma \in \Gamma$ such that

$$\max_{u \in \gamma[-1,1]} \tilde{J}_s(u) < \beta, \tag{4.3}$$

which contradicts with the definition of $c(s)$ as the minimax value (2.26).

In order to construct γ, we will first show the existence of function $v \in H$ which changes sign and which satisfies

$$\frac{\int_{\mathbb{R}^N} |\nabla v^+|^2 + V(x)(v^+)^2}{\int_{\mathbb{R}^N} (v^+)^2} < \alpha \text{ and } \frac{\int_{\mathbb{R}^N} |\nabla v^-|^2 + V(x)(v^-)^2}{\int_{\mathbb{R}^N} (v^-)^2} < \beta. \tag{4.4}$$

By regularity, we know $u \in C(\mathbb{R}^N)$, so let us take a component Ω_1 of $\{x \in \mathbb{R}^N : u(x) > 0\}$, and a component Ω_2 of $\{x \in \mathbb{R}^N : u(x) < 0\}$. We claim that

$$\lambda_1(\Omega_1) < \alpha, \text{ and } \lambda_1(\Omega_2) \leq \beta, \tag{4.5}$$

where $\lambda_1(\Omega_i)$ denotes the first eigenvalue of $-\Delta + V$ on $W_0^{1,2}(\Omega_i)$ with the inner product

$$(u, v) := \int_{\Omega_i} (\nabla u \cdot \nabla v + V(x)uv)dx.$$

By (4.1) and $\lambda_1 < \alpha' < \alpha$ and the fact that the restriction $u|_{\Omega_i}$ belongs to $W_0^{1,2}(\Omega_i)$, we have

$$\frac{\int_{\Omega_1} |\nabla u|^2 + V(x)u^2}{\int_{\Omega_1} |u|^2} < \alpha \frac{\int_{\Omega_1} |\nabla u|^2 + V(x)u^2}{\int_{\Omega_1} \alpha' |u|^2} = \alpha,$$

which implies $\lambda_1(\Omega_1) < \alpha$. The other inequality in (4.5) is proved similarly. Similarly to the proof of Lemma 5.3 of [CFG] (they are bounded domains in [CFG]), we can modify a little bit the open sets Ω_1, Ω_2 so as to get two open sets in $\mathbb{R}^N$, $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$, with empty intersection and such that

$$\lambda_1(\tilde{\Omega}_1) < \alpha \textit{ and } \lambda_1(\tilde{\Omega}_2) < \beta. \tag{4.6}$$

The desired function v is then obtained by putting $v = v_1 - v_2$, where v_i denotes the extension by zero outside $\tilde{\Omega}_i$ of the positive eigenfunction associated to $\lambda_1(\tilde{\Omega}_i)$.

Then we can construct a path exactly as in the proof of Theorem 3.1, using v instead of the critical point u. One starts from $v / \int_{\mathbb{R}^N} v^2$ and goes successively to $v^+ / \int_{\mathbb{R}^N} (v^+)^2$ and to $-v^- / \int_{\mathbb{R}^N} (v^-)^2$. Using (4.4), one verifies that the levels of $\tilde{J}_s$ (noticing $s = \alpha - \beta$) remains $< \beta$; moreover the level of $-v^- / \int_{\mathbb{R}^N} (v^-)^2$ is $\beta - s$. One then goes on to, say, e_1 using Lemma 3.2 with $r = \beta - s$ (one also use here the fact that since (α, β) belongs to the first curve, the only critical points of $\tilde{J}_s$ at level $< \beta$ are e_1 and $-e_1$). One then returns from $-e_1$ to $v^- / \int_{\mathbb{R}^N} (v^-)^2$ and finally to $v / \int_{\mathbb{R}^N} v^2$, exactly as in the proof of Theorem 3.1. A path γ satisfying (4.3) is constructed in this way. $\square$

Proposition 4.1. *The curve $s \in \mathbb{R}^+ \to (s + c(s), c(s))$ is continuous and strictly decreasing (in the sense that $s < s'$ implies $s + c(s) < s' + c(s')$ and $c(s) > c(s')$).*

Proof. We use Lemma 4.1, similarly to the proof of Proposition 4.1 of [CFG], to finish the proof. □

Proposition 4.2. *The limit of $c(s)$ as $s \to +\infty$ is equal to λ_1.*

Proof. Assume by contradiction that there exists $\tau > 0$ such that $\max\limits_{u\in\gamma[-1,1]} \tilde{J}_s(u) \geq \lambda_1 + \tau$ for all $\gamma \in \Gamma$ and all $s \geq 0$. We choose $\phi \in H$ such that for any $r \in \mathbb{R}$, $\phi \not\leq re_1$. This is possible, for $N \geq 2$, fixed $x_0 \in \mathbb{R}^N$, take $\phi \in H$ which is unbounded from above in the neighborhood of x_0; for $N = 1$ take a function $\phi \in H$ such that $\limsup\limits_{x\to+\infty} \phi(x) = +\infty$. Consider the path $\gamma \in \Gamma$ defined by

$$\gamma(t) = \frac{te_1 + (1 - |t|)\phi}{\|te_1 + (1 - |t|)\phi\|_{L^2(\mathbb{R}^N)}}, t \in [-1, 1].$$

The maximum of $\tilde{J}_s$ on $\gamma[-1, 1]$ is achieved at say $t = t_s$. Putting $v_{t_s} = t_s e_1 + (1 - |t_s|)\phi$, we thus have

$$\int_{\mathbb{R}^N} |\nabla v_{t_s}|^2 + V(x)v_{t_s}^2 - s\int_{\mathbb{R}^N} (v_{t_s}^+)^2 \geq (\lambda_1 + \tau)\int_{\mathbb{R}^N} v_{t_s}^2 \tag{4.7}$$

for all $s \geq 0$. Letting $s \to +\infty$, we can assume, for a subsequence, $t_s \to \bar{t} \in [-1, 1]$. Since v_{t_s} remains bounded in H as $s \to +\infty$, it follows from (4.7) that $\int_{\mathbb{R}^N} (v_{t_s}^+)^2 \to 0$. Consequently

$$\int_{\mathbb{R}^N} [(\bar{t}e_1 + (1 - |\bar{t}|)\phi)^+]^2 = 0,$$

by the choice of ϕ, the only case is $\bar{t} = -1$. So $t_s \to -1$. Let $s \to +\infty$, by (4.7) we get

$$\lambda_1 \int_{\mathbb{R}^N} e_1^2 = \int_{\mathbb{R}^N} |\nabla e_1|^2 + V(x)e_1^2 \geq (\lambda_1 + \tau)\int_{\mathbb{R}^N} e_1^2,$$

which is a contradiction. □

Now we use the methods as in [BWW] to show that the eigenfunctions corresponding to eigenvalues (α, β) in the first Fucik curve Θ are foliated Schwarz symmetric if $V(x) = V(|x|), \forall x \in \mathbb{R}^N$.

We denote by $\mathbb{H}$ the family of all affine closed halfspaces in $\mathbb{R}^N$, and by $\mathbb{H}_0$ the family of all closed halfspaces in $\mathbb{R}^N$, that is, $H \in \mathbb{H}_0$ if $H \in \mathbb{H}$ and 0 lies in the hyperplane ∂H (Note in this section H denotes differently). For $H \in \mathbb{H}$ we consider the reflection $\sigma_H : \mathbb{R}^N \to \mathbb{R}^N$ with respect to the boundary of H, and we define the polarization of measurable function $u : \mathbb{R}^N \to \mathbb{R}$ with respect to H by

$$u_H(x) = \begin{cases} \max\{u(x), u(\sigma_H(x))\}, & x \in H \\ \min\{u(x), u(\sigma_H(x))\}, & x \in \mathbb{R}^N \setminus H. \end{cases}$$

Let $P \in S^{N-1} = \{x \in \mathbb{R}^N : |x| = 1\}$. We put

$$\mathbb{H}_P = \{H \in \mathbb{H}_0 : P \in int\ H\}.$$

Moreover, let $r > 0$ and let μ_r denote the standard measure on $\partial B_r(0)$. The symmetrization A^P of a set $A \subset \partial B_r(0)$ with respect to P is defined as the closed geodesic ball in $\partial B_r(0)$ centered at rP which satisfies $\mu_r(A^P) = \mu_r(A)$. For a continuous function $u : \mathbb{R}^N \to \mathbb{R}$, the foliated Schwarz symmetrization $u_P : \mathbb{R}^N \to \mathbb{R}$ of u with respect to P is defined by the condition

$$\{u_P \geq t\} \cap \partial B_r(0) = [\{u \geq t\} \cap \partial B_r(0)]^P \quad \text{for every } r > 0, t \in \mathbb{R}.$$

If $u = u_P$, then we say that u is foliated Schwarz symmetric with respect to P. In that case we have $u_H = u$ for every $H \in \mathbb{H}_P$. Clearly, a foliated Schwarz symmetric functions is symmetric with respect to rotations around the axis through P, hence it has an $O(N-1)$-symmetry.

Now we use different minimax characterizations. We consider the functionals defined on Hilbert Space H (see page 2).

$$\Phi : H \to \mathbb{R}, \Phi(u) = \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2,$$

$$\Psi : H \to \mathbb{R}, \Psi(u) = \int_{\mathbb{R}^N} \alpha(u^+)^2 + \beta(u^-)^2,$$

and we set

$$M_{(\alpha,\beta)} := \{u \in H | \Psi(u) = 1\},$$

$$\Gamma_1 := \{\gamma \in ([0,1], M_{(\alpha,\beta)}) |\ \gamma(0) = \frac{1}{\sqrt{\alpha}} e_1 \text{ and } \gamma(1) = -\frac{1}{\sqrt{\beta}} e_1\}.$$

Then the original problems with $(\alpha, \beta) \in \Theta$

$$-\Delta u + V(x)u = \alpha u^+ + \beta u^-,\ x \in \mathbb{R}^N, \tag{4.8}$$

can be considered as a eigenvalue problem $-\Delta u + V(x)u = \lambda(\alpha u^+ + \beta u^-),\ x \in \mathbb{R}^N$, with eigenvalue $\lambda_2(\alpha, \beta) = 1$.

By the proofs above, we know that

$$\lambda_2(\alpha, \beta) = \inf_{\gamma \in \Gamma_1} \max_{t \in [0,1]} \Phi(\gamma(t)) = 1. \tag{4.9}$$

Proposition 4.3. *Put* $\Gamma_2 := \{\gamma \in ([0,1], M_{(\alpha,\beta)}) |\ \gamma(0) \geq 0$ *and* $\gamma(1) \leq 0\}$. *Then*

$$\lambda_2(\alpha, \beta) = \inf_{\gamma \in \Gamma_2} \max_{t \in [0,1]} \Phi(\gamma(t)) = 1. \tag{4.10}$$

Proof. Let $d = \inf_{\gamma\in\Gamma_2} \max_{t\in[0,1]} \Phi(\gamma(t))$. Clearly $d \le \lambda_2(\alpha,\beta)$. Assume by contradiction $d < \lambda_2(\alpha,\beta)$. Take μ with $d < \mu < \lambda_2(\alpha,\beta)$ and choose a path $\gamma \in \Gamma_2$ which remains at levels $< \mu$. Similarly to the end of the proof of Theorem 3.1, we can construct a path γ_1 in Γ_1 which also remains at level $\max_{t\in[0,1]} \Phi(\gamma_1(t)) < \mu$, then get a contradiction with (4.9). (Noticing $\Phi(u) = \frac{1}{\alpha}(\tilde{J}_s(v) + s)$ for $u \in M_{(\alpha,\beta)}, u \ge 0$, $v = \sqrt{\alpha}u, v \in S$, since $M_{(\alpha,\beta)} \cap \{u : u(x) \ge 0\} = \frac{1}{\sqrt{\alpha}} S \cap \{u : u(x) \ge 0\}$). $\square$

In order to investigate the symmetry of the corresponding eigenfunctions, we need yet another minimax characterization of $\lambda_2(\alpha,\beta) = 1$.

Proposition 4.4. *There holds*

(4.11)
$$1 = \lambda_2(\alpha,\beta) = \inf_{\substack{u \in H \\ \Psi(u^\pm)=1}} \max_{0\le t\le\pi/2} \left(\cos^2 t \int_{\mathbb{R}^N} [|\nabla u^+|^2 + V(x)(u^+)^2] + \sin^2 t \int_{\mathbb{R}^N} [|\nabla u^-|^2 + V(x)(u^-)^2]\right).$$

Moreover, for every $u \in H$ with $\Psi(u^\pm) = 1$ the following are equivalent:

i) $\max_{0\le t\le\pi/2} \left(\cos^2 t \int_{\mathbb{R}^N} [|\nabla u^+|^2 + V(x)(u^+)^2] + \sin^2 t \int_{\mathbb{R}^N} [|\nabla u^-|^2 + V(x)(u^-)^2]\right) = \lambda_2(\alpha,\beta).$

ii) There exists $t \in (0,\pi/2)$ such that $(\cos t)u^+ + (\sin t)u^-$ is a sign changing eigenfunction of (4.8) corresponding to the eigenvalue $\lambda_2(\alpha,\beta) = 1$.

Proof. Let

$$\lambda_2'(\alpha,\beta) := \inf_{\substack{u \in H \\ \Psi(u^\pm)=1}} \max_{0\le t\le\pi/2} \left(\cos^2 t \int_{\mathbb{R}^N} [|\nabla u^+|^2 + V(x)(u^+)^2] + \sin^2 t \int_{\mathbb{R}^N} [|\nabla u^-|^2 + V(x)(u^-)^2]\right).$$

Proposition 4.3 yields
$$\lambda_2'(\alpha,\beta) \ge \lambda_2(\alpha,\beta) = 1.$$
Let v be the eigenfunction of (4.8) corresponding to $\lambda_2(\alpha,\beta) = 1$. Multiplying both sides of (4.8) with $v^\pm$ gives

$$\int_{\mathbb{R}^N} |\nabla v^+|^2 + V(x)(v^+)^2 = \alpha \int_{\mathbb{R}^N} (v^+)^2, \ \int_{\mathbb{R}^N} |\nabla v^-|^2 + V(x)(v^-)^2 = \beta \int_{\mathbb{R}^N} (v^-)^2.$$

We can assume that $\Psi(v) = 1$, so that for some $t \in (0,\pi/2)$

$$\Psi(v^+) = \cos^2 t, \ \Psi(v^-) = \sin^2 t.$$

Setting $u = (\cos t)^{-1} v^+ + (\sin t)^- v^-$ we obtain

$$\int_{\mathbb{R}^N} |\nabla u^+|^2 + V(x)(u^+)^2 = 1 = \int_{\mathbb{R}^N} |\nabla u^-|^2 + V(x)(u^-)^2.$$

Thus we have proved that $1 = \lambda_2(\alpha, \beta) = \lambda'_2(\alpha, \beta)$ and that $ii)$ implies $i)$.

In order to prove that $i)$ implies $ii)$, consider $u \in H$ with $\Psi(u^{\pm}) = 1$ and such that $i)$ holds. We set

$$M_\alpha = \{v \in H : \int_{\mathbb{R}^N} \alpha v^2(x)dx = 1\}, M_\beta = \{v \in H : \int_{\mathbb{R}^N} \beta v^2(x)dx = 1\},$$

we note that $u^+ \in M_\alpha$ and $u^- \in M_\beta$. Since M_α, M_β are $C^{1,1}$-manifolds, there exist global semiflows

$$\eta_\alpha : [0, \infty) \times M_\alpha \to M_\alpha, \eta_\beta : [0, \infty) \times M_\beta \to M_\beta$$

defined by

$$\frac{\partial \eta_\alpha(t, u)}{\partial t} = -\nabla[\Phi|_{M_\alpha}](u), \quad \eta_\alpha(0, u) = u,$$
$$\frac{\partial \eta_\beta(t, u)}{\partial t} = -\nabla[\Phi|_{M_\beta}](u), \quad \eta_\beta(0, u) = u.$$

It is well known that

$$u \in M_\alpha, u \geq 0 \quad \Rightarrow \quad \eta_\alpha(t, u) \geq 0, \text{ for } t \geq 0,$$

$$u \in M_\beta, u \leq 0 \quad \Rightarrow \quad \eta_\beta(t, u) \leq 0, \text{ for } t \geq 0.$$

Since $\frac{1}{\sqrt{\alpha}} e_1$ is the only positive critical point of $\Phi|_{M_\alpha}$, we know that

$$\eta_\alpha(t, u^+) \to \frac{1}{\sqrt{\alpha}} e_1 \quad \text{as} \quad t \to \infty,$$

and similarly

$$\eta_\beta(t, u^-) \to -\frac{1}{\sqrt{\beta}} e_1 \quad \text{as} \quad t \to \infty,$$

since $-\frac{1}{\sqrt{\beta}} e_1$ is the only negative critical point of $\Phi|_{M_\beta}$. We now define a path $\gamma \in \Gamma_1$ by

$$\gamma(t) = \begin{cases} \frac{1}{\sqrt{\alpha}} e_1, & t = 0, \\ \eta_\alpha(\frac{1}{3t} - 1, u^+), & t \in (0, \frac{1}{3}), \\ \cos((3t-1)\frac{\pi}{2})u^+ + \sin((3t-1)\frac{\pi}{2})u^-, & t \in [\frac{1}{3}, \frac{2}{3}], \\ \eta_\beta(\frac{1}{1-t} - 3, u^-), & t \in (\frac{2}{3}, 1), \\ -\frac{1}{\sqrt{\beta}} e_1, & t = 1. \end{cases}$$

Then by construction

$$\max_{t \in [0,1]} \Phi(\gamma(t)) = \max_{t \in [\frac{1}{3}, \frac{2}{3}]} \Phi(\gamma(t)) = \lambda_2(\alpha, \beta) = 1.$$

That is $\gamma \in \Gamma_1$ is a extremal path for the minimax characterization (4.9). Hence by Lemma 26 of [ACC], there exisits $t_0 \in [\frac{1}{3}, \frac{2}{3}]$ such that $\gamma(t_0)$ is a critical point of $\Phi|_M$ and hence a solution of (4.8) according to $\lambda_2(\alpha, \beta) = 1$. In fact, $t_0 \in (\frac{1}{3}, \frac{2}{3})$, since $\gamma(t)$ has to changes sign. Hence $\gamma(t_0) = (\cos t)u^+ + (\sin t)u^-$ for some $t \in (0, \frac{\pi}{2})$. $\square$

Proposition 4.5. *Let $H \in \mathbb{H}$ and u be an eigenfunction of (4.8). Then one of the following holds:*

(i) $u(x) > u(\sigma_H(x))$ for all $x \in int\ H$.

(ii) $u(x) < u(\sigma_H(x))$ for all $x \in int\ H$.

(iii) $u(x) = u(\sigma_H(x))$ for all $x \in \mathbb{R}^N$.

Proof. By standard elliptic regularity, $u \in C^2(\mathbb{R}^N)$. If

$$u(x) \le u(\sigma_H(x)) \text{ for all } x \in int\ H, \tag{4.12}$$

then we find that $v = u - u \circ \sigma_H \in C^2(H)$ satisfies that for $x \in H$

$$-\Delta v(x) = \alpha[u^+(x) - (u \circ \sigma_H)^+(x)] + \beta[u^-(x) - (u \circ \sigma_H)^-(x)] \le 0, \tag{4.13}$$

hence either $v \equiv 0$ or $v < 0$ on *int* H by the maximum principle. Thus either (ii) or (iii) is satisfied.

It remains to consider the case where

$$u(x_0) > u(\sigma_H(x_0)) \text{ for some } x_0 \in H. \tag{4.14}$$

Since u is continuous, there exists $\delta > 0$ such that

$$u(x) > u(\sigma_H(x)) \text{ for } x \in B_\delta(x_0)\ (B_\delta(x_0) := \{x \in \mathbb{R}^N : |x - x_0| = \delta\}).$$

Therefore $u_H(x) = u(x)$ for $x \in B_\delta(x_0)$. By Lemma 2.2, 2.3 of [BWW], we get that

$$\begin{aligned} 1 = \lambda_2(\alpha,\beta) &= (\cos^2 t)\frac{\int_{\mathbb{R}^N} |\nabla u^+|^2 + V(x)(u^+)^2}{\int_{\mathbb{R}^N} \alpha|u^+|^2} + (\sin^2 t)\frac{\int_{\mathbb{R}^N} |\nabla u^-|^2 + V(x)(u^-)^2}{\int_{\mathbb{R}^N} \alpha|u^-|^2} \\ &= (\cos^2 t)\frac{\int_{\mathbb{R}^N} |\nabla u_H^+|^2 + V(x)(u_H^+)^2}{\int_{\mathbb{R}^N} \alpha|u_H^+|^2} + (\sin^2 t)\frac{\int_{\mathbb{R}^N} |\nabla u_H^-|^2 + V(x)(u_H^-)^2}{\int_{\mathbb{R}^N} \alpha|u_H^-|^2} \end{aligned}$$

for any $t \in (0, \frac{\pi}{2})$. Hence by Proposition 4.4 there exist $a > 0, b > 0$ such that $w := au_H^+ + bu_H^-$ is also an eigenvalue of (4.8). We consider the following two cases:

(1) $a \ge b$.

(2) $a < b$.

In case (1) we calculate

$$w - au \ge 0 \text{ on } H, \quad -\Delta(w - au) \ge 0 \text{ on } H, \tag{4.15}$$

$$w - bu \ge 0 \text{ on } H, \quad -\Delta(w - bu) \ge 0 \text{ on } H. \tag{4.16}$$

Now suppose first that $u \ge 0$ on $B_\delta(x_0)$. Then $w \equiv au$ on $B_\delta(x_0)$, and hence $w \equiv au$ on H by (4.15) and the maximum principle. From this we deduce that

$$au_H = au_H^+ + au_H^- \le au_H^+ + bu_H^- = w = au \text{ on } H,$$

and by the definition of u_H we get that $u = u_H$. Hence $u \geq u \circ \sigma_H$ on H, and by (4.14) and the maximum principle we conclude that alternative (i) holds.

Next suppose that $u(x) < 0$ for some $x \in B_\delta(x_0)$. Then $w \equiv bu$ on a neighborhood of x in $B_\delta(x_0)$. Hence (4.16) and the maximum principle imply that $w \equiv bu$ on H. From this we get that

$$bu_H = bu_H^+ + bu_H^- \leq au_H^+ + bu_H^- = w = bu \text{ on } H,$$

and again we conclude $u = u_H$. Again (i) holds by (4.14) and the maximum principle.

It remains to consider case (2). Here we compute

$$w - au \leq 0 \;\text{ on } \mathbb{R}^N \setminus H, \quad -\Delta(w - au) \leq 0 \;\text{ on } \mathbb{R}^N \setminus H, \tag{4.17}$$

$$w - bu \leq 0 \;\text{ on } \mathbb{R}^N \setminus H, \quad -\Delta(w - bu) \leq 0 \;\text{ on } \mathbb{R}^N \setminus H, \tag{4.18}$$

We suppose first that $u \geq 0$ on $B_\delta(\sigma_H(x_0))$. Then $w \equiv au$ on $B_\delta(\sigma_H(x_0))$, and hence $w \equiv au$ on $\mathbb{R}^N \setminus H$ by (4.17) and the maximum principle. From this we deduce that

$$au_H = au_H^+ + au_H^- \geq au_H^+ + bu_H^- = w = au \text{ on } \mathbb{R}^N \setminus H,$$

and by the definition of u_H we get that $u = u_H$. As above we now conclude that alternative (i) holds.

Finally suppose that $u < 0$ for some $x \in B_\delta(\sigma_H(x_0))$. Then $w \equiv bu$ on a neighborhood of x in $B_\delta(\sigma_H(x_0))$, and hence $w \equiv bu$ on $\mathbb{R}^N \setminus H$ by (4.18) and the maximum principle. From this we deduce that

$$bu_H = bu_H^+ + bu_H^- \geq au_H^+ + bu_H^- = w = bu \text{ on } \mathbb{R}^N \setminus H,$$

and by the definition of u_H we get that $u = u_H$ too. As above we now conclude that alternative (i) holds. $\square$

Theorem 4.1. *Every eigenfunction u of (4.8) is foliated Schwarz symmetric.*

Proof. Pick $x_0 \in \mathbb{R}^N$ with

$$u(x_0) = \max\{u(x) : x \in \mathbb{R}^N, |x| = |x_0|\},$$

and put $P := \frac{x_0}{|x_0|}$. By Proposition 4.5 we infer that $u_H = u$ for every $H \in \mathbb{H}_P$. By Lemma 2.4 of [BWW](that is: Let $1 < q < \infty, u \in C(\mathbb{R}^N)$ and $P \in S^{N-1}$. If $u \neq u_P$, then there exists $H \in \mathbb{H}_P$ such that $\|u_H - u_P\|_q < \|u - u_P\|_q$), we know that u is foliated Schwarz symmetric with respect to P. $\square$

5 Asymptotically linear and jumping nonlinearities

In this section,we consider the existence of solutions for nonlinear time-independent Schrödinger equations of the form

$$-\Delta u + V(x)u = f(x, u) \text{ in } \mathbb{R}^N \tag{5.1}$$

which satisfy $u(x) \to 0$ as $|x| \to \infty$. This type of equations arise also from study of standing wave solutions of time-dependent nonlinear Schrödinger equations. $f(x, s)$ satisfies that

$(f1)$ $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}), f(x, s)s \geq 0$;

$(f2)$ $\limsup\limits_{|s|\to\infty} \frac{f(x,s)}{s} := a(x) < a_0 < \lambda_1$;

$(f3)$ $\lim\limits_{s\to 0^+} \frac{f(x,s)}{s} = \alpha$, $\lim\limits_{s\to 0^-} \frac{f(x,s)}{s} = \beta$, (α, β) is above the curve Θ in $\mathbb{R}^2$.

Another case we assume

$(f4)$ $\limsup\limits_{|s|\to 0} \frac{f(x,s)}{s} := a(x) < a_0 < \lambda_1$;

$(f5)$ $\lim\limits_{s\to +\infty} \frac{f(x,s)}{s} = \alpha$, $\lim\limits_{s\to -\infty} \frac{f(x,s)}{s} = \beta$, $(\alpha, \beta) \notin \Sigma$, (α, β) is above the curve Θ in $\mathbb{R}^2$.

For the jumping problems $(f2)(f3)$ or $(f4)(f5)$ we need study the following functional:

$$J(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x, u(x))dx \tag{5.2}$$

Standard arguments ensure that the functional J is of class C^1 on H (see introduction for the Hilbert space H), and the critical points of J are distributional solutions of (5.1). The gradient of J has the form $\nabla J : H \to H, \nabla J = id_H - A$ with $A : H \to H$ given by $A(u) := (-\Delta + V)^{-1}[f(\cdot, u(\cdot))]$ for $u \in H$. In other words, $A(u)$ is uniquely determined by the relation

$$\langle A(u), \varphi\rangle = \int_{\mathbb{R}^N} f(x, u(x))\varphi dx, \text{ for all } \varphi \in H. \tag{5.3}$$

By $(V_1)(V_2)$, we know A is compact.

Since f has jumping property, we need study the related functional:

$$J_{(\alpha,\beta)}(u) = \frac{1}{2}\|u\|^2 - \frac{\alpha}{2}\int_{\mathbb{R}^N} (u^+)^2 dx - \frac{\beta}{2}\int_{\mathbb{R}^N} (u^-)^2 dx, \forall u \in H. \tag{5.4}$$

We know that

$$\langle J'(u), v\rangle = \int_{\mathbb{R}^N} [\nabla u \nabla v + V(x)uv - f(x, u)v], \forall v \in H,$$

$$\langle J'_{(\alpha,\beta)}(u), v\rangle = \int_{\mathbb{R}^N} [\nabla u \nabla v + V(x)uv - \alpha u^+ v - \beta u^- v], \forall v \in H.$$

Lemma 5.1. *J satisfies the P.S. (Palais-Smale) condition on H under the assumptions $(f1),(f2)$.*

Proof. Suppose that $\{u_n\}_1^{+\infty}$ satisfies

$$|J(u_n)| \leq C, \quad \|J'(u_n)\|_* \to 0.$$

Here and in the following $C > 0$ are different constants. By $(f1),(f2)$, we have that there exist $0 < \varepsilon < \frac{\lambda_1 - a_0}{2}$ and $t_0 > 0$ such that

$$|f(x,u) - a_0 u| < \varepsilon |u|, \forall |u| > t_0,$$

thus for all $u \in \mathbb{R}$,

$$|F(x,u)| \leq \frac{(a_0+\varepsilon)u^2}{2} + C|u|.$$

Therefore, by Sobolev imbedding inequality [WM]

$$\begin{aligned} C \geq J(u_n) &= \frac{1}{2}\|u_n\|^2 - \int_{\mathbb{R}^N} F(x,u_n(x))dx \\ &\geq \frac{1}{2}\|u_n\|^2 - \int_{\mathbb{R}^N} \frac{(a_0+\varepsilon)u^2}{2} - C\int_{\mathbb{R}^N} |u_n| \\ &\geq \frac{1}{2}\|u_n\|^2 - \frac{(\lambda_1-\varepsilon)}{2}\int_{\mathbb{R}^N} |u_n|^2 - C\|u_n\| \\ &\geq \frac{1}{2}\frac{\varepsilon}{\lambda_1}\|u_n\|^2 - C\|u_n\|. \end{aligned} \tag{5.5}$$

So $\{u_n\}$ is bounded in H. Passing to a subsequence, we may assume that $u_n \rightharpoonup u \in H$, $u_n \to u$ in $L^s(\mathbb{R}^N)$ for $s \in [2, 2^*)$. In order to establish strong convergence it suffices to show

$$\|u_n\| \to \|u\|. \tag{5.6}$$

Since $\langle J'(u_n), u_n - u\rangle \to 0$, we have

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty}(\|u_n\|^2 - \|u\|^2) \\ &= \limsup_{n\to\infty}(u_n, u_n - u) \\ &= \limsup_{n\to\infty} \textstyle\int_{\mathbb{R}^N} f(x,u_n)(u_n-u). \end{aligned} \tag{5.7}$$

By $(f2)$, $\exists C > 0$ such that

$$\int_{\mathbb{R}^N} f(x,u_n)(u_n-u) \leq \int_{\mathbb{R}^N} (C|u_n|)|u_n-u| \leq C\|u_n\|_{L^2(\mathbb{R}^N)}\|u_n-u\|_{L^2(\mathbb{R}^N)} \to 0. \tag{5.8}$$

Hence (5.6) follows from (5.7). $\square$

Theorem 5.1. *under assumptions $(f1)-(f3)$, then (5.1) has at least 3 nontrivial solutions, at least one positive, one negative.*

Proof. By truncation method, it is easy to know that there exists one positive local minimum critical point, and one local minimum negative critical point, and a mountain pass type critical point. By $(f3)$ we know there exists paths which are convergent to 0 and connect the positive cone and negative cone, on which the functional values are negative, then 0 is not a mountain pass type critical point. Thus (5.1) has at least 3 nontrivial solutions □

Lemma 5.2. *J satisfies the P.S. (Palais-Smale) condition on H under the assumptions $(f1),(f4),(f5)$.*

Proof. Suppose that $\{u_n\}_1^{+\infty}$ satisfies

$$|J(u_n)| \leq C, \quad \|J'(u_n)\|_* \to 0.$$

We first get $\{u_n\}$ is bounded in H. If not, assume that $\|u_n\| \to +\infty$, let $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$, and

$$\begin{aligned} v_n &\rightharpoonup v \text{ in } H, \\ v_n &\to v \text{ in } L^2(\mathbb{R}^N), \\ v_n &\to v \ a.e. \text{ in } \mathbb{R}^N. \end{aligned}$$

First we show $v \neq 0$. In fact,

$$\langle J'(u_n), v_n \rangle = \int_{\mathbb{R}^N} [\nabla u_n \nabla v_n + V(x) u_n v_n - f(x, u_n) v_n] \to 0,$$

then

$$\int_{\mathbb{R}^N} [\nabla \frac{u_n}{\|u_n\|} \nabla v_n + V(x) \frac{u_n}{\|u_n\|} v_n - \frac{f(x, u_n)}{u_n} \frac{u_n}{\|u_n\|} v_n] \to 0.$$

Thus

$$1 = \|v_n\| = \int_{\mathbb{R}^N} \frac{f(x, u_n)}{u_n} v_n^2 + o(1) \tag{5.9}$$

By $(f1),(f4),(f5)$, $\frac{f(x,u_n)}{u_n} \leq C, C > 0$, thus

$$\|v_n\|_{L^2(\mathbb{R}^N)} \not\to 0, \text{ and } v \neq 0. \tag{5.10}$$

On the other hand, by $\|J'(u_n)\|_* \to 0$, we know that

$$-\Delta v_n + V(x) v_n = \frac{f(x, u_n)}{u_n} v_n + o(1). \tag{5.11}$$

i.e.,

$$-\Delta v_n + V(x) v_n = \frac{f(x, u_n^+)}{u_n^+} v_n^+ + \frac{f(x, u_n^-)}{u_n^-} v_n^- + o(1). \tag{5.12}$$

Noticing that $u_n^+(x) \to +\infty, u_n^-(x) \to -\infty$ as $n \to +\infty$, thus we have

$$-\Delta v + V(x)v = \alpha v^+ + \beta v^- \text{ in } \mathbb{R}^N, \tag{5.13}$$

which contradicts the assumption $(\alpha, \beta) \notin \Sigma$. Therefore, $\{u_n\}$ is bounded in H, then

$$u_n \rightharpoonup u \text{ in } H,$$

$$u_n \to u \text{ in } L^2(\mathbb{R}^N),$$
$$u_n \to u \text{ } a.e. \text{ in } \mathbb{R}^N.$$

Similarly to the proof of Lemma 5.1, we get that a convergent subsequence of $\{u_n\}$. This competes the proof. □

As in [BLW], define

$$D_\varepsilon^+ := \{u \in H : \|u^-\| < \varepsilon\}, D_\varepsilon^- := \{u \in H : \|u^+\| < \varepsilon\},$$

and

$$D_\varepsilon = \overline{D_\varepsilon^+} \cup \overline{D_\varepsilon^-} \quad \text{for} \quad \varepsilon > 0.$$

D_ε^+ and D_ε^- are open convex subsets of H, where D_ε is a closed and symmetric subset of H, Moreover, $H \setminus D_\varepsilon$ contains only sign changing functions.

Lemma 5.3. *There exists a $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$ there holds*

(i) $A(\partial D_\varepsilon^-) \subset D_\varepsilon^-$, and every nontrivial solution $u \in D_\varepsilon^-$ of (1.1) is negative;

(ii) $A(\partial D_\varepsilon^+) \subset D_\varepsilon^+$, and every nontrivial solution $u \in D_\varepsilon^+$ of (1.1) is positive.

Proof. By $(f4), (f5)$ we have that

$$|f(x,t)| \leq a_0|t| + C|t|^2 \text{ for } x \in \mathbb{R}^N, t \in \mathbb{R}. \tag{5.14}$$

Let $u \in H$ and $v = A(u)$. From (5.3) and $(f1)$ we get

$$\begin{aligned} \|v^+\|^2 &= \langle v, v^+\rangle = \int_{\mathbb{R}^N} f(x,u)v^+ \\ &\leq \int_{\mathbb{R}^N} f(x,u)^+v^+ = \int_{\mathbb{R}^N} f(x,u^+)v^+ \\ &\leq \int_{\mathbb{R}^N} (a_0|u^+| + C|u^+|^2)v^+ \\ &\leq a_0\|u^+\|_{L^2(\mathbb{R}^N)}\|v^+\|_{L^2(\mathbb{R}^N)} + C\|u^+\|^2_{L^3(\mathbb{R}^N)}\|v^+\|_{L^3(\mathbb{R}^N)} \\ &\leq (\frac{a_0}{\lambda_1}\|u^+\| + C\|u^+\|^2)\|v^+\| \end{aligned} \tag{5.15}$$

Hence,

$$\|A(u)^+\| \leq \frac{a_0}{\lambda_1}\|u^+\| + C\|u^+\|^2.$$

So, there exists $\varepsilon_0 > 0$ such that $\|A(u)^+\| \le (\frac{a_0}{\lambda_1} + \delta_0)\|u^+\|$ for every $u \in D_\varepsilon^-$ with $0 < \varepsilon \le \varepsilon_0$, where $\delta_0 < 1 - \frac{a_0}{\lambda_1}$. In particular we have $A(\partial D_\varepsilon^-) \subset D_\varepsilon^-$. If moreover $u \in D_\varepsilon^-$ satisfies $A(u) = u$, then $u^+ = 0$. If finally $u \ne 0$, we conclude $u(x) < 0$ for all x by the maximum principle. This completes the proof of (i).

(ii) can be proved analogously. □

Denote $H_0 = \{u \in H | J'(u) \ne 0\}$. We recall that a continuous map $\Upsilon : H \to H$ is said to be a pseudogradient vector field for J if $J|_{H_0} : H_0 \to H$ is locally Lipschitz continuous and if the following two conditions are satisfied:

(i) $\langle J'(u), \Upsilon(u)\rangle \ge \frac{1}{2}\|J'(u)\|^2$ for all $u \in H_0$;

(ii) $\|\Upsilon(u)\| \le 2\|J'(u)\|$ for all $u \in H_0$.

If Υ is a pseudogradient vector field for J, we can integrate $-\Upsilon$ and obtain a flow $\varphi : \Pi \to H$ satisfying

$$\begin{cases} \dfrac{d}{dt}\varphi(t,u) = -\Upsilon(\varphi(t,u)), t \ge 0, \\ \varphi(0,u) = u \end{cases} \tag{5.16}$$

for all $(t,u) \in \Pi$, where $\Pi = \{(t,u) : u \in H, 0 \le t < T(u)\}$, $T(u) \in (0,\infty]$ is the maximal existence time for the trajectory $\varphi(t,u)$. We call φ the descending flow associated with Υ. A subset $D \subset H$ is positive invariant for the flow φ if

$$\varphi(t,u) \in D \text{ for every } u \in D \text{ and for every } t \in [0, T(u)).$$

We also consider the domain of attraction of a positive invariant subset D of H defined by:

$$\mathbb{A}(D) := \{u \in H : dist(\varphi(t,u), D) \to 0 \text{ as } t \to T(u)\}.$$

Theorem 5.2. *under assumptions* $(f1),(f4),(f5)$*, then* (5.1) *has at least one positive, one negative, and one sign-changing solution.*

Proof. Let $0 < \varepsilon \le \varepsilon_0$, in view of Lemma 5.3, there is a pseudogradient vector field Υ for J such that $D_\varepsilon^+, D_\varepsilon^-$ are invariant for the associated descending flow. Moreover, $\mathbb{A}(D_\varepsilon^+) \supset \partial D_\varepsilon^+$, $\mathbb{A}(D_\varepsilon^-) \supset \partial D_\varepsilon^-$.

By the standard methods for invariant set of descent flow, we get the conclusion. (see the proof of Theorem 2 of [LZ] or the Theorem 2 of [ZL], here we use $J_{(\alpha,\beta)}$ to construct a path L connecting the positive cone and the negative cone, which are the interior point set of D_ε^- and D_ε^+ respectively, and on tL as t sufficiently large, $J(tu) < 0$ uniformly for $u \in L$). □

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