Toward double affine flag varieties and Grassmannians

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Finite v. single affine v. double affine settings

	Finite	Single affine	Double affine
Groups	$\mathbf{G}(\Bbbk)$	$\widehat{\mathbf{G}}(\Bbbk)$ $\mathbf{G}(\Bbbk((\pi)))$	$\widehat{\mathbf{G}}(\Bbbk((\pi)))$
Hecke algebras	$\mathbb{C}[\mathbf{B}\backslash\mathbf{G}/\mathbf{B}]$	$\mathbb{C}_c[\widehat{\mathbf{B}}\backslash\widehat{\mathbf{G}}/\widehat{\mathbf{B}}]$ or $\mathbb{C}_c[\mathrm{I}\backslash\mathbf{G}(\Bbbk((\pi)))/\mathrm{I}]$	?
Weyl groups	W	W_{aff} or $W \ltimes Q^\vee$	?
Schubert varieties	$\overline{\mathbf{B}w\mathbf{B}/\mathbf{B}}$	$\overline{\mathbf{B}w\mathbf{B}/\mathbf{B}}$ or $\overline{\mathrm{I}w\mathrm{I}/\mathrm{I}}$	?

Here $\Bbbk = \mathbb{F}_q$, $\mathbf{G}$ is a finite-type Kac-Moody group (think $\mathbf{SL}_n$), and $\widehat{\mathbf{G}}$ is the affinization of $\mathbf{G}$, an affine Kac-Moody group, (think $\widehat{\mathbf{SL}_n}$).

Notation

We will shift notation slightly and drop hats.

- An (affine) Kac-Moody group $\mathbf{G}$.
- Positive and negative Borels $\mathbf{B}$ and $\mathbf{B}^-$. Torus $\mathbf{A} = \mathbf{B} \cap \mathbf{B}^-$.
- Weyl group W, coweight lattice $\Lambda = \mathrm{Hom}(\mathbb{G}_m, \mathbf{A})$, dominant cone Λ^{++}, Tits cone $\mathcal{T} = W \cdot \Lambda^{++}$.
- Local field $\mathcal{K} = \mathbb{k}((\pi))$ and ring of integers $\mathcal{O} = \mathbb{k}[[\pi]]$. ($\mathcal{K} = \mathbb{Q}_p$ okay when it makes sense)
- Groups of $\mathcal{K}$-points: $G = \mathbf{G}(\mathcal{K})$, $A = \mathbf{A}(\mathcal{K})$, etc.
- "Maximal compact" $K = \mathbf{G}(\mathcal{O})$ and Iwahori subgroup $I = \{g \in K \mid g \in \mathbf{B}(\mathbb{k}) \mod \pi\}$.
- For $\mu \in \Lambda$, we can consider $\mu : \mathcal{K}^\times \to A \hookrightarrow G$. Write π^μ for the image of $\pi \in \mathcal{K}^\times$. We get an embedding $\Lambda \hookrightarrow G$.

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The Cartan semigroup

- The Cartan decomposition fails [Garland]:

$$\bigsqcup_{\lambda \in \Lambda^{++}} K\pi^{\lambda}K \subsetneq G$$

- Let $G^{+} = \bigsqcup_{\lambda \in \Lambda^{++}} K\pi^{\lambda}K$.
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Spherical Hecke algebras and Satake isomorphism

Theorem (Braverman-Kazhdan, Gaussent-Rousseau)

There is a completion $\widehat{\mathcal{H}_K}$ of $\mathbb{C}_c[K\backslash G^+/K]$ that is an algebra under convolution. Furthermore, there is an isomorphism:

$$\mathsf{Sat} : \widehat{\mathcal{H}_K} \to \mathsf{Rep}(\mathbf{G}^\vee)$$

- The completion is natural because $V(\mu) \otimes V(\lambda)$ is an infinite sum of irreps.
- Braverman-Kazhdan proof: an interpretation of the problem via bundles on algebraic surfaces.
- Gaussent-Rousseau proof: via the theory of masures (a.k.a hovels, a.k.a. double affine buildings).
- One needs to understand the structure coefficients $\mathbb{1}_{K\pi^\lambda K} \star \mathbb{1}_{K\pi^\mu K} = \sum_\nu a_{\lambda,\mu}^\nu \mathbb{1}_{K\pi^\nu K}$.
- They later (with Patnaik and Bardy-Panse resp.) show that $\mathsf{Sat}(\mathbb{1}_{K\pi^\lambda K})$ is an affine Hall-Littlewood function (up to an important correction factor).

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Work of Braverman-Kazhdan-Patnaik

Theorem (BKP)

There is an Iwahori decomposition

$$G^+ = \bigsqcup_{\pi^\mu w \in W \ltimes \mathcal{T}} I w \pi^\mu I$$

The vector space $\mathbb{C}_c[I \backslash G^+ / I]$ *is an algebra under convolution.*

Theorem (BKP)

The algebra $\mathbb{C}_c[I \backslash G^+ / I]$ *has a Bernstein presentation* $\langle \Theta_\mu, T_w \mid \mu \in \mathcal{T}, w \in W \rangle$, *so it is almost equal to Cherednik's DAHA.*

- There are two bases, the Bernstein basis and the basis consisting of vectors $T_{w\pi^\mu} = \mathbb{1}_{I w \pi^\mu I}$.
- A question raised by BKP is to understand the $\{T_{w\pi^\mu}\}$ basis combinatorially and the dependence of this basis on the local field $\mathcal{K}$.

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The double coset basis and Iwahori-Matsumoto formula (1)

Theorem (M—,Gaussent–Rousseau–Bardy-Panse)

Let $\pi^\mu w \in W \ltimes \mathcal{T}$. Let s_i be a simple reflection in W. Then we have:

$$T_{\pi^\mu w s_i} = \begin{cases} T_{\pi^\mu w} T_{s_i} & \text{if } \langle \mu, w(\alpha_i) \rangle > 0 \text{ or if } \langle \mu, w(\alpha_i) \rangle = 0 \text{ and } w(\alpha_i) > 0 \\ T_{\pi^\mu w} T_{s_i}^{-1} & \text{if } \langle \mu, w(\alpha_i) \rangle < 0 \text{ or if } \langle \mu, w(\alpha_i) \rangle = 0 \text{ and } w(\alpha_i) < 0 \end{cases}$$

Let $\mu \in \mathcal{T}$. Write $\mu = v(\mu_+)$ for $\mu_+ \in \Lambda^{++}$ and $v \in W$. Then we have:

$$T_{\pi^\mu} = T_{v^{-1}}^{-1} T_{\pi^{\mu_+}} T_{v^{-1}}$$

The double coset basis and Iwahori-Matsumoto formula (2)

Corollary

The structure coefficients of the basis $\{T_{\pi^\mu w} \mid \pi^\mu w \in W \ltimes \mathcal{T}\}$ *are specialization of universal polynomials (independent of* $\mathbb{k}$*) at* $q = \#\mathbb{k}$.

Perspective on the double affine flag variety

- We consider G^+/I as if it were the $\mathbb{k}$-points of the "double affine flag variety".
- The Schubert cells IxI/I are indexed by $x \in W \ltimes \mathcal{T}$.
- Our goal is to probe the geometry of G^+/I indirectly.
- For example, we would like to compute Kazhdan-Lusztig polynomials.

Double affine roots and Bruhat order [BKP] (1)

- Let Φ_{re} denote the roots of $\mathbf{G}$, and let Φ_{re}^+ denote the subset of positive real roots.
- We define the double affine (real) roots to be formal linear combinations of the form

$$\beta + n\pi$$

where $\beta \in \Phi_{re}$ and $n \in \mathbb{Z}$.

- Define $\beta + n\pi > 0$ if $n > 0$, or $n = 0$ and $\beta \in \Phi_{re}^+$.
- Given $\beta \in \Phi_{re}^+$ and $n \in \mathbb{Z}$, we define

$$\beta[n] = \begin{cases} \beta + n\pi & \text{if } \beta + n\pi > 0 \\ -(\beta + n\pi) & \text{if } \beta + n\pi < 0 \end{cases}$$

- Associated to $\beta[n]$, we define a reflection (we drop the $\vee$):

$$s_{\beta[n]} = \pi^{n\beta^\vee} s_\beta = \pi^{n\beta} s_\beta$$

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Double affine roots and Bruhat order [BKP] (2)

- Given $\pi^\mu w \in W \ltimes \mathcal{T}$ and a double affine root $\beta + n\pi$, we define

$$\pi^\mu w(\beta + n\pi) = w(\beta) + (n + \langle \mu, \beta \rangle))\pi$$

- BKP define a preorder (the double affine Bruhat order) on $W \ltimes \mathcal{T}$ by declaring

$$\pi^\mu w < \pi^\mu w s_{\beta[n]}$$
$$\text{if } \pi^\mu w(\beta[n]) > 0$$

- They conjecture it to be partial order.

The function ℓ_ε (1)

Let ε be a formal symbol, and consider $\mathbb{Z} \oplus \mathbb{Z}\varepsilon$ ordered lexicographically. So, for example, $1 + 10\varepsilon < 2 + 1\varepsilon$.

Theorem (M—)

There is a function $\ell_\varepsilon : W \ltimes \mathcal{T} \to \mathbb{Z} \oplus \mathbb{Z}\varepsilon$ that is strictly compatible with the Bruhat order and the lexicographic order on $\mathbb{Z} \oplus \mathbb{Z}\varepsilon$.

Corollary

The Bruhat preorder on $W \ltimes \mathcal{T}$ is a partial order.

The function ℓ_ε (2)

The function ℓ_ε is characterized by

$$\ell_\varepsilon(\pi^\mu w s_i) =$$

$$\begin{cases} \ell_\varepsilon(\pi^\mu w) + 1 \text{ if } \langle \mu, w(\alpha_i) \rangle > 0 \text{ or if } \langle \mu, w(\alpha_i) \rangle = 0 \text{ and } w(\alpha_i) > 0 \\ \ell_\varepsilon(\pi^\mu w) - 1 \text{ if } \langle \mu, w(\alpha_i) \rangle < 0 \text{ or if } \langle \mu, w(\alpha_i) \rangle = 0 \text{ and } w(\alpha_i) < 0 \end{cases}$$

and:

$$\ell_\varepsilon(\pi^\mu) = \ell_\varepsilon(\pi^{\mu_+}) = \langle 2\rho, \mu_+ \rangle$$

Compare: the conditions in the Iwahori-Matsumoto formula.

- Deciding whether $\beta + n\pi > 0$ is a “lexicographic” procedure.
- Deciding whether $\ell_\varepsilon(x) > \ell_\varepsilon(y)$ is also a “lexicographic” procedure.
- The proof involves matching them up.
- Even for usual affine Weyl groups this is a new and useful result.

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Specializing $\varepsilon = 1$

Define $\ell : W \ltimes \mathcal{T} \to \mathbb{Z}$ to be the composition of:

$$W \ltimes \mathcal{T} \xrightarrow{\ell_\varepsilon} \mathbb{Z} \oplus \mathbb{Z}\varepsilon \xrightarrow{\varepsilon \mapsto 1} \mathbb{Z}$$

Theorem (M-Orr)

The function $\ell : W \ltimes \mathcal{T} \to \mathbb{Z}$ is strictly compatible with the Bruhat order and the usual order on $\mathbb{Z}$.

Corollary

Let $x, y \in W \ltimes \mathcal{T}$ with $x < y$. The chains from x to y are no longer than $\ell(y) - \ell(x)$.

A sample application (even for usual affine Weyl groups)

Consider $\mathbf{G} = \widehat{\mathbf{SL}_2}$, $x = \pi^{\Lambda_0 + \delta}$, and $y = \pi^{\Lambda_0} t^{3\alpha}$. We compute

$$\ell_\varepsilon(x) = 4 \text{ and } \ell_\varepsilon(y) = 6\varepsilon$$

so $\ell_\varepsilon(x) > \ell_\varepsilon(y)$, and

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so $\ell(x) < \ell(y)$. Therefore: x and y are not comparable.

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A toy theorem

Toy Theorem

Let $w \in W$, $\beta \in \Phi^+_{\mathrm{re}}$. *If* $w(\beta) > 0$, *then:*

$$\ell(ws_\beta) > \ell(w)$$

The usual proof is closely related to the Strong Exchange Condition for Coxeter groups. Instead, we will proceed differently.

Proof.

We can check that:

$$\ell(ws_\beta) = \ell(w) + \#\{\gamma \in \mathrm{Inv}(s_\beta) \mid w(\gamma) > 0\} - \#\{\gamma \in \mathrm{Inv}(s_\beta) \mid w(\gamma) < 0\}$$

Notice that $\mathrm{Inv}(s_\beta)$ has an involution ι defined by $\iota(\gamma) = -s_\beta(\gamma)$; β is the unique fixed point of ι (which, by the way, implies $\mathrm{Inv}(s_\beta)$ has odd cardinality).

Moreover, $w(\beta) > 0$ implies $w(\gamma) > 0$ or $w(\iota(\gamma)) > 0$ for all $\gamma \in \mathrm{Inv}(s_\beta)$.

Therefore:

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The set on the right has at least one element, namely β, so we are done.

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$$\ell(ws_\beta) = \ell(w) + \#\{\gamma \in \mathrm{Inv}(s_\beta) \mid w(\gamma) > 0 \text{ and } w(\iota(\gamma)) > 0\}$$

The set on the right has at least one element, namely β, so we are done.

□

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We can check that:

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Notice that $\mathrm{Inv}(s_\beta)$ has an involution ι defined by $\iota(\gamma) = -s_\beta(\gamma)$; β is the unique fixed point of ι (which, by the way, implies $\mathrm{Inv}(s_\beta)$ has odd cardinality).
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Idea of our proof

- Let $x \in W \ltimes \mathcal{T}$ and $s_{\beta[n]} = \pi^{n\beta} s_\beta$, such that $x s_{\beta[n]} > x$.
- Write

$$\mathrm{Inv}_x^{++}(s_{\beta[n]}) = \left\{ \gamma[m] \in \mathrm{Inv}(s_{\beta[n]}) \mid x(\gamma[m]) > 0 \text{ and } x(\iota(\gamma[m])) > 0 \right\}$$

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$$\ell(x s_\beta) = \ell(x) + \#\mathrm{Inv}_x^{++}(s_{\beta[n]})$$

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$$\text{"}\#\mathrm{Inv}((x s_{\beta[n]})^{-1}) = \#\mathrm{Inv}(x^{-1}) + \#\mathrm{Inv}_x^{++}(s_{\beta[n]})$$

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A subtlety about inversion sets

- Consider $\mathbf{G} = \widehat{\mathbf{SL}_2}$. Then we have $\mathrm{Inv}(\pi^{\Lambda_0}) = \mathrm{Inv}(\pi^{\Lambda_0+\delta})$.
- But $\ell(\pi^{\Lambda_0}) = 0$, and $\ell(\pi^{\Lambda_0+\delta}) = 4$.
- In fact, $\pi^{\Lambda_0} < \pi^{\Lambda_0+\delta}$.
- Our proof constructs a injection

$$\mathrm{Inv}(\pi^{\Lambda_0}) \hookrightarrow \mathrm{Inv}(\pi^{\Lambda_0+\delta})$$

whose image omits exactly 4 elements.

Classification of covers

Conjecture

Let $x, y \in W \ltimes \mathcal{T}$*. Then* x *covers* y *if and only if*

$$x > y \text{ and } \ell(x) = \ell(y) + 1$$

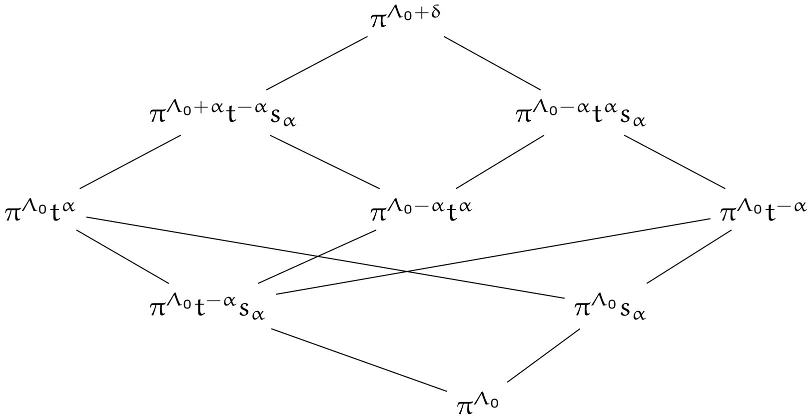
Theorem (M-Orr)

The above conjecture is true in affine ADE type. Additionally, each $x \in W \ltimes \mathcal{T}$ *is covered by finitely many elements.*

Corollary

Intervals in the Bruhat order are finite.

A Bruhat interval for affinized $\widehat{SL_2}$



Schubert slices (a.k.a Kazhdan-Lusztig varieties) (1)

- Let $I_\infty = \left\{ g \in \mathbf{G}(\Bbbk[\pi^{-1}]) \mid g \in \mathbf{B}^-(\Bbbk) \mod \pi^{-1} \right\}$
- The group I_∞ serves as the "opposite" double affine Borel.
- Let $x, y \in W \ltimes \mathcal{T}$, then we consider the open Kazhdan-Lusztig variety:

$$(I_\infty y I \cap I x I)/I \subset G^+/I$$

Schubert slices (a.k.a Kazhdan-Lusztig varieties) (2)

Conjecture

The set $(I_\infty y I \cap I x I)/I \neq \varnothing$ *if and only if* $y \leqslant x$.

Conjecture

We have $\dim(I_\infty y I \cap I x I)/I = \ell(x) - \ell(y)$ *(counting dimension).*

Conjecture

There exists a universal polynomial $R_{y,x} \in \mathbb{Z}[v]$ *(independent of* $\Bbbk$*), such that:*

$$\#(I_\infty y I \cap I x I)/I = R_{y,x}(q)$$

The polynomials $R_{y,x}$, if they exist, are the Kazhdan-Lusztig R-polynomials. Combined with the Bruhat order, we can get the usual Kazhdan-Lusztig polynomials.

Aside: (double) affine Grassmannian slices (1)

Fix $\mu, \lambda \in \Lambda^{++}$. Four perspectives on affine Grassmannian slices Gr^{λ}_{μ}:

1. Bundle theoretic: $\mathbf{G}_{\circ}$-bundles on curves/surfaces (Braverman-Finkelberg in double affine case).
2. Group theoretic: $I_{\infty}\pi^{\mu}K \cap \overline{K\pi^{\lambda}K}/K$.
3. Quantization: shifted truncated Yangians.
4. Symplectic duality: the Braverman-Finkelberg-Nakajima "Coulomb branch".

- $1 = 2$: classical in single affine setting (e.g. Beauville-Laszlo theorem). Unknown in double affine.
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Conjecture (Braverman-Finkelberg)

Intesection cohomology stalks (i.e. KL polynomials) of double affine Grassmannian slices are q*-analogues of* $\dim V(\lambda)_\mu$

- In the single affine case, this is usual called the Kato-Lusztig formula.
- This is a concrete manifestation of geometric Satake correspondence.
- For the rest of the talk we will work with G^+/K for simplicity of exposition.
- We will study $I_\infty \pi^\mu K \cap K\pi^\lambda K/K$. Replacing K by I is no more difficult, but the exposition gets more complicated.

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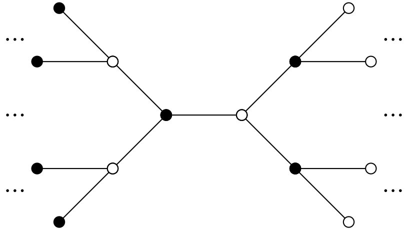
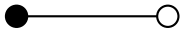
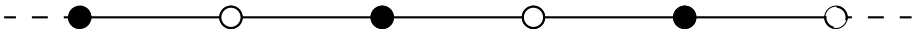
Brief recollection of affine buildings (1)

A portion of the $\mathbf{SL}_2(\mathbb{F}_2)$ affine building:

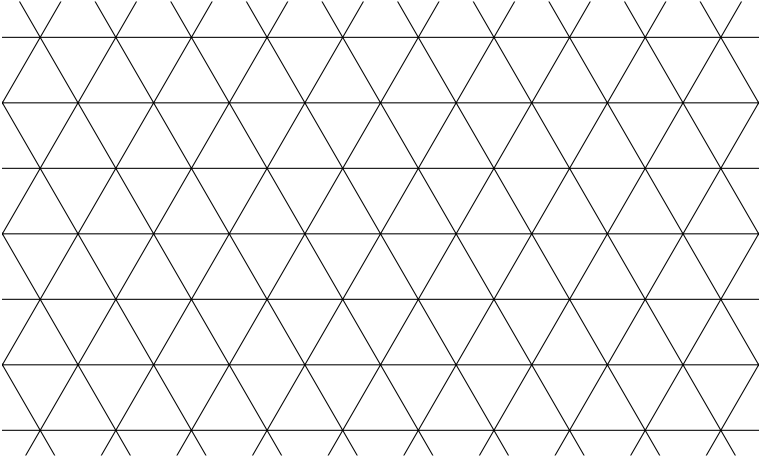


Two perspectives:

- Glue facets: 
- Glue apartments $\mathbb{A}$: 

Brief recollection of affine buildings (2)

The standard apartment $\mathbb{A}$ for $\mathbf{SL}_3(\mathbb{k})$:



Masures (1)

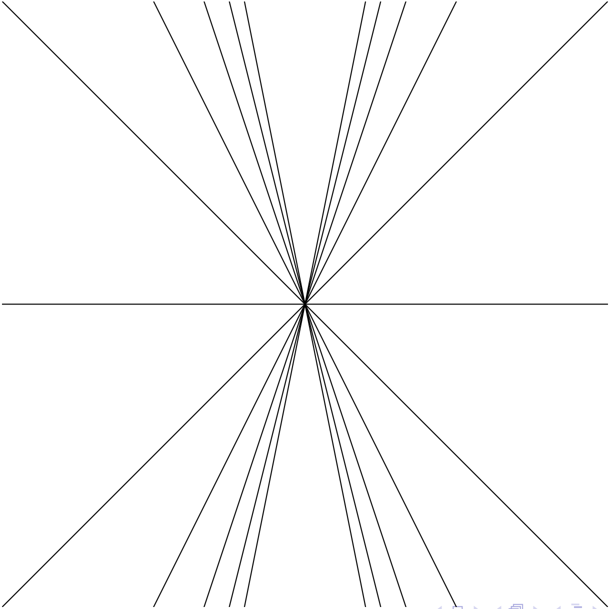
- Gaussent and Rousseau have developed a theory of affine buildings for Kac-Moody group, which they call the theory of masures (a.k.a. hovels).
- Affine buildings $\mathcal{J}$ contains a distinguished copy of $\mathbb{A} = \Lambda \otimes \mathbb{R}$, the "standard apartment".
- The group G will act on $\mathcal{J}$.
- For each $x \in \mathbb{A}$, let P_x be the fixator of x.
- Then $\mathcal{J} \cong G \times \mathbb{A}/\sim$
- We have $(p, x) \sim (1, x)$ for all $x \in \mathbb{A}$ and $p \in P_x$.
- We have $(\pi^\mu w, x) \sim (1, \pi^\mu w.x)$ for all $\pi^\mu w \in W \times \mathcal{T}$ and $x \in \mathbb{A}$.

Idea: start from P_x and define $\mathcal{J}$ by the above recipe.

Masures (2)

- Roughly, P_x is the group generated by $e_\beta(f)$ for real roots β, where $\mathrm{val}(f) \geqslant \langle \beta, x \rangle$.
- Here $e_\beta : \mathcal{K} \to G$ is the one-parameter subgroup.
- In the single-affine case, P_x is always conjugate to a standard parahoric.
- Not so in the double-affine case.

Affine $\widehat{SL_2}$ fundamental apartment



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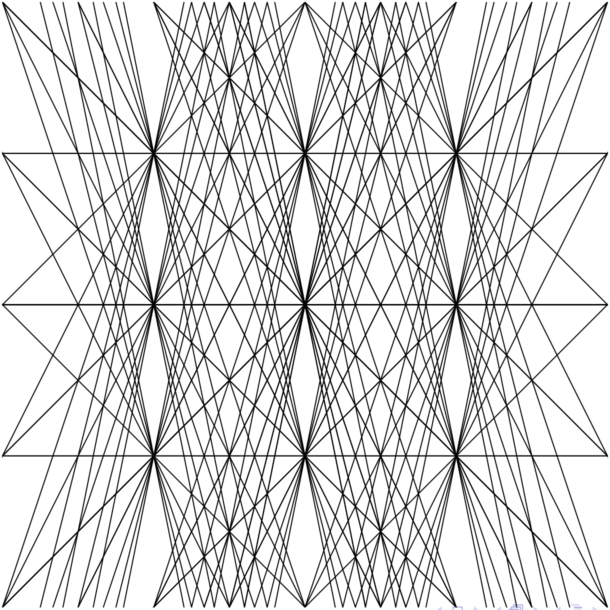
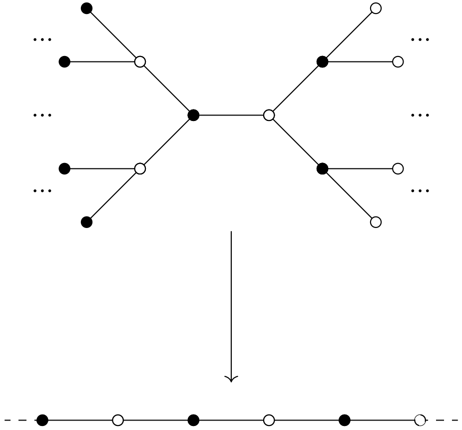


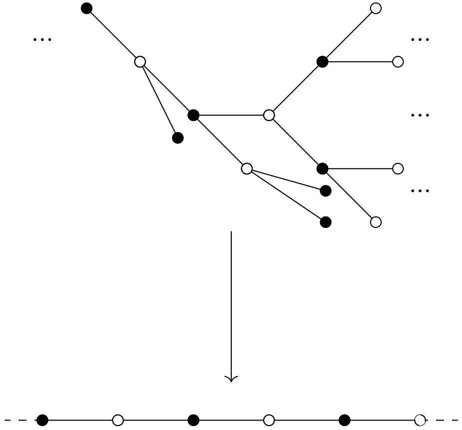
Retractions (1)

Retraction ρ_I centered at the fundamental alcove for $\mathbf{SL}_2(\mathbb{F}_2)$:



Retractions (2)

Retraction ρ_{U^-} centered at $-\infty$ (anti-dominant chamber) $\mathbf{SL}_2(\mathbb{F}_2)$:



Retractions (3)

- Let $\mathcal{J}^+ = G^+ \times \mathcal{T}_{\mathbb{R}} / \sim \subseteq \mathcal{J}$ denote the "positive part" of the masure.
- The map $\rho_I : \mathcal{J}^+ \to \mathcal{T}_{\mathbb{R}}$ is characterized by
  - ρ_I is a retraction.
  - ρ_I is I-invariant.
- The map $\rho_{U^-} : \mathcal{J} \to \mathbb{A}$ is characterized by
  - ρ_{U^-} is a retraction.
  - ρ_{U^-} is U^--invariant.
- Both exist for general masures by work of Gaussent and Rousseau.

Retractions along I_∞

Assumption (Work in progress with Patnaik)

There exists a retraction $\rho_{\mathrm{I}_\infty} : \mathcal{J}^+ \to \mathcal{T}_{\mathbb{R}}$ *along* I_∞.

This is closely related to Birkhoff decomposition and the theory of twin buildings.

I_∞-Hecke paths (1)

- Fix $\lambda \in \Lambda^{++}$ and $\nu \in \mathcal{T}$.
- Let $\varphi_\circ : [0,1] \to \mathbb{A}$ be the straight path starting at 0 and ending at λ.
- We can identify $K \cdot \varphi_\circ = K\pi^\lambda K/K$.
- Let $\varphi \in I \cdot \varphi_\circ$, which we think of as a straight line in the masure that starts at 0 and ends at a point in $K\pi^\lambda K/K$.
- The retraction $\rho_{I_\infty}(\varphi)$ is a piecewise linear path.
- We call such piecewise linear paths I_∞-Hecke paths. (Replacing I_∞ with U^-, we get the usual notion of Hecke path)
- If the endpoint $\rho_{I_\infty}(\varphi)(1) = \nu$, then we have $\varphi \in I_\infty \pi^\nu K/K$.

I_∞-Hecke paths (2)

Denote the set of I_∞-Hecke paths ending at ν by:

$$^{I_\infty}\mathcal{H}^\lambda_\nu = \{\rho_{I_\infty}(\varphi) \mid \varphi \in K\pi^\lambda K/K \text{ and } \varphi(1) = \nu\}$$

Let $\tau \in {}^{I_\infty}\mathcal{H}^\lambda_\nu$. Then there is a sequence of folding times

$$0 = t_0 < t_1 < \cdots t_N < t_{N+1} = 1$$

and folding directions

$$x_0, \ldots, x_N \in W \ltimes Q$$

such that for all $i \in \{0, 1, \ldots, N\}$

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Counting points in Schubert slices (1)

Theorem (M)

The set $\{\varphi \in K\pi^\lambda K/K \mid \rho_{I_\infty}(\varphi) = \tau\}$ *is in bijection with:*

$$\prod_{k=0}^{N} (((x_{k-1}^{-1} I_\infty x_{k-1}) \cap P_{t_k\lambda}) x_k P_{[t_k, t_k+\varepsilon)\lambda)} \cap P_{(t_k-\varepsilon, t_k]\lambda} P_{[t_k, t_k+\varepsilon)\lambda)}) / P_{[t_k, t_k+\varepsilon)\lambda}$$

- The k-th factor above is a subset of $P_{t_k\lambda}/P_{[t_k,t_k+\varepsilon)\lambda}$, which is in canonical bijection with the $\Bbbk$-points of a Kac-Moody partial flag variety.
- The set we are looking at is an intersection of (translates of) opposite Schubert cells.
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- The above set is non empty only if (for some $w \in W$):

$$\pi^\lambda \geqslant_{W_{t_0}} \pi^\lambda x_0 >_{W_{t_1}} > \cdots >_{W_{t_N}} \pi^\lambda x_N = \pi^\nu w$$

- Here W_t is the group generated by double affine reflections that fix $t \cdot \lambda \in \mathbb{A}$. The notation $x >_{W_t} y$ means $x > y$ and $xy^{-1} \in W_t$.
- This implies $\ell(\pi^\lambda) \geqslant \ell(\pi^\nu w)$, which only allows finitely many possible w. Therefore, in affine ADE, there are only finitely many possible $(x_0, \ldots, x_N)$.
- Both conclusions are by my work with Orr on double affine Bruhat order.

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- For G^+/K the situation is more subtle (the subtlety is related to the Uhlenbeck compactification of instanton spaces and the non-trivial stratification of affine Coulomb branches).

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Modulo the assumption about the existence of ρ_{I_∞}, *we have*

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References (1)

My work:

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Thank you!