Introduction to Kac-Moody symmetric spaces

Talk 1: Finite-dimensional symmetric spaces

Talk 2: Structure of Kac-Moody symmetric spaces

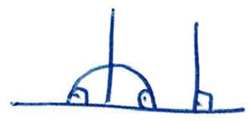
Talk 3: Boundary rigidity for Kac-Moody groups

I] Finite-dimensional symmetric spaces

Perspectives on symm. spaces
- (a) differential geometric
- (b) group theoretic
- (c) combinatorial (reflection spaces) } generalize to KM setting

§1 The hyperbolic plane

(a)] $\mathbb{H}^2 = \left(\{z = x+iy \in \mathbb{C} \mid y>0\},\ ds^2 = \frac{dx^2+dy^2}{y}\right)$



hyperbolic plane $(K \equiv -1)$

Features: •) $\forall x,y\ \exists!$ geodesic γ through x,y,
geodesics $\cong (\mathbb{R}, d)$ global ($\Rightarrow$ contractible)

•) geodesics diverge exponentially ($K<0$), metric convex ($K \leq 0$)

•) unique midpoints : (in fact: unique barycenters)

Visual boundary: $\gamma_1, \gamma_2 : [0,\infty) \to \mathbb{H}^2$ (quasi-)geodesic $\rightsquigarrow$ $\gamma_1 \sim \gamma_2 :\Leftrightarrow d(\gamma_1(t), \gamma_2(t)) < C$
asymptotic

$\partial \mathbb{H}^2 = \{\gamma : [0,\infty) \to \mathbb{H}^2 \text{ quasi-geodesic}\}/\sim\ \cong \mathbb{R} \cup \{\infty\}$.

$f : \mathbb{H}^2 \to \mathbb{H}^2$ quasi-isometry $\rightsquigarrow$ $\partial f : \partial\mathbb{H}^2 \to \partial\mathbb{H}^2$ boundary extension

(b)] •) $G := PSL_2(\mathbb{R}) \curvearrowright \mathbb{H}^2$, $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right].z := \frac{az+b}{cz+d}$ proper, isometric, transitive

$K := \mathrm{Stab}(i) = PSO_2(\mathbb{R}) \rightsquigarrow \mathbb{H}^2 \cong G/K$

$A := \{a_t = \left(\begin{smallmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{smallmatrix}\right)\}$, $N^+ := \{n_s = \left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right)\}$, $B^+ := AN^+ = N^+A$

Note: $a_t.z = e^t z$, $n_s.z = z+s$

$\mathfrak{g}, \mathfrak{k}, \mathfrak{a}, \mathfrak{n}^+, \mathfrak{b}^+$ Lie algebras

$G = N^+AK$ Iwasawa decomposition

•) $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{p} = \{X \in \mathrm{Sym}_2(\mathbb{R}) \mid \mathrm{tr}(X) = 0\}$
$G = \exp(\mathfrak{p})K$, $g = (gg^T)^{1/2} \cdot [(gg^T)^{-1/2} g]$ } polar decomposition

$\Rightarrow \mathbb{H}^2 \cong \exp(\mathfrak{p})$, $T_i \mathbb{H}^2 \cong \mathfrak{p} \overset{\mathrm{Ad}}{\curvearrowleft} K$

•) G-inv. Riem. metric on $\mathbb{H}^2$ $\hat{=}$ K-inv. scalar product on $\mathfrak{p}$: $\langle X, Y \rangle = \mathrm{tr}(XY)$.

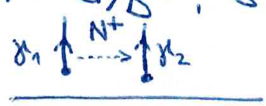
geodesics through i: $\gamma(t) = \exp\left(t \frac{X}{\|X\|}\right).i$, $X \in \mathfrak{p}$

general geodesics: $\gamma(t) = g \exp\left(t \frac{X}{\|X\|}\right).i$, $X \in \mathfrak{p}$

•) $\mathfrak{p} = \bigcup_{k \in K} \mathrm{Ad}(k)(\mathfrak{a})$ (real spectral theorem) $\Rightarrow$ $K \curvearrowright$ geod. through i transitively
$\hat{=}$ geod. through i, $\exp(\mathfrak{p}) = \bigcup_{k \in K} kAk^{-1}$

$\Rightarrow G \curvearrowright \{(p, \gamma) \mid p \in \gamma(\mathbb{R})\}$ transitively, KAK Cartan decomposition
geodesic; "All pointed geodesics are equal."

•) $\partial \mathbb{H}^2 = G/B^+$, $B^+ = \mathrm{Stab}(\infty)$



$\gamma_1 \sim \gamma_2$ asymptotic $\Longleftrightarrow$ Can slide from γ_1 to γ_2 along horosphere.

(c) $\cdot$) $s_i : \mathbb{H}^2 \to \mathbb{H}^2$, $\exp(\underset{\underset{\mathfrak{p}}{\cap}}{X}).i \mapsto \exp(-X).i$  geodesic reflection in i

$\cdot$) $s_{gi} : \mathbb{H}^2 \to \mathbb{H}^2$, $s_{gi} = g s_i g^{-1}$ geod. reflection in gi

$\rightsquigarrow$ $\mu : \mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{H}^2$, $(x,y) \mapsto s_x(y) := x \cdot y$ s.t.

$$\left.\begin{array}{l} (S1)\ x\cdot x = x \\ (S2)\ x\cdot(x\cdot y) = y \\ (S3)\ x\cdot(y\cdot z) = (x\cdot y)\cdot(x\cdot z) \end{array}\right\} \text{reflection space}$$

$\cdot$) $\mathrm{Is}(\mathbb{H}^2, d) = \mathrm{Aut}(\mathbb{H}^2, \mu) = \underbrace{PSL_2(\mathbb{R})}_{= \langle s_x \circ s_y \mid x,y \in \mathbb{H}^2 \rangle} \rtimes \langle s_i \rangle$. transvection group

$\bullet$) Geodesics = closed $\overset{\text{connected}}{\text{reflection}}$ subspaces $\neq \{*\}, \mathbb{H}^2$

§ 2 Riemannian symmetric spaces

A reductive group is an algebraic subgroup $G < GL_n(\mathbb{R})$ s.t. $G^T = G$.

Goal: Every reductive group acts properly, isom., transitively on a NPC Riem. mfd. $(K \leq 0)$

$\boxed{\text{Notation:}}$ $\Theta : G \to G$, $g \mapsto g^{-T}$ Cartan involution, $K = G^\Theta = G \cap O_n(\mathbb{R})$

$\Theta : \mathfrak{g} \to \mathfrak{g}$, $X \mapsto -X$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{p} = \mathfrak{g} \cap \mathrm{Sym}_n(\mathbb{R})$ (-1)-eigenspace

Polar decomposition: $G = \exp(\mathfrak{p}) K$

(a) $X_G = G/K \cong \exp(\mathfrak{p}) = G \cap \mathrm{Pd}_n(\mathbb{R})$, $o = eK$ basepoint

$T_o X_G \cong \mathfrak{p}$, $\langle X, Y \rangle = \mathrm{tr}(XY)$ K-inv. inner product on $\mathfrak{p}$

$\rightsquigarrow$ G-inv. Riem. metric on X_G Riem. symm. space of X

Features: non-positive curvature $(K \leq 0)$

($\Rightarrow$ global geod., contractible, unique barycenters, geodesics diverge) $\to$ application: K unique max. compact subgp. up to conjugation (Bruhat-Tits)

$\rightsquigarrow$ non-trivial visual boundary ∂X_G.

(b) $\cdot$) Geodesics through o: $\gamma(t) = \exp(t \frac{X}{\|X\|}).o$, $X \in \mathfrak{p}$.

Geodesics: $\gamma(t) = g \exp(t \frac{X}{\|X\|}).o$

$\cdot$) $\mathfrak{a} \subset \mathfrak{p}$ subalgebra of $\mathfrak{g}$ $\Rightarrow$ $[\mathfrak{a},\mathfrak{a}] \subset \mathfrak{p} \cap \underbrace{[\mathfrak{p},\mathfrak{p}]}_{=\mathfrak{k}} = \{0\} \Rightarrow \mathfrak{a}$ abelian (Cartan $\overset{\text{split}}{\text{subalgebra}}$)

$\xRightarrow{\text{spectral theorem}}$ All such $\mathfrak{a}$ are conjugate under $Ad(K)$

and $\mathfrak{p} = \bigcup_{k \in K} Ad(k)(\mathfrak{a}_0)$

Fix $\mathfrak{a} \subset \mathfrak{p}$ subalgebra $\rightsquigarrow$ $A < G$ split torus $\rightsquigarrow$ $\underset{\text{choice of pos. roots}}{\Delta^+(\mathfrak{g},\mathfrak{a})} \subset \Delta(\mathfrak{g},\mathfrak{a})$

$\rightsquigarrow$ $N^+ = \langle U_\alpha = \exp(\mathfrak{g}_\alpha) \mid \alpha \in \Delta^+(\mathfrak{g},\mathfrak{a}) \rangle$, $B^+ = MAN^+ = N^+AM$, $M = N_K(A)$

·) Decompositions: $G = N^+AK = KAK$ - geometric meaning?
(Iwasawa) (Cartan)

·) $\varphi: \mathbb{R}^n \xhookrightarrow{\text{isometric}} X_G \rightsquigarrow F := \varphi(\mathbb{R}^n)$ n-flat in X_G — 0-flats = points, 1-flats = geodesics, $\nexists$ 2-flats in $\mathbb{H}^2$ - triangles have bounded volume

·) $X, Y \in \mathfrak{a} \Rightarrow \exp(Y).o = \exp(X)\exp(Y-X).o = \exp(X)\exp\left(\|Y-X\| \frac{Y-X}{\|Y-X\|}\right).o$ — geod. of length $\|Y-X\|$

$\Longrightarrow d(\exp(X).o, \exp(Y).o) = \|Y-X\| \Rightarrow A.o = \exp(\mathfrak{a}).o$ is an n-flat

$\rightsquigarrow$ {max flats through o} $\hat{=}$ {max. split tori} $\hat{=}$ {max. split CSA} $\circlearrowleft Ad(K)$ transitively

$F \quad \hat{=} \quad A.o \quad = \exp(\mathfrak{a}).o$

Consequences: (i) $G \overset{\text{trans.}}{\curvearrowright}$ {pointed max. flats (p, F)} strong transitivity

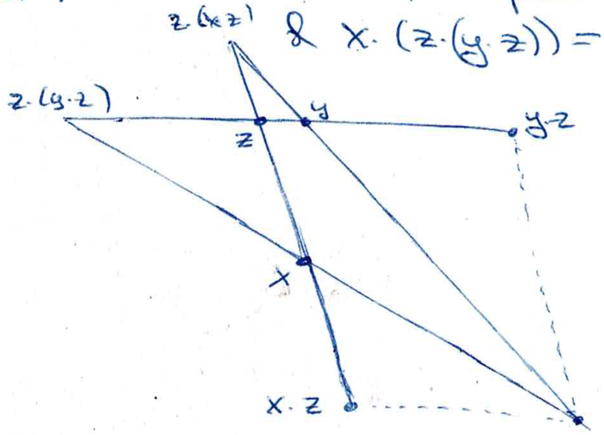
(ii) Every geodesic lives in a max. flat of dim. $\mathrm{rk}_{\mathbb{R}}(G)$ (rank)

(c) $\mu: X_G \times X_G \to X_G$ reflection structure. $s_o(g.o) = (g^{-T}.o)$, $s_{g.o} = g s_o g^{-1}$ geodesic reflections

·) $\mathrm{Is}(X_G, d) = \mathrm{Aut}(X_G, \mu) \supset G(X_G, \mu) = \langle s_x \rangle \supset \mathrm{Trans}(X_G, \mu) = \langle s_x \circ s_y \rangle \supset G$.

·) Theorem (Loos '69, FHHK '18)

$F \subset X_G$ flat $\Longleftrightarrow$ F closed, midpoint-convex ($\forall y, z \in F \exists x \in F: x.y = z$)
& $x.(z.(y.z)) = y.(z.(x.z))$



'flatness'

§3 Non-Riemannian symmetric spaces

Theorem (Loos)

The following classes of spaces coincide: ("affine symmetric spaces")

(a) X = 1-connected fin.-dim. manifold with connection ∇, s.t. $\forall x \in X \exists s_x \in \mathrm{Aut}(X, \nabla)$ geodesic reflection

(b) $X = G/H$, G red. Lie group, $(G^\Theta)^\circ \subset H \subset G^\Theta$ for an involution Θ of G

(c) (X, μ) fin.-dim. reflection space of class C^2 s.t. (S4) $\forall x \in X \exists U_x$ nbhd. of x $\forall y \in U_x$ $x.y = y \Rightarrow y = x$.

X is Riemannian $\Leftrightarrow$ H is compact.

Rem. (i) A reflection space (X, μ) is irreducible if $(X, \mu) \overset{p}{\twoheadrightarrow} (\bar{X}, \bar{\mu}) \Rightarrow X = \bar{X}$, $p = \mathrm{Id}$

(ii) Every affine sym. space is a product of irreducibles.

(iii) Irreducible Non-compact Rien. sym. spaces $< \begin{matrix} \mathbb{R}^n \\ G/K, \text{ G reductive} \end{matrix} \rightsquigarrow$ classification

§4 The spherical building at ∞

•) $G \overset{\text{trans.}}{\curvearrowright} \{\text{pointed max. flats in } X_G\}$

$\boxed{\mathrm{rk}_{\mathbb{R}} G = 1} \Rightarrow G \overset{\text{trans.}}{\curvearrowright} \{\text{pointed geod. in } X_G\}$ "all geodesics are equal"

•) <u>Key observation</u>: In higher rank, <u>not</u> all geodesics are equal.

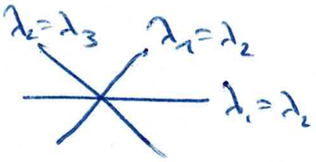
A geodesic ray γ is <u>regular</u> if it is contained in a unique max. flat through $\gamma(0)$ (otherwise <u>singular</u>)

$\boxed{\text{Example}}$ $G = SL_3(\mathbb{R})$, $\mathfrak{a} = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} \middle| \sum \lambda_i = 0 \right\} \overset{\exp}{\underset{\cong}{\longrightarrow}} F \subset X_G$

$X_1 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} \in \mathfrak{a} \cap \mathrm{Ad}\begin{pmatrix} R_\theta & \\ & 1 \end{pmatrix} \mathfrak{a} \Rightarrow \gamma_{X_1}(t) = \exp\left(t \frac{X_1}{\|X_1\|}\right).o$ <u>singular</u>

$X_2 = \begin{pmatrix} 1 & & \\ & 2 & \\ & & -3 \end{pmatrix} \Rightarrow \underset{\text{(spectral theorem)}}{\mathrm{cent}_{\mathfrak{g}}(X_2) \cap \mathfrak{p} = \mathfrak{a}} \Rightarrow \gamma_{X_2}(t)$ <u>regular</u>

$\rightsquigarrow \mathfrak{a}^{\text{sing}} := \{X \mid \gamma_X \text{ singular}\} = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} \middle| \exists i \neq j : \lambda_i = \lambda_j \right\}$



<u>Theorem</u> (General case)

(i) $\mathfrak{a}^{\text{sing}} = \bigcup_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})} \ker(\alpha) \subset \mathfrak{a}$ (hyperplane arrangement)

(ii) $H_\alpha := \ker(\alpha)$, s_α orth. refl. at $H_\alpha \Rightarrow W = \langle s_\alpha \rangle < O(\mathfrak{a})$ preserves $\mathfrak{a}^{\text{sing}}$

(iii) $\Pi \subset \Delta(\mathfrak{g}, \mathfrak{a})$ ~~root~~ basis $\Rightarrow (W, \{s_\alpha \mid \alpha \in \Pi\})$ Coxeter system.

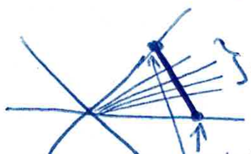
•) $\boxed{\text{Local action}}$ $G \curvearrowright X_G \rightsquigarrow K \curvearrowright (X_G, o) \rightsquigarrow \underline{\mathrm{Stab}(F, o) \curvearrowright (F, o)}$

↑ independent of choice of (F, o) (strong transitivity)

<u>Theorem</u> (Cartan-Weyl)

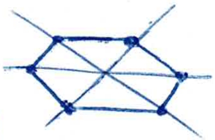
$\left(\mathrm{Stab}(F, o) / \mathrm{Fix}(F, o) \curvearrowright (F, o) \right) \cong (W \curvearrowright \mathfrak{a})$, i.e. the local action is given by a Coxeter group (<u>Weyl group</u>)

•) $\boxed{G\text{-orbits of } \underline{\text{geodesic rays}}}$



•) <u>Simplicial structure on $\partial F \subset \partial X_G$</u>

chambers (max. simplices) $\hat{=}$ conn. components of $\mathfrak{a}^{\text{reg}}$ (Weyl chambers)



(bijection preserves codim. 1 adjacency)

The underlying abstract simplicial complex is isomorphic to the <u>Coxeter complex</u> $\Sigma(W, S_\Pi) = (\{w \langle T \rangle \mid w \in W, T \subset S_\Pi\}, \supset)$.

In particular, its chamber graph is the <u>Cayley graph</u> $\mathrm{Cay}(W, S_\Pi)$.

·) Simplicial structure on ∂X_G:

$\partial X_G = \bigcup_{\sigma\in F} \partial F$ and $\partial F \cap \partial F' \overset{\subset \partial F}{\subset \partial F'}$ simplicial subcomplex $\rightsquigarrow$ glue!

Corollary (i) $\Delta = \partial X_G$ is a simplicial complex.

(ii) Δ is covered by subcomplexes (apartments) $\cong \Sigma(W,S)$, which are stabilized by split-tori.

(iii) $G \curvearrowright \Delta$ chamber-transitively by simplicial automorphisms, $Ch(\Delta) = G/B^+$.

(More generally, $Is(X_G, d) = Aut(X_G, \mu) \to Aut(\Delta)$.)

Theorem (Boundary rigidity, Tits) $\quad$ [G semisimple, $rk(G) \geq 2$]

$Aut(X_{G}, \mu) \to Aut(\Delta)$ is an isomorphism.
$\quad f \mapsto \partial f$

$\blacksquare$

(This fails for $\mathbb{R}^n$)

Example $\quad G = SL_n(\mathbb{R}),\ X_G = Pd_n(\mathbb{R})_1 = \{X \in \mathbb{R}^{n\times n} \mid X \text{ pos. def.},\ \det(X) = 1\}$

$Ch(\Delta) = \mathcal{F}\ell(\mathbb{RP}^{n-1})$

$\varphi \in Aut(\Delta) \rightsquigarrow \varphi$ incidence-preserving map of $PG(\mathbb{R}^n)$

$\rightsquigarrow$ fundamental thm. of proj. geometry.

§ 5 Causal symmetric spaces and causal boundary

·) Instead of invariant Riem. metric on G/H, take an invariant cone field $\mathcal{C}_x \subset T_x(G/H)$, i.e. $\mathcal{C}_x$ open cone and $g_* \mathcal{C}_x \subset \mathcal{C}_{gx}$. (e.g. Lorentzian case)

·) $\gamma: [0,\infty) \to G/H$ causal geod. ray $:\Leftrightarrow \forall t: \dot\gamma(t) \in \mathcal{C}_{\gamma(t)}$.

$\rightsquigarrow \partial^+ G/H := \{[\gamma] \in \partial G/H \mid \gamma \text{ causal}\}$ causal boundary of G/H

·) $g \in G \subset Aut(G/H, \mu, \mathcal{C}_\cdot) \rightsquigarrow \partial_g: \partial^+ G/H \to \partial^+ G/H$ causal boundary map

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Outlook: G Kac-Moody group

$\rightsquigarrow G \hookrightarrow Aut(\underbrace{G/K}_{\text{top reflection space}}, \mu, \mathcal{C}^{\pm}_\cdot)$ $\leftarrow$ pair of causal structures $\quad \boxed{\text{Kac-Moody symmetric space}}$

$\rightsquigarrow G \curvearrowright (\partial^- G/K, \partial^+ G/K)$ $\quad \boxed{\text{twin building at } \infty}$

II | Structure of Kac-Moody symmetric spaces

<u>Recall</u>: G reductive $\rightsquigarrow$ $G \xrightarrow{\rho_i} \mathrm{Aut}(X_{G,\mu})$, $G \overset{\text{trans.}}{\curvearrowright} \{$ pointed max. flats $\}$

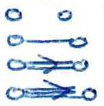
$\rightsquigarrow$ local action given by $W \curvearrowright \mathfrak{a}$,

∂X_G building modelled on $\Sigma(W,S)$

What about infinite Coxeter groups?

§1 Split real Kac-Moody groups

.) <u>Basic examples</u>: $\overset{\circ}{SL_2(\mathbb{R})}$



$SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$, $SL_3(\mathbb{R})$, $Sp_4(\mathbb{R})$, $G_2(\mathbb{R})$ $\Big\}$ generated by two $SL_2(\mathbb{R})$s + canonical Lie group topology

.) 2-spherical Dynkin diagram  $\mathcal{D}$ 

associated top. group $G = G_{\mathcal{D}} = \varinjlim \begin{pmatrix} G_1 \rightarrow G_{12} \\ G_2 \rightarrow G_{2n} \\ G_n \rightarrow G_{1n} \end{pmatrix}$

$(\cong SL_2(\mathbb{R}))$

.) <u>Properties</u> (i) G is an alg. simply-connected Kac-Moody group $/\mathbb{R}$ (Abramenko-Mühlherr)

(ii) The topology is Hausdorff & "k_ω". (Kac-Peterson, GHKM)

For $\mathcal{D}$ non-spherical it is not metrizable.

E.g.  $\cong \underbrace{SL_3(\mathbb{R}[t,t^{-1}]) \rtimes \mathbb{R}^\times}_{\text{box topology}}$.

(iii) Analogous decompositions hold for Coxeter groups and symmetrizable Lie algebras (Gabber-Kac).

§2 Subgroups

.) $G_i = \varphi_i(SL_2(\mathbb{R}))$ <u>fundamental rank one subgroups</u>

$G_{ij} = \langle G_i, G_j \rangle$ — <u>two</u> — $\Big\}$ carry their Lie group topology ($\Rightarrow$ closed)

.) $T_i := \varphi_i\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$, $T := \prod_{i=1}^{n} T_i \cong (\mathbb{R}^\times)^n$, $A := T^\circ \cong (\mathbb{R}^{>0})^n$

<u>conjugates = max. split tori</u>   <u>standard max. Θ-split torus</u> $M := (\mathbb{Z}/2\mathbb{Z})^n < T$

.) $U_{\alpha_i} := \varphi_i\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$, $U^+ := \langle U_{\alpha_i} \rangle$, $B^+ := T \ltimes U^+$ <u>standard Borel subgroups</u>

$U_{-\alpha_i} := \varphi_i\begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}$, $\underbrace{U^- := \langle U_{-\alpha_i} \rangle}_{\text{max unipotent subgroups}}$, $B^- := T \ltimes U^-$

.) $\theta_i : G_i \to G_i$, $g \mapsto g^{-T}$ $\rightsquigarrow$ $\theta : G \to G$ <u>Cartan-Chevalley involution</u>

$K := G^\theta = \langle G_i^\theta \rangle = \varinjlim (K_i \to K_{ij})$.

$\tilde{W} := N_K(T) = \langle \underbrace{\varphi_i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{s_{\alpha_i}} \rangle$ <u>extended Weyl group</u>

$W := N_K(T)/T$ <u>Weyl group</u>   <u>Fact</u>: $(W, \underbrace{(s_{\alpha_i})}_{=S})$ Coxeter system

.) $\Phi(W,S)$ roots of (W,S) ("real roots")

$\rightsquigarrow$ $\exists$ subgroups $(U_\alpha)_{\alpha \in \Phi}$ of G s.t. $\forall \tilde{w} \in \tilde{W}$: $\tilde{w} U_\alpha \tilde{w}^{-1} = U_{w\alpha}$ (<u>root groups</u>)

and $U_\alpha \subset U^\pm \Leftrightarrow \alpha \in \Phi^\pm(W,S)$.

§3 The symmetric space

- $\tau : G \to G,\ g \mapsto g\theta(g^{-1})$ twist map ("$g \mapsto gg^T$")
- $X_G := G/K,\ \mu : X_G \times X_G \to X_G,\ (gK, hK) \mapsto \tau(g)\theta(h)K$

Prop. (FHHK)

(i) X_G is a top. symmetric space: $\left\{\begin{array}{l} x\cdot x = x \\ x\cdot(x\cdot y) = y \\ x\cdot(y\cdot z) = (x\cdot y)\cdot(x\cdot z) \\ x\cdot y = y \Rightarrow x = y \end{array}\right\}$

(ii) τ induces $\hat{\tau} : X_G \hookrightarrow G$ s.t. $\hat{\tau}(x\cdot y) = \hat{\tau}(x)\hat{\tau}(y)^{-1}\hat{\tau}(x)$ ∎

(iii) $G \curvearrowright X_G$ by automorphisms and $\hat{\tau}(gx) = g * x = gx\theta(g)^{-1}$.

$\rightsquigarrow X_G$ - 'symm. space of G', $\tau(G) \cong X_G$ - 'group model' ($\hat{=}\ G \cap \mathrm{Pdh}(\mathbb{R})$ if $G \subset Gl_n(\mathbb{R})$ red.dr.)

- $\mathbb{R}^n \xrightarrow{\exp} (\mathbb{R}^{>0})^n \cong A \xrightarrow{\cong} AK \subset X_G$ is a reflection isomorphism
  $\Rightarrow F_0 := AK = TK$ is a flat in X_G (i.e. closed, midpoint-convex, $x\cdot(z\cdot(y\cdot z)) = y\cdot(z\cdot(x\cdot z))$)
  $\Rightarrow F_g := gF_0$ is a flat in X_G standard flats

- **Theorem A** (FHHK)
  Every flat in X_G is contained in a standard flat.

  Corollary
  (i) $G \overset{\text{trans.}}{\curvearrowright} \{$pointed max. flats $(=$std. flats$)\}$; in particular, all max. flats have $\dim = n$.
  (ii) $G/N_G(T) \cong \{$max. flats$\} \cong \{$max. θ-split tori$\}$.
  (iii) $\{$geodesics$\} = \{\gamma(t) = ga(t)K \mid g \in G,\ a(t) \subset A$ one-parameter subgroup$\}$

- Write $x \rightsquigarrow y$ if $\exists$ geod. ray from x to y in X_G.

  Theorem B (Horn)
  (i) $\exists x, y \in X_G : x \not\rightsquigarrow y$. $\boxed{\Sigma \text{ not spherical}}$
  (ii) $\forall x, y \in X_G\ \exists x_1, \ldots, x_n \in X_G : x \rightsquigarrow x_1 \rightsquigarrow \ldots \rightsquigarrow x_n \rightsquigarrow y$.

- For $G \subset Gl_n(\mathbb{R})$ reductive: $G = KAK$, $p = \bigcup_{h\in K} \mathrm{Ad}(h)\mathfrak{a}, \ldots$
  Uses: Elements of $\tau(G)$ (symm. matrices) are diagonalizable matrices.
  $\rightsquigarrow$ FAILS here!

§4 Diagonalizable elements of Kac-Moody groups

- $N := N_G(T) = A \rtimes N_K(T) = A \rtimes \tilde{W}$
  $\Rightarrow G = \bigsqcup_{n\in N} U_\pm n U_\pm = \bigcup_{w\in W} B_\pm w B_\pm \Rightarrow (B^+, B^-, N)$ twin BN pair
  $= \bigsqcup_{n\in N} U_\pm n U_\mp = \bigcup_{w\in W} B_\pm w B_\mp$

- $\Delta_\pm := G/B_\pm$ buildings with distance $\delta_\pm : \Delta_\pm \times \Delta_\pm \to W$, $B_\pm g^{-1}hB_\pm = B_\pm \delta_\pm(gB_\pm, hB_\pm) B_\pm$
  $\delta^* : \Delta_+ \times \Delta_- \to W$ analogously, $C^+ \text{ op } C^- :\Leftrightarrow \delta^*(C^+, C^-) = e$
  $\rightsquigarrow \Delta = (\Delta_\pm, \delta_\pm, \delta^*)$ twin building of G

c) Θ induces $\Theta: \Delta_\pm \to \Delta_\mp$ (since $\Theta(B_\pm) = B_\mp$)

$\mathrm{Sym}(\Theta) := \{g \in G \mid \Theta(g) = g^{-1}\} \supset \tau(G)$ ($= G \cap \mathrm{Sym}_n(\mathbb{R})$ in the reductive case)

<u>Lemma</u> For $g \in \mathrm{Sym}(\Theta)$ TFAE

(i) g fixes a (Θ-stable) twin apartment. (i') $g \in xTx^{-1}$, $x \in G$.

(ii) $\exists C_\pm \in \Delta_\pm$: $gC_\pm = C_\pm$, $C_+ \text{ op } C_-$. (g is <u>diagonalizable</u>)

(iii) g fixes a chamber.

(iv) $\langle g \rangle$ has $\delta_\pm$-bounded orbits

(v) g stabilizes spherical residues $R_\pm \subset \Delta_\pm$. (g is <u>bounded</u>)

<u>Proof</u>: (iv) $\Rightarrow$ (v) Bruhat-Tits fixpoint theorem, (v) $\Rightarrow$ (ii) $g \in \underbrace{\mathrm{Stab}(R_+, \Theta(R_+))}_{\text{reductive}} < GL_n(\mathbb{R})$

g symmetric $\Rightarrow$ diagonalizable. ∎

<u>Note</u>: $\mathcal{D}$ spherical $\Leftrightarrow$ Every $g \in \mathrm{Sym}(\Theta)$ is diagonalizable.

<u>Cor.</u> (Horn)

(i) $\tau(G) \supset \tau^2(G) \supset \dots \supset \bigcap_{n \in \mathbb{N}} \tau^n(G) = \{x \in \tau(G) \mid x \text{ diagonalizable}\}$

(ii) $u = \begin{pmatrix} 1 & 1+t \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{R}[t, t^{-1}]) \Rightarrow \tau(u)$ not diagonalizable.

(iii) $\mathcal{D}$ not spherical $\Rightarrow$ $\tau(G) \supsetneq \bigcap \tau^n(G)$. ∎

§5 On Theorems A and B

<u>Proof of Theorem B</u>: We work in the group model $\tau(G) \cong X_G$.

(i) Let $x \in \tau(G) \setminus \bigcap \tau^n(G)$ be non-diagonalizable. Assume $\{e, x\} \subset \underset{\subset \tau(G)}{F}$ flat.

$$\left.\begin{aligned} &x^{(1)} = \underset{\in F}{\mathrm{mid}(x,e)} \Rightarrow x = x^{(1)} \cdot e = \tau(x^{(1)}) \\ &x^{(2)} = \underset{\in F}{\mathrm{mid}(x^{(1)}, e)} \Rightarrow x = \tau^2(x^{(2)}) \quad \text{etc.} \end{aligned}\right\} \Rightarrow x \in \bigcap \tau^n(G) \ \lightning.$$

(ii) If $u \in U_\alpha$, $\alpha \in \Phi^+$, $t \in A$, then $\tau(ut) = ut^2\Theta(u)^{-1} \in \underbrace{A\langle U_\alpha, U_{-\alpha}\rangle}_{\text{diagonalizable!}}$

$\Rightarrow e \rightsquigarrow \tau(ut) \Rightarrow e \rightsquigarrow \tau(u_1 t_1 \cdots u_n t_n)$

Now the <u>Iwasawa decomposition</u> $\underset{G = KAU^+}{}$ implies $\tau(G) = \{\tau(u_1 t_1 \cdots u_k t_k)\}$. ∎

<u>Proof of Theorem A</u>

Consider a flat $F \subset \underset{\cong X_G}{\tau(G)}$, wlog $e \in F$. ($\hat{=}$ Θ-split tori VS. split tori)

$\boxed{\text{Step 1}}$ $\forall x, y \in F$: $[xy, yx] = e$ (Computation from $x \cdot e \cdot y \cdot e = y \cdot e \cdot x \cdot e$.)

$\boxed{\text{Step 2}}$ $\forall x, y \in F$: xy, yx diagonalizable.

$\{e, x\} \subset F$, $x' := \mathrm{mid}(e, x)$, $\Rightarrow x = (x')^2 \Rightarrow x'yx' = S_{x'} \circ S_e(y) \in F$

$\overset{\text{Th. B}}{\Rightarrow}$ $x'yx'$ diagonalizable, fixes $\Sigma_\pm$.

$c \in \Sigma$, $(\Sigma', c') := x'(\Sigma, c) \Rightarrow xy(\Sigma', c') \overset{(1)}{=} (\Sigma', c') \Rightarrow xy$ diagonalizable

$\boxed{\text{Step 3}}$ $\exists$ sph. residues R_+ op $R_- = \Theta(R_+)$: $xy R_\pm \subset R_\pm$ (Step 1 & 2)

<u>refinement</u>: $xy \cdot d_+ = \underset{\in R_+}{d_+}$, yx fixes $\hat{d} = \mathrm{proj}_{R_+} \Theta(d)$

$\boxed{\text{Step 4}}$ $xy, yx \in H \hookrightarrow GL_n(\mathbb{R})$, $yx = \Theta(xy)^{-1} = (xy)^T$

$$xy = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix}, \quad yx = \begin{pmatrix} * & & \\ \vdots & \ddots & \\ * & \cdots & * \end{pmatrix} \xrightarrow[\text{computation}]{\text{matrix}} xy, yx \text{ fix a common chamber } d' \in \mathcal{R}_+$$

$\boxed{\text{Step 5}}$ $\Theta(d') = \Theta(xy.d') = x^{-1}y^{-1}\Theta(d') \Rightarrow \Theta(d') = yx\,\Theta(d')$

$\Rightarrow xy, yx$ fix $d', \Theta(d') \Rightarrow xy, yx \subset \Theta$-split torus

$\Rightarrow (xy)^{-1} = \Theta(xy) = \Theta(x)\Theta(y) = x^{-1}y^{-1} = (yx)^{-1} \Rightarrow \boxed{xy = yx}$

Refinement: F is a commutative group of diagonalizable elements

$\boxed{\text{Step 6}}$ Deduce that $F \subset$ max. split torus.

§ 6 ~~[illegible]~~ Irreducibility and contractibility

-) Assume $\mathcal{D}$ irreducible, let $A = A_{\mathcal{D}}$ (GCM).

  ~~Prop~~ Conj. $rk(A) = n$ (i.e. A invertible) $\Longrightarrow X_G$ irreducible.

  If $rk(A) < n$, then $\dim Z(G) = \mathrm{cork}(A) > 0$ and we have:

  Prop. $X_G \twoheadrightarrow \overline{X}_G$ with fiber $\cong \mathbb{R}^{\mathrm{cork}(A)}$ (non-split extension!), (and $\overline{X}_G$ irreducible?)

  $G \circlearrowright X_G$, $\mathrm{Ad}(G) = G/Z(G) \circlearrowright \overline{X}_G$

  $\overline{X}_G$ reduced Kac-Moody sym. space for $\mathcal{D}$.

-) $G \longrightarrow \overline{G} \xrightarrow{\text{finite}} \mathrm{Ad}(G) \rightsquigarrow \overline{U}^{\pm}, \overline{A}, \overline{B}^{\pm}, \ldots < \overline{G}$

  $T = (\mathbb{R}^\times)^n \subset G \to \overline{T} = (\mathbb{R}^\times)^{rk(A)} \subset \overline{G}$

-) Theorem (FHHK)

  $\overline{K} \times \overline{A} \times \overline{U}^+ \longrightarrow \overline{G}$ homeo.

  (Problem: $K \times A \times U^+ \to G$ open?)

  (Uses: $\overline{K} \curvearrowright \overline{G}/\overline{B}^+ = \overline{K}/\overline{M}$ properly + topology of the building)

-) Theorem (HK)

  $\overline{U}^+$ is contractible.

  (Uses: Dilation structure on $\overline{U}^+$ from $\overline{A}$-action.)

  Corollary

  X_G is contractible.

-) Corollary $\overline{K} \hookrightarrow \overline{G}$ homotopy equivalence, $K \hookrightarrow G$ weak homotopy equivalence.

  In particular, $\pi_1(K) = \pi_1(G)$.

  Rem. $\pi_1(G)$ have recently been computed for all 2-spherical G by P. Harring. If $\mathcal{D}$ is simply-laced, the $\pi_1(G) \cong \mathbb{Z}/2\mathbb{Z}$.

III | Boundary rigidity for Kac-Moody groups

Setting: $\mathcal{D} = \mathcal{D}_A$ 2-spherical diagram with GCM A irreducible 

$G = G_{\mathcal{D}} = \varinjlim (G_i \hookrightarrow G_{ij})$ real split Kac-Moody group

$K = G^{\Theta}$, Θ Cartan-Chevalley involution, $\tau : G \to G$, $g \mapsto g\,\Theta(g)^{-1}$

$(X_G = G/K, \mu) \cong (\tau(G), \tilde{\mu})$ symm. space $[\tilde{\mu}(x,y) = x y^{-1} x]$

$\tilde{G} := \mathrm{Aut}(X_G, \mu) > G$

§1 ~~[illegible]~~ The singular set

·) $\mathcal{F} = \{\text{max. flats in } \tau(G)\}$, $\mathcal{F}_* = \{\text{max pointed flats in } \tau(G)\}$

Recall: $G \curvearrowright \mathcal{F}_*$ transitively with basepoint $(e, A) \in \mathcal{F}_*$

·) **Def**: A geod. ray γ is regular if $\exists!\, F \in \mathcal{F} : \gamma(\mathbb{R}_{\geq 0}) \subset F$, else singular.

(Notation: $F_x^{sing} = \bigcup_{\substack{\gamma \text{ singular} \\ \subset F,\, \gamma(0)=x}} \gamma(\mathbb{R}_{\geq 0})$.)

·) $(x, F) \in \mathcal{F}_* \rightsquigarrow \tilde{G}_{(x,F)} = \mathrm{Stab}_{\tilde{G}}(x,F) / \mathrm{Fix}_{\tilde{G}}(x,F) \curvearrowright (F, F_x^{sing})$ local action

Note: independent of $(x,F) \rightsquigarrow$ wlog $(x,F) = (e, A)$

Goal: Describe the local action in a "linear model".

·) $\mathfrak{g}(A)$ - Kac-Moody algebra of A [Kac, §1]

$\mathfrak{g}(A) = \mathfrak{h}(A) \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ root space decomposition, $\Delta = \Delta^{re} \sqcup \Delta^{im}$

$\mathfrak{g}_{\mathcal{D}} = [\mathfrak{g}(A), \mathfrak{g}(A)]$ $(= \varinjlim (\mathfrak{g}_i \hookrightarrow \mathfrak{g}_{ij})$ if A symmetrizable$)$

$= \mathfrak{a} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$, $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{g}_{\mathcal{D}} < \mathfrak{h}$ **Note**: $\mathfrak{a} = \bigoplus_{i=1}^{n} \mathfrak{a}_i < \bigoplus_{i=1}^{n} \mathfrak{g}_i$

$\alpha \in \Delta^{re} \cong \Phi(W,S) \rightsquigarrow \alpha|_{\mathfrak{a}} : \mathfrak{a} \to \mathbb{R} \rightsquigarrow H_\alpha := \ker(\alpha|_{\mathfrak{a}}) \subset \mathfrak{a}$ root hyperplane

·) $\exp : \mathfrak{a} \xrightarrow{\cong} A$,

$$\begin{array}{ccc} \mathfrak{sl}_2(\mathbb{R}) & \xrightarrow{\exp} & SL_2(\mathbb{R}) \\ \cong & & \cong \\ \mathfrak{g}_i & & G_i \\ \cup & & \\ \mathfrak{a}_i & \longrightarrow & A_i \end{array}$$

Recall: This is a reflection isomorphism w.r.t. $X \cdot Y = 2X - Y$ on $\mathfrak{a}$. (Immediate from formula for $\tilde{\mu}$.)

Prop. (FHHK)

$\exp^{-1}(A_e^{sing}) = \mathfrak{a}^{sing} = \bigcup_{\alpha \in \Delta^{re}} H_\alpha$ is a hyperplane arrangement.

Sketch of proof: (1) Strong transitivity $\Rightarrow \mathfrak{a}^{sing} = \bigcup_{\{k \in K \mid \mathrm{Ad}(k)\mathfrak{a} \neq \mathfrak{a}\}} \mathfrak{a} \cap \mathrm{Ad}(k)\mathfrak{a}$.

(2) $X \in H_\alpha \rightsquigarrow \exists k \in K \cap \langle U_\alpha, U_{-\alpha} \rangle$ not normalizing T

$\rightsquigarrow \exp(X) \in A \cap kAk^{-1}$ $\Big\} \Rightarrow$ „$\supset$".

(3) Let $X \in \mathfrak{a} \cap \mathrm{Ad}(k)\mathfrak{a} \subsetneq \mathfrak{a}$, $(\neq 0)$, set $H := T \cap \underbrace{kTk^{-1}}_{T'} \supset \exp(tX)$.

Set $\Phi^H := \{\alpha \in \Phi \mid [U_\alpha, H] = 1\}$, $G^H = T\langle U_\alpha \mid \alpha \in \Phi^H \rangle$.

[Caprace '09]: $(G^H, (U_\alpha)_{\alpha \in \Phi^H})$ locally $\mathbb{R}$-split twin root datum with Weyl group $W^H = \langle s_\alpha \mid \alpha \in \Phi^H \rangle$ and

tor: [illegible] T, T'

(4) Δ^H twin building of G^H $\Rightarrow$ $K^H \curvearrowright \Delta^H$ trans. on twin apartments

$\Rightarrow \exists k_H \in K^H : T' = k_H T k_H^{-1}$, $k_H = \prod_{i=1}^{N} k_i$, $k_i \in \langle U_{-\beta_i}, U_{\beta_i} \rangle \cap K$

(5) Not all k_i normalize T $\rightarrow \exists k_i : H^\circ \leq (T \cap k_i T k_i^{-1})^\circ \leq T^\circ$

$\Rightarrow \exp(X) \in H^\circ < \exp(H_{\beta_i})$ $= \exp(H_{\beta_i})$, since codim 1. ◻

Thus every local automorphism induces $f \in GL(\mathfrak{a}, \mathfrak{a}^{sing})$.

<u>Rem</u>: Same argument shows: $\forall F_1, F_2 \in \mathcal{F}\ \exists g \in G : g(F_1) = F_2,\ g|_{F_1 \cap F_2} = Id$.

§ 2 The Tits cone

•) $C := \{ v \in \mathfrak{a} \mid \alpha_i(v) \geq 0$ for $i = \{1 \ldots n\} \} \subset \mathfrak{a}$ <u>fundamental chamber</u>

$\tilde{W} = N_K(A) \curvearrowright A \rightsquigarrow W = \tilde{W}/T \curvearrowright \mathfrak{a}$

$\mathcal{T} := \bigcup_{w \in W} w.C \subset \mathfrak{a}$ <u>Tits cone</u>, $\mathcal{T}^\circ := Int(\mathcal{T}) \subset \mathfrak{a}$

<u>Fact</u>: $\mathcal{D}$ sph. $\Rightarrow \mathcal{T}^\circ = \mathfrak{a}$; $\mathcal{D}$ non-sph. $\Rightarrow \mathcal{T}^\circ \cap (-\mathcal{T}^\circ) = \emptyset$.

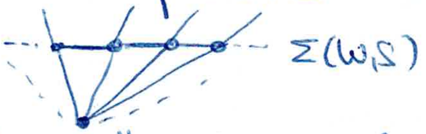
$\boxed{\text{Temporary assumption}}$ A symmetrizable, $\det(A) \neq 0$. $(*)$

•) <u>Consequences</u>: (i) $\exists$ non-deg. bilinear form $(\cdot|\cdot)$ on $\mathfrak{a}$ s.t. $(\mathfrak{a}, (\cdot|\cdot), \{\gamma_1, \ldots, \gamma_n\})$ is a 'root basis' for (W,S). This implies:

normalized coroots $\gamma_j = \frac{\alpha_j^\vee}{(\alpha_j^\vee|\alpha_j^\vee)}$

(a) $W \subset O(\mathfrak{a}, (\cdot|\cdot))$, s_α acts by orth. refl. at H_α w.r.t. $(\cdot|\cdot)$.

(b) $\mathcal{T}$ is a simplicial cone over the simplicial complex $\Sigma(W,S)$.



(chambers = conn. components of $\mathfrak{a}^{reg}$)

(c) $\mathcal{T}^\circ$ is the "ideal simplicial cone" formed by simplices with spherical stabilizer.

$(\Rightarrow W \curvearrowright \mathcal{T}^\circ$ is proper$)$

•) $Aut(\Sigma(W,S)) = W \rtimes Aut(\mathcal{D}_W) \curvearrowright \mathcal{T} \curvearrowleft \mathbb{R}^{>0}$

<u>Theorem</u> (Mühlherr)

$GL(\mathfrak{a}, \mathfrak{a}^{sing}) = \mathbb{R}^{>0} \times \overbrace{Aut(\Sigma(W,S)) \times \underbrace{\mathbb{Z}/2\mathbb{Z}}_{\text{if } \mathcal{D} \text{ not spherical}}}^{= O(\mathfrak{a}, \mathfrak{a}^{sing}, (\cdot|\cdot))}$

<u>Idea</u>: $f \in GL(\mathfrak{a}, \mathfrak{a}^{sing})$ preserves hyperplanes $\Rightarrow$ normalizes W

hyperplanes $\overset{(\cdot|\cdot)}{\Longleftrightarrow}$ reflections ◻

§ 3 Global and local automorphisms

•) <u>Theorem</u> (Caprace)

$Aut(G) = \langle Inn(G), Aut(\mathcal{D}), \theta, \text{"diagonal aut."} \rangle$

<u>Corollary</u>

$Aut(G) \hookrightarrow Aut(\Delta)$ with equality if $Aut(\mathcal{D}) = Aut(\mathcal{D}_W)$.

•) $\{s_x \circ s_y \mid x, y \in X_G\} = \tau(G) C_k(G) \Rightarrow \mathrm{Trans}(X, \mu) = \langle s_x \circ s_y \rangle = G$

$\Rightarrow \langle s_x \rangle = G \rtimes \langle \theta \rangle$

•) Trick: $\alpha \in \mathrm{Aut}(X, \mu) \Rightarrow \alpha \circ s_x \circ \alpha^{-1} = s_{\alpha(x)} \Rightarrow c_\alpha \in \mathrm{Aut}(\langle s_x \rangle)$

$(R4) \Rightarrow \mathrm{Aut}(X, \mu) \hookrightarrow \mathrm{Aut}(G \rtimes \langle \theta \rangle) \overset{(*)}{=} \mathrm{Aut}(G)$

$\alpha \mapsto c_\alpha$

Corollary A (Assume (*))

The local action is given by

$$\widetilde{G}_{(x,F)} \cong (W \rtimes \mathrm{Aut}(\mathcal{D})) \times \underbrace{\mathbb{Z}/2\mathbb{Z}}_{\text{non-sph}} \xhookrightarrow{\mathbb{R}^{>0}x} (W \rtimes \mathrm{Aut}(\mathcal{D}_W)) \times \mathbb{Z}/2\mathbb{Z} = GL(\mathfrak{a}, \mathfrak{a}^{sg})$$

$$G_{(x,F)} \cong W$$

If $\mathcal{D}$ is simply-laced, any $f \in O(\mathfrak{a}, \mathfrak{a}^{sg})$ can be realized by $\varphi \in \widetilde{G}_{(x,F)}$.

Corollary B (Assume (*))

$\exp(\pm \mathcal{J}^\circ)$ ~~are~~ invariant under $\widetilde{G}_{(x,F)}$.

Remark (Removing (*))

(i) We need A symmetrizable to get an invariant form.

(ii) If $\det(A) = 0$, pass to $\overline{\mathfrak{a}} = \mathfrak{a}/\mathrm{Rad}(\cdot|\cdot)$ and replace X_G by the irreducible quotient $\overline{X}_G$.

Problem: Is $(\overline{\mathfrak{a}}, (\cdot|\cdot), \{\overline{\gamma}_1, \dots, \overline{\gamma}_n\})$ still a root basis?

Answer (Kraummer, Mühlherr) YES, iff $\mathcal{D}$ non-affine!

(iii) $\mathcal{J}$ is a polyhedral (non-simplicial) realization of $\Sigma(W, S)$.

Assume A symmetrizable, non-spherical, non-affine from now on.

§4 Causal structures and tangent twin building

•) Prop. (FHHK)

There exists unique families $(C_x^\pm)_{x \in \overline{X}_G}$ of subsets of $\overline{X}_G$ s.t. $g C_x^\pm = C_{gx}^\pm$ for all $g \in \widetilde{G}$ and $C_e^\pm \cap A = \exp(\pm \mathcal{J}^\circ)$.

Proof: Set $C_e^\pm := \bigcup_{k \in K} k \exp(\pm \mathcal{J}^\circ) k^{-1}$ and $C_{g.e}^\pm := g.C_e^\pm$. This is well-defined by Cor. B. ☐

•) $C_x^\pm$ intersects each flat through x in a cone

⇝ $(C_x^\pm)$ are invariant causal structures (cone fields).

•) Geodesic rays γ in $C^+_{\gamma(0)}$ are called causal rays

$C^-_{\gamma(0)}$ anti-causal rays

(We ignore all other rays, as usual in causal symmetric spaces.)

•) $\Delta_x := (\Delta_x^+, \Delta_x^-)$ where $\Delta_x^\pm = \{$causal / anti-causal rays emanating from $x\}$

$\Delta_x^\pm(F) := \{\gamma \in \Delta_x^\pm \mid \gamma(\mathbb{R}^{\geq 0}) \subset F\} \rightsquigarrow \Delta_x^\pm = \bigcup_{x \in F} \Delta_x^\pm(F)$

•) $\left.\begin{array}{l} (x, F) \cong (e, A) \cong (0, \mathfrak{a}) \\ C_x^+ \cap F \;\widehat{=}\; J^\circ \end{array}\right\} \Rightarrow \Delta_x^\pm(F) \cong J^\circ/\mathbb{R}^{>0}$

$\Rightarrow \Delta_x^\pm(F)$ ideal polyhedral complex (cells $\widehat{=}$ spherical simplices in $\Sigma(W,S)$)

"polyhedral Davis complex" (Davis complex if $\det(A) \neq 0$)

$\Delta_x^\pm(F) \cap \Delta_x^\pm(F') \subset \Delta_x^\pm(F)$ subcomplex $\rightsquigarrow$ glue!

Corollary

(i) Δ_x^+, Δ_x^- are ideal polyhedral buildings.

(ii) Combinatorially, $\Delta_x^\pm \cong \Delta^\pm$ (twin building of G). $\mathrm{Ad}(K_x)$-equivariant.

$\rightsquigarrow \Delta_x$ tangent twin building at x.

§ 5 Boundary rigidity

•) $\Delta_\circ^\pm := \bigcup_{x \in \overline{X}_G} \Delta_x^\pm$ municipality

Fact: There exists an unique equivalence relation $\|$ on $\Delta_\circ^\pm$ s.t.

(A1) $\forall r \in \Delta_\circ^\pm, x \in \overline{X}_G \;\exists!\, r_x \in \Delta_\circ^\pm : r \| r_x$.

(A2) $\|$ is $\tilde{G}$-invariant.

(A3) $r_1, r_2 \subset \mathbb{H}^2 \subset X_G \Rightarrow r_1 \| r_2$ iff asymptotic in the hyperbolic sense.

$r_1, r_2 \subset F_{flat} \subset X_G$ ... Euclidean

(A4) $\|$ ~~Parallelity~~ classes are orbits of parabolic subgroups.

($\sim$ sliding along horospheres)

$\rightsquigarrow \partial^\pm \overline{X}_G := (\Delta_\circ^\pm/\|)$ causal / anti-causal boundary ($\frac{\text{future}}{\text{past}}$ boundary)

$\partial \overline{X}_G = \partial^+ \overline{X}_G \cup \partial^- \overline{X}_G$

•) Theorem (Boundary rigidity, FHHK)

(i) The bijections $\Delta_\circ^\pm \hookrightarrow \Delta_\circ^\pm \to \partial^\pm \overline{X}_G$ induce the same ideal polyhedral structure on $\partial^\pm \overline{X}_G$.

(ii) Every $f \in \mathrm{Aut}(\overline{X}_G)$ induces a polyhedral boundary map $\partial f : \partial \overline{X}_G \circlearrowleft$.

(iii) $\partial^\pm \overline{X}_G \cong \Delta^\pm$ combinatorially & G-equivariantly

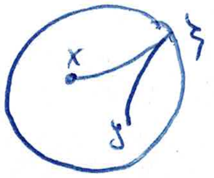
(iv) $\forall f_1, f_2 \in \mathrm{Aut}(\overline{X}_G)$: $\partial f_1 = \partial f_2 \Rightarrow f_1 = f_2$

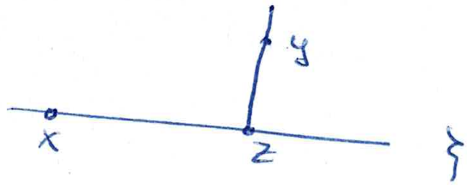
Proof: (i)-(iii) are routine modulo §4.

Then (iv) follows, since $\mathrm{Aut}(\overline{X}_G) \to \mathrm{Aut}(\partial \overline{X}_G) \xrightarrow{(iii)} \mathrm{Aut}(\Delta)$ is the embedding from §3.

§6 Comparison to masures

- For X $\frac{\text{symm-space}}{\text{masure}}$, $x \in X$, $\xi \in \partial X$ there exists a unique ray $r_{x,\xi}$ from x to ξ. Write $x \overset{\circ}{\leq} y$ if there is a $\overset{\text{flat of}}{\text{causal}}$ ray from x to y.
- For $x, y \in X$, $r_{x,\xi} \parallel r_{y,\xi}$, but there is a difference:





In a masure $\forall x,y \; \exists z: x \overset{\circ}{\leq} \cancel{z} \overset{\circ}{\geq} \cancel{z}\, y \Rightarrow x \sim z \sim y$
and $\overset{\circ}{\leq}$ is a ~~special~~ partial order, since $x \overset{\circ}{\leq} y$ can be seen in an apartment.

It is not known whether $\leq$ is a partial order in ~~the~~ a symmetric space.
(FHHK: Either $\overset{\circ}{\leq} =$ "$=$" or it is a partial order!)
Also, there are flats of distance $\gg 2$: