

Motivation
○○○
○○○

Definition and properties
○○○

Affine case
○○

Results beyond affine type
○○○○○

Further
○○○○
○○

A correction factor for Kac-Moody groups and t -deformed root multiplicities

Anna Puskás

Kavli IPMU

Workshop on Kac-Moody Geometry
Besse-et-Saint-Anastaise, 24 May 2019

Motivation
○○○
○○○

Definition and properties
○○○

Affine case
○○

Results beyond affine type
○○○○○

Further
○○○○
○○

Joint work with Dinakar Muthiah and Ian Whitehead;
arXiv: 1806.05209.

1 Motivation

2 Definition and properties

3 Affine case

4 Results beyond affine type

5 Further

Macdonald's identity (1972, *The Poincaré series of a Coxeter group*):

$$\sum_{w \in W} w \left(\prod_{\alpha \in \Phi^+} \frac{1 - te^\alpha}{1 - e^\alpha} \right) = \sum_{w \in W} t^{\ell(w)}$$

W Weyl group, $\ell : W \rightarrow \mathbb{Z}_{\geq 0}$ length function, Φ^+ positive roots.

Kac-Moody root systems:

$$\mathfrak{m} \cdot \sum_{w \in W} w \left(\prod_{\alpha \in \Phi_{\text{re}}^+} \frac{1 - te^\alpha}{1 - e^\alpha} \right) = \sum_{w \in W} t^{\ell(w)}$$

$$\mathfrak{m}' \cdot \sum_{w \in W} w \left(\prod_{\alpha \in \Phi^+} \left(\frac{1 - te^\alpha}{1 - e^\alpha} \right)^{\text{mult}(\alpha)} \right) = \sum_{w \in W} t^{\ell(w)}$$

We wish to write:

$$\mathfrak{m} = \prod_{\lambda \in Q_{\text{im}}^+} \prod_{n \geq 0} (1 - t^n e^\lambda)^{-m(\lambda, n)}$$

where Q_{im}^+ positive imaginary root cone; and

$$m_\lambda(t) = \sum_{n \geq 0} m(\lambda, n) t^n$$

are polynomials with constant term: $m_\lambda(0) = \text{mult}(\lambda)$:

$$\mathfrak{m}|_{t=0} = \prod_{\alpha \in \Phi_{\text{im}}^+} (1 - e^\alpha)^{-\text{mult}(\alpha)}$$

$$\Delta_{\text{re}} = \prod_{\alpha \in \Phi_{\text{re}}^+} (1 - e^\alpha), \quad \Delta_{t,\text{re}} = \prod_{\alpha \in \Phi_{\text{re}}^+} (1 - te^\alpha), \quad P(t) = \sum_{w \in W} t^{\ell(w)}$$

$$\mathfrak{m} \cdot \sum_{w \in W} w \left(\frac{\Delta_{t,\text{re}}}{\Delta_{\text{re}}} \right) = P(t)$$

Specializing at $t = 0$:

$$\mathfrak{m}|_{t=0} \cdot \frac{1}{\Delta_{\text{re}}} \cdot \sum_{w \in W} (-1)^{\ell(w)} \cdot \prod_{\alpha \in \Phi(w^{-1})} e^\alpha = 1$$

$$\mathfrak{m}|_{t=0} = \prod_{\alpha \in \Phi_{\text{im}}^+} (1 - e^\alpha)^{-\text{mult}(\alpha)}$$

Remark: sometimes consider $\alpha \mapsto -\alpha$ (and omit \vee s).

Formulae of p -adic Kac-Moody groups

$$S(\mathbb{1}_{K\pi^\lambda K}) = \frac{1}{\mathfrak{m}} \cdot \frac{t^{\langle \rho, \lambda \rangle}}{P_\lambda(t)} \cdot \sum_{w \in W} w \left(e^\lambda \frac{\Delta_{t,\text{re}}}{\Delta_{\text{re}}} \right)$$

- Macdonald's formula for the spherical function
- Generalizations: Braverman–Kazhdan–Patnaik (affine),
Bardy-Panse–Gaussent–Rousseau (Kac–Moody)

Taking a limit in λ , this converges to the Gindikin–Karpelevich formula,
 \mathfrak{m} persists. (Braverman–Garland–Kazhdan–Patnaik, Hébert, Ali)

Remark. Here \mathfrak{m} (not \mathfrak{m}') appears, factors corresponding to the
multiplicities of imaginary roots were included in \mathfrak{m}

Formulae of p -adic Kac-Moody groups, continued

- Casselman-Shalika formula for the spherical Whittaker function in affine type [Patnaik]

$$\mathcal{W}(\pi^\lambda) = t^{-\langle \rho, \lambda \rangle} \mathfrak{m}' \cdot \prod_{\alpha \in \Phi^+} (1 - te^\alpha) \chi_\lambda$$

- Metaplectic analogue in Kac-Moody type [Patnaik, P.]

$$\mathcal{W}(\pi^\lambda) = \mathfrak{m}'_{R_n} \Delta_{R_n} \sum_{w \in W} (-1)^{\ell(w)} \left(\prod_{\tilde{a} \in R_n(w)} e^{-\tilde{a}} \right) w * e^\lambda$$

The factor \mathfrak{m} relates Hecke symmetrizers to Weyl symmetrizers.

Geometry of flag varieties

Macdonald's second proof of $1 \cdot \sum_{w \in W} w \left(\frac{\Delta_t}{\Delta} \right) = P(t)$ for finite W

Computation of the Betti numbers of a flag variety using Hodge theory.

- Right hand side: counting Schubert cells
- Left hand side: a computation of Dolbeault cohomology using localization at fixed points for the action of the maximal torus.
- The flag variety is smooth and projective Dolbeault cohomology is equal to Betti cohomology by the Hodge theorem.

Failure of $\mathfrak{m} = 1$ beyond finite type \Rightarrow Kac-Moody flag varieties are not smooth; they are homogeneous \Rightarrow everywhere singular.

Fishel-Grojnowski-Teleman explicitly compute the Dolbeault cohomology of the affine flag variety, prove *Strong Macdonald Conjecture*.

Preparations

Recall: we wish to define $\textcolor{blue}{m} \sum_{w \in W} w \left(\frac{\Delta_{t,\text{re}}}{\Delta_{\text{re}}} \right) \stackrel{?}{=} P(t)$

$$\Delta_{\text{re}} = \prod_{\alpha \in \Phi_{\text{re}}^+} (1 - e^\alpha), \quad \Delta_{t,\text{re}} = \prod_{\alpha \in \Phi_{\text{re}}^+} (1 - te^\alpha), \quad P(t) = \sum_{w \in W} t^{\ell(w)}$$

$Q^+ \supseteq Q_{\text{im}}^+$ cones graded by height;

Laurent series units on Q^+ have form $ue^{\lambda_0} \prod_{\lambda \in Q^+ \setminus \{0\}} \prod_n (1 - t^n e^\lambda)^{m(\lambda, n)}$

W acts on a multiplicative subset containing $\frac{\Delta_{t,\text{re}}}{\Delta_{\text{re}}}$

$\sum_{w \in W} w \left(\frac{\Delta_{t,\text{re}}}{\Delta_{\text{re}}} \right)$ unit in $\mathbb{Z}[[t]][t^{-1}]^{[[Q^+]]}$, regular at $t = 0$,
constant coefficient $P(t)$.

Motivation
○○○
○○○

Definition and properties
○●○

Affine case
○○

Results beyond affine type
○○○○○

Further
○○○○
○○

Definition of \mathfrak{m}

$\sum_{w \in W} w \left(\frac{\Delta_{t,\text{re}}}{\Delta_{\text{re}}} \right)$ and $P(t)$ unit in $\mathbb{Z}[[t]][t^{-1}][[Q^+]]$, regular at $t = 0$.

We may define \mathfrak{m} by $\mathfrak{m} \sum_{w \in W} w \left(\frac{\Delta_{t,\text{re}}}{\Delta_{\text{re}}} \right) = P(t)$

The factor \mathfrak{m} is Weyl-invariant and therefore supported on Q_{im}^+ .

Both $\mathfrak{m}, \mathfrak{m}^{-1}$ units in $\mathbb{Z}[t, t^{-1}][[Q^+]]$, regular at $t = 0$,
constant coefficient 1.

$$\left(\mathfrak{m}^{-1} \frac{\Delta_{\text{re}}}{\Delta_{t,\text{re}}} \right) \Big|_{Q_{\text{im}}^+} = 1$$

Motivation
○○○
○○○

Definition and properties
○○●

Affine case
○○

Results beyond affine type
○○○○○

Further
○○○○
○○

Caution!

$$\left(\mathfrak{m}^{-1} \frac{\Delta_{\text{re}}}{\Delta_{t,\text{re}}} \right) \Big|_{Q_{\text{im}}^+} = 1.$$

In the affine case, this implies “constant term property”

$$\left(\frac{\Delta_{\text{re}}}{\Delta_{t,\text{re}}} \right) \Big|_{Q_{\text{im}}^+} = \mathfrak{m}$$

In the Kac-Moody case, this is **not true!**

$$\text{Supp}(\mathfrak{a}) \subseteq Q_{\text{im}}^+ \not\Rightarrow (\mathfrak{a} \cdot \mathfrak{b})|_{Q_{\text{im}}^+} = \mathfrak{a} \cdot (\mathfrak{b})|_{Q_{\text{im}}^+}$$

Cherednik's solution of Macdonald's Constant Term Conjecture:
 \mathfrak{m} is known for Φ of affine type.

For untwisted, simply laced affine types:

$$\mathfrak{m} = \prod_{i=1}^{\infty} \left(\left(\frac{1 - t \cdot e^{i \cdot \delta}}{1 - e^{i \cdot \delta}} \right)^r \cdot \prod_{j=1}^r \frac{1 - t^{m_j} \cdot e^{i \cdot \delta}}{1 - t^{m_j+1} \cdot e^{i \cdot \delta}} \right)$$

where r is the rank, m_j exponents of underlying finite-dimensional root system, δ the minimal imaginary root.

$$\mathfrak{m} = \prod_{i=1}^{\infty} \prod_{j=1}^r \left(\prod_{k=1}^{m_j} \frac{(1 - t^k \cdot e^{i \cdot \delta})^2}{(1 - t^{k-1} e^{i \cdot \delta})(1 - t^{k+1} e^{i \cdot \delta})} \right)$$

$$-m_{i \cdot \delta}(t) = \sum_{j=1}^r \left(\sum_{k=1}^{m_j} t^{k-1} \cdot (-1 + 2t - t^2) \right) = -(1-t)^2 \cdot \sum_{j=1}^r \frac{t^{m_j} - 1}{t - 1}$$

Motivation
○○○
○○○

Definition and properties
○○○

Affine case
○●

Results beyond affine type
○○○○○

Further
○○○○
○○

More generally, by work of Cherednik and Macdonald, for any affine Φ :

$$\mathfrak{m} = \prod_{\lambda \in Q_{\text{im}}^+} \left(\prod_{\beta \in S(\lambda)} \frac{(1 - t^{\text{ht}(\beta)} e^\lambda)^2}{(1 - t^{\text{ht}(\beta)-1} e^\lambda)(1 - t^{\text{ht}(\beta)+1} e^\lambda)} \right)$$

where

$$S(\lambda) = \{\beta \in Q_{\text{fin}}^+ \mid \beta + \lambda \in \Phi_{\text{re}}\},$$

Q_{fin}^+ is a root lattice corresponding to a finite root subsystem $\Phi_{\text{fin}} \subseteq \Phi$ determined by omitting an appropriate simple root.

$$m_\lambda = (1 - t)^2 \cdot \sum_{\beta \in S(\lambda)} t^{\text{ht}(\beta)-1}$$

Generalized Petersen algorithm

We wish to write

$$\mathfrak{m} = \prod_{\lambda \in Q_{\text{im}}^+} \prod_{n=0}^{N_\lambda} (1 - t^n e^\lambda)^{-m(\lambda, n)}$$

starting from

$$\left(\mathfrak{m}^{-1} \frac{\Delta_{\text{re}}}{\Delta_{t,\text{re}}} \right) \Big|_{Q_{\text{im}}^+} = 1$$

- power series inverse with respect to Q_{im}^+
- by induction on height
- algorithm polynomial in height
- generalization of the Petersen algorithm for $\text{mult}(\lambda)$
- suffices to compute for one λ per W -orbit, i.e. on antidominant cone

Motivation
○○○
○○○

Definition and properties
○○○

Affine case
○○

Results beyond affine type
○●○○○

Further
○○○○
○○

Sketch of algorithm

$$\mathfrak{m} = \prod_{\lambda \in Q_{\text{im}}^+} \prod_{n=0}^{N_\lambda} (1 - t^n e^\lambda)^{-m(\lambda, n)}$$

Set $N_\alpha = 1$, $m(\alpha, 0) = 1$, $m(\alpha, 1) = -1$ for $\alpha \in \Phi_{\text{re}}$;
 $m(\lambda, n) = 0$ if λ is not an imaginary vector or a real root.

$$\left(\mathfrak{m}^{-1} \frac{\Delta_{\text{re}}}{\Delta_{t,\text{re}}} \right) \Big|_{Q_{\text{im}}^+} = 1; \quad \mathfrak{m}^{-1} \frac{\Delta_{\text{re}}}{\Delta_{t,\text{re}}} = \prod_{\lambda \in Q^+} \prod_{n=0}^{N_\lambda} (1 - t^n e^\lambda)^{m(\lambda, n)} = \prod_{\lambda \in Q^+} \mathfrak{m}_\lambda^{-1}$$

$$\left. \mathfrak{m}_\lambda^{-1} \cdot \prod_{\substack{\mu \in Q^+ \\ \text{ht}(\mu) < \text{ht}(\lambda)}} \mathfrak{m}_\mu^{-1} \right|_\lambda = 0$$

Sketch of algorithm (2)

$$\left| \mathfrak{m}_\lambda^{-1} \cdot \prod_{\substack{\mu \in Q^+ \\ \text{ht}(\mu) < \text{ht}(\lambda)}} \mathfrak{m}_\mu^{-1} \right|_\lambda = \prod_{n=0}^{N_\lambda} (1 - t^n e^\lambda)^{m(\lambda, n)} \cdot \left| \prod_{\substack{\mu \in Q^+ \\ \text{ht}(\mu) < \text{ht}(\lambda)}} \mathfrak{m}_\mu^{-1} \right|_\lambda = 0$$

Set $\mathfrak{m}_0 = 1$.

Assume $\text{ht}(\lambda) > 0$, and \mathfrak{m}_μ known for $\text{ht}(\mu) < \text{ht}(\lambda)$

$\mu \in Q_{\text{im}}^+$: from previous steps of the induction

$\mu \notin Q_{\text{im}}^+$: by computing real roots up to $\text{ht}(\lambda)$

The coefficient of e^λ in $\prod_{\substack{\mu \in Q^+ \\ \text{ht}(\mu) < \text{ht}(\lambda)}} \mathfrak{m}_\mu^{-1}$ is a polynomial in $t :=$

$$m(\lambda, 0) + m(\lambda, 1)t + \cdots + m(\lambda, N_\lambda)t^{N_\lambda}$$

Generalized Berman-Moody formula

Theorem [Muthiah-P-Whitehead] For all $\lambda \in Q^+$, we have:

$$m_\lambda(t) = \sum_{\kappa|\lambda} \mu(\lambda/\kappa) \left(\frac{\lambda}{\kappa}\right)^{-1} \sum_{\underline{\kappa} \in \text{Par}(\kappa)} (-1)^{|\underline{\kappa}|} \frac{B(\underline{\kappa})}{|\underline{\kappa}|} \prod_{i=1}^{|\underline{\kappa}|} P_{\kappa_i}(t^{\lambda/\kappa_i})$$

- For $t = 0$ recovers the Berman-Moody formula for $\text{mult}(\lambda) = m_\lambda(0)$
- $\lambda, \kappa \in Q^+$, $\lambda = k \cdot \kappa$, then $\kappa|\lambda$, $\frac{\lambda}{\kappa} = k \in \mathbb{Z}$, $\mu(\lambda/\kappa)$ Möbius function
- $\text{Par}(\lambda)$ vector partitions of λ , $|\underline{\kappa}|$, $B(\underline{\kappa})$
- $P_{\kappa_i}(t^{\lambda/\kappa_i}) = P_{\kappa_i}(t^k)$ given in terms of Kostant partitions of $\kappa_i \in Q^+$.
- $\frac{\Delta_{\text{re}}}{m\Delta_{t,\text{re}}}$ logarithm, differential operator $\sum_i e^{\alpha_i} \frac{\partial}{\partial e^{\alpha_i}}$, Möbius transform
- For any $\mu \in Q^+$, $\mu \neq 0$ $P_\mu(1) = 0$

Properties of \mathfrak{m}

Theorem [Muthiah-P-Whitehead] For $\lambda \in Q_{\text{im}}^+$, $m_\lambda(t) \neq 0 \Leftrightarrow \lambda \in \Phi_{\text{im}}$.

- If $\Phi_1 \subseteq \Phi$ root subsystem, $Q_1 \subseteq Q$, \mathfrak{m}_1 , \mathfrak{m} ; then $\mathfrak{m}|_{Q_1} = \mathfrak{m}_1$.
- If $\Phi_1, \Phi_2 \subseteq \Phi$, simple roots $\Delta_1 \perp \Delta_2$, then $\mathfrak{m} = \mathfrak{m}_1 \mathfrak{m}_2$.
- If $\lambda \in Q_{\text{im}}^+ \setminus \Phi_{\text{im}}$ antidominant, then $\text{Supp}_\Delta \lambda$ disconnected.

Theorem [Muthiah-P-Whitehead] For $\lambda \in Q_{\text{im}}^+$, $(1-t)^2 | m_\lambda(t)$.

- Use Generalized Berman-Moody formula
- Observation: $\lambda \in Q_{\text{im}}^+$ as sum over Φ_{re} has at least two terms.
- For any $\mu \in Q^+$, $\mu \neq 0$: $P_\mu(1) = 0$

$$m_\lambda(t) = \sum_{\kappa|\lambda} \mu(\lambda/\kappa) \left(\frac{\lambda}{\kappa}\right)^{-1} \sum_{\underline{\kappa} \in \text{Par}(\kappa)} (-1)^{|\underline{\kappa}|} \frac{B(\underline{\kappa})}{|\underline{\kappa}|} \prod_{i=1}^{|\underline{\kappa}|} P_{\kappa_i}(t^{\lambda/\kappa})$$

An illustration...

$$\chi_\lambda(t) = \frac{m_\lambda(t)}{(1-t)^2} = \frac{\sum_{i=0}^{N_\lambda} m(\lambda, n) \cdot t^n}{(1-t)^2}$$

Using the Generalized Petersen algorithm, compute this for the hyperbolic root systems with Cartan matrices

$$\begin{bmatrix} 2 & -3 \\ -2 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

$\chi_\lambda(t)$, Cartan matrix $\begin{bmatrix} 2 & -3 \\ -2 & 2 \end{bmatrix}$

λ	$\chi_\lambda(t)$
(1, 1)	1
(2, 2)	$-t + 1$
(3, 2)	$t^2 + 0t + 2$
(3, 3)	$-t^3 - 2t + 2$
(4, 3)	$t^4 - t^3 + 2t^2 - 3t + 3$
(4, 4)	$-t^5 + t^4 - 2t^3 + 3t^2 - 6t + 3$
(5, 4)	$t^6 - 2t^5 + 4t^4 - 6t^3 + 9t^2 - 9t + 6$
(5, 5)	$-t^7 + t^6 - 4t^5 + 6t^4 - 10t^3 + 13t^2 - 13t + 7$
(6, 4)	$t^6 - 4t^5 + 5t^4 - 8t^3 + 11t^2 - 13t + 6$
...	...
(10, 9)	$t^{16} - 7t^{15} + 29t^{14} - 91t^{13} + 248t^{12} - 584t^{11} + 1197t^{10} - 2170t^9 + 3505t^8 - 5039t^7 + 6437t^6 - 7253t^5 + 7042t^4 - 5618t^3 + 3405t^2 - 1372t + 272$

$\chi_\lambda(t)$, Cartan matrix $\begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}$

λ	$\chi_\lambda(t)$
(1, 1)	1
(2, 2)	$-2t + 1$
(2, 3)	$t^2 - t + 2$
(3, 3)	$-2t^3 + 3t^2 - 4t + 3$
(3, 4)	$t^4 - 3t^3 + 6t^2 - 6t + 4$
(4, 4)	$-2t^5 + 7t^4 - 12t^3 + 17t^2 - 16t + 6$
(4, 5)	$t^6 - 5t^5 + 15t^4 - 26t^3 + 30t^2 - 23t + 9$
(4, 6)	$t^6 - 8t^5 + 19t^4 - 31t^3 + 36t^2 - 28t + 9$
(5, 5)	$-2t^7 + 9t^6 - 30t^5 + 58t^4 - 82t^3 + 77t^2 - 50t + 16$
...	...
(10, 9)	$t^{16} - 15t^{15} + 135t^{14} - 811t^{13} + 3535t^{12} - 11729t^{11} + 30615t^{10} - 64282t^9 + 110096t^8 - 154852t^7 + 178868t^6 - 168420t^5 + 127110t^4 - 74539t^3 + 32094t^2 - 9070t + 1267$

$\chi_\lambda(t)$, Cartan matrix

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

λ	$\chi_\lambda(t)$
(1, 1, 0)	1
(2, 2, 0)	1
(2, 2, 1)	2
(3, 3, 0)	1
(3, 3, 1)	$-t + 3$
(3, 4, 2)	$-2t + 5$
(4, 4, 0)	1
(4, 4, 1)	$-2t + 5$
(4, 4, 2)	$-t^2 - 6t + 7$
(4, 5, 2)	$t^3 + t^2 - 9t + 11$
(5, 5, 0)	1

λ	χ_λ
(5, 5, 1)	$-5t + 7$
(5, 5, 2)	$2t^3 + 2t^2 - 17t + 15$
(5, 6, 2)	$-t^4 + 3t^3 + 6t^2 - 26t + 22$
(5, 6, 3)	$-3t^4 + 6t^3 + 13t^2 - 43t + 30$
(6, 6, 0)	1
(6, 6, 1)	$t^2 - 8t + 11$
(6, 6, 2)	$-2t^4 + 5t^3 + 11t^2 - 43t + 30$
(6, 6, 3)	$-6t^4 + 8t^3 + 23t^2 - 65t + 42$
(6, 7, 2)	$-5t^4 + 6t^3 + 22t^2 - 63t + 42$
(7, 7, 0)	1
(7, 7, 1)	$2t^2 - 15t + 15$

Questions

Conjecture The polynomials χ_λ have alternating sign coefficients in rank two hyperbolic type.

Problem Interpret all coefficients of χ_λ in terms of the Kac-Moody Lie algebra.

Problem Give upper bounds for the degree and coefficients of $\chi_\lambda(t)$.

Question Relationship of $m_\lambda(t)$ and Kac polynomials?

Question What is the Dolbeault cohomology of Kac-Moody flag varieties? (A two-parameter generalization of \mathfrak{m} .)

Motivation
○○○
○○○

Definition and properties
○○○

Affine case
○○

Results beyond affine type
○○○○○

Further
○○○○
○●

Further Questions and Remarks

Thank you!

