# Wilson loops in Hamiltonian form

arXiv: 2202.11122

Inspired by T. Cohen, 2202.08745

Wilson/Polyakov loop, $W \sim e^{ig\int A_0 dx}$

But $\bar{c}$ $H$, go to $A_0 = 0$ gauge

Where does $W$ show up?

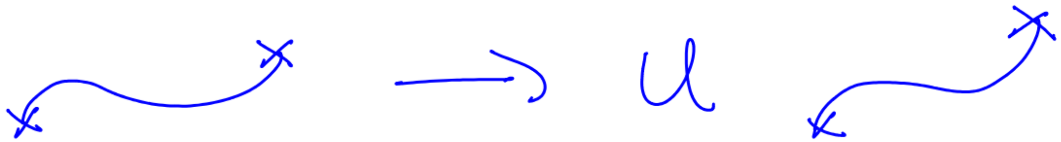
Obvious, after the fact...

# Global q-form symmetries

Usually, act on a point (0-form), e.g., spin $\bar{s}$

$\bar{s} \to \mathcal{U}\bar{s}$ $\qquad$ $\mathcal{U}$ = Abelian <u>or</u> non-Abelian

1-form: acts on line




2-form: acts on surface


For $q \geq 1$, $\mathcal{U}$ must be Abelian

Gaiotto, Kapustin, Seiberg & Willett 1412.5148

# SU(N) gauge

<u>Assume</u> no dynamical quarks (will relax later)

Gauge transf. :

$$A_\mu \to \frac{1}{-ig}\Omega^\dagger \underbrace{D_\mu}_{\partial - igA}\Omega$$

$\Omega \in SU(N) \Rightarrow \det \Omega = \mathbb{1}$.

Have both local <u>and</u> global gauge transf.'s

local : $\Omega = \Omega(x) \longrightarrow$ const. as $x \to \infty$

global : $\Omega =$ const.

Normally, don't worry about global $\Omega$'s

# Global $\Omega$

Under arbitrary global gauge transf.,

$$A_\mu^\Omega = \Omega^\dagger A_\mu \Omega \neq A_\mu$$

So what. Special subset

$$\Omega_j = \omega_j \mathbb{1}_N \qquad \omega_j = e^{2\pi i j/N}$$

$$\det \Omega_j = \omega_j^N = 1$$

$$j = 1, \ldots, N = \mathbb{Z}_N = \text{center of } SU(N)$$

$$A_\mu^{\Omega_j} = \omega_j^\dagger \mathbb{1}_N A_\mu \omega_j \mathbb{1}_N = A_\mu$$

Global $\mathbb{Z}_N$ symmetry "hidden" in local $SU(N)$

# Add test quark

With mass $M \to \infty$.

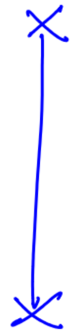
$$\mathcal{L}_{test} = \bar{\chi} D_0 \chi$$

Choose $\chi$ = particle, going forward in time
$\Rightarrow$ eliminate anti-particles

Can neglect spin, as effects $\sim 1/M$

$\Rightarrow$ $\chi$ = single component, only carries color

$$\langle \bar{\chi}(t)\chi(0)\rangle = \frac{1}{D_0} = \underbrace{\mathbb{P}\, e^{ig\int_0^x A_0\, dx}}_{\text{Wilson line}}\ \theta(x)$$



$\mathbb{L}(\bar{x}; 0, x)$

# Symmetries

Global 0-form sym. – for $\mathcal{L}$, no big deal.

1-form: consider temperature $T \neq 0$.

gluons = bosons $\Rightarrow$ $A_\mu(\bar{x}, 1/T) = + A_\mu(\bar{x}, 0)$

Choose $\Omega(\bar{x}, 1/T) = \omega_j \Omega(\bar{x}, 0)$ $\omega_j \in Z(N)$

$A_\mu$ still periodic, but

$$\mathbb{L}(\bar{x}, x) \to \Omega^\dagger(\bar{x}, 0)\, \mathbb{P} e^{i\int A_0}\, \Omega(\bar{x}, 1/T)$$

$$= \omega_j \mathbb{L}$$ = 1-form $Z_N$ sym.

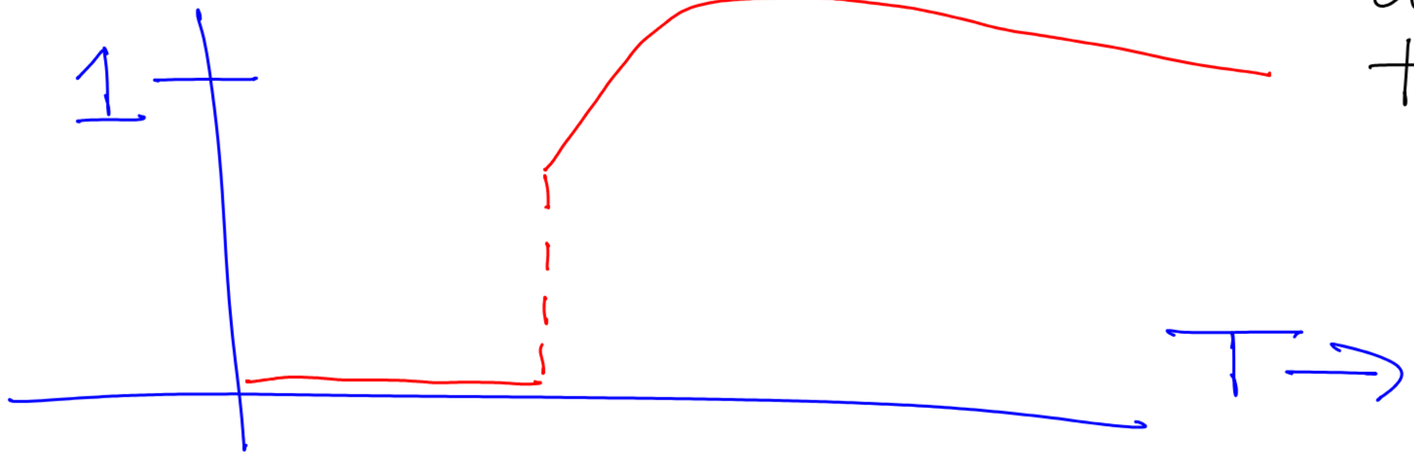
# Polyakov loop

Wilson line $\mathbb{L} \sim$ propagator of test quark

Polyakov loop $P(\bar{x}) = \mathrm{tr}_{color}\, \mathbb{L}(\bar{x})$

$$\int \frac{d^3x}{V} \langle P(\bar{x}) \rangle \sim \omega_j \left( 1 + \# g^3 + \ldots \right) \qquad T > T_{deconf}$$

$$= 0 \qquad T < T_d$$




deconfining transition

# Hamiltonian

Lagrangian:

$$\mathcal{L} = \frac{1}{2}\,\mathrm{tr}\, G_{\mu\nu}^2 + \bar{\chi} D_0 \chi(y)$$

@ <u>one</u> point, $y$

Go to $A_0 = 0$ gauge

$$\mathcal{H} = \mathrm{tr}\left(\vec{E}^2 + \vec{B}^2\right) + 0_q$$

Nothing from qks. $\bar{\chi} D_0 \chi = \bar{\chi}\,\partial_0 \chi$.

Momenta conjugate to $\chi$ is $\bar{\chi}$, $\chi$'s completely drop out of $\mathcal{H}$. Like coupling to external current $\sim \int j_0 A_0 = 0$ for $A_0 = 0$

# Gauss' law

Sure, need to impose conservation of electric charge

$$\mathcal{H}_{Gauss} = i \, \mathrm{tr}_{color} \left( \chi \left( D \cdot E - g Q \right) \right)$$

$$Q^a = \bar{\psi} t^a \psi$$ = adjoint field @ 1 point

$\chi$ = constraint field

in adjoint representation;

on lattice, lives on sites

Also: need to fix residual gauge symmetry

under time-independent gauge transf.'s

# Partition function

As usual,

$$Z = \sum_{\text{states}} \int \mathcal{D}\chi \; e^{-(\mathcal{H} + \mathcal{H}_{Gauss})/T}$$

$A_i$ (or $E_i$) and $\chi, \overline{\psi}$ $\Leftarrow$ only carry color @ $y$

But:

above $= \sum \ldots e^{-\mathrm{tr}_{color}\, \chi Q \ldots}$ upstairs

$$W = \mathrm{tr}_{color}\, e^{ig \int A_0 \, dx}$$ downstairs

$\chi \sim A_0$. Sure, But how does $\mathrm{tr}_{color}$ go from "upstairs" in $\mathcal{H}$ to "downstairs" in $W$?

# Test states

With test charges, need to expand the Hilbert space

Easy to do! Can always choose $\chi$'s to be diagonal by gauge rotation

$$\bar{\chi}, \chi = \text{color } 1 \Rightarrow \bar{\chi}\, t^a \chi = t^a_{11}$$

$$\text{" \quad " \quad } 2 \Rightarrow \text{ " } = t^a_{22}$$

$$\vdots \qquad\qquad \vdots$$

$$N \qquad\qquad = t^a_{NN}$$

$$\Rightarrow \sum_{\chi, \bar{\chi}} e^{-ig \int \mathrm{tr}_{\text{color}} \chi Q / T} = \underbrace{\mathrm{tr}_{\text{color}}\, e^{-ig\, \chi(y)/T}}_{\text{Polyakov loop}}$$

In all

Partition fnc.:

$$Z = \sum_{\{A,E\}} \int \mathcal{D}X \; e^{-\frac{1}{T}\int \mathcal{H} + ig\, \mathrm{tr}\, X D\cdot E}$$

Test chg like "defect" @ $y$: $\quad \mathrm{tr}\, e^{-ig X(y)}$

In $\mathcal{H}$-form, need to add extra states

↳ Gervais & Sakita PRD 18 '78

Character expansion in group theory $\left\{ \begin{array}{l} \text{Susskind PRD 20 '79} \\ \text{W. Greiner \& B. Muller} \\ \quad \text{Sym.'s of Quantum Mechanics} \end{array} \right.$

## Gauge transf.'s

Int. by parts:

$$\int \mathrm{tr}\, \chi D\cdot E = \int_{\infty} \mathrm{tr}\, \chi E_i - \int \mathrm{tr}\, \underbrace{D_i \chi}_{} E_i$$

spatial $\infty$

generates $\Omega = e^{i\chi}$ on $A_i$

Divide $\chi^a$ into global (const.) and local (falls off @ $\infty$) components

For <u>local</u> $\chi$,

$$P_\chi = \int \mathcal{D}\chi \; e^{ig\int \mathrm{tr}\, \chi D\cdot E} = \delta(D\cdot E)$$

= projector, $\quad P_\chi^2 = P_\chi$

Sure...

# Global transf.'s

Total color charge $= \int_\infty E^a$

$\Rightarrow$ integrating over all constant $\chi^a$

imposes total color chg $= 0$

Ok in confined phase, <u>not</u> necessary in deconfined

For $\chi_\infty =$ const.,

$$\underbrace{e^{ig\int \mathrm{tr}\, \chi_\infty E/T}}_{} |A\rangle = |A^{\Omega_\infty}\rangle$$

$e^{ig\chi_\infty/T}$

<u>not</u> a projector,

but a global gauge rotation

# More global

For const. $\Omega_\infty$,

$$Z(\Omega_\infty) = \sum e^{-\mathcal{H}/T} \delta(D \cdot E)\, e^{ig \int \chi_\infty E/T}$$

$$= \sum \langle A_i | e^{-\mathcal{H}/T} \delta(D \cdot E) | A_i^{\Omega_\infty} \rangle$$

$$\neq Z(1) \qquad |A_i^{\Omega_\infty}\rangle \neq |A_i\rangle$$

if $\Omega_\infty \neq \omega_j$

= twisted partition function

Like for a spin system:

$$\langle \uparrow | e^{-\mathcal{H}/T} | \uparrow \rangle = \langle \downarrow | e^{-\mathcal{H}/T} | \downarrow \rangle$$ if $\uparrow$ & $\downarrow$ degenerate

$$\neq \langle \uparrow | e^{-\mathcal{H}/T} | \rightarrow \rangle$$ if $|\rightarrow\rangle \neq$ degenerate vacuum

# Global degeneracy

By global $Z(N)$ sym., if $\Omega_\infty = \omega_j$,

$$|A_i^{\omega_j}\rangle = |A_i\rangle \Rightarrow Z(\omega_j) = Z(\mathbb{1})$$

Easy to work out explicitly

$$\tilde{\chi}_\infty^j = \frac{2\pi T}{gN} j\, t_N \,, \qquad t_N = \begin{pmatrix} \mathbb{1}_{N-1} & \\ & -(N-1) \end{pmatrix}$$

$$\omega_j = e^{ig\tilde{\chi}_\infty^j/T}$$

N.B.: $\omega_j$ commutes $\bar{c}$ all $\Omega$, but $\tilde{\chi}_\infty^j \sim t_N$ does <u>not</u>

# Polyakov loop

Now add a defect. Const. $\chi$ shifts the propagator by that phase

$$\mathrm{tr}\,\mathbb{L} = \sum_{A_i} e^{-\mathcal{H}/T} \underbrace{\delta(D\cdot E)}_{\text{locally}} \,\mathrm{tr}\, e^{ig\chi(y)}$$

$$= \sum \langle A_i | e^{-\mathcal{H}/T} e^{ig\chi(y)} | A_i \rangle$$

$$\xrightarrow[\chi \to \chi + \tilde{\chi}^j_\infty]{} \sum \langle A_i | e^{-\mathcal{H}/T} e^{ig\chi} | A_i^{\omega_j} \rangle \, \omega_j$$

In $\mathcal{H}$ form, <u>no</u> 1-form symmetry, only usual global 0-form sym. for defect.

# Quarks

With dynamical quarks: $\chi(\bar{x}, 1/T) = -\chi(\bar{x}, 0)$

But $\chi^{\Omega} = \Omega \chi \neq \chi$ for $\underline{\text{any}}$ $\Omega \neq 1$, even $\Omega = \omega_j$

So $Z(N)$ sym. is lost. In $\mathcal{H}$ form, for constant $\chi_\infty$,

$$Z(\omega_j) = \sum_{A,\chi} \langle A, \chi | \, e^{-\mathcal{H}/T} \delta(D \cdot E) \, | \underbrace{A^{\omega_j}, \chi^{\omega_j}}_{= |A, \chi^{\omega_j}\rangle} \rangle$$

$$\neq Z(1)$$

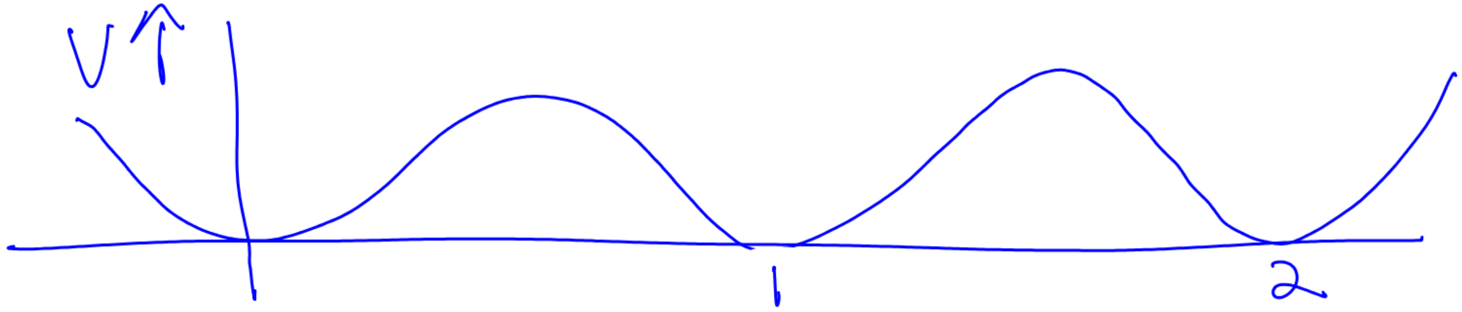
Result for Wilson/Polyakov loops unchanged, just no $Z(N)$ sym.

# Holonomy & twist

In $\mathcal{L}$ form, can compute "holonomous" potential at constant $A_0$. E.g.,

volume

$$A_0^{cl} = \tilde{X}^1_\infty \, q \qquad V^{1\text{-loop}}_{hol.} \sim T^4 q^2 (1-q)^2 V$$




$q = n \Rightarrow \omega_n$

$q \longrightarrow$

In $\mathcal{H}$ form, $Z(N)$ sym. $\Rightarrow$ degeneracy of twisted partition function

$$\sum \langle A | e^{-\mathcal{H}/T} | A \rangle = \sum \langle A | e^{-\mathcal{H}/T} | A^{\omega^j} \rangle$$

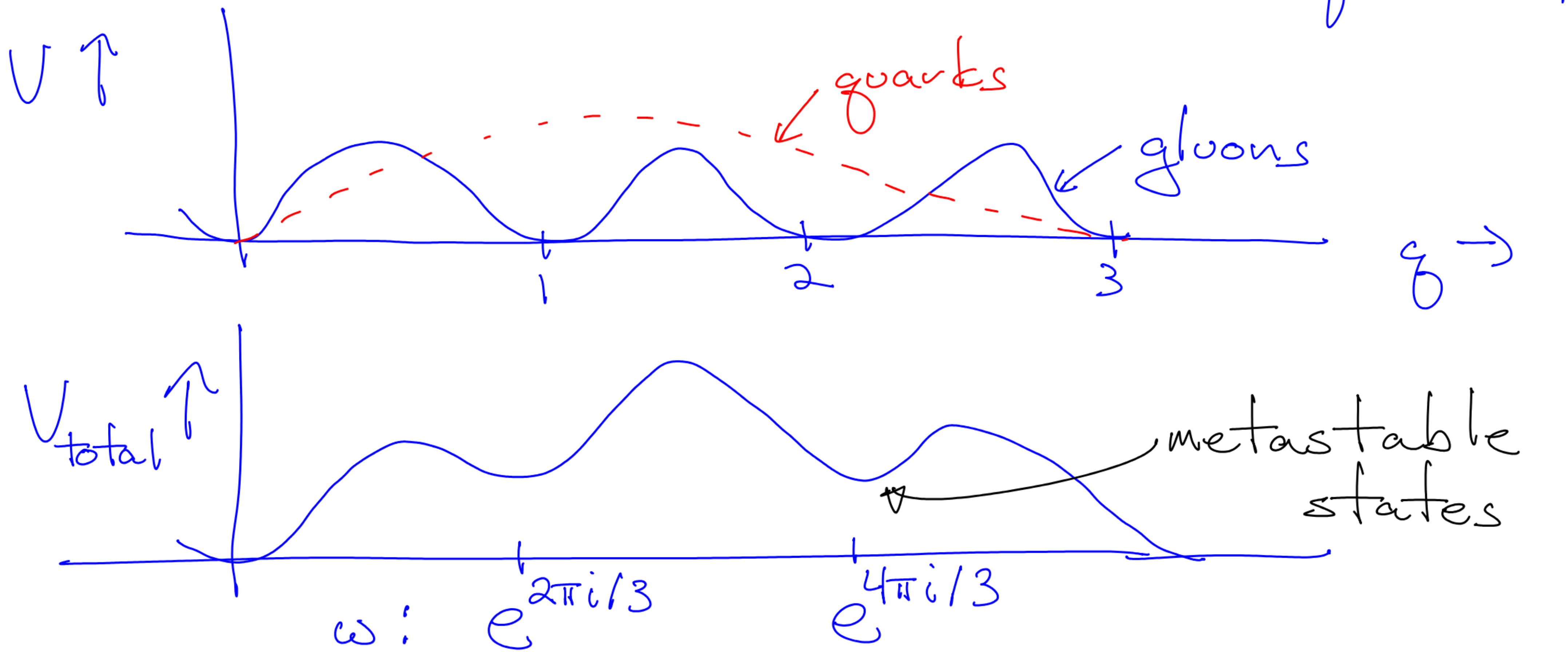
# Twist $\bar{c}$ quarks

Adding quarks, $Z(N)$ global sym. is lost.

Can compute either $\bar{c}$ $\mathcal{L}$ or $\mathcal{H}$ form.

E.g., for $SU(3)$:

$q = n : \omega_n$



# Summary

Is the Polyakov loop an artifact of an overly active Euclidean imagination? No.

Is the "holonomous" potential @ constant $A_0$ physical? Yes = "twisted" partition function.

Could metastable states be there? Yes

Measurable even in U(1) gauge theory $\bar{c}$ matter in 1+1 dimensions. Most doable.

# Fractional Topological Charge

RDP & V.P. Nair, 2206.

In $SU(N)$ gauge theories withOUT dynamical quarks,
at large $N$, need objects not $\bar{c}$ $Q_{top} = \pm 1, \pm 2 \ldots$
but

$$Q_{top} = \pm \frac{1}{N}, \pm \frac{2}{N} \ldots$$

Arise immediately $\bar{c}$ $Z(N)$ twisted b.c.'s
`t Hooft '80 + ...

On "femto-slab": width $L \ll \Lambda^{-1}_{QCD}$
Unsal 2007.03880; Poppitz 2111.10423

Today: "quantum" instantons $\bar{c}$ size $\sim 1/\Lambda^{-1}_{QCD}$

# Instantons

Topological charge $Q \sim \int d^4x \, \mathrm{tr}\, G_{\mu\nu} \tilde{G}^{\mu\nu}$ $\qquad \epsilon^{\mu\nu\alpha\beta} G^{\alpha\beta}$

Instantons – solutions to classical equations of motion

self-dual, $G_{\mu\nu} = \pm \tilde{G}_{\mu\nu}$

As classical eqs. scale invariant, instantons come in all scale sizes, $\rho : 0 \to \infty$

For $Q = n =$ integer, $S = \int \mathrm{tr}\, G_{\mu\nu}^2 = \frac{8\pi^2}{g^2} n$

Construction of all instantons known – collective coor.'s (moduli space) involved

Atiyah, Hitchin, Drinfeld, Manin Phys. Lett. A65 '78

# Instantons$^2$

Inst.'s valid semi-classically, when $g^2 \ll 1$.

E.g., temperature $T \gg \Lambda_{QCD}$.

By asymptotic freedom, $g^2(T) \sim \# / \ln T$

$$\Rightarrow Z_{inst.} \sim e^{-8\pi^2/g^2(T)} \sim \#' / T^c \qquad c = \frac{11 N_c - 2 N_f}{3}$$

$\underline{T \gg \Lambda_{QCD}}$

Lattice, $N_c = 3$;

Instantons dominant for $T > 300$ MeV ! $\sim \Lambda_{QCD}$

$\#' \sim 10 * 1$-loop result - need 2-loop!

Borsanyi +... 1606.07494     Lombardo & Trunin

Petreczky +... 1606.03145 ...     2005.06547

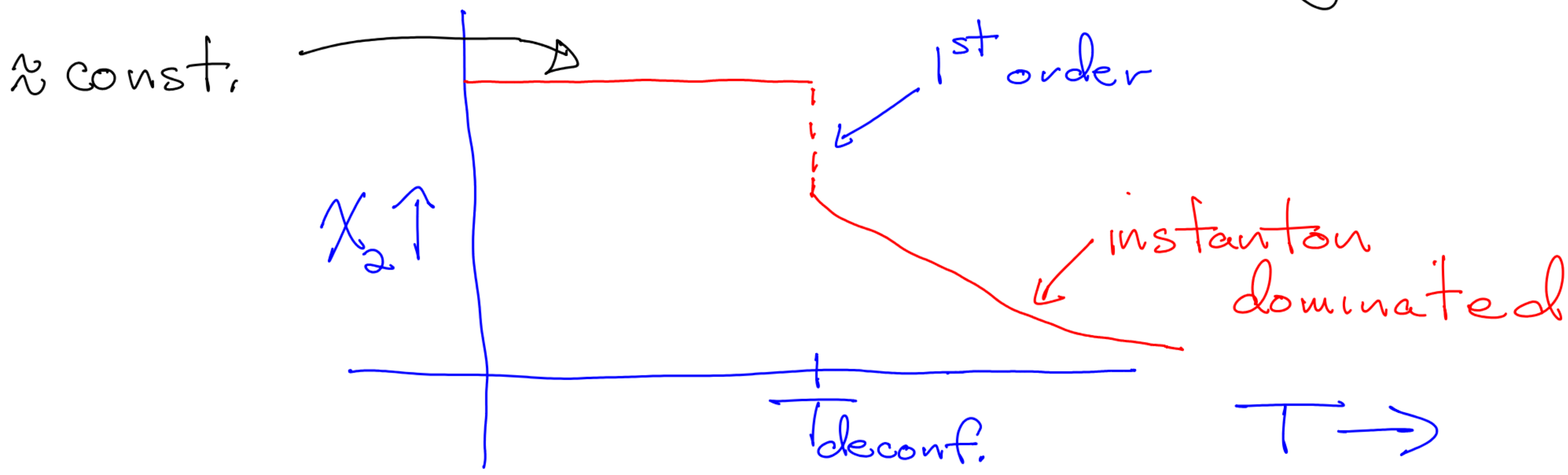
# Lattice - pure glue

Compute top. susceptibility:

$$S_\theta = S_0 + i\theta Q \qquad \chi_2 = \frac{\partial^2 \ln Z}{\partial \theta^2} \sim \langle Q^2 \rangle$$

For pure SU(N) glue, for $N \geq 3$, weak dependence on N. First order transition @ $T_{deconfinement} \sim 270$ MeV

E.g., $T_c/\sqrt{\sigma} \sim$ const. $\simeq$ N ($\sigma$ = T=0 string tension)



# Lattice – $\bar{c}$ quarks

Three regimes: $T > 300$ MeV – instanton dominated

QCD – $T_\chi \sim 155$ MeV (crossover)

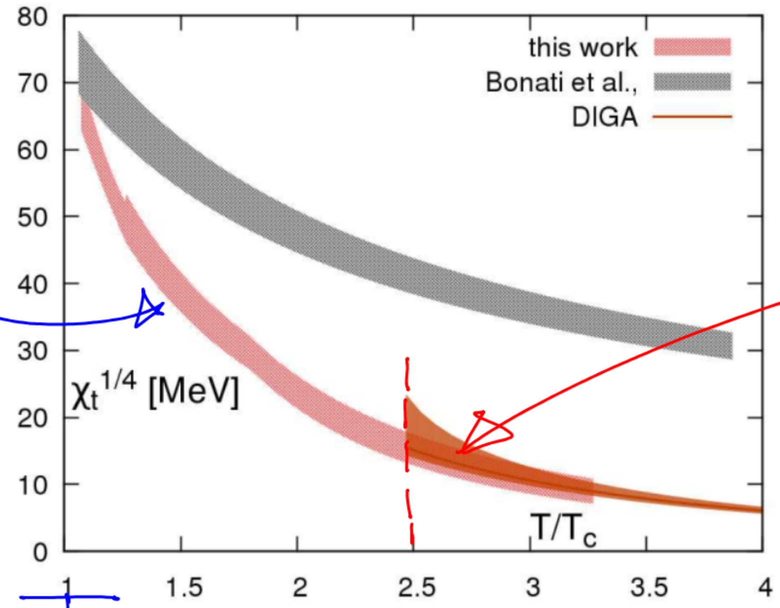
Intermediate T: $300 > T > 155$ MeV

<u>not</u> instanton – slower fall off for $\chi_2$

Low T: $\chi_2 \approx$ const, $T < 155$ MeV

$\chi_2^{1/4} \uparrow$

<u>not</u> instantons




instanton dominated

Petreczky + ...
1606.03145

$T_\chi$

$T \longrightarrow$

# Instantons @ large N

Veneziano '79 & Witten '79:
instantons exponentially suppressed @ large N!
hold $g^2 N \equiv \lambda$ fixed as $N \to \infty$,

$$\chi_2 \sim e^{-8\pi^2/g^2} \sim e^{-(8\pi^2/g^2 N) N}$$

Even so, argue $\chi_2 \sim 1$ as $N \to \infty$, $T = 0$
Can <u>not</u> be instantons

N.B.: or - instanton liquid? Liu, Shuryak, Zahed, 1802.00540

Perhaps - but need to show term in $S_\theta \sim N = 0$
Not obvious if instantons arise with
<u>all</u> scale sizes, $\rho$

# "Fractional" instantons

IF there are objects $\bar{c}$ $Q_{top} \sim 1/N$, then no problem!

$$\chi_2 \sim e^{-8\pi^2/(g^2 N)} \sim 1 \quad \text{as} \quad N \to \infty$$

't Hooft '80, van Baal '82, Sedlacek CMP 86 '82

$Q_{top} \sim 1/N$ $\bar{c}$ $Z(N)$ twisted boundary conditions

manifestly finite volume

only analytic soln's for finite box

$\bar{c}$ sizes in certain ratio, etc.

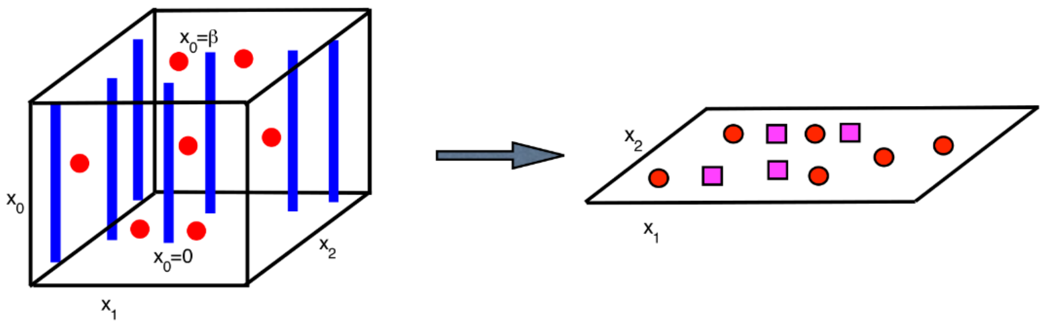
Do they persist in infinite volume?

# Femto-slab

Unsal, Poppitz, Anber ....

Take one spatial dimension, $L$, $\ll \Lambda_{QCD}^{-1}$, so $g^2(L\Lambda_{QCD}) \ll 1$. Using results in 2+1 dim.'s, over large distances, confining thy $\overline{c}$

"monopole-instantons", $Q_{top} \sim 1/N$




Poppitz, 2111.10423

Size of monopole-inst.'s $\sim L$

What happens as $L \sim \Lambda_{QCD}^{-1}$ ?

# Punch line

Size of monopole-instanton gets stuck at $L \sim \Lambda_{QCD}^{-1}$

Dominant configurations $\simeq$ one size

Measurable on lattice $\bar{c}$ adjoint (not fund.) quark prop. as external probe

not dynamical

# $CP^{N-1}$ model

In 1+1 dim.'s: $N$ component field $z^i$, $\bar{z}^i z^i = 1$, inv.:

$$z^i(x) \to e^{i\alpha(x)} z^i(x)$$

local $U(1)$

$$\mathcal{L} = \frac{1}{g^2}\int d^2x\, |D_\mu z^i|^2 \qquad D_\mu = \partial_\mu - iA_\mu$$

Global sym.: $z^i \to U^{ij} z^j \Rightarrow SU(N)$

But: if $U = \omega_1 = e^{2\pi i/N}$, $z^i \to \underbrace{e^{2\pi i/N}}_{\text{part of } U(1)} z^i$

$\Rightarrow$ global sym. $SU(N)/Z(N)$

$g^2$ asymptotically free, soluble as $N \to \infty$

Witten '79 d'Adda, Lüscher, Di Vecchia '78

## $CP^{N-1}$ inst.'s

Topological chg.

$$Q_{top} = \frac{1}{2\pi}\int d^2x \; \varepsilon^{\mu\nu}\partial_\mu A_\nu$$

<u>All</u> classical soln's known; self-dual,

$$D_\mu z = \varepsilon^{\mu\nu} D_\nu z \qquad z^i \sim \frac{(x+iy)\,v^i}{\sqrt{x^2+y^2+\rho^2}}$$

Can compute 1-loop fluc's about all instantons

$$S_{top} \sim \frac{1}{g^2} \sim \Big(\underbrace{\frac{1}{g^2 N}}_{\text{fixed}}\Big) N \Rightarrow e^{-S_{top}} \sim e^{-\# N}$$

But: large $N$ soln. shows that

$$\chi \sim \langle Q_{top}^2 \rangle \sim \frac{1}{N} \quad \underline{\text{not}} \quad e^{-N} \;!$$

# Frac. inst.'s in $CP^{N-1}$

Consider

$$z^1(r,\theta) = e^{i\varphi/N} h(r) \qquad z^{2,\ldots} = 0$$

Not single-valued:

$$z^1(r,2\pi) = e^{2\pi i/N} h(r) \sim z^1(r,0)$$

by $Z(N)$ sym. Then the corresponding

$$A_\varphi \sim \frac{1}{rN}, \quad r \to \infty \qquad \Rightarrow \qquad Q_{top} = \frac{1}{N}$$

To obtain frac. Q, require multi-valued soln.'s allowed by $Z(N)$

# $CP^N$ @ $N \to \infty$

Introduce a constraint field $\lambda(|z|^2 - 1)$, integrate out $z$'s

$$S_{eff} = N \,\mathrm{tr}\ln(-D_\mu^2 + i\lambda) - i\int \frac{\lambda}{g^2}\, d^2x$$

Vacuum: $i\lambda = m^2$ (dim. trans.), $A_\mu = 0$

Frac. inst.: action non-local, only limiting behavior

But: scale sym. of classical action lost

frac. inst. has <u>one</u> size $\sim 1/m$ = confinement distance

# SU(N) gauge, NO quarks

Back to 3+1 dim.'s, <u>no</u> quarks, $A_0 = 0$ gauge

Parametrize gauge field as function of arbitrary parameter $\xi$,

$$A_i(\bar{x}, \xi) = (1-\xi)\, A_i(\bar{x}) + \xi\, A_i^{\Omega}(\bar{x})$$

gauge transf. of $A_i$

If $\Omega \to \mathbb{1}$ @ $\bar{x} \to \infty$,

$$Q_{top} = \frac{1}{8\pi^2} \int \mathrm{tr}\, G_{\mu\nu} \tilde{G}^{\mu\nu} = \frac{1}{24\pi^2} \int d^3x \;\; \mathrm{tr}\left(\Omega^{\dagger} \partial_i \Omega\right)^3$$

$$= \text{integer}$$

Above doesn't give soln. $\bar{c}$ minimal action, but gets $Q_{top}$ right

# $Z(N)$ vacua

But alternately, we can choose $\Omega_\infty = \omega_j = e^{2\pi i j/N} \mathbb{1}$

$$\Omega_\infty(\xi) = e^{i\tilde{\chi}^j_\infty \xi} \qquad \tilde{\chi}^j_\infty = \frac{2\pi}{N} j\, t_N, \quad t_N = \begin{pmatrix} \mathbb{1}_{N-1} & \\ & -(N-1) \end{pmatrix}$$

For finite $r = \sqrt{x^2+y^2+z^2}$, need more involved ansatz

$$Y_{ij} = \frac{\sigma\cdot\hat{x}}{2} + \frac{1}{N} - \frac{1}{2} \quad i,j = 1,2 \qquad Y_{ij} = \frac{\delta^{ij}}{N} \quad i,j = 3\ldots N$$

$$\Omega(\bar{x},\xi) = e^{iY\theta(r,\xi)}$$

$\theta = 0$ @ $r = 0$, $\xi = 0$

$= 2\pi\xi$ @ $r = \infty$

Only illustrative construction; not minimal action

# Topological Chg.

$$G_{\mu\nu} \to \Omega^{+} G\, \Omega + \frac{\partial}{\partial \xi} A^{\Omega} = \Omega^{+} (G - Da)\, \Omega$$

Variation in $\xi$ like "time" $\quad a = \frac{d\Omega}{d\xi}\, \Omega^{-1}$

"a" analogous to $A_0$

$$Q_{top} = \frac{1}{8\pi^2} \int tr (G - Da)^2\, d^4x$$

$$= \frac{1}{4\pi^2} \int_{x=\infty} d^2 s^i \int dx\; tr (a B^i)$$

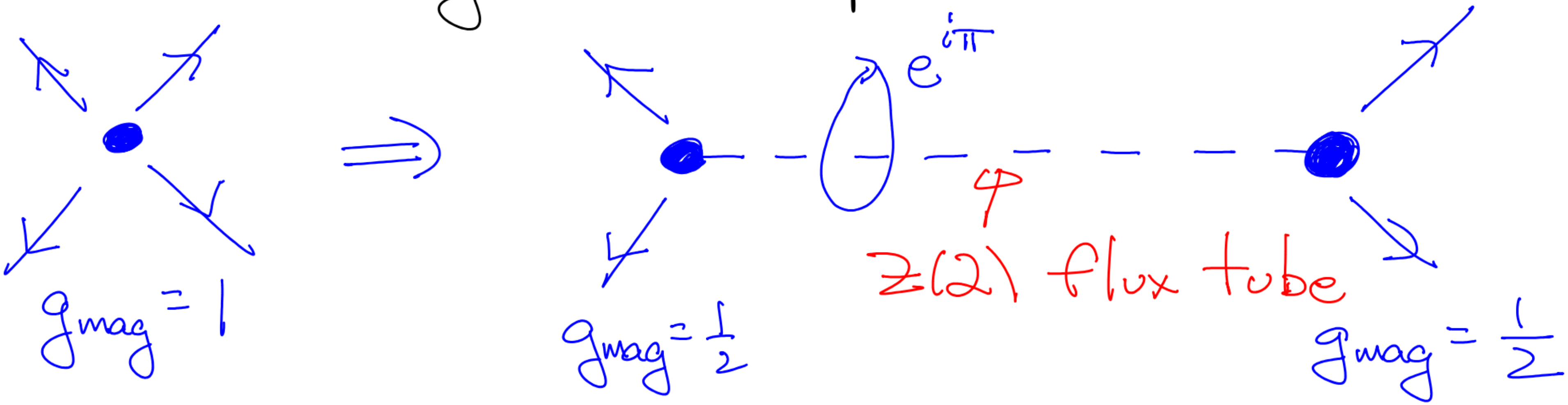
<u>Need</u> magnetic chg. For $SU(2)$:

$$G_{ij} \sim -\frac{i}{2} \vec{\sigma} \cdot \hat{x}\; \varepsilon_{ijk} \frac{\hat{x}^k}{r^2} \quad \Rightarrow \quad Q_{top} = 1$$

$S_0$?

# "Split" $Z_2$ monopole

For a SU(2) magnetic monopole




Without quarks, Z(2) flux tube invisible

Each end has $Q_{top} = +\frac{1}{2}$.

To observe each end seperately, do need Z(2) twisted boundary conditions

# Vacuum of frac. inst.'s

For $SU(N)$, instanton $\bar{c}$ $Q_{top} = 1$ $= N$ frac.'s

Natural length scale of $Z(N)$ flux tubes $\sim \Lambda_{QCD}^{-1}$

Configurations manifestly non-perturbative.

So what? Vacuum = sum of $I$'s & $\bar{I}$'s

or - better described as sum of $Q_{top} = +\frac{1}{N}$ & $-\frac{1}{N}$

Unsal: on femto-slab, $\theta$-dependence is $f(\theta/N)$, not $f(\theta)$. Presumably carries over

2007.03880

## $Z(N)$ dyon

A explicit construction, @ $T > T_{deconf}$. Now $A_0 \neq 0$

$$A_0^\infty = \frac{2\pi T}{gN} k, \qquad k = t_N = \begin{pmatrix} \mathbb{1}_{N-1} & \\ & -(N-1) \end{pmatrix}$$

$$\mathbb{L} = e^{ig\int_0^{1/T} A_0 dx} = e^{\frac{2\pi i}{N} k} \qquad \text{or} \qquad t_N' = \begin{pmatrix} -(N-1) & \\ & \mathbb{1}_{N-1} \end{pmatrix}$$

$k$ = nontrivial holonomy. Above value is a minimum of the holonomous potential

$$A_0 = \frac{2\pi T}{gN} q k \Rightarrow V_{holonomous} \sim T^4 q^2 (1-q)^2$$

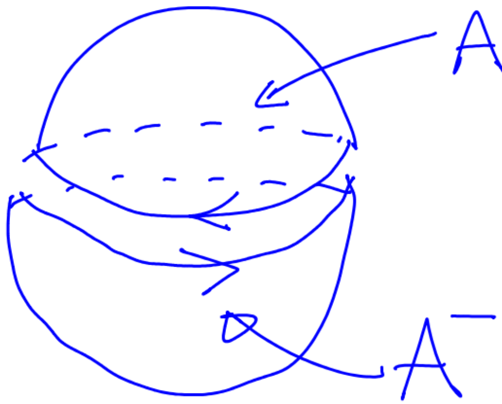
$|q| \bmod 1$

# $Z(N)$ monopole

At spatial $\infty$,

$$A_\varphi^{\pm} = \frac{1}{N\,2r}\, m\, \frac{(\pm 1 - \cos\theta)}{\sin\theta} = \frac{1}{N} * \text{Dirac monopole}$$

$m =$ magnetic chg, $\sim k_1$ or $k_2$




$$e^{i\oint A^+} e^{-i\oint A^-} = e^{\frac{2\pi i}{N} m}$$

Above $A_\varphi$ at spatial $\infty$. For $A_0$

$$A_0(r) \underset{r\to\infty}{\sim} \frac{2\pi T}{N} k - \frac{1}{2Nr} m + \ldots$$

# $Z(N)$ dyon

Assume there is a regular solution for all $r$, esp. $r=0$. For simplicity, take it to be static

$$Q_{top} = \frac{1}{4\pi^2} \int \partial_i \, \mathrm{tr}\, A_0 B_i = \frac{1}{N^2} \mathrm{tr}(mk)$$

$m=k$: $Q_{top} = \frac{N-1}{N}$ $\qquad$ $m \neq k$: $Q_{top} = \frac{1}{N}$

Identical to 't Hooft, but $Z(N)$ twist not in a box, but in radial direction

# Dyons vs calorons

Lee & Lu: th/9802108
Kraan & van Baal: th/9805168 } KvBLL

Show instanton @ $T \neq 0$ composed of
N constituents $\bar{c}$ $Q_{top} = 1/N$

| | KvBLL | Z(N) dyon |
| --- | --- | --- |
| magnetic chg | integer | $1/N$ |
| $V_{hol}(q)$ | maximum $\sim 1/N$ | minimum $\Rightarrow$ integer |

But max. of $V_{hol.}$ $\Rightarrow$ @ 1-loop, free energy
$\sim$ + volume of space!

## Z(N) dyons

$T > T_{deconf}$ : Z(N) electric chg unconfined
but Z(N) magnetic chg confined
$\Rightarrow$ Z(N) dyons bound $> T_d \Rightarrow$ instantons soon dominate

$T < T_{deconf}$ : Z(N) dyons propagate freely
Geometry not obvious - world lines of dyons twisted

Presumably size $\sim \Lambda_{QCD}^{-1}$

# Lattice - pure glue

Edwards, Heller, Narayanan lat/9806011

To measure $Q_{top} \sim 1/N$, use $\chi$-symmetric qk. prop.
in adjoint, <u>not</u> fund., representation

$\bar{c}$ adjoint, $Q_{top} = 1/N$ gives two (L & R) zero modes

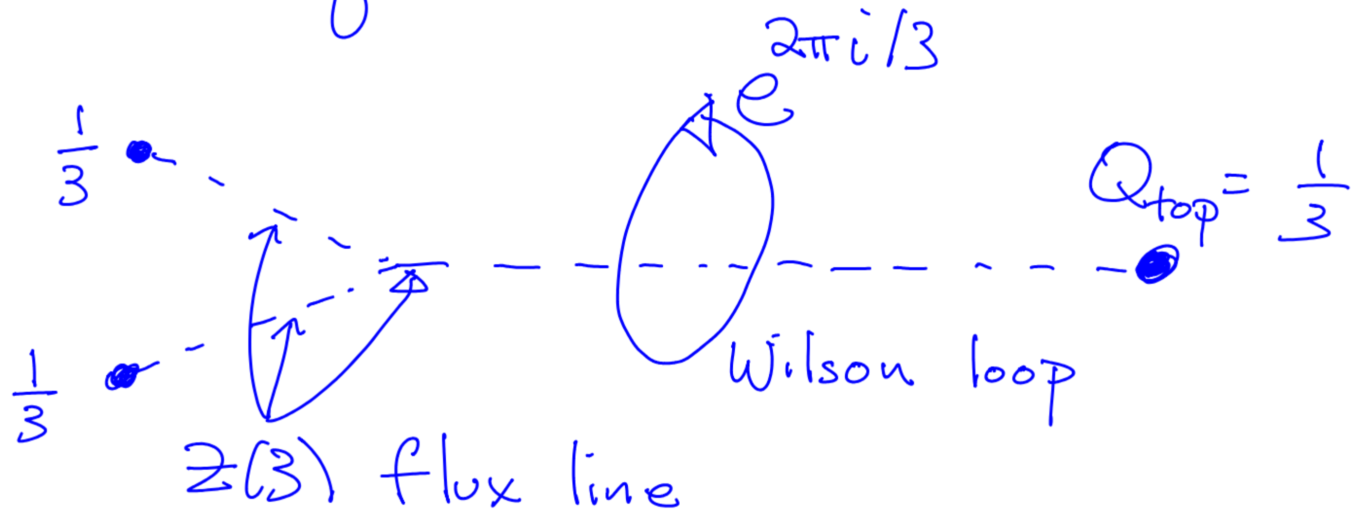
From eigenvector, estimate size

If one finds $Q_{top} \sim 1/N$, are they dilute, or
densely packed?

(Fodor +..., 0905.3586:
SU(3) $\bar{c}$ sextet rep. Do <u>not</u> find frac. $Q_{top}$.
But sextet only sensitive to $Q_{top} = \frac{1}{5}$, not $\frac{1}{3}$)

+ Dynamical quarks

"Split" monopoles in $SU(3)$:




$Z(3)$ flux tube invisible to gluons, but Wilson loop in fund. rep., <u>or</u> dynamical quarks, see it.

$\Rightarrow$ Dynamical quarks <u>confine</u> objects $\bar{c}$ $Q_{top} = \frac{1}{N}$

Yes, quarks confine. Clearly complicated int.'s between qks & $Z(N)$ dyons

## SU(3): $T \neq 0$, $\mu = 0$

Lattice: c/o qks, $T_{deconf} \sim 270$ MeV

2+1 flavors, $T_\chi \sim 155$ "

Perhaps three regimes

$T > 300$: instanton dominated
(Z(3) dyons confined)

$300 > T > T_\chi$: Z(3) dyons + $\approx$ massless qks

$T_\chi > T$: Z(3) dyons + massive qks

Quantitative tests of dynamical qks $\bar{c}$ dyons?

# Three regimes @ $\mu \neq 0$, $T \approx 0$?

Because of # d.o.f., instantons suppressed @ $T \neq 0$ much more than $T = 0$, $\mu \neq 0$

$$m^2_{Debye} = g^2\left(\frac{1}{3}\left(N_c + \frac{N_f}{2}\right)T^2 + 2N_f\left(\frac{\mu_{qk}}{2\pi}\right)^2\right)$$

RDP & Rennecke: instantons don't dominate until $\mu_{qk} > 2$ GeV (or more)

Perhaps:

$\mu_{qk} > 2$ GeV – instantons

$2\text{ GeV} > \mu_{qk} > \mu_\chi$ – $Z(N)$ dyons + massless qks

$\mu_\chi > \mu > 313$ MeV – " " + massive qks

$\ell_2$?

Clearly base speculation

# Summary

For pure gauge, two sources of fluc's in topological charge

instantons in weak coupling – all sizes

$Z(N)$ dyons " strong " – one size

→ confine? world line of $Z(N)$ dyon $\sim Z(N)$ vortex

With quarks: much more complicated

$\mu = 0$, $T > 300$ MeV – dyons confined into inst's

$T <$ " " – dyons & quarks int'g

$T = 0$, $\mu \neq 0$: dyons & qks relevant for all densities in neutron stars

instantons never matter

# QCD in 1+1 dim.'s

Soluble in several ways.

't Hooft '74: $SU(N_c)$ soluble for massive quarks as $N_c \to \infty$, $m_{qk} \gg g$

Bringoltz 0901.4035 : @ $\mu \neq 0$, kink crystal:
$\langle \bar\Psi \Psi \rangle$ oscillates about $\langle \bar\Psi \Psi \rangle \neq 0$

Baluni '80, Steinhardt '80...Armoni, Frishman, Sonnenschein, th/9709097
NJL: Azaria, Konik, Lecheminant, Palmai, Takacs, Tsvelik, 1601.02979
QCD: Lajer, Konik, Tsvelik, RDP, to appear

For arbitrary $N_c$, # flavors $N_f$, complicated
At strong coupling, $m_{qk} \ll g$ ($g \sim$ mass in 1+1 dim.'s)
Wess-Zumino-Witten models arise. Sure.
At $\mu \neq 0$, near Fermi surface, just WZW plus Luttinger liquid
Very simple!

# Bosonization in 1+1 dim.'s

For small $m_{qk}$, bosonize.

$N_f = 1$ : Abelian bosonization $\quad j_\mu = \bar\psi \gamma_\mu \psi = \varepsilon_{\mu\nu} \partial^\nu \phi$

Coleman '74 $\quad \bar\psi \psi \sim e^{i\# \phi}$

$N_f \geq 2$ : non-Abelian boson'n

Witten '84...James, Konik, Lecheminant,Robinson,Tsvelik, 1703.046002

$N_f = 1$ simplest. Baluni '80: take $A_0 = 0$.

For $A_1 = A$ = $SU(N_c)$ matrix, clever choice of gauge:

$A$ = off-diagonal, $\quad E$ = electric field = diagonal

$A^{aa} = 0$ (no sum!) $\quad E^{ab} = e^a \delta^{ab} \quad a, b = 1 \ldots N_c$

Yeah, so....

## $QCD_{1+1}$, $N_f = 1$

Gauss' law: $\partial_x e^a = j_0^{aa}$ $\quad ig(e^a - e^b) A^{ab} = j_0^{ab}$ $\quad a \neq b$

Integrating out A, get $S_{int} \sim \int\int j_0(x) \frac{1}{|x-y|} j_0(y)$

With Abelian boson'n, $N_c - 1$ sine-Gordon models; ($\pi$ = conj. mom.)

$$\mathcal{H}_0 = \frac{1}{2} \sum_{a=1}^{N_c} \pi_a^2 + \tilde{m}\left(1 - \cos(2\sqrt{\pi}\, e^a)\right) \qquad \tilde{m} \sim m_{qk}$$

$$\mathcal{H}_g = \sum_{a,b=1}^{N_c} \frac{g^2}{8\pi N_c} (e^a - e^b)^2 + \frac{\Lambda^2}{g} \frac{\sin(2\sqrt{\pi}(e^a - e^b))}{e^a - e^b}$$

from normal ordering, so $\Lambda \sim g$

Still, $N_c - 1$ s-G models: not easy! Plus $\mathcal{H}_g$ from gauge int.'s

## $QCD_{1+1}$, $N_f = 1$, $\mu \neq 0$

Remember $j_0 \sim \partial_z \varphi$. So constant $j_0$ is like $\varphi \sim \mu z$

Writing $e^a \to \varphi^a$, $\varphi \equiv \frac{1}{\sqrt{N_c}} \sum_{a=1}^{N_c} e^a$, $\pi =$ conj. mom.,

$$\mathcal{H}_{eff}(\varphi) = \frac{1}{2}\left(\pi^2 + (\partial_x \varphi)^2\right) + \tilde{m}\left(1 - \cos\sqrt{\frac{4\pi}{N_c}}\,\varphi\right)$$

At $\mu \neq 0$, <u>only</u> $\varphi$ matters. <u>All</u> $V(\varphi^a - \varphi^b)$ drop out.

$$\mathcal{L}_{eff} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{\tilde{m}}{2\pi}\cos\left(\sqrt{\frac{4\pi}{N_c}}\,\varphi + 2k_0 x\right)$$

$\hookrightarrow k_0 = \frac{\mu}{N_c}$

Spectra: soliton $\bar{c}$ mass $m_s$
anti-soliton " "
breathers = soliton + $\overline{\text{soliton}}$

# Luttinger liquid

$\mu$ = baryon $\mu$. For Fermi sea, $\mu > m_{\text{soliton}}$

At low energies,

$$\mathcal{L}_{\text{eff}} = \frac{K}{2}\left(\frac{1}{v_F}(\partial_0\varphi)^2 + v_F(\partial_x\varphi)^2\right)$$

Just single, massless boson $\varphi \sim U(1)$ current

$K$ = Luttinger parameter $= K(\mu)$

$v_F$ = Fermi velocity $= v_F(\mu)$

non-Fermi liquid - excitations near Fermi surface are <u>not</u> baryons, but $\varphi$

# Soln. of Luttinger liquid

M. Lajer, R. Konik, A. Tsvelik, RDP, to appear

Use Thermodynamic Bethe Ansatz:

$\mu \to m_s$: $K \to 1$, $v_f \to 0$

$\varphi$ doesn't propagate

$\mu \gg m_s$: $K \to \frac{1}{N_c}$, $v_f \to 1$

$\varphi$ relativistic, $K(\infty)$?

# Soln. of Luttinger liquid

M. Lajer, R. Konik, A. Tsvelik, RDP, to appear

Both $K$, $v_F$ are functions of $\mu$.

When Fermi surface first appears:

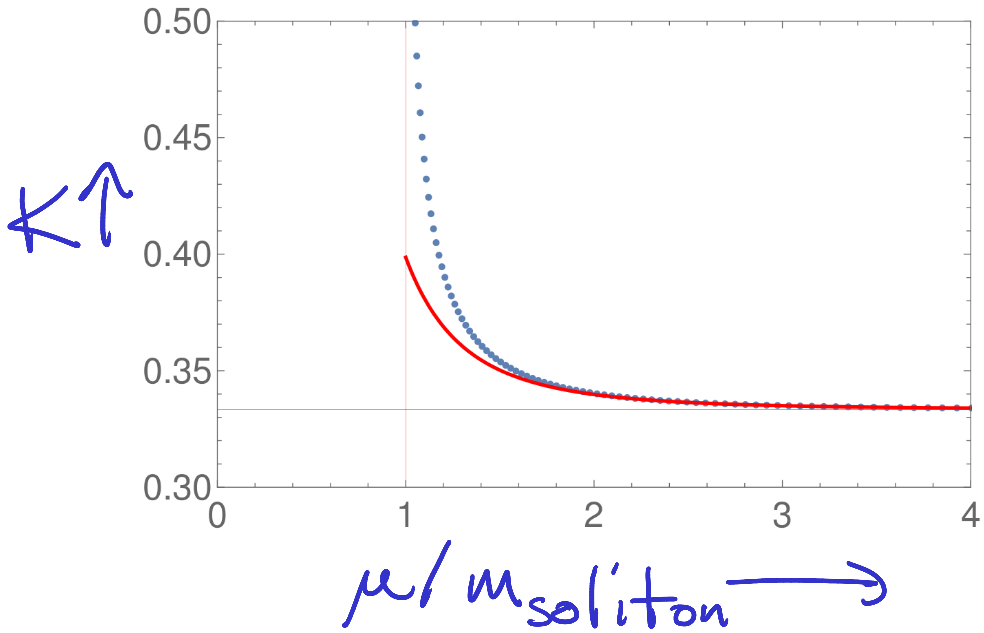
$\mu \to m_{soliton}$: $K \to 1$, $\underbrace{v_F \to 0}_{\varphi \text{ doesn't propagate}}$

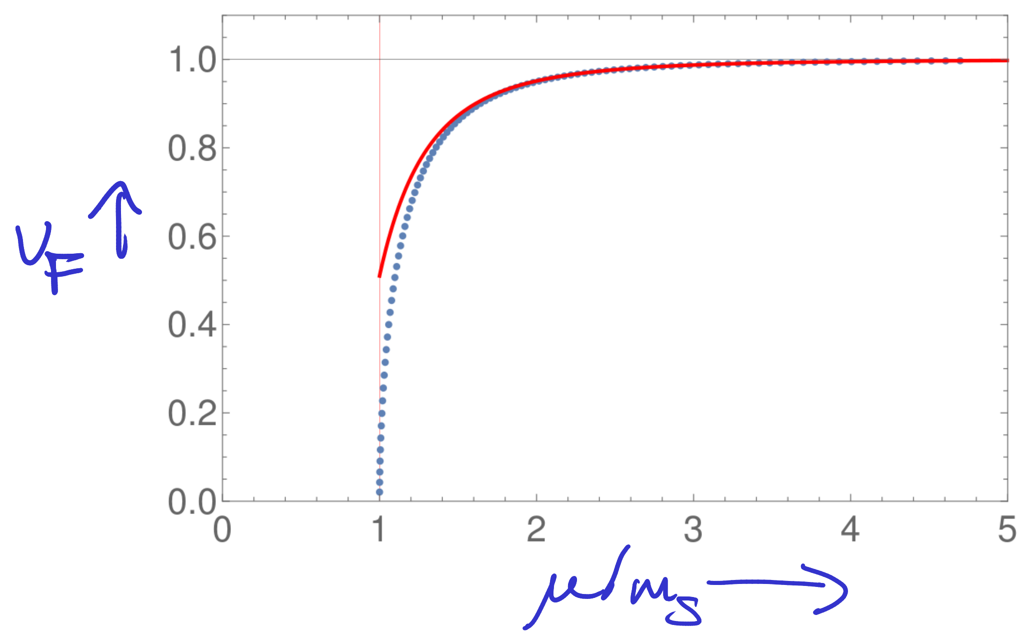
At high $\mu$:

$\mu \gg m_{soliton}$ $\underbrace{K \to \frac{1}{N_c}}_{!}$, $\underbrace{v_F \to 1}_{\text{relativistic propagation}}$

For any $\mu$

Using Thermodynamic Bethe Ansatz, (blue dots) vs pert. theory in mass (red line) $N_c = 3$, $N_f = 1$



# Open questions

1+1 dim.'s: compute for small mass.

$N_f \geq 2$: WZW + Luttinger liquid

Computed $K$, $v_F$ for $N_c = 3$, $N_f = 2$

NJL: Luttinger liquid only for small mass;

at large, "strange" metal (non-Fermi)

QCD: guess non-Fermi $\forall$ masses

Luttinger liquid " "

Previous work: computed spectrum of baryons = solitons

complicated

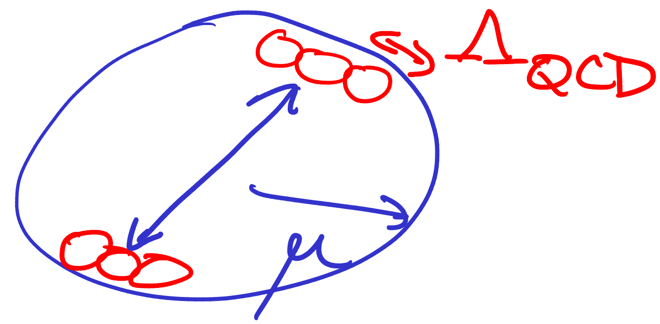
Here: near Fermi surface, much simpler

# Is Quarkyonic matter a non-Fermi liquid?

In 3+1 dim's: nuclear matter Fermi liquid @ low density

In quarkyonic regime? Can reduce to effectively 1-dim

$$\int \frac{d^4k}{(k^2)^2} \underset{\int d^2k_\perp}{\sim} \int \frac{d^2k}{k^2} \quad \text{qualitative}$$




On Fermi sphere, width $\Lambda_{QCD}$:

At large $\mu \sim N_c^{1/4}$, many patches

$\Rightarrow$ effectively 1-dim.

In 3+1 dim's, $\bar{c}$ many patches, perhaps get an effective $\phi$,

$$\partial_z \phi \sim j_0 = \text{baryon current!?}$$