HIGHER TOPOLOGICAL CHARGE EFFECTS IN QCD & AXION COSMOLOGY

Fabian Rennecke

JUSTUS-LIEBIG-UNIVERSITÄT GIESSEN

[Pisarski, FR, 1910.14052]
[FR, 2003.13876]

- LUNCH CLUB SEMINAR-

GIESSEN - 27/04/2022

A SIMPLE MODEL

Consider a linear sigma model to describe $N_f = 2$ QCD at low energies in the chiral limit. Meson field:

$$\phi = (\sigma + i\eta') + (\vec{a}_0 + i\vec{\pi}\,)\,\vec{\tau}$$

Classically, the theory has chiral symmetry $SU(2)_L \times SU(2)_R \times U(1)_A$ and the effective potential $V(\phi)$ is a general function of the chiral invariants

$$\Phi_1 = \mathrm{tr}\,\phi^\dagger\phi\,, \quad \Phi_2 = \mathrm{tr}\,(\phi^\dagger\phi)^2$$

Due to the axial anomaly $U(1)_A \to \mathbb{Z}_{N_f}$ and we can write down one more invariant

$$\xi = \det\phi + \det\phi^\dagger$$

't Hooft determinant

['t Hooft (1976)]

Conventional ansatz for the effective potential: $V(\phi) = \overline{V}(\Phi_1, \Phi_2) \; - c_A\,\xi$

anomalous 2-meson correlation (in general: $2N_f$-quark correlation)

$\longrightarrow$ makes η' heavy

HIGHER ORDER ANOMALOUS CORRELATIONS

This is clearly not the most general effective potential.
Instead, it is of the form

$$V(\phi) = \overline{V}(\Phi_1, \Phi_2, \xi) \supset \xi^{Q=1,2,3,\dots}$$

$\longrightarrow$ anomalous $N_f\, Q$ -meson correlations

Simple example: consider the Lagrangian $\mathscr{L} = \mathscr{L}_{\mathrm{cl}} + \mathscr{L}_{\mathrm{A}}$

$$\mathscr{L}_{\mathrm{cl}} = \mathrm{tr}\left(\partial_\mu \phi^\dagger\right)\left(\partial_\mu \phi\right) + m^2\, \mathrm{tr}\, \phi^\dagger \phi + \lambda_1\, \mathrm{tr}\left(\phi^\dagger \phi\right)^2 + \lambda_2 \left(\mathrm{tr}\, \phi^\dagger \phi\right)^2$$

$$\mathscr{L}_{\mathrm{A}} = -\chi_1\left(\det \phi + \det \phi^\dagger\right) - \chi_2\left[\left(\det \phi\right)^2 + \left(\det \phi^\dagger\right)^2\right]$$

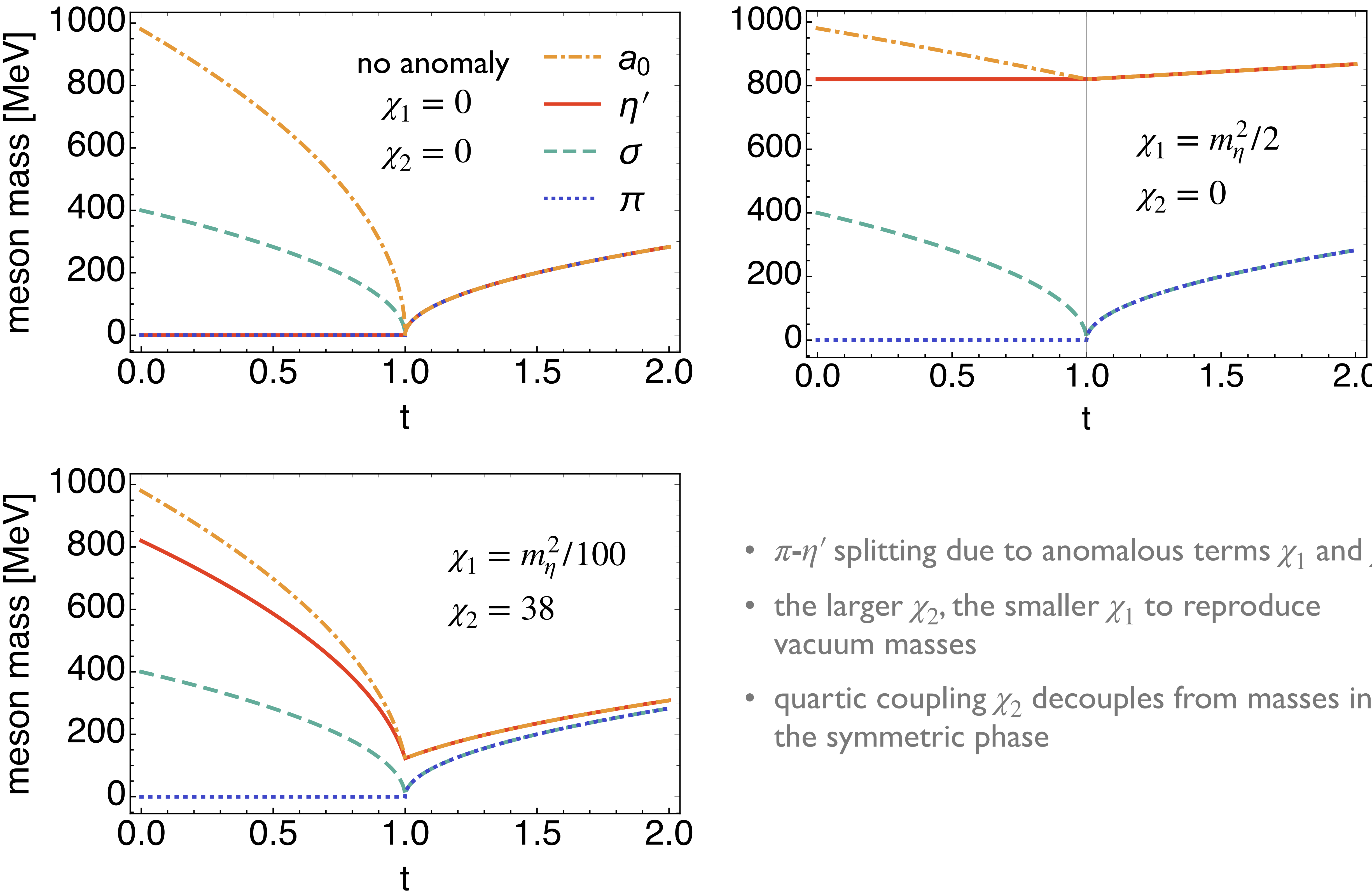
$Q = 1$: quadratic term $\qquad$ $Q = 2$: quartic term

Look at the qualitative mass spectrum in mean-field approximation. Compute masses from $\mathscr{L}$ on the solution of the EoM

$$\frac{\delta \int d^4x\, \mathscr{L}}{\delta \phi} = 0$$

HIGHER ORDER ANOMALOUS CORRELATIONS

How do higher order anomalous couplings affect the mass spectrum?

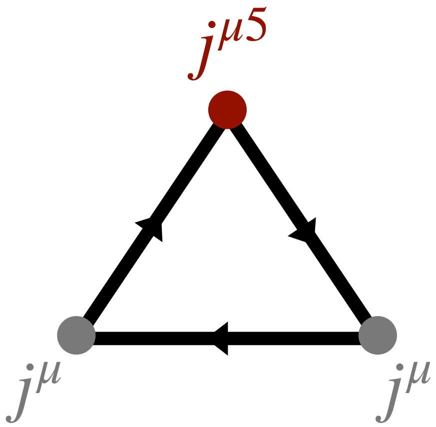


- π-η' splitting due to anomalous terms χ_1 and χ_2
- the larger χ_2, the smaller χ_1 to reproduce vacuum masses
- quartic coupling χ_2 decouples from masses in the symmetric phase

ORIGIN OF ANOMALOUS CORRELATIONS

What is the microscopic origin of these anomalous correlations?

Axial anomaly due to topologically nontrivial fluctuations



[Adler, Bell & Jackiw (1969)]

$\partial_\mu j^{\mu 5} \sim \mathrm{tr}\, F\tilde{F} \sim q =$ topological charge density

axial current $j^{\mu 5} = \bar{\psi}\gamma^\mu\gamma^5\psi$

At weak coupling topological effects are described by **instantons** $A_\mu^{(Q)}$.

Typically, only instantons with topological charge $Q = \pm 1$ are taken into account

- give rise to anomalous $2N_f$-quark correlation function ('t Hooft determinant)

$$\det\left(\bar{\psi}_f\, \mathbb{P}_R\, \psi_g\right) + \det\left(\bar{\psi}_f\, \mathbb{P}_L\, \psi_g\right) \qquad \mathbb{P}_{L/R} = \frac{1 \mp \gamma^5}{2}$$

- distribution of topological charge characterized by θ-dependent free energy

$$F(\theta) \sim \Delta Z_1 \cos\theta$$

OUTLINE

Is there a similar story for higher order anomalous correlations?

What about instantons with higher topological charge? ($|Q| > 1$: multi-instantons)

$\longrightarrow$ topic of this talk

- they give rise to anomalous $2N_f|Q|$-quark correlation functions [Pisarski, FR; 1910.14052]

$$\det\left(\bar{\psi}_f \mathbb{P}_R \psi_g\right)^{|Q|} + \det\left(\bar{\psi}_f \mathbb{P}_L \psi_g\right)^{|Q|}$$

  - outline the derivation

- they yield corrections to the θ-dependence of QCD [FR; 2003.13876]

$$F(\theta) \sim -\sum_{Q=1}^{\infty} \Delta Z_Q \cos(Q\theta)$$

  - study implications for topological susceptibilities
  - explore possible effects on axion dark matter

OUTLINE

Is there a similar story for higher order anomalous correlations?

What about instantons with higher topological charge? ($|Q| > 1$: multi-instantons)

→ topic of this talk

- they ... 14052]

compute in a controlled limit:

semiclassical approximation at large T

- derive effects induced by higher top. charge
- might be small in semiclassical limit, but must be there in general; could be significant at lower energies/temperatures

- they yield corrections to the θ-dependence of QCD [FR; 2003.13876]

$$F(\theta) \sim -\sum_{Q=1}^{\infty} \Delta Z_Q \cos(Q\theta)$$

- study implications for topological susceptibilities
- explore possible effects on axion dark matter

BACKGROUND: INSTANTONS

Minimize the classical action of Yang-Mills theory,

$$S = -\frac{1}{2g^2}\int d^4x \, \mathrm{tr}\, F^2 \,,$$

$$F_{\mu\nu} = [D_\mu, D_\nu]$$
$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$$
$$D_\mu = \partial_\mu + A_\mu$$

requiring that the solution has finite action, one gets (using $\mathrm{tr}\, F^2 = \mathrm{tr}\, \tilde{F}^2$)

$$S = -\frac{1}{4g^2}\int d^4x \Big[\underbrace{\mathrm{tr}\big(F_{\mu\nu} \mp \tilde{F}_{\mu\nu}\big)^2}_{\geq 0} \pm 2\, \mathrm{tr}\, F\tilde{F}\Big] \geq -\frac{1}{2g^2}\left|\int d^4x \, \mathrm{tr}\, F\tilde{F}\right|$$

⟶ action minimized by (anti) selfdual gauge fields $F = \pm\tilde{F}$ $\left(S_{\min} = -\frac{1}{2g^2}\int d^4x \, \mathrm{tr}\, F\tilde{F}\right)$

⟶ they always solve classical EoM $D_\mu F^{\mu\nu} = 0$ (Bianchi identity $D_\mu \tilde{F}^{\mu\nu} = 0$)

Relevant example for such solutions: **instantons** $A_\mu^{(Q)}$

For topological charge $Q = 1$:

$$\sigma^\mu = (-i, \vec{\sigma})^\mu$$
$$\bar{\sigma}^\mu = (i, \vec{\sigma})^\mu$$
$$\bar{\sigma}^{\mu\nu} = \frac{1}{2}\left(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu\right)$$

$$A_\mu^{(1)}(x) = U_1\, \bar{\sigma}^{\mu\nu}\, U_1^\dagger \frac{\rho_1^2}{(x-z_1)^2}\frac{(x-z_1)_\nu}{(x-z_1)^2+\rho_1^2}$$

[Belavin, Polyakov, Schwartz, Tyupkin (1975)]

BACKGROUND: TOPOLOGY & INSTANTONS

For the action to be finite, F has to vanish for $|x| \to \infty$

→ there is a large sphere S^3_∞ with $F\Big|_{S^3_\infty} = 0$

$F = 0$ implies that the gluon is "pure gauge"

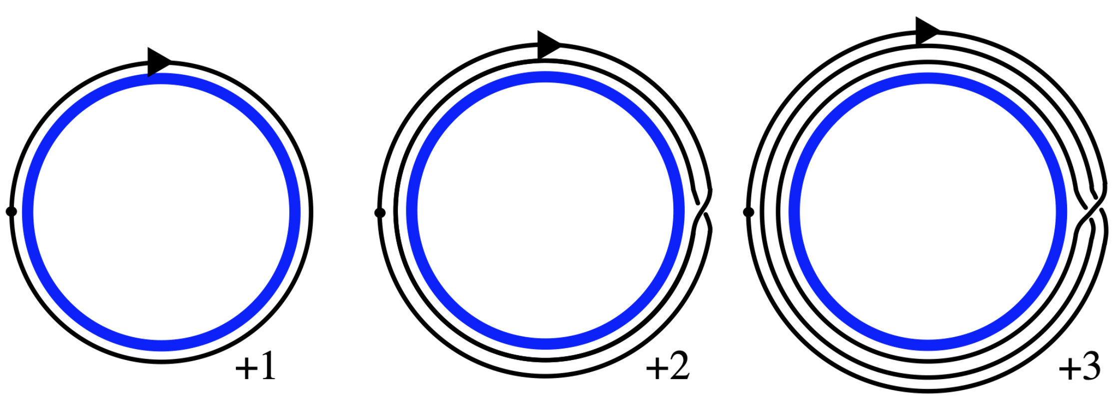
$$A_\mu\Big|_{S^3_\infty} = U(x)^\dagger \partial_\mu U(x)\Big|_{S^3_\infty}$$

$U \in SU(N_c)$

→ gauge field defines map $S^3 \to SU(N_c)$ which allows for a topological classification: homotopy class

$$\pi_3\big(SU(N_c)\big) = \mathbb{Z}$$

winding number



$\pi_1\big(U(1)\big) = \mathbb{Z}$ [wikipedia.org/wiki/Homotopy_groups_of_spheres]

"winding number" of gluons:

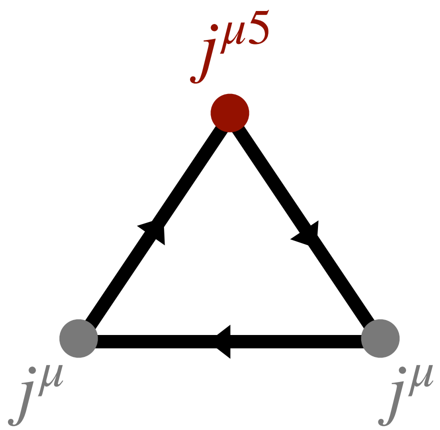
topological charge (2nd Chern number)

$$Q = -\frac{1}{16\pi^2}\int d^4x \,\mathrm{tr}\, F\tilde{F} \;\in \mathbb{Z}$$

topological charge density: $q = -\dfrac{1}{16\pi^2}\,\mathrm{tr}\, F\tilde{F}$

BACKGROUND: AXIAL ANOMALY

Anomaly of the axial current $j^{\mu 5} = \bar{\psi}\gamma^{\mu}\gamma^{5}\psi$:



$\partial_{\mu} j^{\mu 5} = 2 i N_f \, q$

[Adler, Bell & Jackiw (1969)]

$\longrightarrow$ non-conservation of axial charge:

$$\Delta Q_5 = Q_5(t = +\infty) - Q_5(t = -\infty) = 2 N_f \, Q$$

$\longrightarrow$ $U(1)_A$ anomaly due to topological field configurations

Quarks on a topological background acquire zero modes with net chirality (index theorem)

$$\gamma_{\mu}\left(\partial_{\mu} + A_{\mu}^{(Q)}\right)\psi^{(Q)} = 0 \qquad\qquad N_f \, Q = n_L - n_R$$

# of left- and right-handed quark zero modes

solution for $Q = 1$: ['t Hooft (1976)]

$$\psi^{(1)}(x) = \nu \, \frac{U_i \, \rho_i}{\left[(x - z_i)^2 + \rho_i^2\right]^{3/2}} \, \frac{\gamma_{\mu}\,(x - z_i)_{\mu}}{|\, x - z_i \,|} \, \varphi_R \quad \xrightarrow{|x - z_i| \gg \rho_i} \quad \sim \rho_i \, U_i \, \Delta(x - z_i) \, \varphi_R$$

RH spinor

free quark propagator!

PARTITION FUNCTION IN A MULTI-INSTANTON BACKGROUND

- get the ingredients: instantons and the corresponding quark zero modes
- compute semi-classically: small fluctuations around instanton background

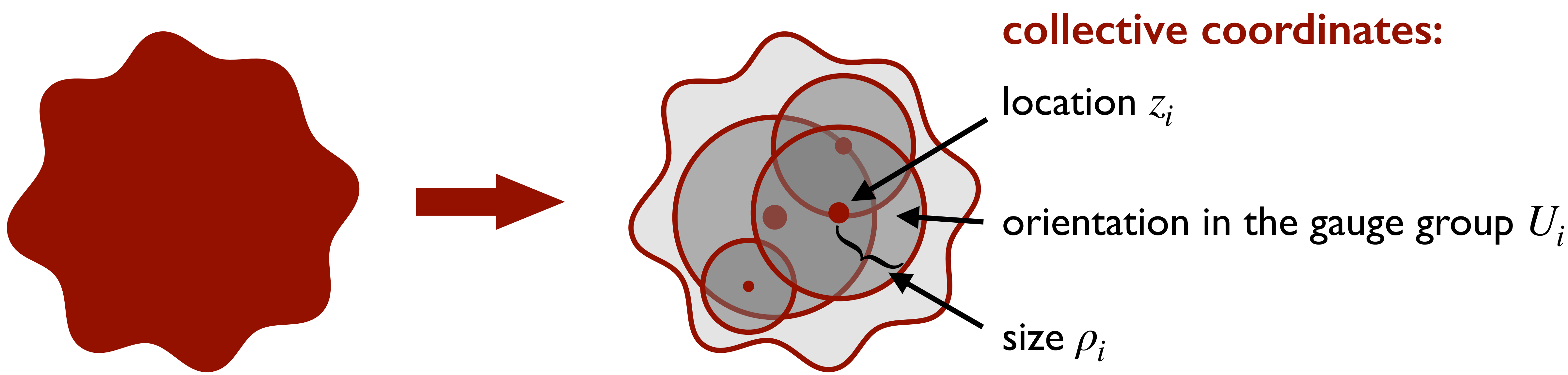
CONSTRUCTION OF INSTANTONS

General construction of instantons with arbitrary topological charge: **ADHM**

[Atiyah, Drinfeld, Hitchin, Manin (1978)]

- reduces classical self-dual YM equations to a set of nonlinear algebraic equations
- still, explicit solutions for larger Q are unknown
- use approximate solutions here

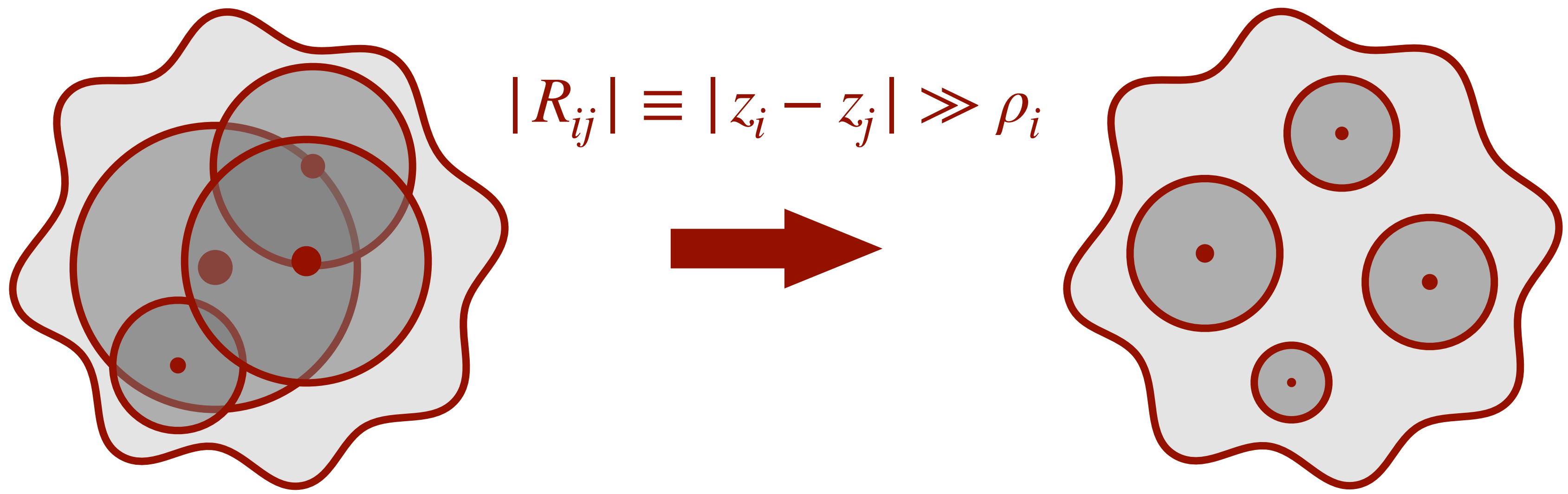
To this end, exploit that Q-instantons can be viewed as composition of constituent-instantons with $Q = 1$



- $4N_c|Q|$ collective coordinates describe a Q-instanton
- arise from symmetries that yield inequivalent instanton solutions

CONSTRUCTION OF INSTANTONS

Solve ADHM by expanding in the limit of small constituent-instantons (SCI)



Results to order $\rho^4/|R|^4$:

- Q-instanton: $A_\mu^{(Q)}(x) = \dfrac{1}{\xi_0(x, \{z_i, \rho_i\})} \displaystyle\sum_{i=1}^{Q} A_\mu^{(1)}(x; z_i, \rho_i, U_i) + \mathcal{O}\left(\dfrac{\rho^4}{|R|^4}\right)$

[Christ, Weinberg, Stanton (1978)]

- $N_f\, Q$ quark zero modes: $(\, f = 1,\ldots, N_f\,,\ i = 1,\ldots, Q\,)$

$$\psi_{fi}^{(Q)}(x) = \psi_{fi}^{(1)}(x, z_i\,, \rho_i, U_i) - \sum_{j \neq i} \mathbb{X}_{ij}(x, z_i\,, \rho_i, \rho_j)\, \psi_{fj}^{(1)}(x, z_j\,, \rho_j, U_i) + \mathcal{O}\left(\frac{\rho^4}{|R|^4}\right)$$

[Pisarski, FR]

$\sim \rho_i\, U_i\, \Delta(x - z_i)\, \varphi_R$ for $|x - z_i| \gg \rho_i$

free quark propagator!

PARTITION FUNCTION

Partition function in a Q-instanton background

$$Z_Q[J] = \int \mathscr{D}\Phi \exp\left\{ - S[\Phi + \bar{\Phi}^{(Q)}] + \int_x \bar{\psi} J \psi \right\}$$

$\Phi = (A, c, \bar{c}, \psi, \bar{\psi})$

$\bar{\Phi}^{(Q)} = (A^{(Q)}, 0, 0, 0, 0)$

source for quark-antiquark pairs (e.g. mass term)

- consider small fluctuations around topological background $A_\mu = A_\mu^{(Q)}$
- collective coordinates correspond to symmetries: resulting gauge field zero modes need to be treated exactly
- replace integral over zero modes by integral over collective coordinates:

$$Z_Q[J] = \int \left[N \prod_{i=1}^{Q} d^4 z_i \, d\rho_i \, dU_i \right] n_Q\left(\{z_i, \rho_i, U_i\}\right) \det{}_0(J)$$

Q-instanton density

- gluon and ghost determinant
- quark determinant over nonzero modes
- Jacobian of coordinate change from zero modes to collective coordinates

quark zero mode determinant

$$\det \int d^4x \, \psi^{(Q)\dagger}(x) \, J(x) \, \psi^{(Q)}(x)$$

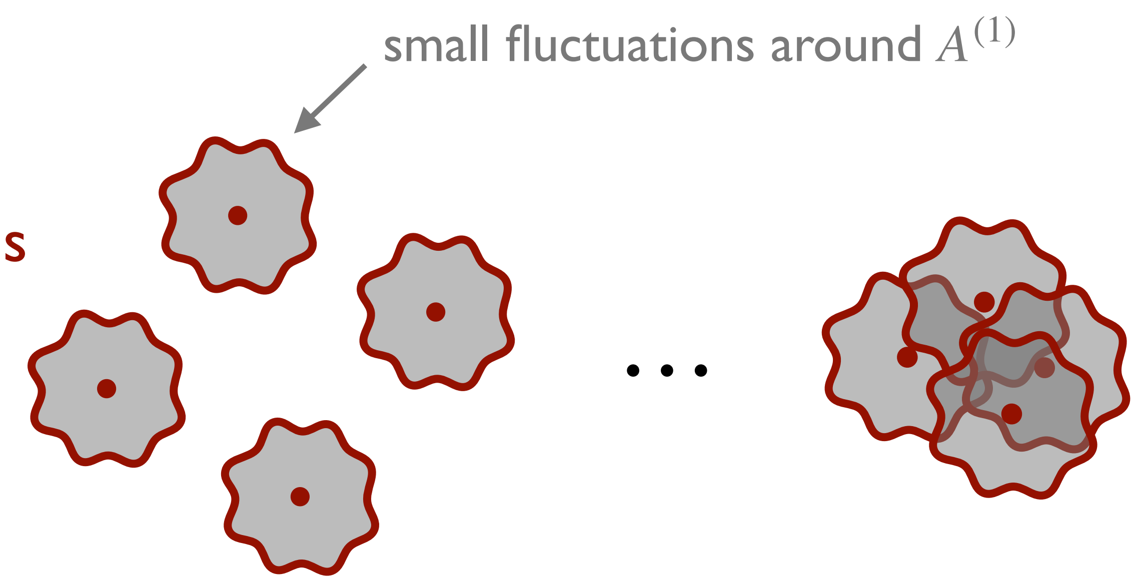
PARTITION FUNCTION IN THE SCI LIMIT

Qualitatively different contributions to Z_Q in the SCI limit due to integration over constituent-instanton locations z_i :

constituents far apart:
~ dilute single-instantons

$$Z_Q^{(1)} = \frac{1}{Q!} Z_1^Q$$

parametrically: $\mathscr{V}^Q$



constituents close together:
genuine multi-instanton contributions

$$\Delta Z_Q$$

parametrically: $\mathscr{V}^1$

Various nonlocal contributions of q-instantons ($q < Q$) and one local Q-instanton contribution:

$$Z_Q = \sum_{q=1}^{Q-1} Z_Q^{(q)} + \Delta Z_Q \longrightarrow$$

compute only ΔZ_Q
for local contributions

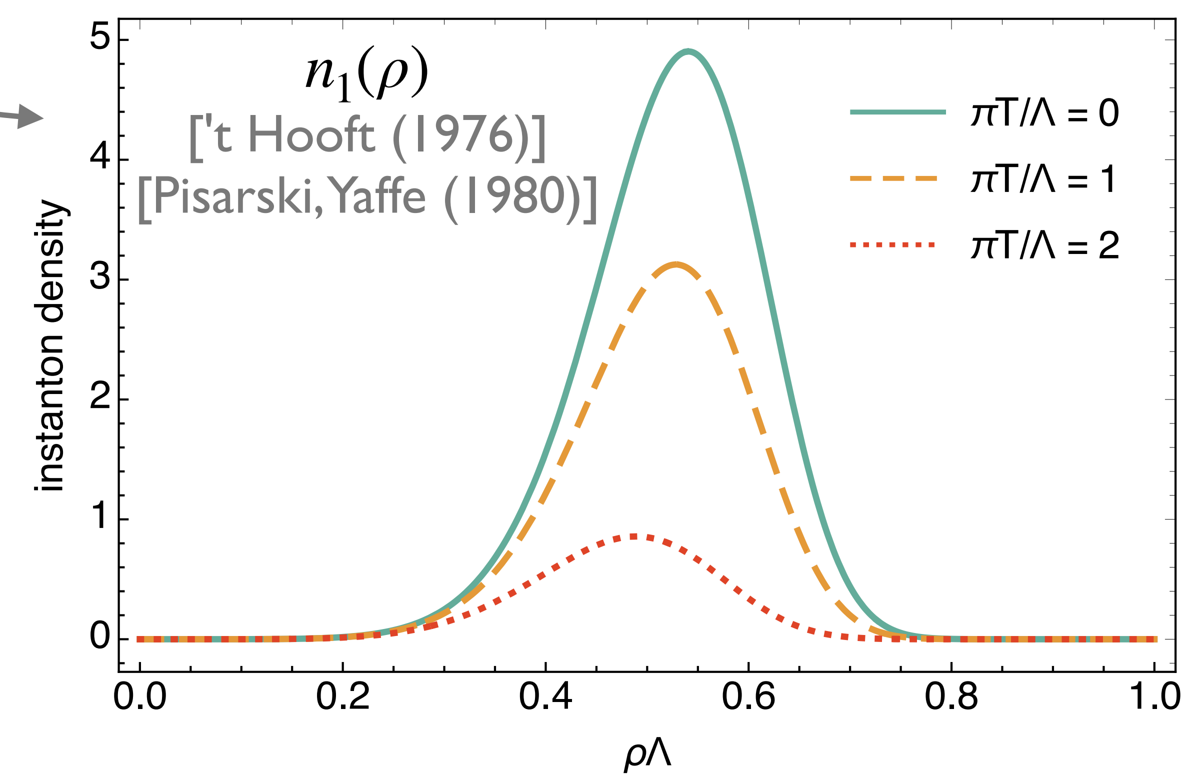
PARTITION FUNCTION IN THE SCI LIMIT

Consider only the contribution of the quark determinant to ΔZ_Q. Then:
(gauge part: accurate to order $\rho^4/|R|^4$ [Brown, Creamer & Bernard (1978)])

$$\Delta Z_Q[J] = \frac{1}{Q!}\int d^4z \left[\prod_{i=1}^{Q} d\rho_i\, n_1(\rho_i)\, \det{}_0^{(1)}(J)\right] I_Q\big(\{\rho_i\}\big) \qquad I_Q(\rho_1,\rho_2) = c_{N_f}\sum_{i=1}^{Q}\prod_{i\neq j}\rho_i^{2N_f}\rho_j^{4-2N_f}$$

Gory details in [FR, 2003.13876].
Intuitive picture:

- instanton density peaks about certain size: effective instanton size $\bar\rho$
- "geometric probability" for overlap of Q constituent-instantons: $\sim (\bar\rho^4/\mathscr{V})^{Q-1}$



geometric overlap

$Z_1 = \int d^4z\, \bar Z_1$

renormalization scale

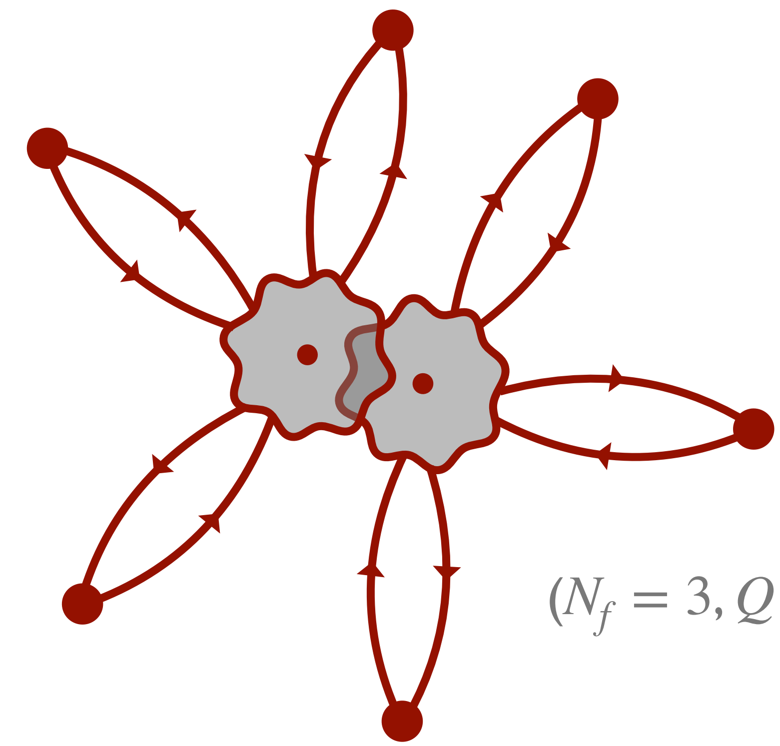
$$\Delta Z_Q \approx \mathscr{V}^Q\left(\frac{\bar c_{N_f}\,\bar\rho^4}{\mathscr{V}}\right)^{Q-1}\frac{\bar Z_1^Q}{(Q-1)!} \approx \mathscr{V}\,\frac{\Lambda^4}{(Q-1)!}\,\hat Z_1^Q \qquad \hat Z_1 = \bar Z_1/\Lambda^4$$

from integration over all constituent positions

from integration over average multi-instanton position

EFFECTIVE PARTITION FUNCTION

Use the large-distance form of the quark zero modes:



$(N_f = 3, Q = 2)$

$$\det{}_0(J) \sim \prod_i \prod_f \int d^4x_{fi}\, \Delta(z_i - x_{fi})\, J(x_{fi})\, \Delta(x_{fi} - z_i)$$

ΔZ_Q is identical to the effective partition function
(details in [Pisarski, FR, 1910.14052])

no instanton background!

$$\Delta Z^{\text{eff}}_{+Q}[\bar{J}] = \int \mathscr{D}\Phi\, e^{-S[\Phi] + \int_x \bar{\psi}\bar{J}\psi}\, \Delta S^{\text{eff}}_{+Q}$$

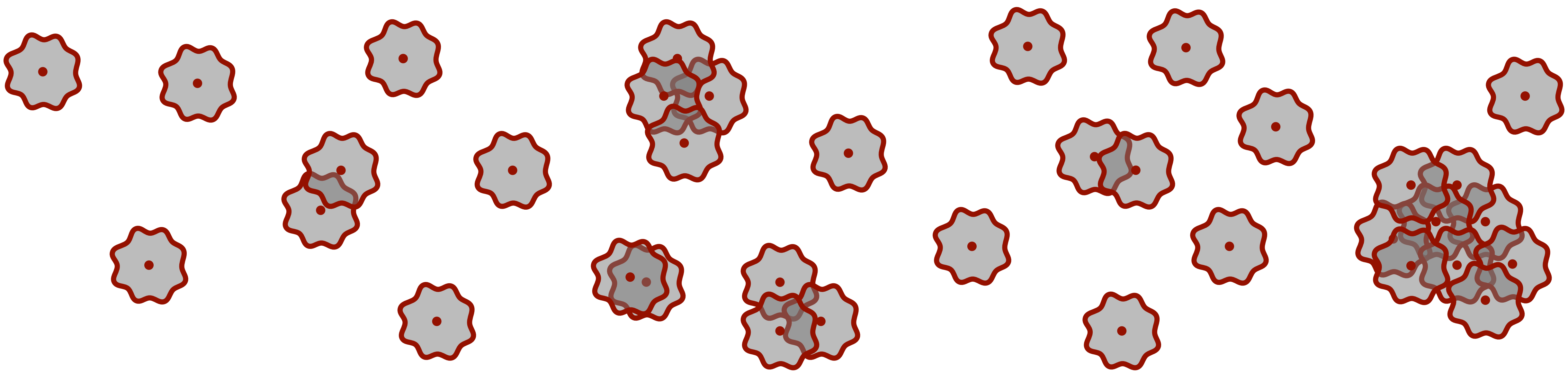
$$\Delta S^{\text{eff}}_{+Q}\Big|_{\text{singlet}} \sim \int d^4z\, \kappa_Q \det_{fg}\left[\bar{\psi}_f(z)\, \mathbb{P}_R\, \psi_g(z)\right]^Q$$

- local $2N_f\, Q$-quark correlation function
- for anti-instantons ($Q < 0$): $\mathbb{P}_R \to \mathbb{P}_L$

DILUTE MULTI-INSTANTON GAS

- so far: partition function with one Q-instanton in the background
- but all possible gluon configurations contribute to the path integral
- assume that topological fluctuations are described by a dilute instanton gas (DIGA)
- reasonable at large enough temperature due to thermal screening of instanton density: (constituent-) instantons are small at large T, $\bar{\rho} \ll 1/(\pi T)$ [Gross, Pisarski, Yaffe (1981)]

Since the path integral involves integrations over all instanton locations, there are genuine multi-instanton corrections to the DIGA



$$\longrightarrow \quad \mathcal{Z} = \prod_{Q>0} \sum_{n_Q} \sum_{\bar{n}_Q} \frac{1}{n_Q!\,\bar{n}_Q!} \Delta Z_{+Q}^{n_Q} \Delta Z_{-Q}^{\bar{n}_Q} = e^{\sum_{Q>0}\left(\Delta Z_{+Q}+\Delta Z_{-Q}\right)}$$

ANOMALOUS QUARK CORRELATIONS

$\Delta S_Q^{\mathrm{eff}}$ is exponentiated in the dilute gas:

$$\mathcal{Z}^{\mathrm{eff}} = \int \mathcal{D}\Phi\, e^{-S[\Phi] + \sum_{Q>0}\left(\Delta S_{+Q}^{\mathrm{eff}} + \Delta S_{-Q}^{\mathrm{eff}}\right)} \longrightarrow$$

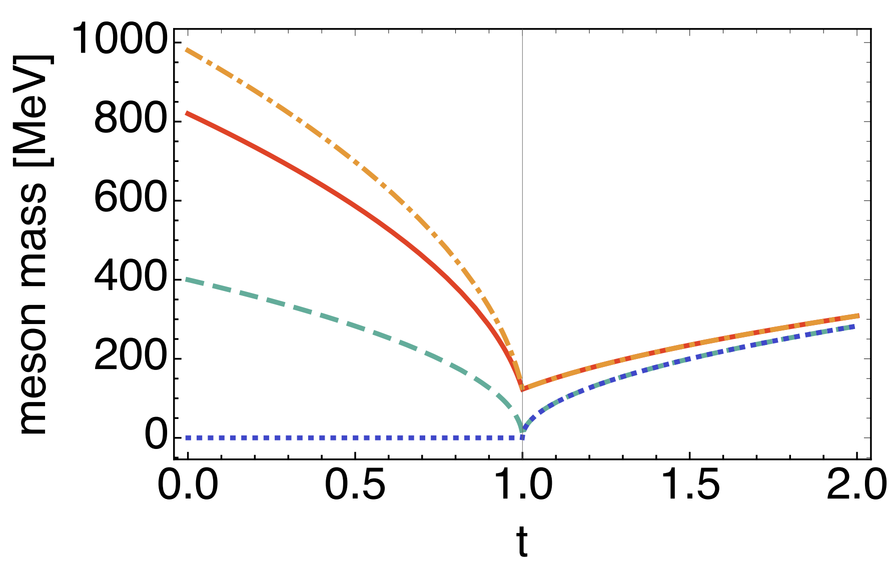
$\det\left(\bar{\psi}_f\, \mathbb{P}_{R/L}\, \psi_g\right)^{|Q|}$ terms in the effective action

$\longrightarrow$ **anomalous $2N_f|Q|$-quark correlation functions**

Bosonization of $Q = 1, 2$ terms lead to LSM from the beginning!

Axial anomaly is also encoded in higher order correlation functions.
Their microscopic origin is instantons with higher topological charge.

Use this to look for new signatures of the anomaly in quark/hadron correlations.



- $U(1)_A$ restoration in **all** anomalous correlations
- $N_f = 1, Q = 2$: anomalous meson mass
- $N_f = 2, Q = 2$: anomalous tetraquark mass?
- ...

θ-DEPENDENCE AND TOPOLOGICAL SUSCEPTIBILITIES

Θ-DEPENDENCE FROM DILUTE INSTANTONS

Topological structure of QCD necessitates the existence of a topological Θ-term:

[Jackiw, Rebbi & Callan, Dashen, Gross (1976)]

free parameter

$$\mathscr{L} = \bar{\psi}\,\gamma_\mu D_\mu \psi + \frac{1}{2}\mathrm{tr}\,FF + \frac{i\theta}{16\pi^2}\,\mathrm{tr}\,F\tilde{F}$$

θ-dependence in a dilute multi-instanton gas via simple substitution:

$$\Delta Z_Q \longrightarrow \Delta Z_Q\, e^{iQ\theta}$$

Resulting **free energy density**

$$F(\theta) = -\frac{1}{\mathscr{V}}\ln \mathscr{Z}(\theta) = -\frac{2}{\mathscr{V}}\sum_Q \Delta Z_Q \cos(Q\theta)$$

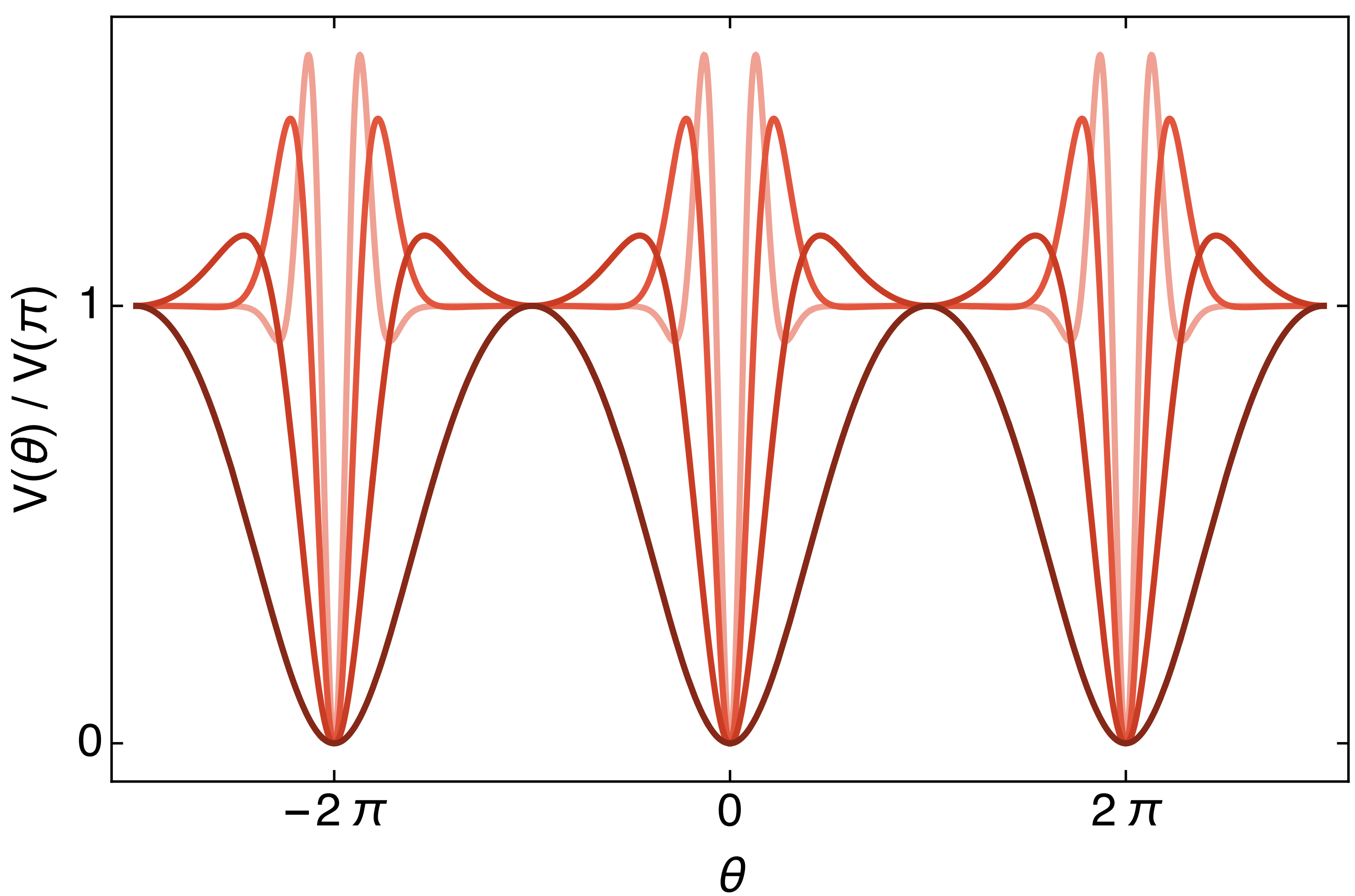
→ describes the distribution of topological charge

Θ-DEPENDENCE FROM DILUTE INSTANTONS

Use simple solution to compute the normalized free energy $\Delta F(\theta) = F(\theta) - F(0)$

$$\Delta F(\theta) = 2\Lambda^4 \sum_Q \frac{\hat{Z}_1^Q}{(Q-1)!}\left[1-\cos(Q\theta)\right] = 2\Lambda^4\,\hat{Z}_1\left[e^{\hat{Z}_1} - \cos\left(\hat{Z}_1 \sin\theta\right) e^{\hat{Z}_1\cos\theta}\right]$$

- recover well-known single-instanton result for $\hat{Z}_1 \ll 1$: $\Delta F(\theta) = 2\Lambda^4 \hat{Z}_1 \left(1-\cos\theta\right) + \mathcal{O}(\hat{Z}_1^2)$
- multi-instantons modify the simple $\cos\theta$ - behavior!



$\Delta F(\theta)/\Delta F(\pi)$ for
$\hat{Z}_1 = 0.1, 1, 3, 6$
(from darkest to lightest red)

TOPOLOGICAL SUSCEPTIBILITIES

$\Delta F(\theta)$ describes the distribution of topological charge in QCD

topological susceptibilities: $$\chi_{2n} = \frac{\partial^{2n} \Delta F(\theta)}{\partial \theta^{2n}}\bigg|_{\theta=0} \sim \langle Q^{2n} \rangle_c$$

dilute single-instantons gas:

$$\chi_{2n}\Big|_{Q=1} = 2(-1)^{n+1}\,\bar{Z}_1 = (-1)^{n+1}\,\chi_2$$

dilute multi-instanton gas:

$$\chi_{2n} = \frac{2(-1)^{n+1}}{\mathscr{V}} \sum_Q Q^{2n}\,\Delta Z_Q$$

$\Delta Z_Q > 0$ $\longrightarrow$ enhanced topological correlations from multi-instanton corrections

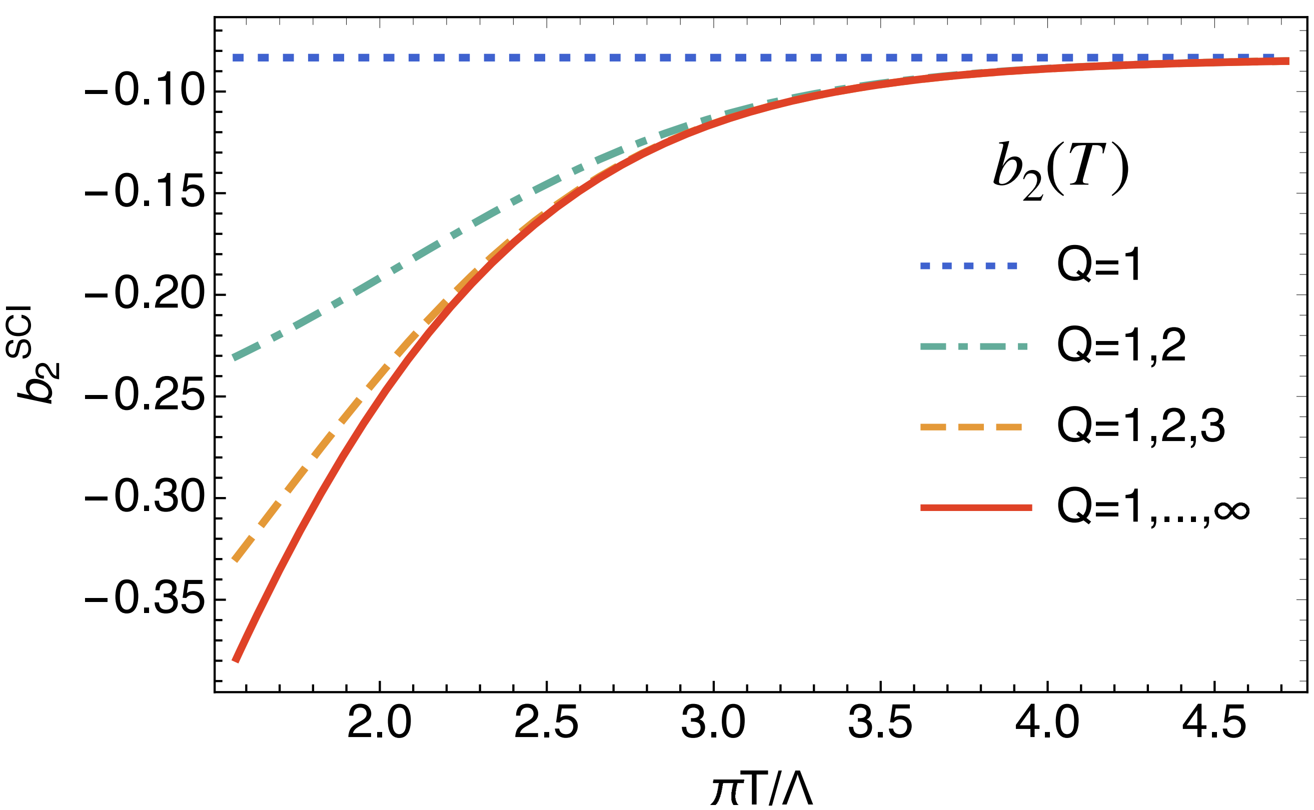
TOPOLOGICAL SUSCEPTIBILITIES

Deviation of higher susceptibilities from χ_2: anharmonicity coefficients

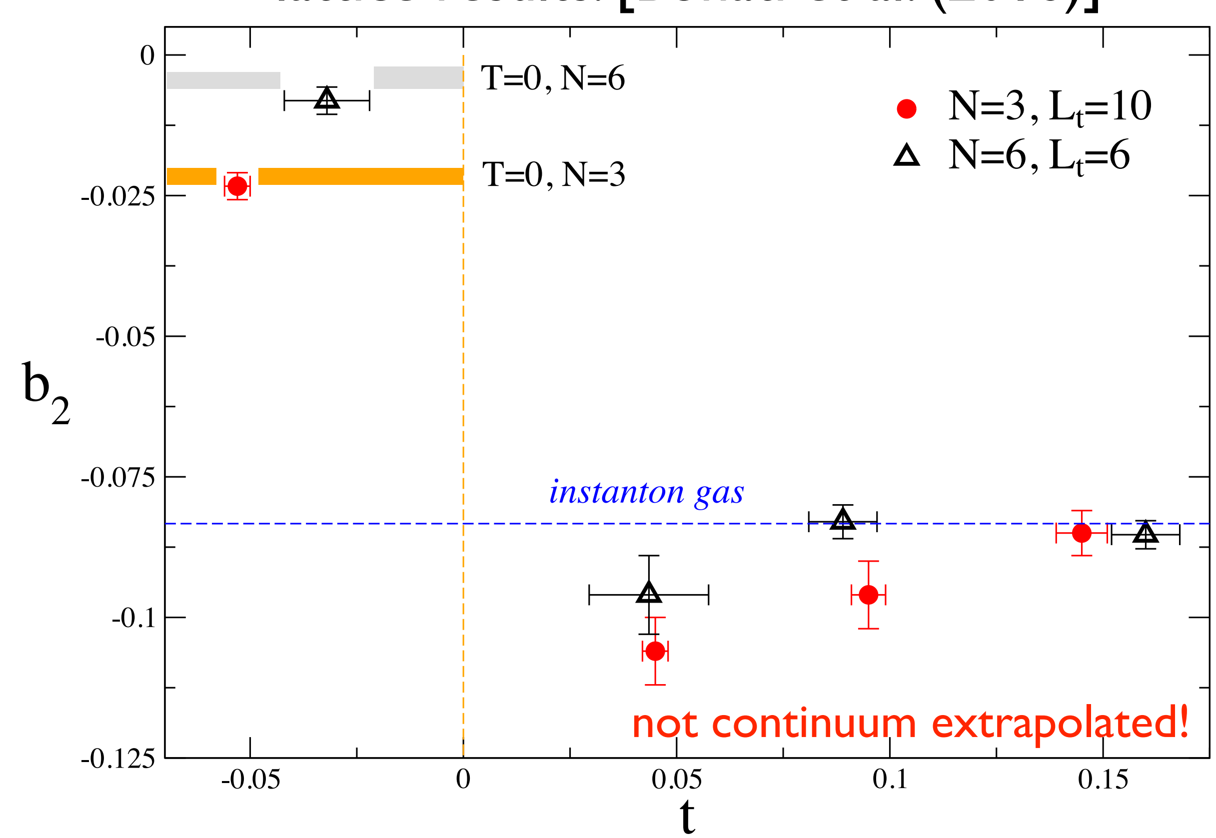
$$b_{2n} = \frac{2}{(2n+2)!}\frac{\chi_{2n+2}}{\chi_2}$$

Multi-instantons yield T-dependent corrections to constant single-instanton prediction!

Result for **quenched QCD** at large temperatures (no dynamical quarks, $m_q \sim 1/\bar{\rho}$)



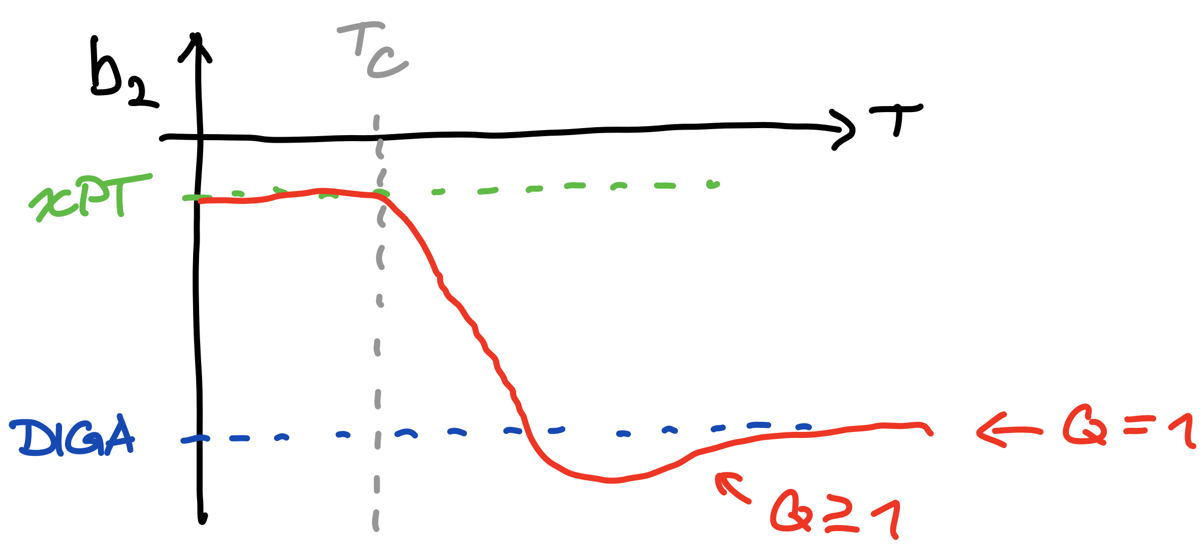
lattice results: [Bonati et al. (2016)]

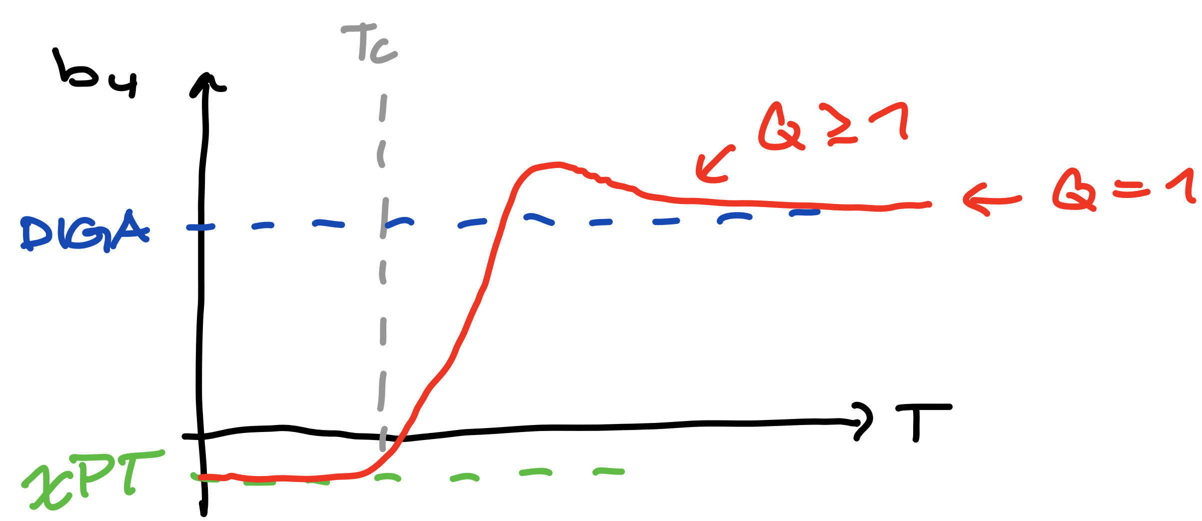


TOPOLOGICAL SUSCEPTIBILITIES

prediction for the global structure of anharmonicity coefficients

- $Q = 1$ behavior at asymptotically high T
- chiral perturbation theory well below T_c [Grilli di Cortana et al. (2015)]
- intermediate $Q \geq 1$ regime above T_c
- ??? around T_c (instanton/dyon liquid, bion condensation, ... [Diakonov, Shuryak, Ünsal ...])





AXION COSMOLOGY

THE STRONG CP PROBLEM

Topological effects give rise to CP violation in QCD

topological θ parameter violates CP for $\theta \neq 0, \pi$

$$\mathscr{L} = \bar{\psi}\gamma_\mu D_\mu \psi + \frac{1}{2}\mathrm{tr}\, FF + \frac{i\theta}{16\pi^2}\,\mathrm{tr}\, F\tilde{F}$$

neutron electric dipole moment

$$d_n \sim e\theta \frac{m_u m_d}{f_\pi^2(m_u + m_d)}$$

[Crewther, Di Vecchia, Veneziano, Witten (1979)]

Most recent measurements yield $|d_n| < 1.8 \times 10^{-26}\, e$cm $\longrightarrow$ $\theta \lesssim 10^{-10}$

[Abel et al. (nEDM) (2020)]

Strong CP problem:

why is $\theta \approx 0$?

PECCEI-QUINN MECHANISM

Possible resolution of the CP problem:

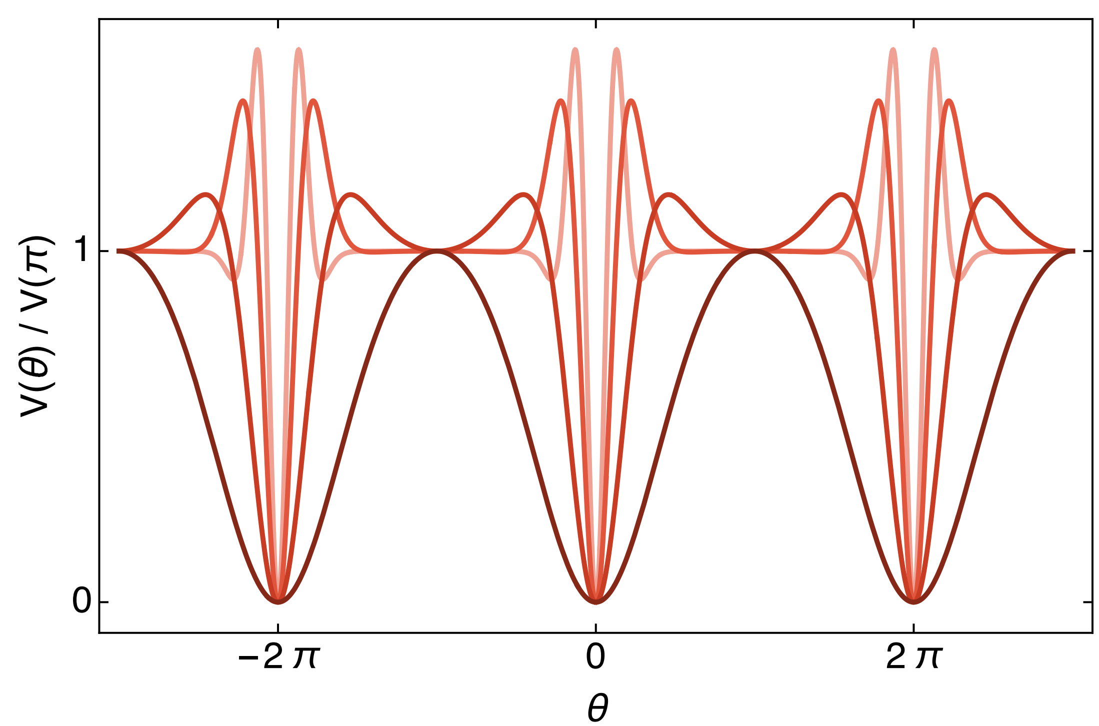
- augment SM with global axial $U(1)_{\rm PQ}$ + charged scalar that couples to quarks
- $U(1)_{\rm PQ}$ is spontaneously broken at scale f_a, resulting Goldstone boson: **axion**
- the axial anomaly in $U(1)_{\rm PQ}$ dictates non-derivative couplings of the axion:

$$\mathscr{L} = \bar{\psi}\,\gamma_\mu D_\mu \psi + \frac{1}{2}\mathrm{tr}\,FF + \frac{i\theta}{16\pi^2}\,\mathrm{tr}\,F\tilde{F} + \frac{a(x)}{f_a}\frac{i}{16\pi^2}\,\mathrm{tr}\,F\tilde{F} + \ldots$$

$\longrightarrow$ 'dynamical' θ-angle $\bar{\theta}(x) = f_a\theta + a(x)$

axion effective potential is given by $F(\theta)$!

$V(\bar{\theta}/f_a) \equiv \Delta F(\bar{\theta}/f_a)$



minimum at $\bar{\theta} = 0$:
CP problem resolved!

AXION DARK MATTER

Axions are also viable dark-matter candidates.

- compatible with observations if light and weakly coupled to SM $(f_a \gtrsim 10^9\,\mathrm{GeV})$

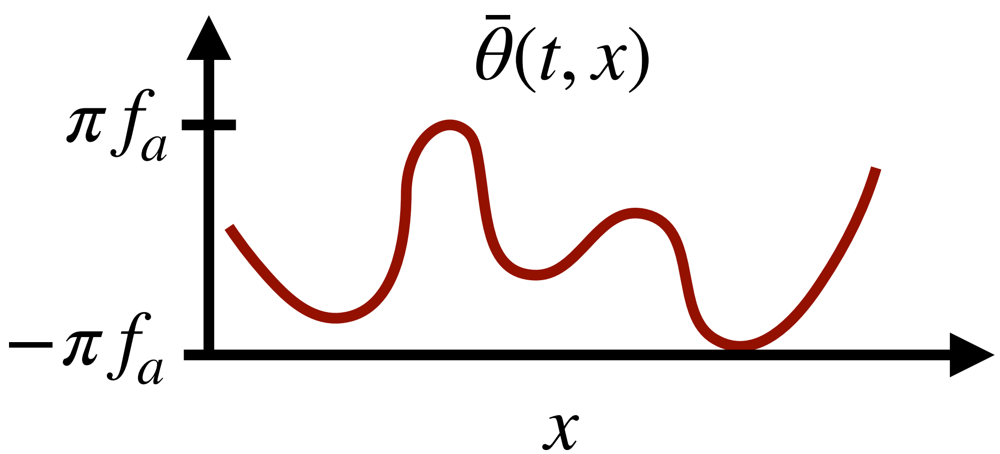
  [Kim & Shifman, Vainshtein, Zakharov & Zhitnitsky & Dine, Fischler, Srednicki (1979+)]

- production of axion dark matter through the **vacuum realignment mechanism**

  [Preskill, Wise, Wilczek & Abbott, Sikivie & Dine, Fischler (1983)]

Assume $U(1)_{\mathrm{PQ}}$ is broken before inflation

axion is very light - fluctuates strongly



AXION DARK MATTER

Axions are also viable dark-matter candidates.

- compatible with observations if light and weakly coupled to SM $(f_a \gtrsim 10^9\,\mathrm{GeV})$

  [Kim & Shifman, Vainshtein, Zakharov & Zhitnitsky & Dine, Fischler, Srednicki (1979+)]

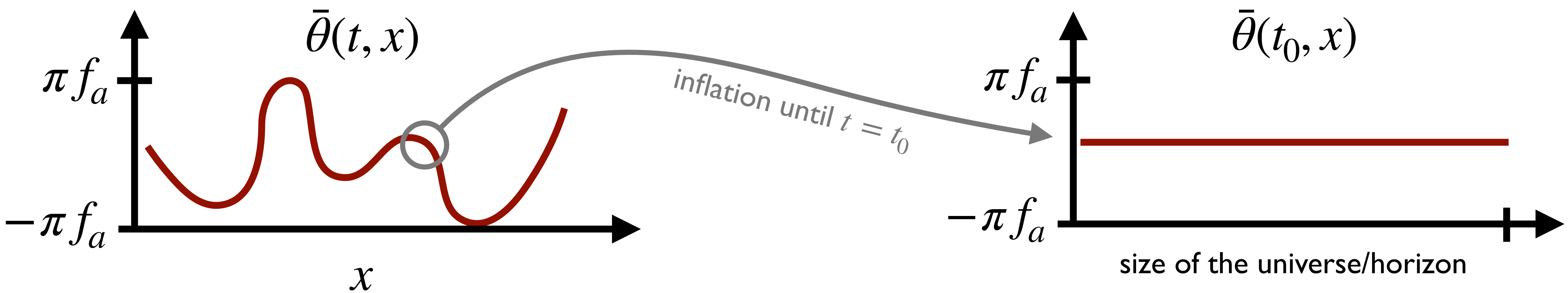
- production of axion dark matter through the **vacuum realignment mechanism**

  [Preskill, Wise, Wilczek & Abbott, Sikivie & Dine, Fischler (1983)]

Assume $U(1)_{\mathrm{PQ}}$ is broken before inflation

axion is very light - fluctuates strongly

homogeneous axion field after inflation



→ vacuum realignment: time evolution of axion dark matter from initial misalignment $\bar{\theta}(t_0)$ until today

AXION DARK MATTER

Time evolution of the axion governed by Einstein's equations.

- assume isotropic, homogenous, expanding & flat universe: Friedmann equations

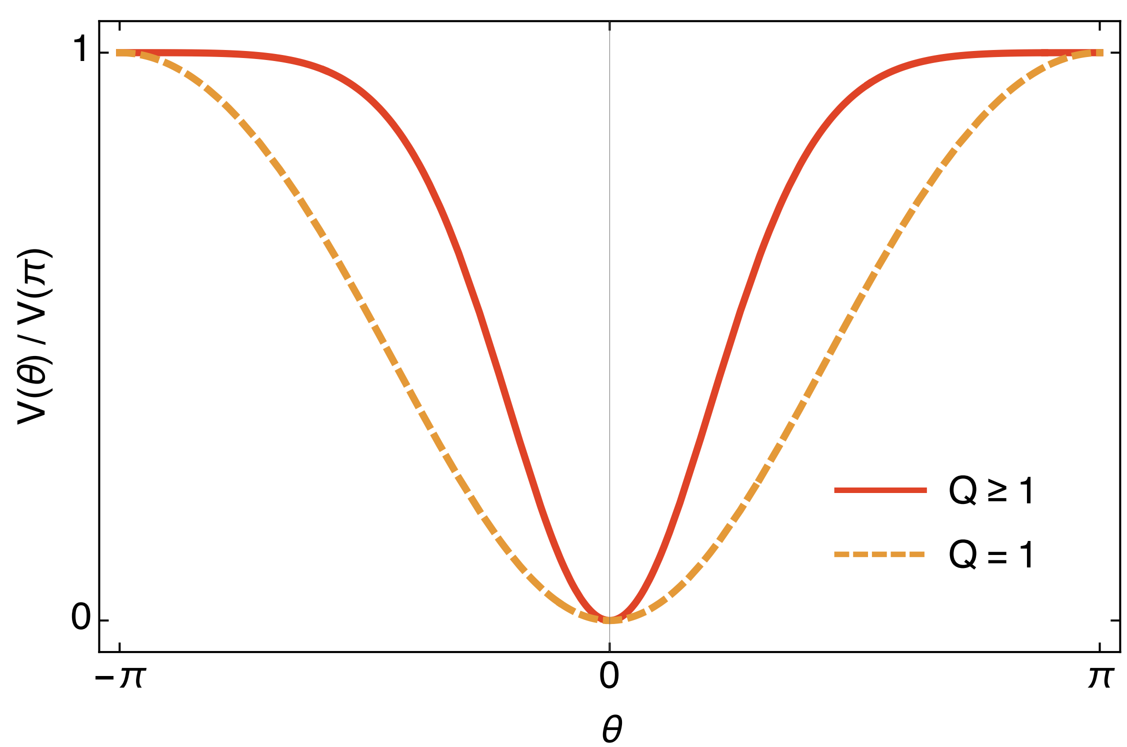
$$\frac{d^2\bar{\theta}}{dt^2} + 3H\frac{d\bar{\theta}}{dt} + \frac{dV(\bar{\theta}/f_a)}{d\bar{\theta}} = 0$$

metric: $g_{\mu\nu} = \mathrm{diag}\big(1, -a^2(t), -a^2(t), -a^2(t)\big)$

Hubble parameter: $H(t) = \frac{\dot{a}(t)}{a(t)}$

Resembles a damped harmonic oscillator

- early t: $\frac{H}{|V''|} \gg 1$: overdamping - axion is frozen in time
- late t: $\frac{H}{|V''|} \ll 1$: underdamping - oscillation around minimum



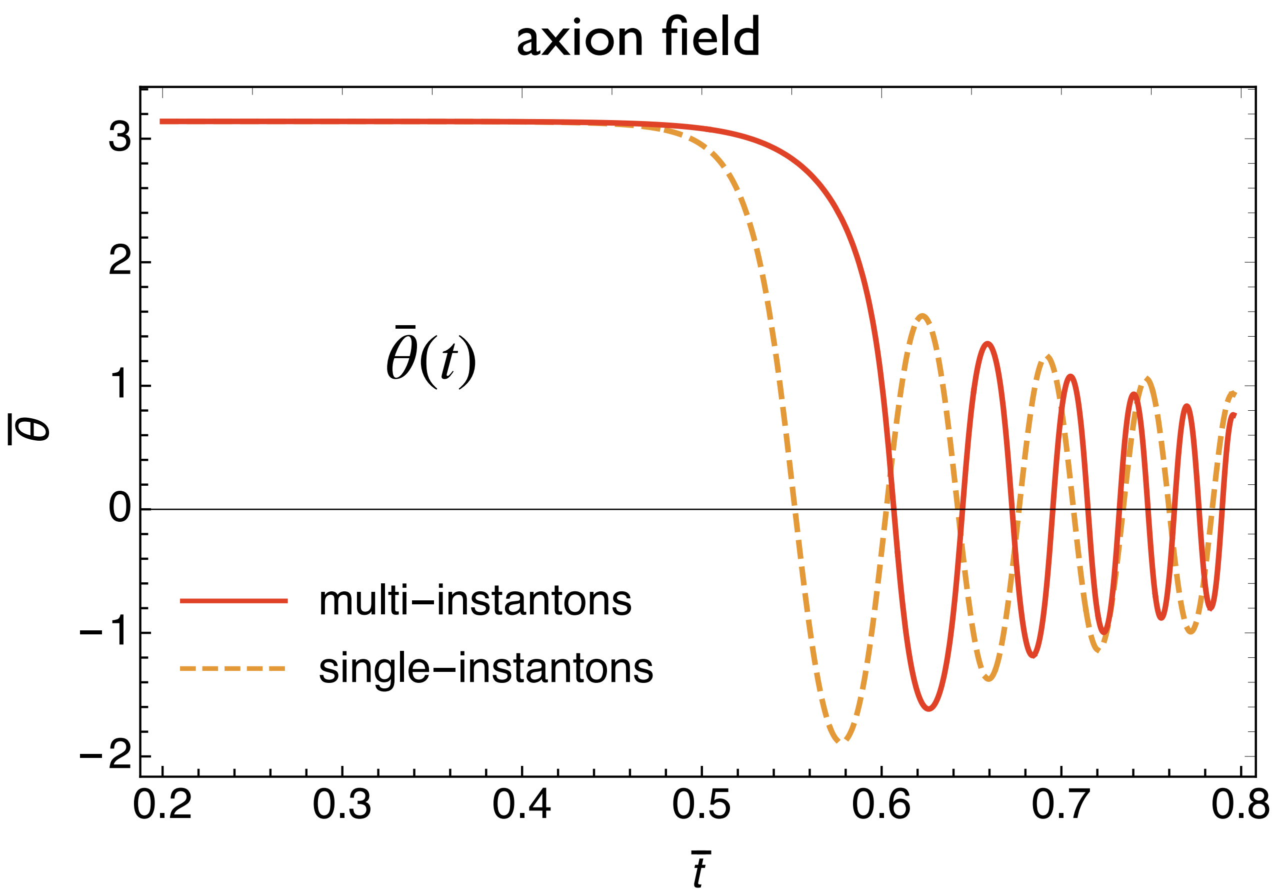
axion DM energy density

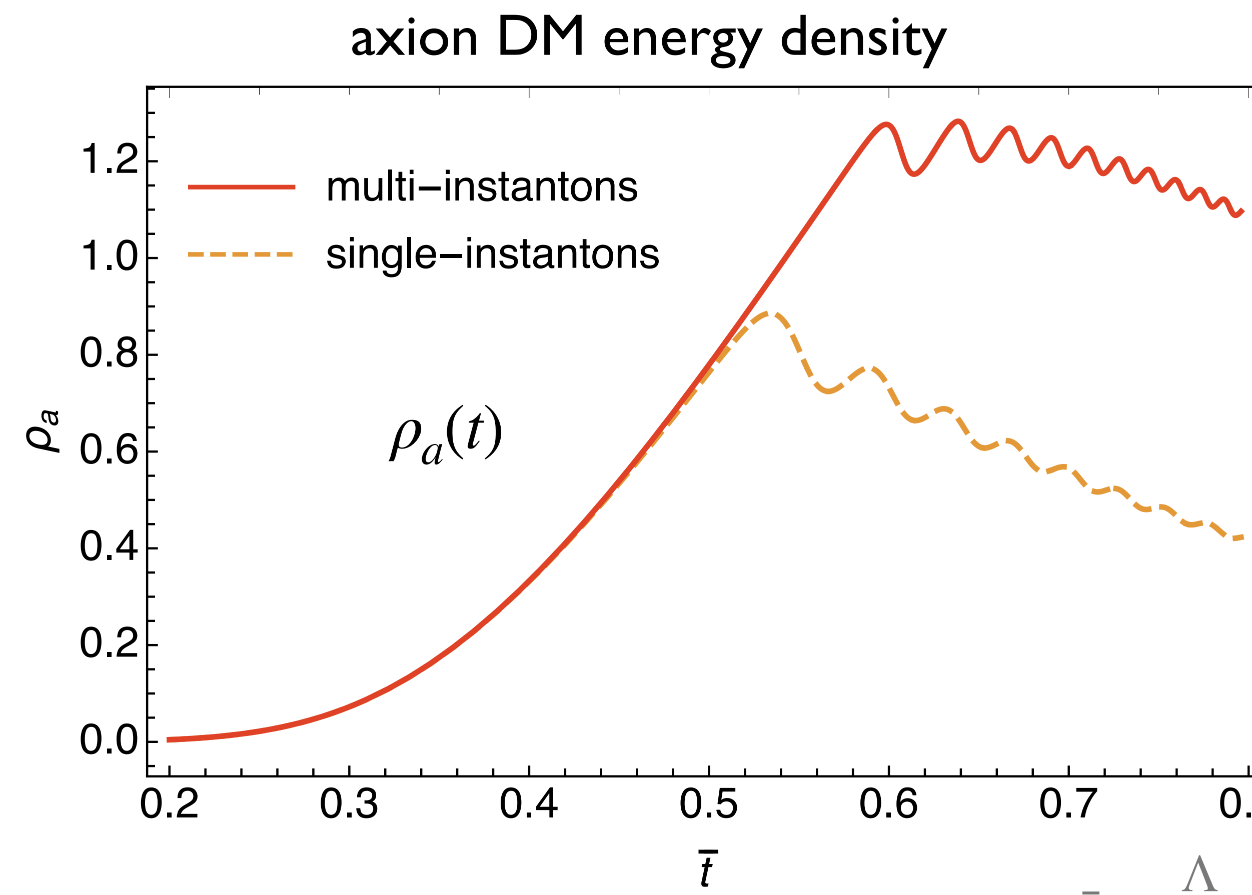
$$\rho_a = \frac{1}{2}\left(\frac{d\bar{\theta}}{dt}\right)^2 + V(\bar{\theta}/f_a)$$

AXION DARK MATTER

Compare single-instanton to 'full' multi-instanton solution $V(\bar{\theta}/f_a)$

(again in the quenched limit)





$\bar{t} = \dfrac{\Lambda}{\pi T}$

→ multi-instanton effects can lead to more axion dark matter

DISCUSSION

Size of the effects studied here depends on the size of multi-instanton effects.

- classically: strong exponential suppression: $n_Q \sim \exp\left(-\frac{8\pi^2}{g^2}|Q|\right)$
- instanton density strongly suppressed by light quarks

$\longrightarrow$ quantum corrections are crucial for multi-instanton effects

How to assess the quantitative relevance of the effects discussed here?

- compute higher-order corrections in the SCI limit: better understanding of ΔZ_Q
- account for interactions between multi-(anti-)instantons: multi-instanton liquid?

Keep in mind: semi-classical picture breaks down eventually at strong coupling!

But: semiclassical analysis shows that all these effects must be there

SUMMARY

What about instantons with higher topological charge?

- they give rise to anomalous $2N_f|Q|$-quark correlation functions [Pisarski, FR; 1910.14052]

$$\det\left(\bar{\psi}_f \mathbb{P}_R \psi_g\right)^{|Q|} + \det\left(\bar{\psi}_f \mathbb{P}_L \psi_g\right)^{|Q|}$$

  - signatures of the axial anomaly through higher order anomalous correlations

- they yield corrections to the θ-dependence of QCD [FR; 2003.13876]

$$F(\theta) \sim -\sum_{Q=1}^{\infty} \Delta Z_Q \cos(Q\theta)$$

  - modified temperature dependence of topological susceptibilities
  - topological mechanism to increase the amount of axion dark matter