JUSTUS-LIEBIG-UNIVERSITÄT GIESSEN

# Real-time functional renormalization group for critical dynamics

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Based on
JR, D. Schweitzer, L. J. Sieke, L. von Smekal, *Real-time methods for spectral functions*, arXiv:2112.12568,
JR, L. von Smekal, in preparation.

DFG

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Strong-interaction matter under extreme conditions

HGS-HIRe for FAIR
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# Outline

1. Real-time . . .
2. . . . functional renormalization group . . .
3. . . . for critical dynamics

# Why *real*-time?

Performing calculations directly in real time (Minkowski spacetime)

- avoids the need of analytic continuation in comparison with the imaginary-time formalism, and
- allows treating phenomena off-equilibrium, e.g. many aspects of heavy-ion collisions, which are very dynamic in nature.

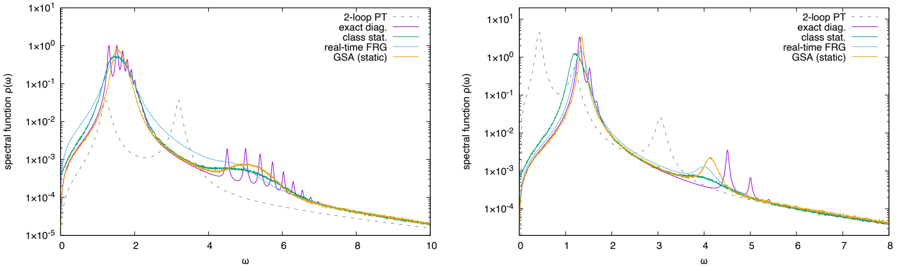



Figure: Spectral functions of the quartic oscillator at finite temperature stemming from various computational techniques, including the real-time FRG. (JR, Schweitzer, Sieke, von Smekal '21)

# Real-time QFT

# Real-time QFT

Time evolution of general mixed state $\hat{\rho}(t)$ is described by von Neumann equation

$$i\frac{d}{dt}\hat{\rho}(t) = [H(t), \hat{\rho}(t)]$$

- which is formally solved by

$$\hat{\rho}(t) = U(t, -\infty)\hat{\rho}_0 U(-\infty, t)$$

- with time-evolution operator

$$U(t, t') = T \exp\left\{-i\int_{t'}^{t} dt''\, H(t'')\right\}$$

- Initial state $\hat{\rho}_0 = \hat{\rho}(-\infty)$ is defined in the distant past (see below)

Expectation value of observable

$$\langle \mathcal{O}(t) \rangle = \frac{\mathrm{Tr}\big(\mathcal{O}\hat{\rho}(t)\big)}{\mathrm{Tr}\,\hat{\rho}(t)}$$ Schrödinger picture

$$= \frac{\mathrm{Tr}\big(\mathcal{O}U(t,-\infty)\hat{\rho}_0 U(-\infty,t)\big)}{\mathrm{Tr}\big(U(t,-\infty)\hat{\rho}_0 U(-\infty,t)\big)}$$ (use cyclicity)

$$= \frac{\mathrm{Tr}\big(U(-\infty,t)\mathcal{O}U(t,-\infty)\hat{\rho}_0\big)}{\mathrm{Tr}\,\hat{\rho}_0}$$ Heisenberg picture

$$= \frac{\mathrm{Tr}\big(U(-\infty,+\infty)U(+\infty,t)\mathcal{O}U(t,-\infty)\hat{\rho}_0\big)}{\mathrm{Tr}\,\hat{\rho}_0}$$ (extend evolution to $+\infty$)

Now the time evolution goes from $-\infty$ to $+\infty$, and then back to $-\infty$, hence the name 'closed time path' (CTP). (Schwinger '60, Kadanoff, Baym '62, Keldysh '64)

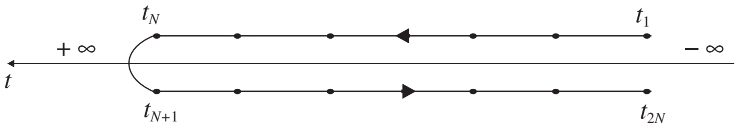
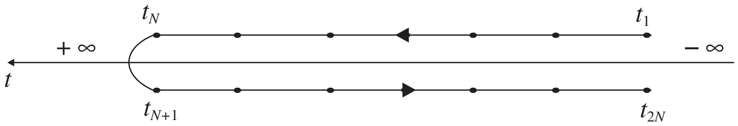



Figure: A. Kamenev, *Field Theory of Non-Equilibrium Systems*, (Cambridge University Press, 2011).

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Define partition function

$$Z \equiv \frac{\mathrm{Tr}\big(U(-\infty,+\infty)U(+\infty,-\infty)\hat{\rho}_0\big)}{\mathrm{Tr}\,\hat{\rho}_0} = 1\,.$$

Expectation values by introducing sources on forward and/or backward branch, e.g.

- to calculate expectation value $\langle \mathcal{O}(t)\rangle$ from above
- replace $H \to H^{\pm} = H \pm V(t)\mathcal{O}$, then

$$Z[V] \equiv \frac{\mathrm{Tr}\big(U_{\mathsf{CTP}}[V]\hat{\rho}_0\big)}{\mathrm{Tr}\,\hat{\rho}_0} \implies \langle \mathcal{O}(t)\rangle = \frac{i}{2}\frac{\delta Z[V]}{\delta V(t)}\bigg|_{V\equiv 0}$$

by functional differentiation.

Digression: Why and when is closing the time path necessary?

- Zero-temperature field theory is concerned with quantities e.g. of the form

$$\langle\Omega|\mathcal{O}(t)|\Omega\rangle$$

with interacting ground state $|\Omega\rangle$.

- Usual trick: Adiabatic switching off interactions in distant past and future
(1) $|\Omega\rangle = U(t_0, -\infty)|0\rangle$ with free ground state $|0\rangle$
(2) $U(+\infty, -\infty)|0\rangle = e^{i\varphi}|0\rangle$
- Then (define Heisenberg picture w.r.t. $t_0$ here, $\mathcal{O}(t) = U(t_0, t)\mathcal{O}U(t, t_0)$)

$$\begin{aligned}
\langle\Omega|\mathcal{O}(t)|\Omega\rangle &\overset{(1)}{=} \langle 0|U(-\infty, t_0)\mathcal{O}(t)U(t_0, -\infty)|0\rangle \\
&= \langle 0|U(-\infty, +\infty)U(+\infty, t_0)\mathcal{O}(t)U(t_0, -\infty)|0\rangle \\
&\overset{(2)}{=} e^{-i\varphi}\langle 0|U(+\infty, t_0)\mathcal{O}(t)U(t_0, -\infty)|0\rangle \\
&\overset{(2)}{=} \frac{\langle 0|U(+\infty, t_0)\mathcal{O}(t)U(t_0, -\infty)|0\rangle}{\langle 0|U(+\infty, -\infty)|0\rangle}
\end{aligned}$$

only needs forward evolution!

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  (2) $U(+\infty, -\infty)|0\rangle = e^{i\varphi}|0\rangle$ — no longer valid! ✗
- Then (define Heisenberg picture w.r.t. $t_0$ here, $\mathcal{O}(t) = U(t_0, t)\mathcal{O}U(t, t_0)$)

$$\begin{aligned}
\langle \Omega | \mathcal{O}(t) | \Omega \rangle &\overset{(1)}{=} \langle 0 | U(-\infty, t_0)\mathcal{O}(t)U(t_0, -\infty)|0\rangle \\
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only needs forward evolution!

**Trick not possible when non-adiabatic changes are present during time evolution!**

Consider harmonic oscillator $H_0 = \omega_0 a^\dagger a$ (zero-point energy subtracted) in thermal equilibrium $\hat{\rho}_0 = e^{-\beta H_0}$.

To arrive at path integral representation of the partition function Suzuki-Trotter-decompose $Z$ in 'coherent' states

$$a|\alpha\rangle = \alpha|\alpha\rangle\,, \qquad\qquad (\alpha \in \mathbb{C})$$

defined as eigenstates of annihilation operator $a$.

- Express in energy eigenstates,

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle\,, \text{ with } H_0|n\rangle = n\omega_0|n\rangle\,.$$

- Calculate inner product,

$$\langle\alpha|\alpha'\rangle = e^{-\frac{1}{2}\left(|\alpha|^2 + |\alpha'|^2 - 2\alpha^*\alpha'\right)}$$

(special case of $\langle\alpha|e^{\rho a^\dagger a}|\alpha'\rangle = e^{-\frac{1}{2}\left(|\alpha|^2 + |\alpha'|^2 - 2e^\rho\alpha^*\alpha'\right)}$ for $\rho \in \mathbb{R}$).

- Form over-complete basis and evaluate traces,

$$\mathbf{1} = \int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha|\,, \quad \mathrm{Tr}\,\mathcal{O} = \int \frac{d^2\alpha}{\pi} \langle\alpha|\mathcal{O}|\alpha\rangle$$

# Real-time QFT

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$$a|\alpha\rangle = \alpha|\alpha\rangle\,, \qquad (\alpha \in \mathbb{C})$$

defined as eigenstates of annihilation operator $a$.
Convenient because discretized partition function is product of exponentials,

$$\langle\alpha_1|\hat{\rho}_0|\alpha_{2N}\rangle = e^{\rho_0\alpha_1^*\alpha_{2N}} \qquad \langle\alpha_{N+1}|\alpha_N\rangle = e^{-\frac{1}{2}\left(|\alpha_{N+1}|^2+|\alpha_N|^2-2\alpha_{N+1}^*\alpha_N\right)}$$

$$\langle\alpha_{n+1}|U(t_n \pm \delta_t, t_n)|\alpha_n\rangle = \langle\alpha_{n+1}|\alpha_n\rangle e^{\mp i\delta_t\omega_0\alpha_{n+1}^*\alpha_n} + \mathcal{O}(\delta_t^2)$$

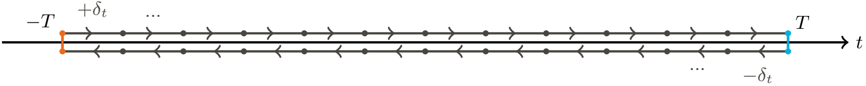


Figure: Discretized CTP.

(Define Boltzmann factor $\rho_0 \equiv e^{-\beta\omega_0}$)

Write partition function now as discretized path integral

$$Z = \frac{1}{\mathrm{Tr}\,\hat{\rho}_0} \int \left( \prod_{j=1}^{2N} \frac{d^2\alpha_j}{\pi} \right) \exp\left\{ iS[\{\alpha_j^*, \alpha_j\}] \right\} = 1$$

with discretized action

$$S[\{\alpha_j^*, \alpha_j\}] = \sum_{j=2}^{2N} \delta t_j \left( i\alpha_j^* \frac{\alpha_j - \alpha_{j-1}}{\delta t_j} - \omega_0 \alpha_j^* \alpha_{j-1} \right) + i\alpha_1^* \left( \alpha_1 - ie^{-\beta\omega_0} \alpha_{2N} \right)$$

$$\xrightarrow{N\to\infty} \int_{\mathrm{CTP}} dt \, \left( \alpha^*(t) i\partial_t \alpha(t) - \omega_0 \alpha^*(t)\alpha(t) \right) + \text{boundary terms}$$

boundary terms are inconvenient in (naive) continuum limit, as they spoil manifest time-translation invariance of a system in thermal equilibrium. (Impractical.)

Goal: Find a continuum action that is time-translation invariant, *and* reproduces free Green functions via rules of Gaussian integration ...

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. . . But before that, simplify the notation:
Introduce fields on the forward $(+)$ and backward $(-)$ branches of the contour,

$$\alpha^+(t) \equiv \alpha(t^+)\,, \quad \alpha^-(t) \equiv \alpha(t^-)$$

Calculate discrete propagators by matrix inversion,

$$G^{T}_{jj'} \equiv G^{++}_{jj'} = i\langle \alpha^+_j \alpha^{+\,*}_{j'} \rangle = \frac{i}{1-\rho_0}(u^+)^{j-j'} \times \begin{cases} 1 & \text{if } j \geq j' \\ e^{-\beta\omega_0} & \text{if } j < j' \end{cases} \quad \text{‘time ordered’}\,,$$

$$G^{\widetilde{T}}_{jj'} \equiv G^{--}_{jj'} = i\langle \alpha^-_j \alpha^{-\,*}_{j'} \rangle = \frac{i}{1-\rho_0}(u^+)^{j-j'} \times \begin{cases} e^{-\beta\omega_0} & \text{if } j > j' \\ 1 & \text{if } j \leq j' \end{cases} \quad \text{‘anti-time-ordered’}\,,$$

$$G^{<}_{jj'} \equiv G^{+-}_{jj'} = i\langle \alpha^+_j \alpha^{-\,*}_{j'} \rangle = \frac{i}{1-\rho_0}(u^+)^{j-j'}\rho_0 \quad \text{‘lesser’}\,,$$

$$G^{>}_{jj'} \equiv G^{-+}_{jj'} = i\langle \alpha^-_j \alpha^{+\,*}_{j'} \rangle = \frac{i}{1-\rho_0}(u^+)^{j-j'} \quad \text{‘greater’}\,,$$

not all independent, but generally interrelated by

$$G^{++}_{jj'} + G^{--}_{jj'} - G^{+-}_{jj'} - G^{-+}_{jj'} = \delta_{jj'} \to 0 \text{ in continuum limit}$$

(Note here: Kronecker-$\delta$, not $\delta$-function!)

Exploit this linear interrelation by orthogonal transformation which sets one of the Green functions identically to zero:

- achieved by ‘Keldysh rotation’

$$\alpha^c(t) \equiv \frac{1}{\sqrt{2}}\Big(\alpha^+(t) + \alpha^-(t)\Big) \ ,$$
$$\alpha^q(t) \equiv \frac{1}{\sqrt{2}}\Big(\alpha^+(t) - \alpha^-(t)\Big) \ ,$$

- with ‘classical’ and ‘quantum’ fields $\alpha^c(t)$, $\alpha^q(t)$

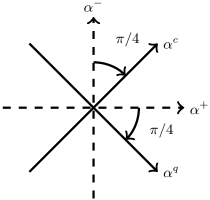


Figure: Keldysh rotation: Clockwise rotation in the $(+,-)$-field space.

- Green functions are ‘rotated’ according to

$$\begin{array}{ccc} \text{time ordered} \qquad \text{lesser} & & \text{Keldysh} \qquad \text{retarded} \\ \begin{pmatrix} G^{++}(t,t') & G^{+-}(t,t') \\ G^{-+}(t,t') & G^{--}(t,t') \end{pmatrix} & \to & \begin{pmatrix} G^K(t,t') & G^R(t,t') \\ G^A(t,t') & 0 \end{pmatrix} \\ \text{greater} \qquad \text{anti-time-ordered} & & \text{advanced} \end{array}$$

Now perform the continuum limit to (statistical function $F(\omega) \equiv 2n_B(\omega) + 1$)

- find Keldysh-rotated propagators ($1^{\text{st}}$ order form),

$$G^R(t,t') = i\theta(t-t')e^{-i\omega_0(t-t')}$$

$$G^A(t,t') = -i\theta(t'-t)e^{-i\omega_0(t-t')}$$

$$G^K(t,t') = iF(\omega_0)e^{-i\omega_0(t-t')}$$

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\begin{aligned}
G^R(t,t') &= i\theta(t-t')e^{-i\omega_0(t-t')} &\quad \to G^R(\omega) &= -\frac{1}{\omega + i\varepsilon - \omega_0}\,,\\
G^A(t,t') &= -i\theta(t'-t)e^{-i\omega_0(t-t')} &\quad \to G^A(\omega) &= -\frac{1}{\omega - i\varepsilon - \omega_0}\,,\\
G^K(t,t') &= iF(\omega_0)e^{-i\omega_0(t-t')} &\quad \to G^K(\omega) &= 2\pi i F(\omega_0)\delta(\omega-\omega_0)\,,
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- discover general requirement of

**Causality**

Retarded (advanced) propagator $G_k^{R(A)}(\omega)$ is analytic in the upper (lower) half $\omega$-plane.

Now perform the continuum limit to (statistical function $F(\omega) \equiv 2n_B(\omega) + 1$)

- find Keldysh-rotated propagators ($1^{\text{st}}$ order form),

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\end{aligned}
$$

- and write down action which reproduces these Green functions by the rules of Gaussian integration,

**Free Keldysh action ($1^{\text{st}}$ order form)**

$$
S = \int_{-\infty}^{\infty} dt \left(\alpha^{c*}(t), \alpha^{q*}(t)\right) \begin{pmatrix} 0 & i\partial_t - i\varepsilon - \omega_0 \\ i\partial_t + i\varepsilon - \omega_0 & 2i\varepsilon F(\omega_0) \end{pmatrix} \begin{pmatrix} \alpha^c(t) \\ \alpha^q(t) \end{pmatrix}
$$

which is manifestly time-translation invariant. (Goal reached!)

Starting with

**Free Keldysh action ($1^{\text{st}}$ order form)**

$$S = \int_{-\infty}^{\infty} dt \left(\alpha^{c*}(t), \alpha^{q*}(t)\right) \begin{pmatrix} 0 & i\partial_t - i\varepsilon - \omega_0 \\ i\partial_t + i\varepsilon - \omega_0 & 2i\varepsilon F(\omega_0) \end{pmatrix} \begin{pmatrix} \alpha^c(t) \\ \alpha^q(t) \end{pmatrix}$$

- introduce canonical oscillator coordinates $\varphi$ and $\pi$ again,

$$\alpha = \frac{1}{\sqrt{2\omega_0}} \left(\omega_0 \varphi + i\pi\right) , \quad \alpha^* = \frac{1}{\sqrt{2\omega_0}} \left(\omega_0 \varphi - i\pi\right) ,$$

- integrate out Gaussian $\pi$'s, to arrive at

**Free Keldysh action ($2^{\text{nd}}$ order form)**

$$S = \frac{1}{2} \int_{-\infty}^{\infty} dt \, \left(\phi^c(t), \phi^q(t)\right) \begin{pmatrix} 0 & (i\partial_t - i\varepsilon)^2 - \omega_0^2 \\ (i\partial_t + i\varepsilon)^2 - \omega_0^2 & -\varepsilon[\partial_t, F] \end{pmatrix} \begin{pmatrix} \phi^c(t) \\ \phi^q(t) \end{pmatrix}$$

(in coordinate space)

(Shorthand notation!
Actually non-local in time . . . )

Include interactions by

- adding potential term to Keldysh action

$$S_V = \int_{-\infty}^{\infty} dt \left( -V_{\text{int}}(\phi^+) + V_{\text{int}}(\phi^-) \right)$$
$$= \int_{-\infty}^{\infty} dt \left( -V_{\text{int}} \left( \frac{\phi^c + \phi^q}{\sqrt{2}} \right) + V_{\text{int}} \left( \frac{\phi^c - \phi^q}{\sqrt{2}} \right) \right)$$

- and imagine that interactions are adiabatically switched off in the distant past, $t \to -\infty$
(but they may stay *finite* in the distant future $t \to +\infty$ (!))
- e.g. quartic coupling $V_{\text{int}}(\varphi) = \lambda\varphi^4/4!$ ,

$$S_V = -\frac{\lambda}{12} \int_{-\infty}^{\infty} dt \, \Big( \underbrace{\phi^c(t)\phi^c(t)\phi^c(t)\phi^q(t)}_{\text{‘classical’ vertex}} + \underbrace{\phi^c(t)\phi^q(t)\phi^q(t)\phi^q(t)}_{\text{‘quantum’ vertex}} \Big)$$

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Why the names 'classical' and 'quantum'?

Perform classical limit of Keldysh action by reintroducing $\hbar$, then take the limit $\hbar \to 0$,

- $S \to S/\hbar$,
- $T \to T/\hbar \implies F(\omega) \to 2T/\hbar\omega + \mathcal{O}(\hbar)$ (Rayleigh-Jeans distribution),
- $\phi^q(t) \to \hbar\phi^q(t)$,

(obtained from dimensional analysis)

$$S[\phi^c,\phi^q] = \frac{1}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,(\phi^c,\phi^q)_{-\omega}\begin{pmatrix} 0 & (\omega-i\varepsilon)^2-\omega_0^2 \\ (\omega+i\varepsilon)^2-\omega_0^2 & 4i\varepsilon\omega\coth(\omega/2T)\end{pmatrix}\begin{pmatrix}\phi^c\\ \phi^q\end{pmatrix}_{\omega}$$
$$-\frac{\lambda}{12}\int_{-\infty}^{\infty} dt\,\left(\phi^c(t)\phi^c(t)\phi^c(t)\phi^q(t)+\phi^c(t)\phi^q(t)\phi^q(t)\phi^q(t)\right),$$

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$$\frac{1}{\hbar}S[\phi^c, \phi^q] = \frac{1}{2\hbar}\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left(\phi^c, \hbar\phi^q\right)_{-\omega} \begin{pmatrix} 0 & (\omega - i\varepsilon)^2 - \omega_0^2 \\ (\omega + i\varepsilon)^2 - \omega_0^2 & 4i\varepsilon\omega \coth\big(\hbar\omega/2T\big) \end{pmatrix} \begin{pmatrix} \phi^c \\ \hbar\phi^q \end{pmatrix}_\omega$$

$$- \frac{\lambda}{12\hbar}\int_{-\infty}^{\infty} dt \, \Big( \hbar\phi^c(t)\phi^c(t)\phi^c(t)\phi^q(t) + \hbar^3\phi^c(t)\phi^q(t)\phi^q(t)\phi^q(t) \Big) ,$$

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$$- \frac{\lambda}{12}\int_{-\infty}^{\infty} dt\,\left(\phi^c(t)\phi^c(t)\phi^c(t)\phi^q(t)\right),$$

i.e. only 'classical' vertex remains in classical limit, hence the name.

$$\frac{1}{\hbar}S[\phi^c,\phi^q] = \frac{1}{2}\int\limits_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left(\phi^c,\phi^q\right)_{-\omega}\begin{pmatrix} 0 & (\omega - i\varepsilon)^2 - \omega_0^2 \\ (\omega + i\varepsilon)^2 - \omega_0^2 & 8i\varepsilon T \end{pmatrix}\begin{pmatrix}\phi^c \\ \phi^q\end{pmatrix}_{\omega}$$
$$- \frac{\lambda}{12}\int_{-\infty}^{\infty} dt\, \left(\phi^c(t)\phi^c(t)\phi^c(t)\phi^q(t)\right)\,,$$

- Arrived at the Martin-Siggia-Rose-Janssen-de Dominicis path integral formulation of classical-statistical systems. (Later)
- From the point of view of the *formalism* non-equilibrium QFT and classical-statistical field theories are virtually indistinguishable.
- One may now linearize action in $\phi^q(t)$ by Hubbard-Stratonovich transformation, integrate linear $\phi^q(t)$ to get $\delta$-functional, enforcing class.-stat. equations of motion

$$\partial_t^2\phi^c + 2\varepsilon\partial_t\phi^c + \omega_0^2\phi^c + \frac{\lambda}{12}(\phi^c)^3 = \xi(t)\,,$$
$$\langle\xi(t)\rangle_\beta = 0\,, \qquad \langle\xi(t)\xi(t')\rangle_\beta = 8\varepsilon T\delta(t-t')$$

for a particle in infinitesimal contact $\varepsilon$ to an external heat bath. (Canonical ensemble)

Generating functional

$$Z[j^c, j^q] = \int \mathcal{D}\phi^c\, \mathcal{D}\phi^q \;\exp\left\{ iS[\phi^c, \phi^q] + i\int_{-\infty}^{\infty} dt\, \left( j^c(t)\phi^q(t) + j^q(t)\phi^c(t) \right) \right\}$$

with Keldysh action

$$S[\phi^c, \phi^q] = \frac{1}{2}\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \left(\phi^c(-\omega), \phi^q(-\omega)\right) \begin{pmatrix} 0 & \omega^2 - i\gamma\omega - \omega_0^2 \\ \omega^2 + i\gamma\omega - \omega_0^2 & 2i\gamma\omega F(\omega) \end{pmatrix} \begin{pmatrix} \phi^c(\omega) \\ \phi^q(\omega) \end{pmatrix}$$
$$- \frac{\lambda}{12}\int_{-\infty}^{\infty} dt\, \left( \phi^c(t)\phi^c(t)\phi^c(t)\phi^q(t) + \phi^c(t)\phi^q(t)\phi^q(t)\phi^q(t) \right),$$

with

- quartic self-interaction
- finite coupling to dissipative external heat bath (Caldeira-Leggett model, later)

Effective action by Legendre transform

$$\Gamma[\bar{\phi}^c, \bar{\phi}^q] = \sup_{j^c, j^q} \left\{ -i\log Z[j^c, j^q] - \int_{-\infty}^{\infty} dt\, \left( j^c(t)\bar{\phi}^q(t) + j^q(t)\bar{\phi}^c(t) \right) \right\}$$

# Real-time functional renormalization group

- Suppose the effective action $\Gamma$ of the theory is known at some momentum/energy scale $k$, denoted $\Gamma_k$, where fluctuations from modes $|\boldsymbol{p}| \gtrsim k$ have been taken into account.
- Realized by modifying the action with an *infrared cutoff* $\Delta S_k[\phi^c, \phi^q]$,

$$S \to S + \Delta S_k$$

suppressing modes with $|\boldsymbol{p}| < k$.
- Has the structure ($D = d + 1$ number of spacetime dimensions)

$$\Delta S_k[\phi] = \frac{1}{2} \int d^D x \int d^D x' \, \phi^T(x) R_k(x, x') \phi(x'), \qquad \phi^T = (\phi^c, \phi^q),$$

with the $2 \times 2$-'regulator' matrix

$$R_k(p) = \begin{pmatrix} 0 & R_k^A(p) \\ R_k^R(p) & R_k^K(p) \end{pmatrix}.$$

in momentum space.

# Idea of the (functional) renormalization group

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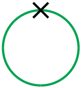
- Change the scale $k \to k + dk$, arrive at ‘flow’ equation
  (Wetterich ’93, Berges, Mesterházy ’12)

$$\partial_k \Gamma_k = -\frac{i}{2} \mathrm{tr}\left(\partial_k R_k \circ G_k\right), \quad G_k = -\left(\Gamma_k^{(2)} + R_k\right)^{-1}$$

- Has the form of a 1-loop integral,
  (Color-coding from Hülsmann, Schlichting, Scior ’20)

$$\partial_k \Gamma_k = -\frac{i}{2}$$



but is exact.

Fully field-dependent propagator $G_k[\phi]$

- Have $\Gamma_k \xrightarrow{k \to \Lambda} S$, classical action.
  (Demonstrated via saddle-point approximation.)

- Regulator changes analytic structure of propagators,

$$G_k^R(\omega,\boldsymbol{p}) = -\frac{1}{\Gamma_k^{qc}(\omega,\boldsymbol{p}) + R_k^R(\omega,\boldsymbol{p})} \qquad \text{(retarded)}$$

$$G_k^A(\omega,\boldsymbol{p}) = -\frac{1}{\Gamma_k^{cq}(\omega,\boldsymbol{p}) + R_k^A(\omega,\boldsymbol{p})} \qquad \text{(advanced)}$$

- What are the consequences?
- Maybe everything fine for $k = 0$?

# Causal regulators?

Test:

- Observe very general property of Keldysh action:

$$S = \frac{1}{2} \int_p (\phi^c(-p), \phi^q(-p)) \begin{pmatrix} \mathbf{0} & \cdots \\ \cdots & \cdots \end{pmatrix} \begin{pmatrix} \phi^c(p) \\ \phi^q(p) \end{pmatrix} + \cdots$$

  follows from that for $\phi^+ = \phi^-$ the action vanishes, $S[\phi^c, 0] = 0$.
- Necessary condition for the correctness of the flow.

Find:

- Popular regulators like sharp/exponential/algebraic/... cutoff produce such an unphysical component during flow.
- Problem of causality not trivial. (Duclut, Delamotte '18)
- An insufficient regulator leads to an incorrect Keldysh action.

What can we do?
(Start with 0+1 dimensional case, i.e. quantum mechanics.)

Most simple regulator which we could write down has form of a purely mass-like shift,
(Callan-Symanzik regulator)

$$R_k^{R/A}(\omega) = -k^2$$

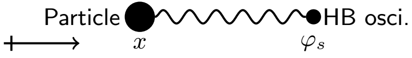
- Trivially causal, only induces mass-shift $m^2 \to m^2 + k^2$ in propagators.
- Too simple?
- Flow no longer consistent with Wilson's idea of integrating out energy (momentum) shells?

# Heat bath regulators



Regulator motivated by physics: (Causality guaranteed!)

- Imagine $\Delta S_k$ is the result of integrating out an external heat bath.
- Heat bath (HB) is modeled as an ensemble of independent harmonic oscillators, attached to the particle. (Caldeira-Leggett model)



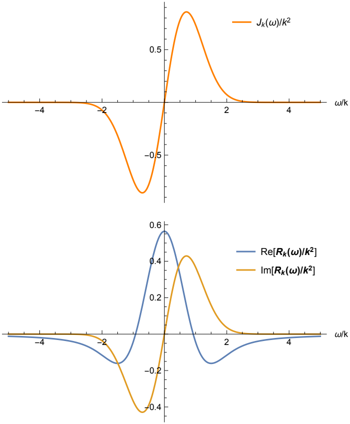

$$H' = \sum_s \left( \frac{\pi_s^2}{2} + \frac{\omega_s^2}{2} \left( \varphi_s - \frac{g_s}{\omega_s^2} x \right)^2 \right)$$

- Integrate out heat bath $\hat{=}$ Particle acquires self-energy $\Sigma^{R/A}(\omega)$

$$\Sigma^R(\omega) = \sum_s \overset{g_s}{\bullet}\,\underset{D_s(\omega)}{-\,-\,-}\,\overset{g_s}{\bullet} = -\int_0^\infty \frac{d\omega'}{2\pi} \frac{2\omega' J(\omega')}{(\omega + i\varepsilon)^2 - \omega'^2}$$

- Fully controlled by a spectral density $J(\omega) = \pi \sum_s \frac{g_s^2}{\omega_s} \delta(\omega - \omega_s)$
- Invert $\rightsquigarrow J(\omega) = 2\mathrm{Im}\,\Sigma^R(\omega)$, but self-energy $\Sigma^R$ also has a non-vanishing real part.

- Now make the spectral density $k$-dependent, $J(\omega) \to J_k(\omega)$, and choose it to *damp* infrared modes.
- The resulting self-energy is the regulator, $\Sigma^{R/A}(\omega) \to R_k^{R/A}(\omega)$.




$$m^2 \to m^2 - \Delta m^2_{\text{HB}}(k)$$

Example:

$$J_k(\omega) = k\omega \exp\left\{-\omega^2/k^2\right\}$$

$$\implies \phi(t) \sim e^{-kt/2} \text{ for } \omega \ll k, \text{damped}$$

But: Heat bath induces *negative* (!) shift in the squared mass

$$\Delta m^2_{\text{HB}}(k) = \int_0^\infty \frac{d\omega}{\pi} \frac{J_k(\omega)}{\omega} = \frac{k^2}{\sqrt{4\pi}}$$

Makes the theory *unstable* and *acausal* for sufficiently large values of $k$ !

# Heat bath regulators

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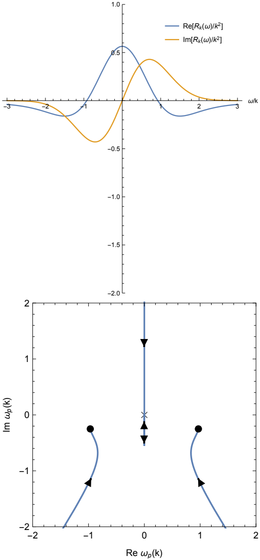
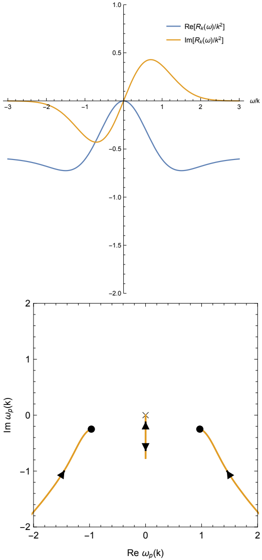
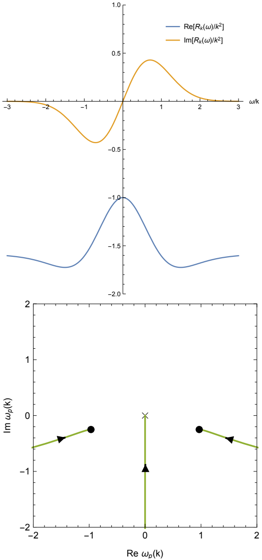
- Way out: We learned that a masslike shift is causal.
  $\rightsquigarrow$ Add mass-like 'counter-term' $-\alpha k^2$ with $\alpha > 0$
  to compensate unwanted shift in squared mass!




$\alpha = 0$




$\alpha = 1/\sqrt{4\pi}$,
(balanced)




$\alpha = 1/\sqrt{4\pi} + 1$,
(balanced + regulated)

- Way out: We learned that a masslike shift is causal.
  $\rightsquigarrow$ Add mass-like 'counter-term' $-\alpha k^2$ with $\alpha > 0$ to compensate unwanted shift in squared mass!

**Heat bath regulator in $1+0d$**

$$R_k^{R/A}(\omega) = -\int_0^\infty \frac{d\omega'}{2\pi} \frac{2\omega' J_k(\omega')}{(\omega \pm i\varepsilon)^2 - \omega'^2} - \alpha k^2$$

(JR, Schweitzer, Sieke, von Smekal '21)

What about a *field theory*?

- Arguably simplest ansatz: Imagine an independent bath of harmonic oscillators for every spatial momentum mode $\boldsymbol{p}$. Then the spectral representation just acquires an additional $\boldsymbol{p}$-dependence,

**Heat bath regulator**

$$R_k^{R/A}(\omega, \boldsymbol{p}) = -\int_0^\infty \frac{d\omega'}{2\pi} \frac{2\omega' J_k(\omega', \boldsymbol{p})}{(\omega \pm i\varepsilon)^2 - \omega'^2} - \alpha_k(\boldsymbol{p}) k^2$$

which still ensures causality.

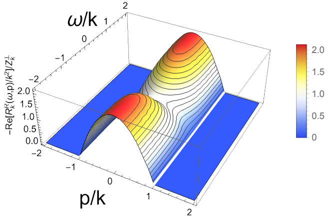



Figure: Real part (Mass shift).

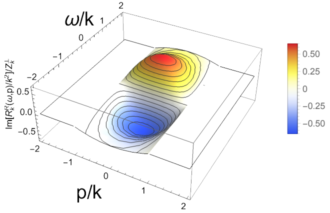
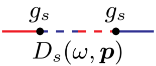



Figure: Imaginary part (Damping).

# Heat bath regulators for field theories

- And when we have no preferred frame of reference, e.g. no external medium? What about *Lorentz invariance*?
- A regulator like the one above would break Lorentz symmetry.
- Imagine the heat bath to be an ensemble of *Klein-Gordon fields* with a relativistic dispersion relation $\omega^2 = \boldsymbol{p}^2 + m_s^2$,
  $\rightsquigarrow$ Our field gains a self-energy
  (Källén-Lehmann representation)

$$\Sigma_k^R(\omega, \boldsymbol{p}) = \sum_s$$




$$= -\int_0^\infty \frac{d\mu^2}{2\pi} \frac{\widetilde{J}_k(\mu^2)}{(\omega + i\varepsilon)^2 - \boldsymbol{p}^2 - \mu^2}$$

  with invariant spectral density $\widetilde{J}(\mu^2) = 2\pi \sum_s g_s^2 \delta(\mu^2 - m_s^2)$ in

$$J(\omega, \boldsymbol{p}) = \operatorname{sgn}(\omega)\ \theta(p^2)\ \widetilde{J}(p^2)$$

- Reintroduce masslike counter-term $-\alpha k^2$, and then find general form of

**Lorentz-invariant heat-bath regulator**

$$R_k^{R/A}(\omega,\boldsymbol{p}) = -\int_0^\infty \frac{d\mu^2}{2\pi}\,\frac{\widetilde{J}_k(\mu^2)}{(\omega \pm i\varepsilon)^2 - \boldsymbol{p}^2 - \mu^2} - \alpha k^2$$

(Special case of spectral representation shown above)

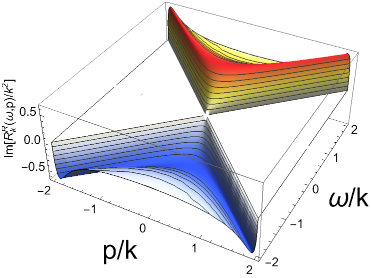



Figure: Imaginary part (damping).

Example:

$$\widetilde{J}_k(\mu^2) = \frac{4k\mu}{(1+\mu^2/k^2)^2}$$

- $p^2$ is a Lorentz scalar.
- $\operatorname{sgn}\omega$ is also a Lorentz scalar, but only if $p$ is timelike and if we restrict ourselves to orthochronous Lorentz transformations.

# Critical dynamics

Consider classical $\lambda\varphi^4$-theory with Landau-Ginzburg free energy (statics) in thermal equilibrium, **Model A**

$$F = \int d^d x \left\{ \frac{1}{2}(\vec{\nabla}\varphi)^2 + V(\varphi) \right\} , \quad Z = \int \mathcal{D}\varphi \, e^{-\beta F} ,$$

and equations of motion (dynamics) with dissipative coupling $\gamma$ to heat bath (Langevin)

$$\partial_t^2 \varphi + \gamma \partial_t \varphi = -\frac{\delta F}{\delta \varphi} + \xi(x) ,$$

with Gaussian white noise(s)

$$\begin{aligned} \langle \xi(x) \rangle_\beta &= 0 , \\ \langle \xi(x) \xi(x') \rangle_\beta &= 2\gamma T \delta(x - x') , \end{aligned}$$

Discrete $Z_2$ ($\varphi \to -\varphi$) symmetry breaks spontaneously for $T < T_c$ when $m^2 < 0$.

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Consider classical $\lambda\varphi^4$-theory with Landau-Ginzburg free energy (statics) **Model B**
with coupling $B$ between conserved density $n(x)$ and $\varphi(x)$ (Son, Stephanov '04)

$$F = \int d^d x \left\{ \frac{1}{2}(\vec{\nabla}\varphi)^2 + V(\varphi) + B\,\varphi\,n + \frac{1}{2\chi_0}n^2 \right\}, \quad Z = \int \mathcal{D}\varphi\mathcal{D}n\, e^{-\beta F},$$

and equations of motion (dynamics) with dissipative coupling $\gamma$ to heat bath (Langevin)

$$\partial_t^2\varphi + \gamma\partial_t\varphi = -\frac{\delta F}{\delta\varphi} + \xi(x)\,,$$

$$\tau_R\partial_t^2 n + \partial_t n = \bar{\lambda}\vec{\nabla}^2\frac{\delta F}{\delta n} + \vec{\nabla}\cdot\vec{\zeta}(x)\,,$$

with Gaussian white noise(s)

$$\langle\xi(x)\rangle_\beta = 0\,, \qquad \langle\xi(x)\xi(x')\rangle_\beta = 2\gamma T\delta(x-x')\,,$$

$$\langle\zeta^i(x)\rangle_\beta = 0\,, \qquad \langle\zeta^i(x)\zeta^j(x')\rangle_\beta = 2\bar{\lambda}T\delta^{ij}\delta(x-x')\,.$$

Discrete $Z_2$ ($\varphi \to -\varphi$) symmetry breaks spontaneously for $T < T_c$ when $m^2 < 0$.

Consider classical $\lambda\varphi^4$-theory with Landau-Ginzburg free energy (statics) **Model C**
with coupling $g$ between conserved density $n(x)$ and $\varphi^2(x)$

$$F = \int d^dx \left\{ \frac{1}{2}(\vec{\nabla}\varphi)^2 + V(\varphi) + \frac{g}{2}\varphi^2 n + \frac{1}{2\chi_0}n^2 \right\}, \quad Z = \int \mathcal{D}\varphi\mathcal{D}n\, e^{-\beta F},$$

and equations of motion (dynamics) with dissipative coupling $\gamma$ to heat bath (Langevin)

$$\partial_t^2\varphi + \gamma\partial_t\varphi = -\frac{\delta F}{\delta\varphi} + \xi(x),$$

$$\tau_R\partial_t^2 n + \partial_t n = \bar{\lambda}\vec{\nabla}^2\frac{\delta F}{\delta n} + \vec{\nabla}\cdot\vec{\zeta}(x),$$

with Gaussian white noise(s)

$$\langle\xi(x)\rangle_\beta = 0, \qquad \langle\xi(x)\xi(x')\rangle_\beta = 2\gamma T\delta(x-x'),$$

$$\langle\zeta^i(x)\rangle_\beta = 0, \qquad \langle\zeta^i(x)\zeta^j(x')\rangle_\beta = 2\bar{\lambda}T\delta^{ij}\delta(x-x').$$

Discrete $Z_2$ ($\varphi \to -\varphi$) symmetry breaks spontaneously for $T < T_c$ when $m^2 < 0$.

Spectral function defined as

$$\rho(\omega) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt\, e^{i\omega t} \int d^d x\, i \langle [\phi(t, \boldsymbol{x}), \phi(0, \mathbf{0})] \rangle \,,$$

which

- behaves like $\rho(\omega) \sim |\omega|^{-\sigma}$ at the critical point, $T = T_c$, with
- scaling exponent $\sigma = (2 - \eta^{\perp})/z$, which is related to
- dynamical critical exponent $z$, defined by $\xi_t \sim \xi^z$.

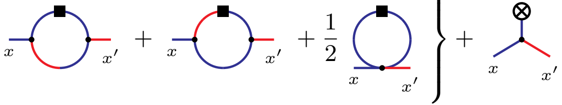
Write down corresponding Keldysh (Martin-Siggia-Rose) action $S[\phi^c, \phi^q]$, then solve via real-time FRG, i.e.

- truncate $\Gamma_k[\phi^c, \phi^q]$,
  (exemplary for Model A)

$$\Gamma_k = \frac{1}{2}\int_p \Delta\phi^T(-p) \begin{pmatrix} 0 & Z_k^{\parallel}(\omega)\,\omega^2 - Z_k^{\perp}\boldsymbol{p}^2 - m_k^2 - i\gamma_k(\omega)\omega \\ \text{c.c. of adv.} & 4i\gamma_k(\omega)T \end{pmatrix} \Delta\phi(p)$$
$$- \frac{\kappa_k}{\sqrt{8}}\int_x \left(\phi^c - \phi^c_{0,k}\right)^2 \phi^q - \frac{\lambda_k}{12}\int_x \left(\phi^c - \phi^c_{0,k}\right)^3 \phi^q \,,$$

  with power-law behavior and finite $(\neq 0)$ anomalous scaling dimension $\eta_k^{\perp} = -k\partial_k \log Z_k^{\perp}$ in mind, and with the fluctuation $\Delta\phi \equiv \phi - \phi_{0,k}$ around the minimum, and then
- solve truncated flow equations numerically,
  (here e.g. for 2-point function)

$$\partial_k \Gamma_k^{cq}(x, x') = -i \left\{ \quad + \quad + \frac{1}{2} \quad \right\} + \quad .$$



Results for critical spectral functions at $T \approx T_c$ **Model A**

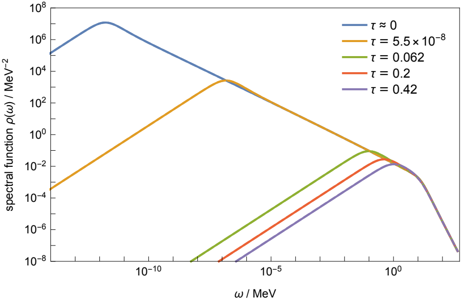



Figure: $d = 2$.

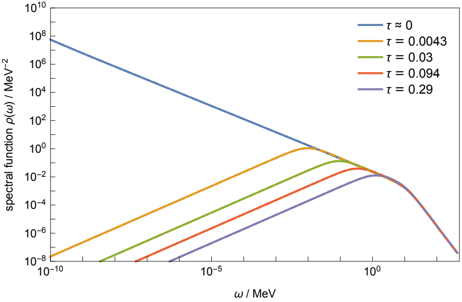



Figure: $d = 3$.

- visible power-law behaviour building up close to the critical point

(Reduced temperature $\tau \equiv (T - T_c)/T_c$)

(JR, von Smekal, in preparation.)

Results for critical spectral functions at $T \approx T_c$ **Model B**

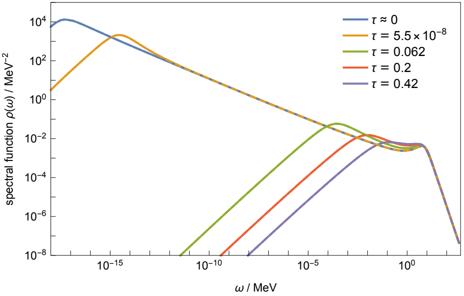



Figure: $d = 2$.

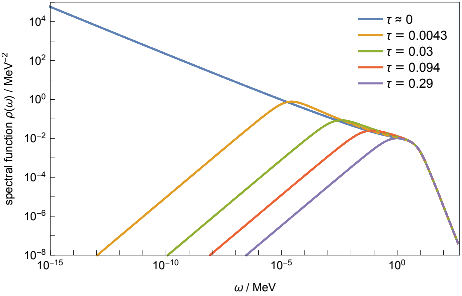



Figure: $d = 3$.

- visible power-law behaviour building up close to the critical point
- conserved density non-critical, but
- non-trivial spectral function at $\boldsymbol{p} = 0$!
  Non-conserved $\varphi$ also resembles critical dynamics of **Model B**

(Reduced temperature $\tau \equiv (T - T_c)/T_c$) (JR, von Smekal, in preparation.)

# Critical spectral functions

Results for critical spectral functions at $T \approx T_c$ **Model C**

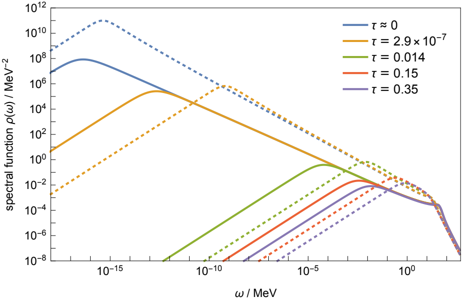



Figure: $d = 2$.

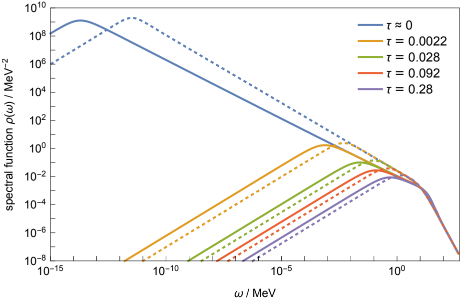



Figure: $d = 3$.

- visible power-law behaviour building up close to the critical point,
- conserved density becomes critical due to non-linear interaction $\sim \varphi^2 n$ with critical $\varphi$-mode, and
- for comparison the **Model A** result indicated as dashed lines.

(Reduced temperature $\tau \equiv (T - T_c)/T_c$) (JR, von Smekal, in preparation.)

Extraction scheme: **Model A**

Look at logarithmic derivative $\sigma = -\omega\, \partial \log \rho(\omega)/\partial\omega$ in scaling regime of critical spectral function to extract dynamical critical exponent $z = (2 - \eta^{\perp})/\sigma$
(also compare against mean-field result $\sigma_{\text{mf}} = 1, \eta^{\perp}_{\text{mf}} = 0 \implies z_{\text{mf}} = 2$)

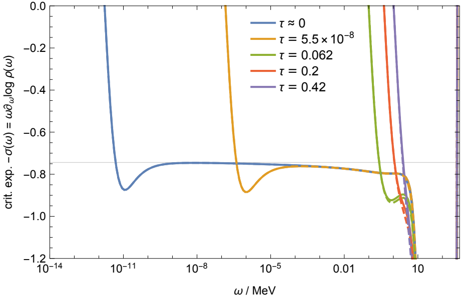



Figure: $d = 2$.

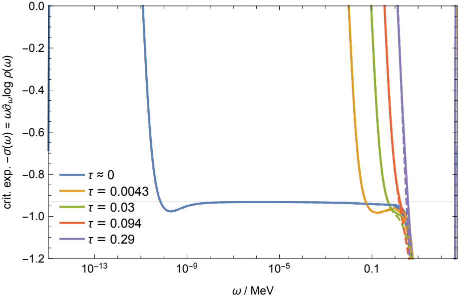



Figure: $d = 3$.

$z \approx 2.094 = 2 + c\eta^{\perp}$

cf. $z \stackrel{?}{=} 2.1665(12)$
Nightingale, Blöte '96
(Monte Carlo)

$z \approx 2.042 = 2 + c\eta^{\perp}$

cf. $z = 2.0245(15)$
Hasenbusch '20
(Monte Carlo)

Extraction scheme: **Model B**

Look at logarithmic derivative $\sigma = -\omega\,\partial \log \rho(\omega)/\partial\omega$ in scaling regime of critical spectral function to extract dynamical critical exponent $z = (2-\eta^{\perp})/\sigma$

(also compare against mean-field result $\sigma_{\text{mf}} = \frac{1}{2}, \eta^{\perp}_{\text{mf}} = 0 \implies z_{\text{mf}} = 4$)

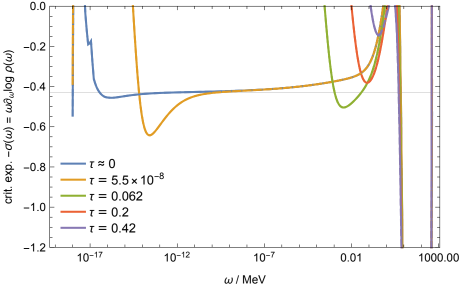



Figure: $d = 2$.

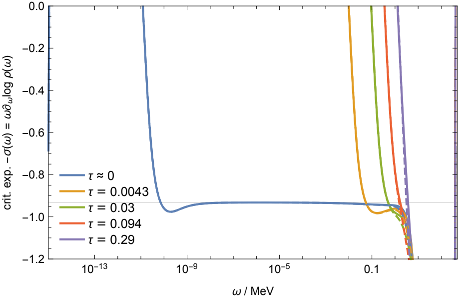



Figure: $d = 3$.

$z \approx 3.55 = 4 - \eta^{\perp}$ cf. $z = 3.75$
Onsager's solution of $2d$ Ising model

$z \approx 3.90 = 4 - \eta^{\perp}$ cf. $z = 3.964$
Kos et al. '16, Komargodski et al. '17 (Conformal bootstrap)

Extraction scheme: **Model C**

Look at logarithmic derivative $\sigma = -\omega\,\partial \log\rho(\omega)/\partial\omega$ in scaling regime of critical spectral function to extract dynamical critical exponent $z = (2-\eta^{\perp})/\sigma$

(also compare against mean-field result $\sigma_{\text{mf}} = 1, \eta^{\perp}_{\text{mf}} = 0 \implies z_{\text{mf}} = 2$)

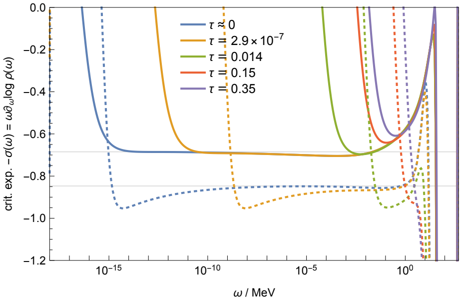



Figure: $d = 2$.

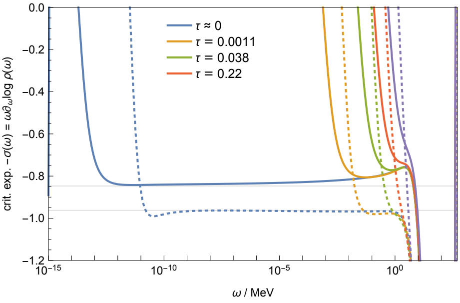



Figure: $d = 3$.

$z \approx 2.56 = 2 + \alpha/\nu$ cf. $z = 2$ Onsager's solution of $2d$ Ising model (problematic...)

$z \approx 2.31 = 2 + \alpha/\nu$ cf. $z = 2.175$ Kos et al. '16, Komargodski et al. '17 (Conformal bootstrap)

# Summary & outlook

We have

- constructed regulators in the real-time FRG which automatically take care of causality and Lorentz invariance, and
- calculated critical spectral functions using one and two-loop self-consistent truncation schemes in Model A, B, and C.

For the future, we plan to

- extract universal scaling functions which describe universal behaviour in close vicinity of critical point,
- inspect real-time dynamics of Model G and H,
- include fermions ($\rightsquigarrow$ low-energy effective models of QCD in real time), and
- analyze non-equilibrium phenomena.

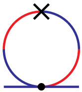
**Thank you for your attention!**

# Appendix

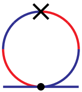
# Causal Regulators?

Diagram(s) that correspond to the unphysical upper left ($cc$) component of the Keldysh action,

$$\partial_k \Gamma_k^{cc} = \frac{-i}{4} \Bigg[$$



$$+$$



$$\Bigg]$$

$$= \frac{i\lambda_k}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \Big( G_k^R(\omega) \partial_k R_k^R(\omega) G_k^R(\omega) + G_k^A(\omega) \partial_k R_k^A(\omega) G_k^A(\omega) \Big)$$

$\stackrel{!}{=} 0$ for a flow that respects the causal structure of the action.

Propagators:

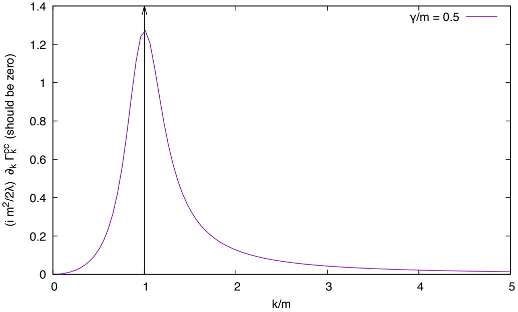
$$G_k^{R(A)}(\omega) = -\frac{1}{\omega^2 \pm i\gamma\omega - m^2 + R_k^{R(A)}(\omega)}$$

- Well-known regulator from the Euclidean FRG (Litim '01)
- Regulator has the form

$$R_k^{R/A}(\omega) = (k^2 - \omega^2)\theta(k^2 - \omega^2),$$

with a sharp cutoff at $\omega = k$.

- Result:



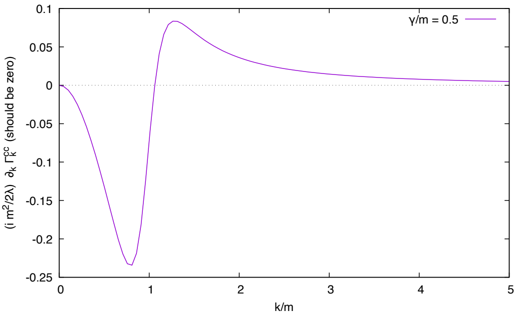

- Flow indeed generates an unphysical $cc$ component in the action.
- Pole at $k = m$ !

- Is it the sign?
- Regulator now has the form

$$R_k^{R/A}(\omega) = -(k^2 - \omega^2)\theta(k^2 - \omega^2),$$

still with a sharp cutoff at $\omega = k$.

- Result:




+ No more singularities in the flow.
− Flow still generates an unphysical $cc$ component in the action.

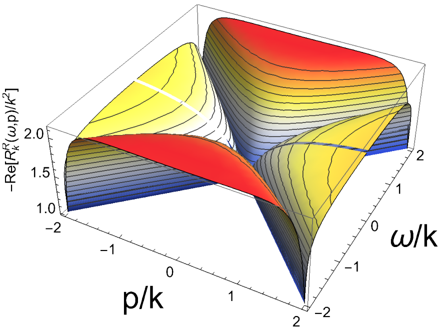



Figure: Real part (Mass shift).

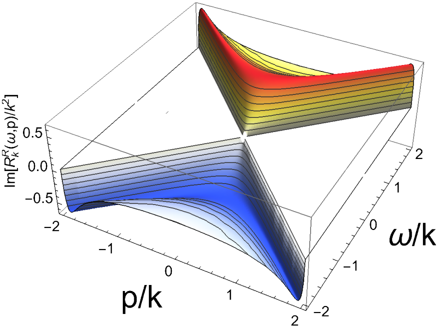



Figure: Imaginary part (Damping).