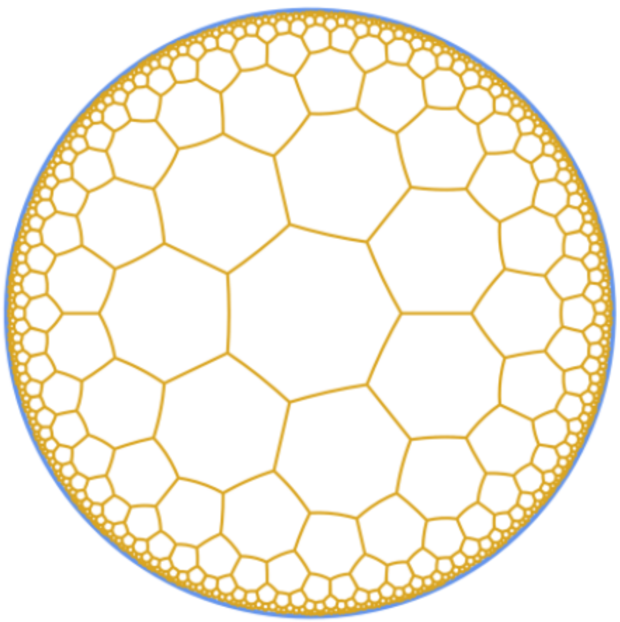
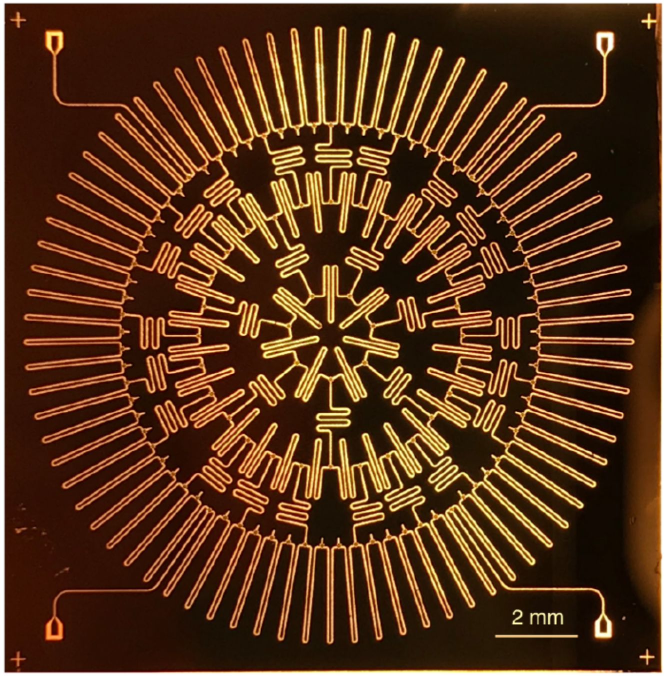
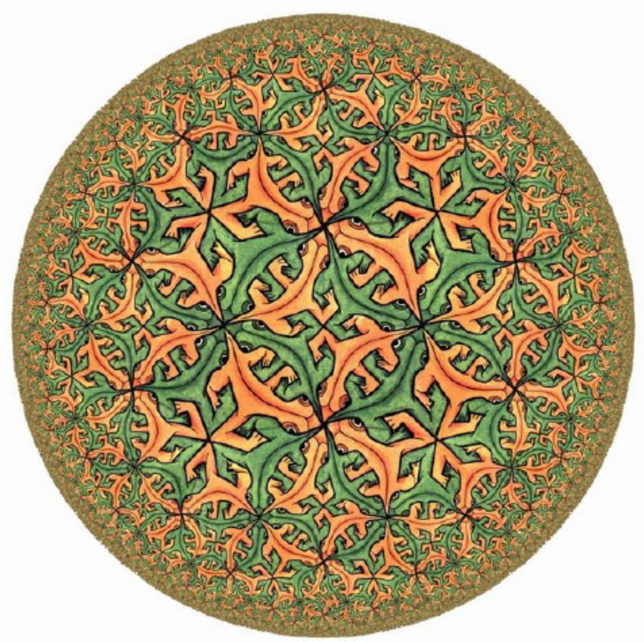
Quantum Simulation of Hyperbolic Space with Circuit QED: From Graphs to Geometry



Igor Boettcher, University of Maryland

IB, Bienias, Belyanksy, Kollár, Gorshkov, arXiv:1910.12318

JULY 12, 2019

Strange warping geometry helps to push scientific boundaries

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by Molly Sharlach, Princeton University

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International journal of science

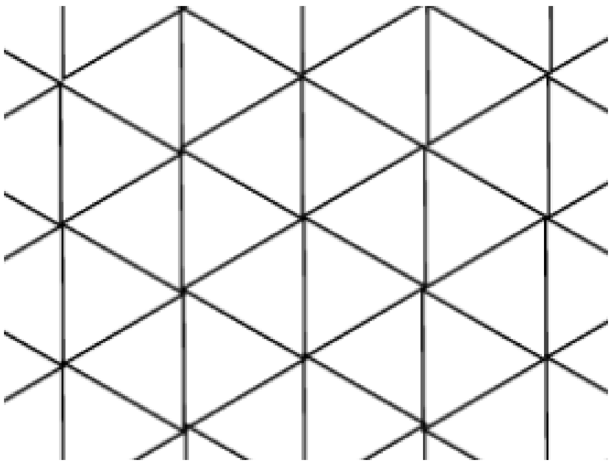
Article | Published: 03 July 2019

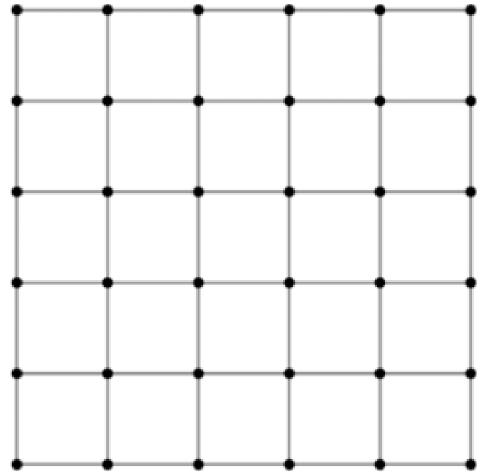
Hyperbolic lattices in circuit quantum electrodynamics

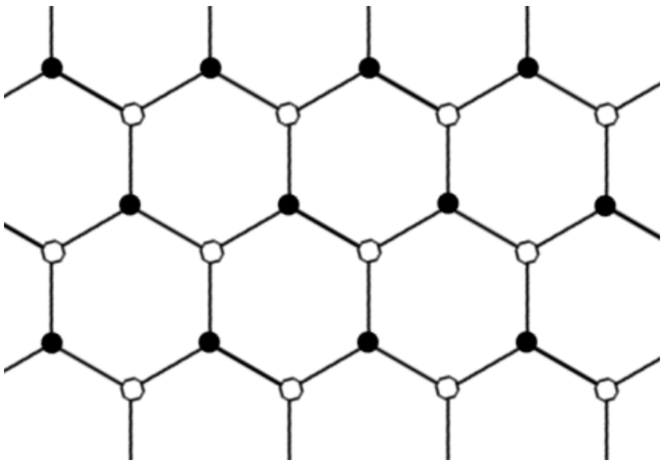
Alicia J. Kollár, Mattias Fitzpatrick & Andrew A. Houck

Nature **571**, 45–50 (2019) | Download Citation

$\{3, 6\}$ $\{4, 4\}$ $\{6, 3\}$



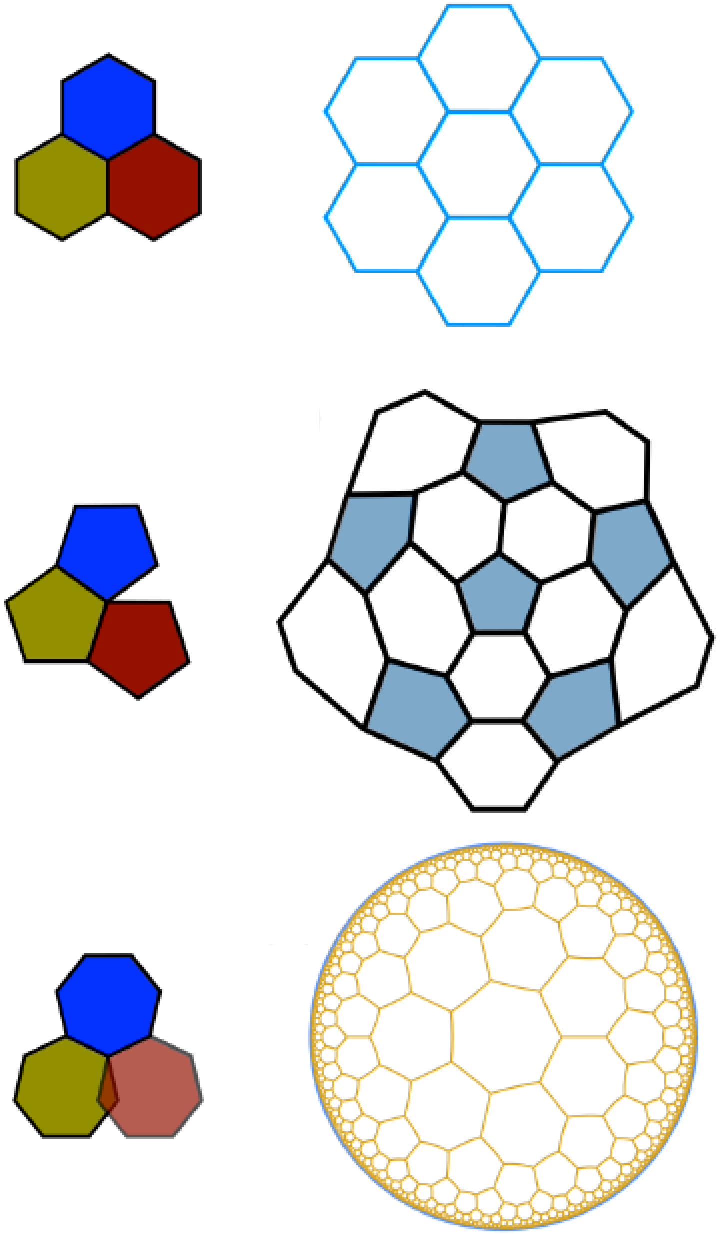


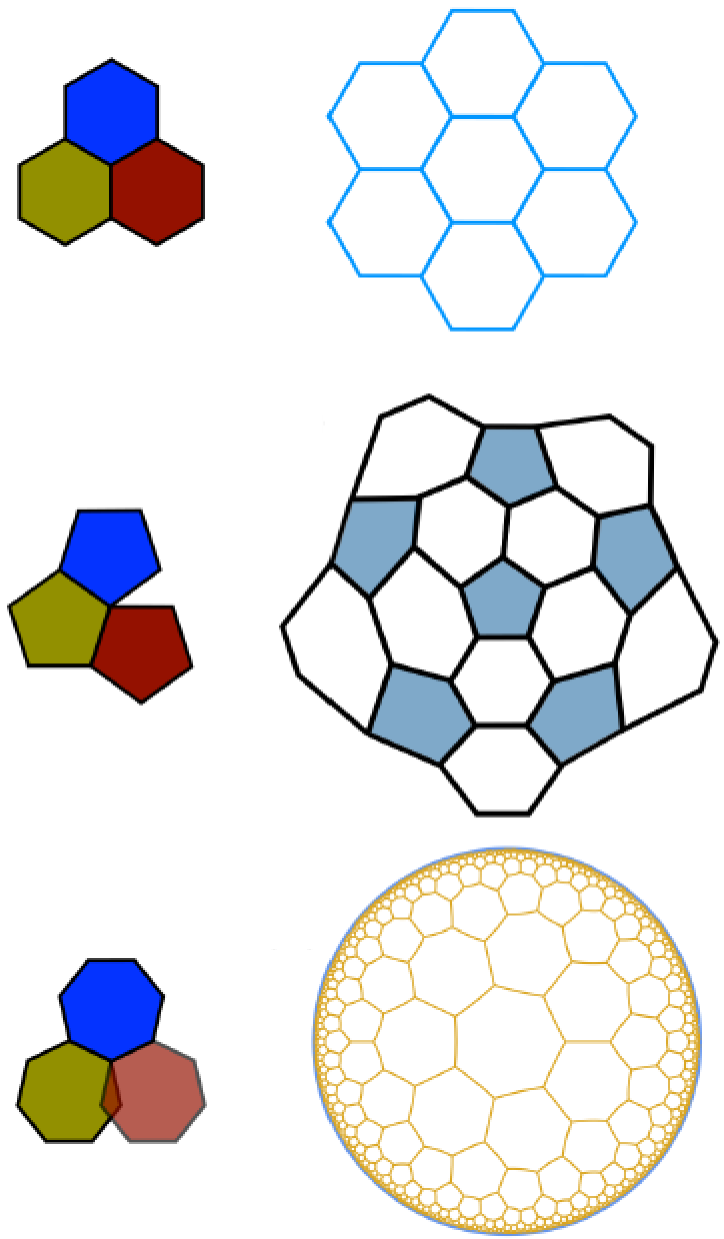


$\{p, q\}$ Schläfli-symbol:
lattice made from regular p-gons,
q lines meet at each vertex

$$(p-2) \cdot (q-2) = 4$$

condition for regular tessellation of the plane

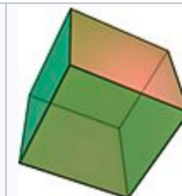
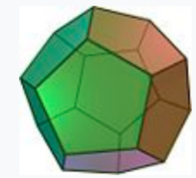


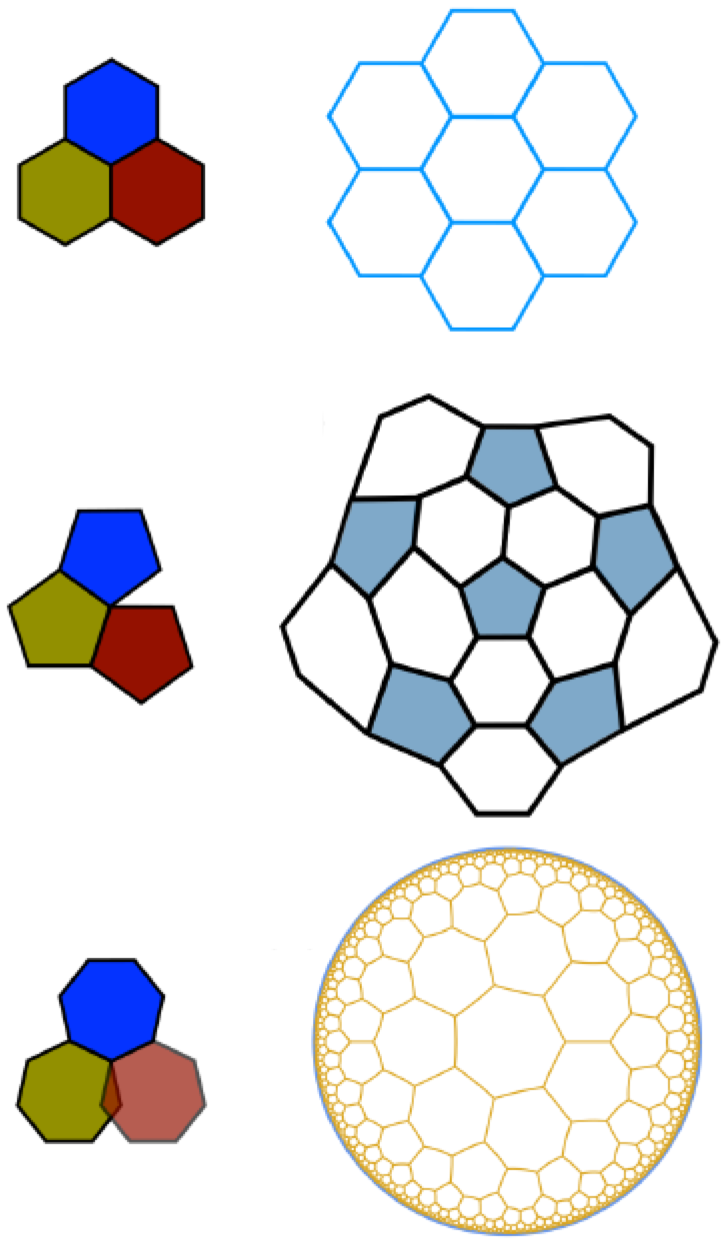


$$(p-2)\cdot(q-2) < 4$$

Platonic solids [edit]

Main article: Platonic solid

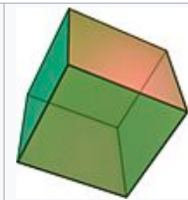
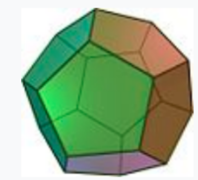
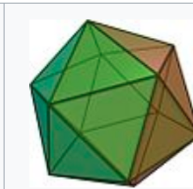
				
Tetrahedron {3, 3}	Cube {4, 3}	Octahedron {3, 4}	Dodecahedron {5, 3}	Icosahedron {3, 5}
$\chi = 2$	$\chi = 2$	$\chi = 2$	$\chi = 2$	$\chi = 2$



$$(p-2)\cdot(q-2) < 4$$

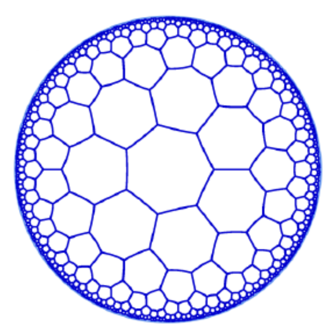
Platonic solids [edit]

Main article: Platonic solid

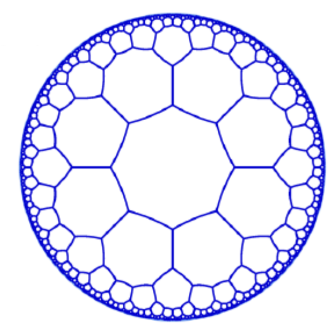
				
Tetrahedron {3, 3}	Cube {4, 3}	Octahedron {3, 4}	Dodecahedron {5, 3}	Icosahedron {3, 5}
$\chi = 2$	$\chi = 2$	$\chi = 2$	$\chi = 2$	$\chi = 2$

hyperbolic tessellations

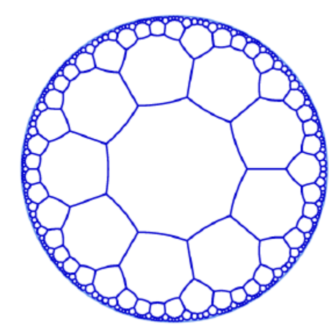
$$(p-2)\cdot(q-2) > 4$$



$\{7,3\}$

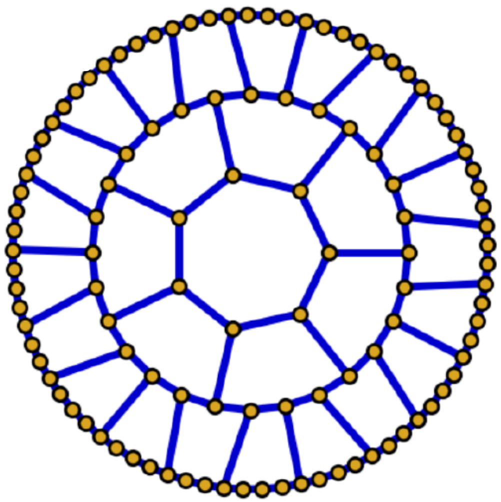


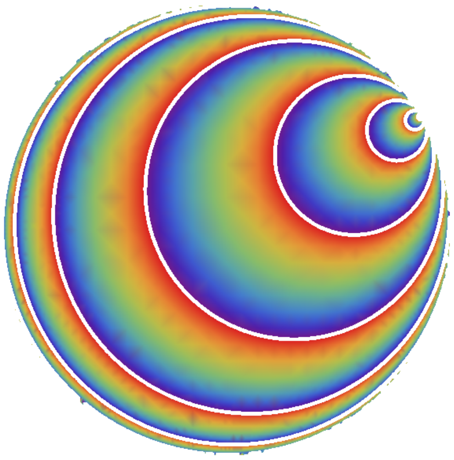
$\{8,3\}$



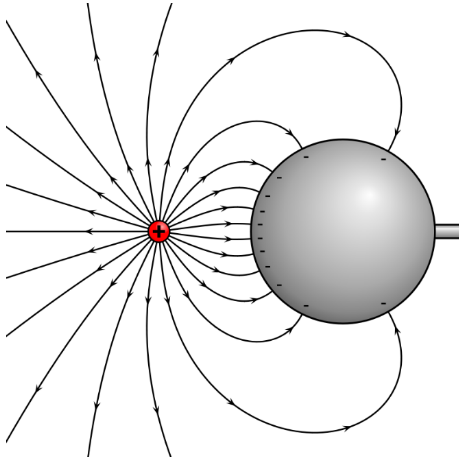
$\{9,3\}$

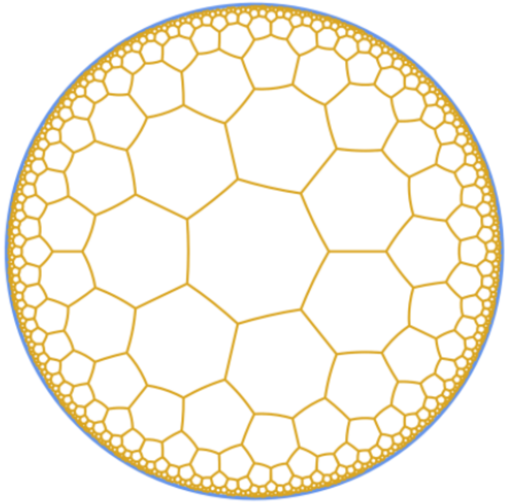
Outline

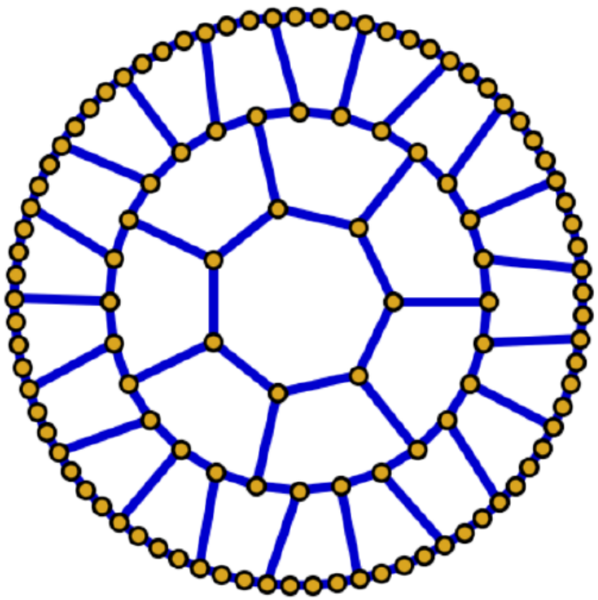




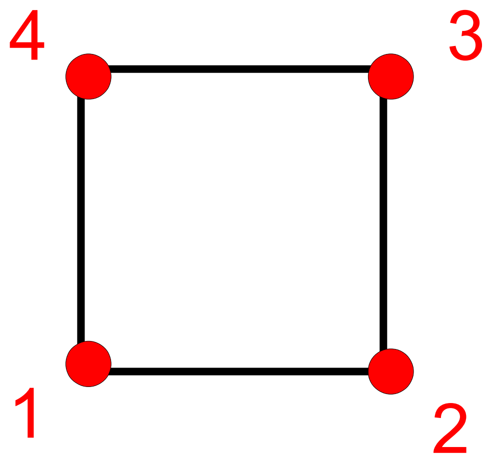
- Graphs
- Geometry
- From Graphs to Geometry
- Applications
- Experimental outlook



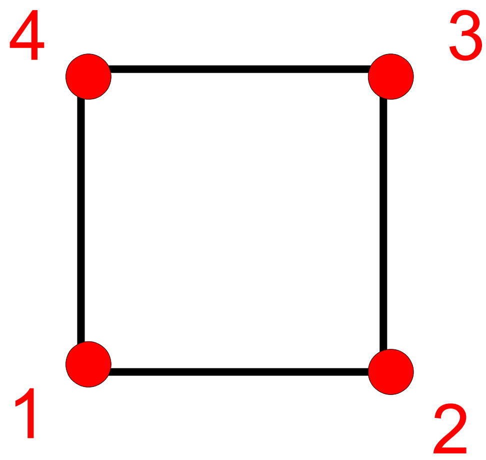




Graphs

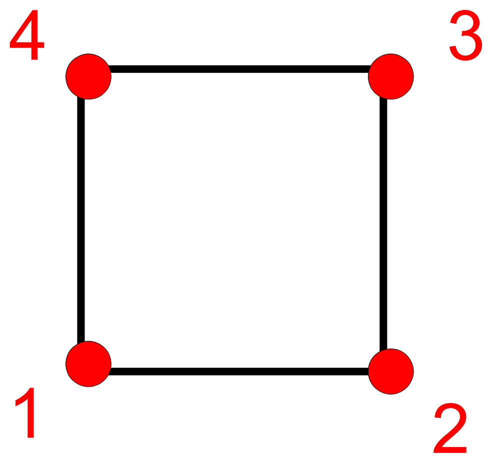


$$H = \sum_{\langle i,j \rangle} (a_i^\dagger a_j + \mathrm{h.c})$$



$$H = \sum_{\langle i,j \rangle} (a_i^\dagger a_j + \text{h.c})$$

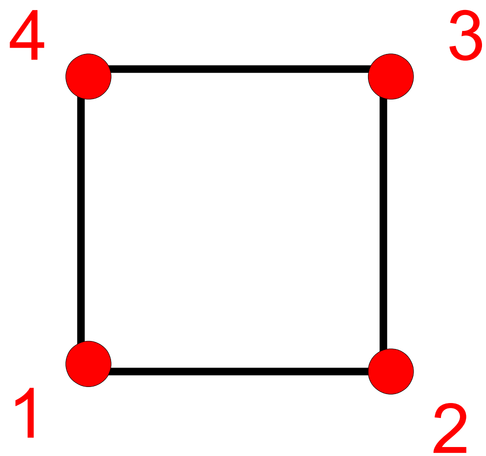
four operators: $a = (a_1, a_2, a_3, a_4)^T$



$$H = \sum_{\langle i,j \rangle} (a_i^\dagger a_j + \text{h.c})$$

$$H = a_1^\dagger a_2 + a_2^\dagger a_3 + a_3^\dagger a_4 + a_4^\dagger a_1 + \text{h.c.}$$

$$= \underbrace{(a_1^\dagger, a_2^\dagger, a_3^\dagger, a_4^\dagger)}_{a^\dagger} \underbrace{\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}}_{a}$$



$$H = \sum_{i,j} a_i^\dagger A_{ij} a_j = a^\dagger A a$$

"adjacency matrix"

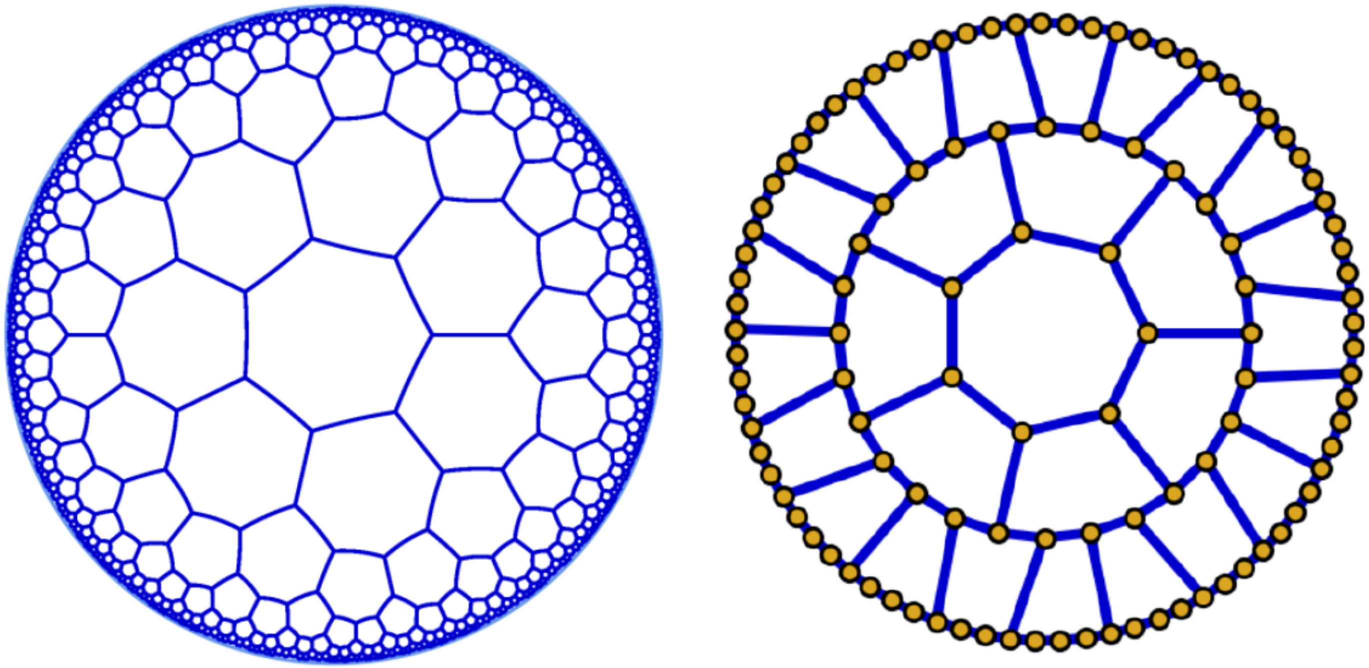
$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

- real and symmetric
- exists orthogonal O:

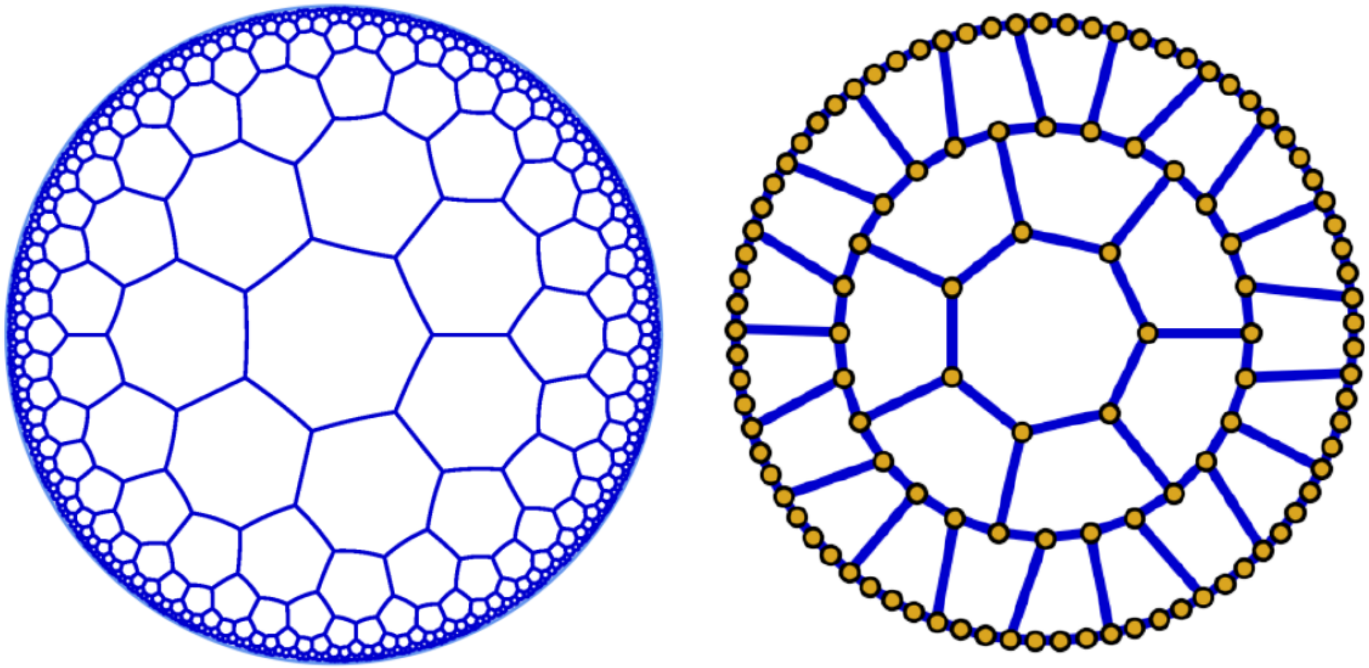
$$OAO^T = D = \text{diagonal}$$

$$H = (a^\dagger O^T)(OAO^T)\underbrace{(Oa)}_{b} = b^\dagger D b = \sum_n D_n b_n^\dagger b_n$$

$$H = \sum_{i,j} a_i^\dagger A_{ij} a_j$$



$$H = \sum_{i,j} a_i^\dagger A_{ij} a_j$$

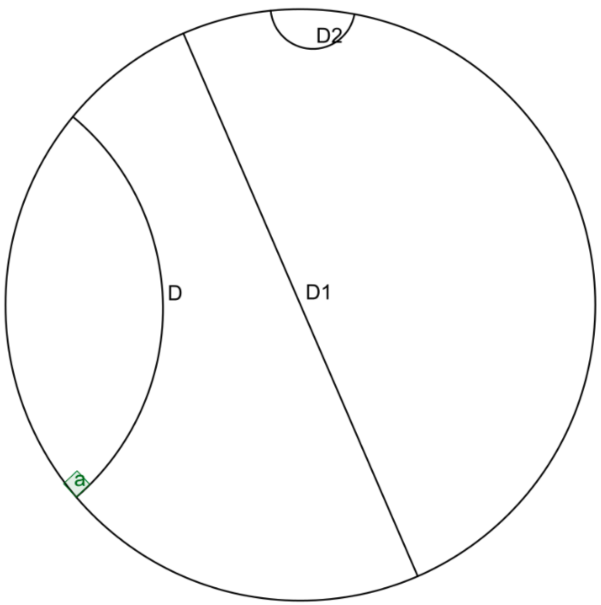


number of rings ℓ	1	2	3	4	5	6	7	8	9	10
number of sites N	7	35	112	315	847	2240	5887	15435	40432	105875

$$N \sim 7\varphi^{2\ell}, \;\; \varphi = \frac{1+\sqrt{5}}{2} = 1.618$$

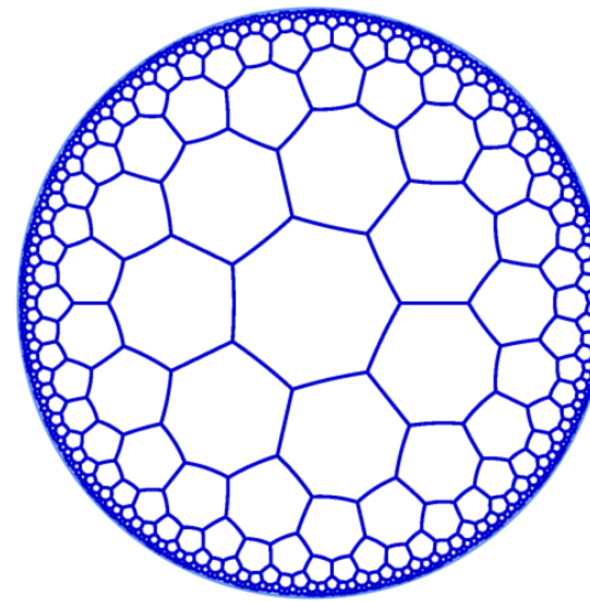
exponential growth
with graph diameter!

Geometry



Poincaré disk model of hyperbolic space

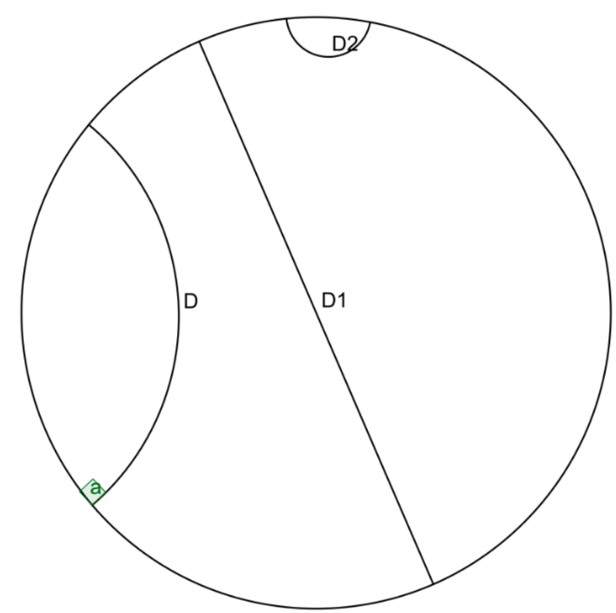
$$\mathbb{D} = \{z \in \mathbb{C},\ |z| < 1\}$$



$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{(1-|z|^2)^2} = \frac{\mathrm{d}r^2 + r^2\mathrm{d}\phi^2}{(1-r^2)^2}$$

$$z = re^{\mathrm{i}\phi} = x + \mathrm{i}y,\ R = 1$$

$$d(z,z') = \frac{1}{2}\mathrm{arcosh}\Big(1 + \frac{2|z-z'|^2}{(1-|z|^2)(1-|z'|^2)}\Big)$$



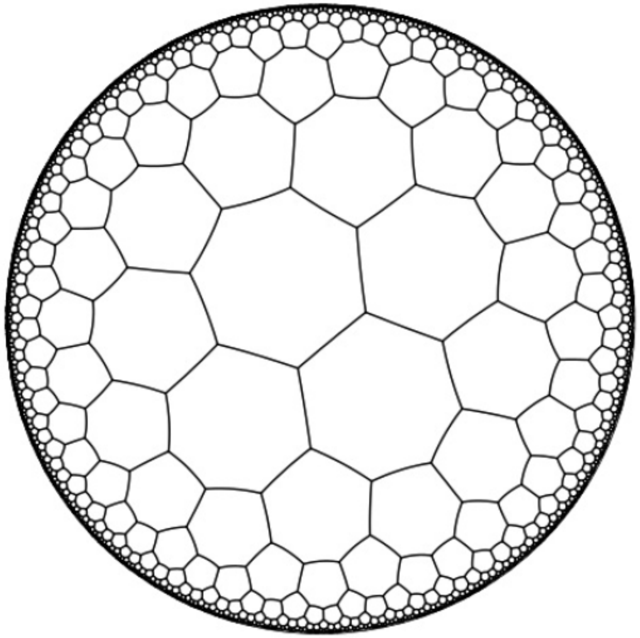
Poincaré disk model

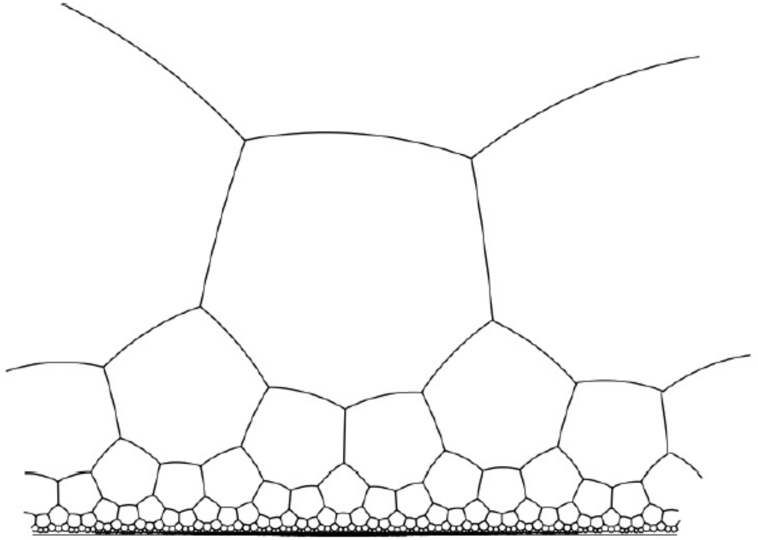
Poincaré half-plane model

$$z \in \mathbb{D} \xrightarrow{\text{Cayley}} \mathbb{H} = \{w \in \mathbb{C},\ \operatorname{Im} w > 0\}$$

$$w = \mathrm{i}\frac{1+z}{1-z}$$

$$ds^2 = \frac{dx^2 + dy^2}{4y^2}$$





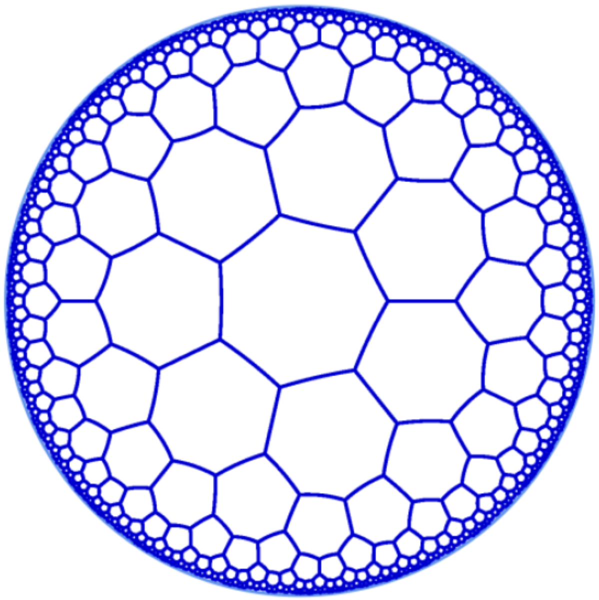
Poincaré disk model of hyperbolic space

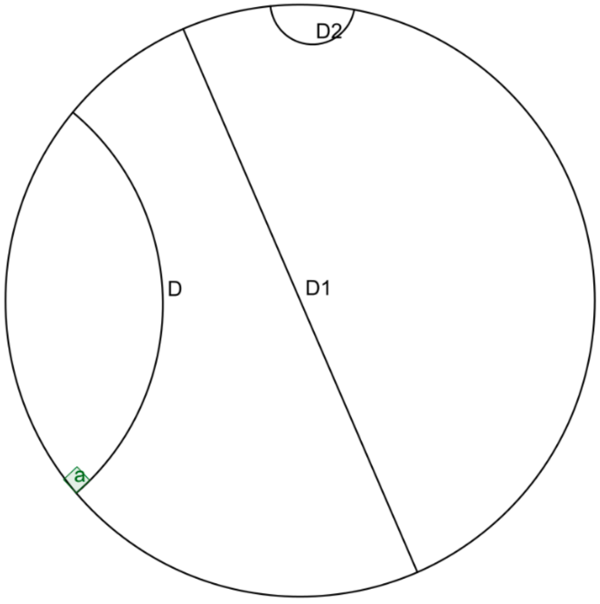
Automorphisms of D

$$z \mapsto e^{\mathrm{i}\alpha} \frac{z - a}{1 - z\bar{a}}, \;\; a \in \mathbb{D}$$

$$\mathrm{PSL}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R}) / \{\pm 1\}$$

Möbius / conformal transformations





non-commutative geometry

$$ds^2 = \frac{dx^2 + dy^2}{(1 - r^2)^2} = g_{ij} dx^i dx^j \qquad g^{ij} := (g^{-1})_{ij}$$

Laplace-Beltrami operator

$$\Delta_g f = \frac{1}{\sqrt{\det g}} \partial_i \left(\sqrt{\det g}\, g^{ij} \partial_j f \right)$$

$$ds^2 = \frac{dx^2 + dy^2}{(1 - r^2)^2} = g_{ij} dx^i dx^j \qquad g^{ij} := (g^{-1})_{ij}$$

Laplace-Beltrami operator

$$\Delta_g f = \frac{1}{\sqrt{\det g}} \partial_i \left(\sqrt{\det g}\, g^{ij} \partial_j f \right)$$

$$\langle f_1, f_2 \rangle = \int \underbrace{d^d x \sqrt{\det g}}_{dV_g} \; f_1^*(\vec{x}) \; f_2(\vec{x})$$

$$\begin{aligned}
\langle f_1, \Delta_g f_2 \rangle &= \int d^d x \sqrt{\det g} \; f_1 \; \frac{1}{\sqrt{\det g}} \partial_i \left(\sqrt{\det g}\, g^{ij} \partial_j f_2 \right) \\
&= - \int d^d x \sqrt{\det g} \; g^{ij} \partial_i f_1 \; \partial_j f_2 \\
&= \langle \Delta_g f_1, f_2 \rangle
\end{aligned}$$

self-adjoint

$$ds^2 = \frac{dx^2 + dy^2}{(1 - r^2)^2} = g_{ij} dx^i dx^j \qquad g^{ij} := (g^{-1})_{ij}$$

Laplace-Beltrami operator

$$\Delta_g f = \frac{1}{\sqrt{\det g}} \partial_i \left(\sqrt{\det g}\, g^{ij} \partial_j f \right)$$

Euclidean $\Delta_g^{(\mathrm{eucl})} = \delta^{ij} \partial_i \partial_j = \Delta$

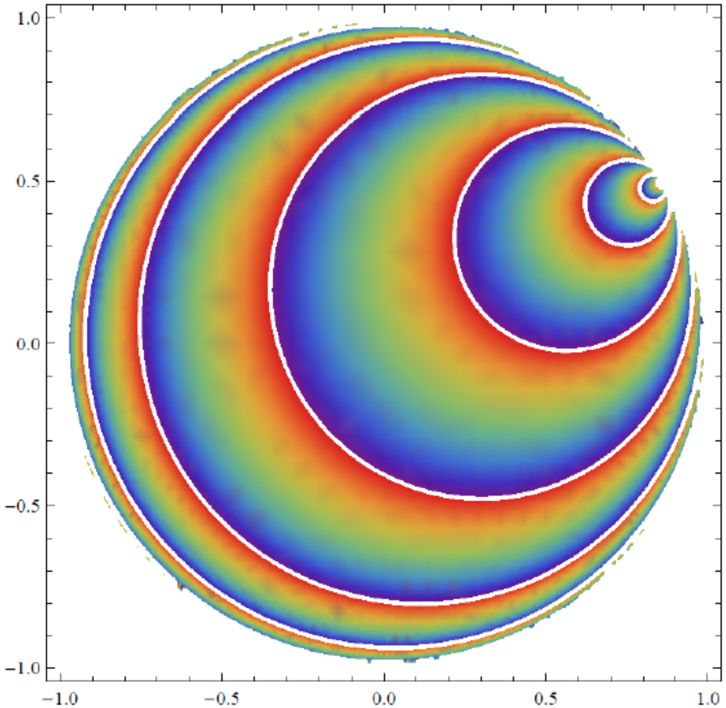
Minkowski $\Delta_g^{(\mathrm{mink})} = \eta^{\mu\nu} \partial_\mu \partial_\nu = \Box$

Poincaré disk* $\Delta_g = (1 - r^2)^2 \Delta$

Spectrum of $-\Delta_g$

$$\varepsilon = 1 + k^2$$

$$\psi_K(z) = \left(\frac{1-|z|^2}{|1-ze^{-i\beta}|^2}\right)^{\frac{1}{2}(1+ik)}$$

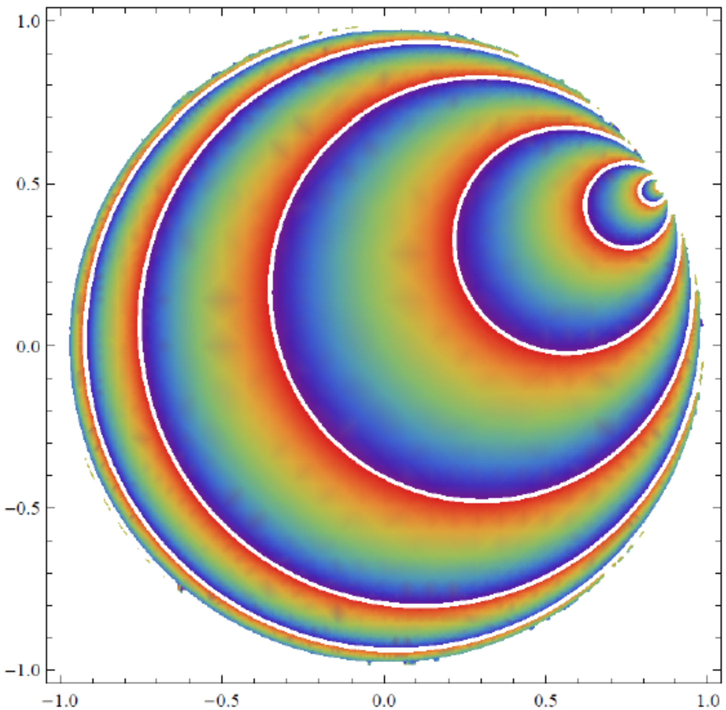


$$K = ke^{i\beta} \in \mathbb{C}$$

Spectrum of $-\Delta_g$

$$\varepsilon = 1 + k^2$$

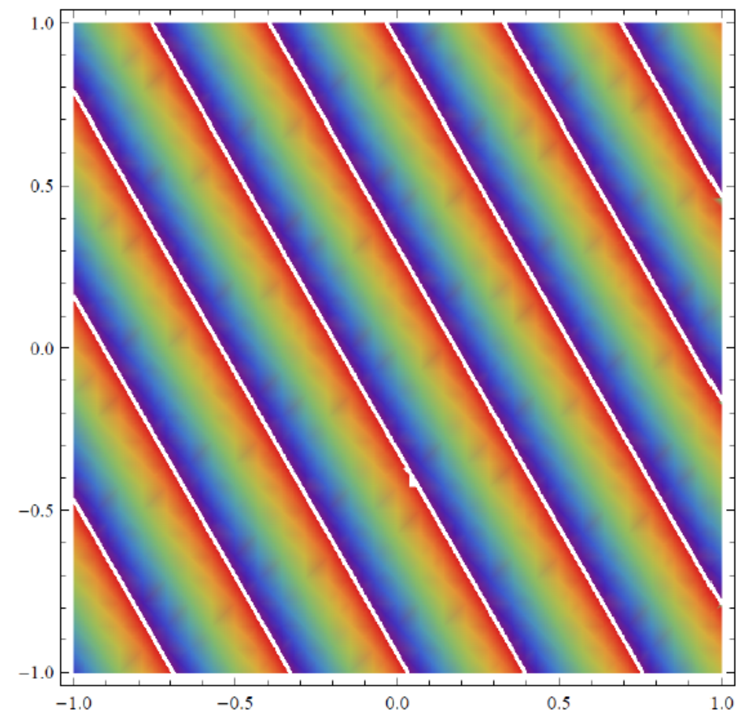
$$\psi_K(z) = \left(\frac{1-|z|^2}{|1-ze^{-i\beta}|^2}\right)^{\frac{1}{2}(1+ik)}$$



Euclidean limit $-\Delta$

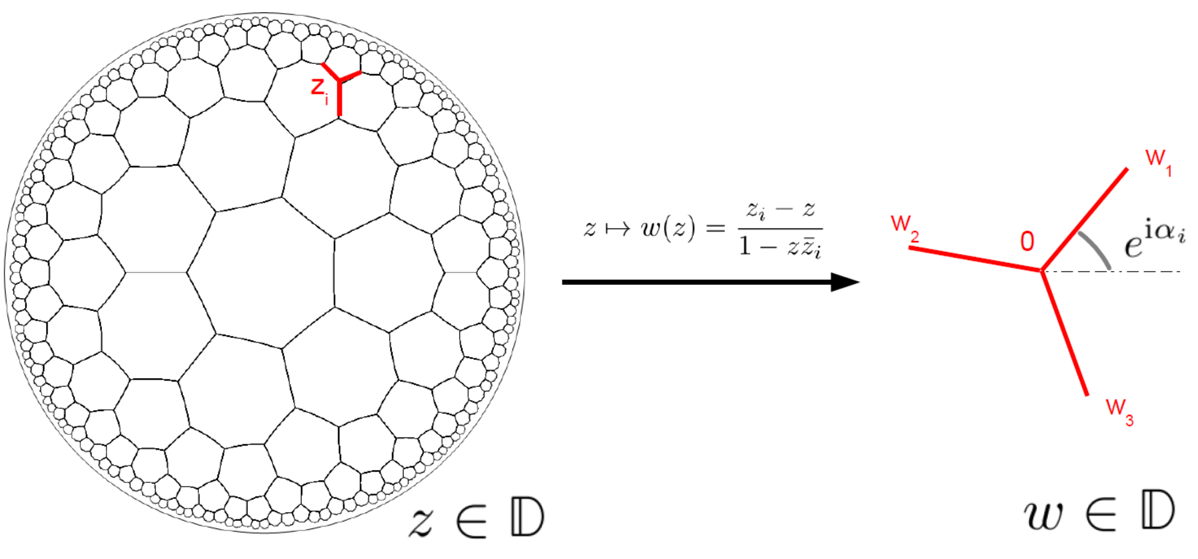
$$\varepsilon = k^2$$

$$e^{i\mathbf{k}\cdot\mathbf{r}} = \exp\left[\frac{i}{2}(K\bar{z} + \bar{K}z)\right]$$



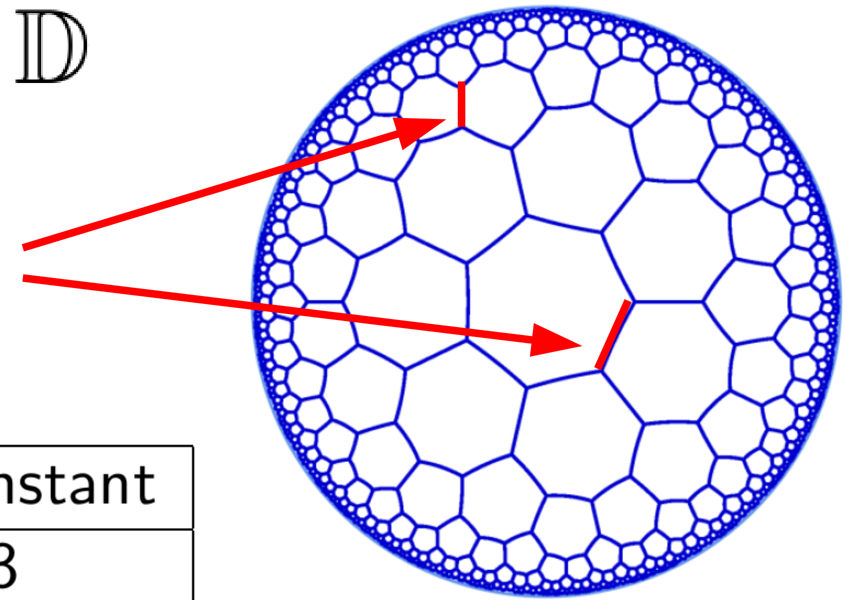
$$K = ke^{i\beta} \in \mathbb{C}$$

From Graphs to Geometry

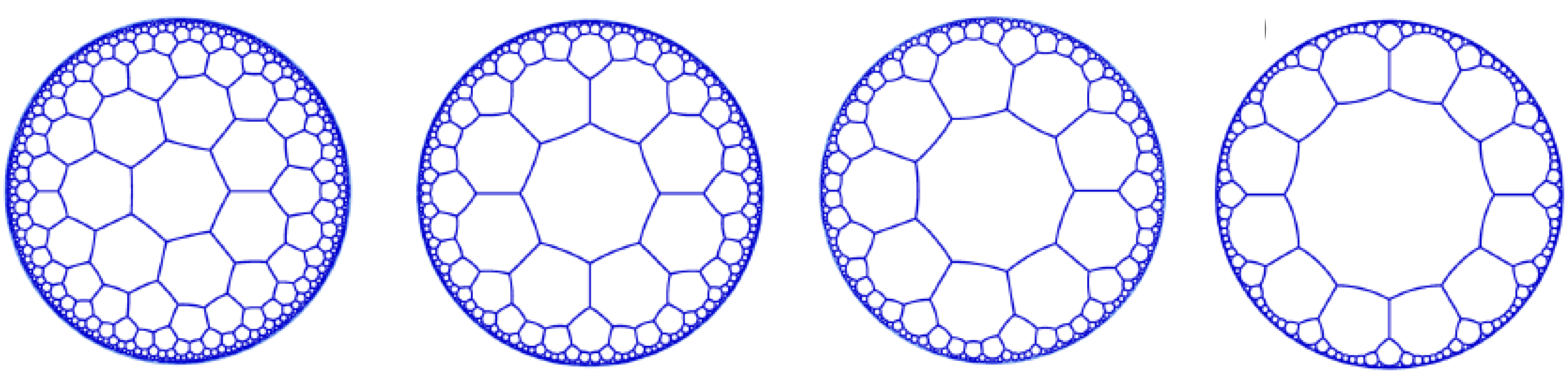


embedding $i \in G \mapsto z_i \in \mathbb{D}$

$$d(z_i, z_{i+e}) = d_0 = 0.283$$



Tiling polygon p	lattice constant
7	0.283
8	0.364
9	0.410
10	0.461

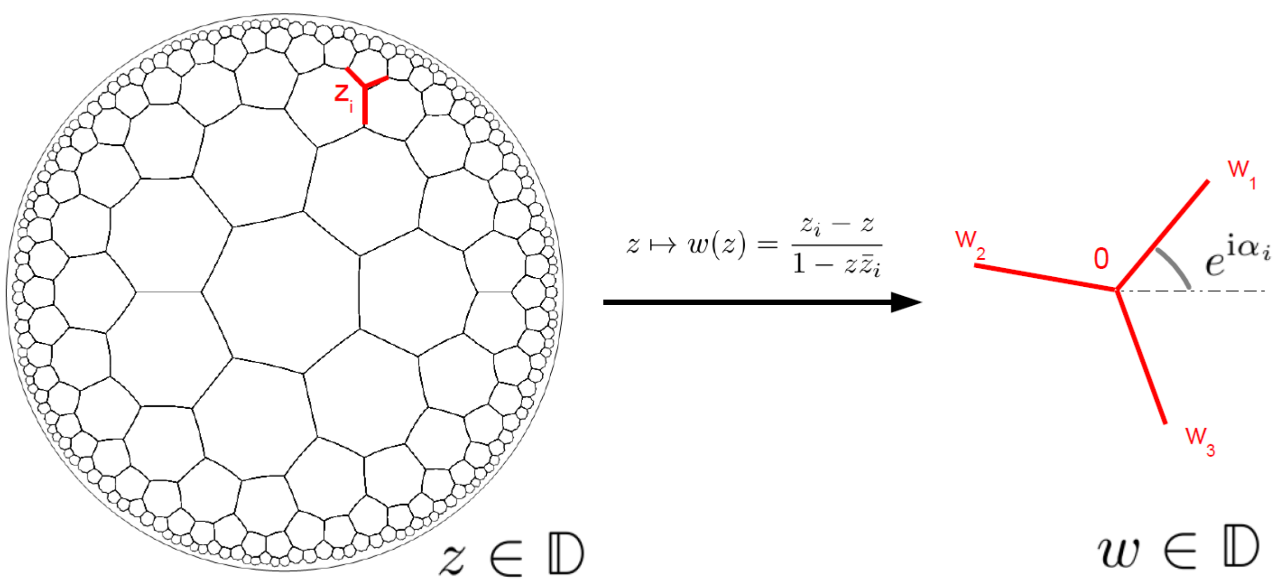


adjacency matrix $H = \sum_{i,j} a_i^\dagger A_{ij} a_j$

$$\sum_j A_{ij} f(z_j) = f(z_{i+e_1}) + f(z_{i+e_2}) + f(z_{i+e_3})$$

adjacency matrix $H = \sum_{i,j} a_i^\dagger A_{ij} a_j$

$$\sum_j A_{ij} f(z_j) = f(z_{i+e_1}) + f(z_{i+e_2}) + f(z_{i+e_3})$$



vertices of equilateral triangle

$$w_1 = w(z_{i+e_1}) = he^{\mathrm{i}\alpha},$$
$$w_2 = w(z_{i+e_2}) = he^{\mathrm{i}2\pi/3}e^{\mathrm{i}\alpha},$$
$$w_3 = w(z_{i+e_3}) = he^{\mathrm{i}4\pi/3}e^{\mathrm{i}\alpha}$$

$$d(h,0) \stackrel{!}{=} d_0 : \quad h = 0.276$$

adjacency matrix $H = \sum_{i,j} a_i^\dagger A_{ij} a_j$

$$\sum_j A_{ij} f(z_j) = f(z_{i+e_1}) + f(z_{i+e_2}) + f(z_{i+e_3})$$

$$= f\left(\frac{z_i - w_1}{1 - w_1 \bar{z}_i}\right) + f\left(\frac{z_i - w_2}{1 - w_2 \bar{z}_i}\right) + f\left(\frac{z_i - w_3}{1 - w_3 \bar{z}_i}\right)$$

$$= 3f(z_i) + \frac{3}{4} h^2 \Delta_g f(z_i) + \mathcal{O}(h^3)$$

$h = 0.276$

lattice sums $H = \sum_{i,j} a_i^\dagger A_{ij} a_j$

$$\sum_{i \in G} f(r_i) \approx \int \mathrm{d}\mathcal{N}(r) f(r)$$

$$\mathcal{N}(r) = \sum_{i \in G} \Theta(r - r_i)$$

counting function

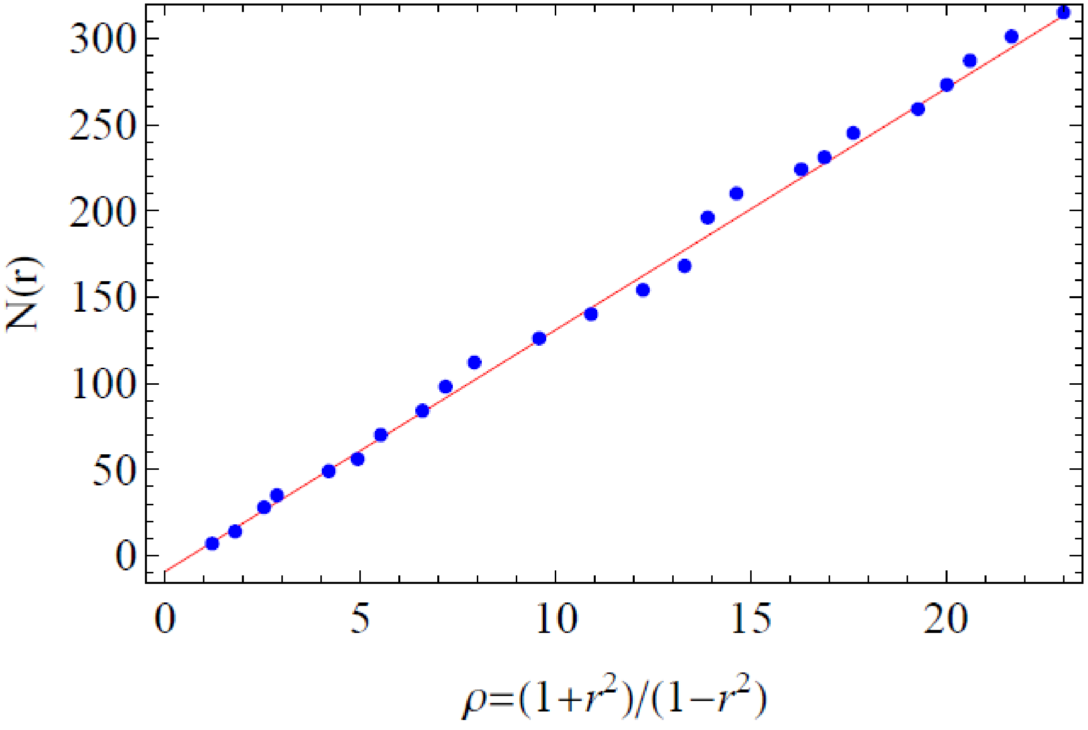
lattice sums $H = \sum_{i,j} a_i^\dagger A_{ij} a_j$

$$\sum_{i \in G} f(r_i) \approx \int \mathrm{d}\mathcal{N}(r) f(r)$$

$$\mathcal{N}(r) = \sum_{i \in G} \Theta(r - r_i)$$

$$\approx 14\rho + \text{const}$$

$$\mathrm{d}\rho = 4 \frac{\mathrm{d}r \; r}{(1-r^2)^2}$$



lattice sums $H = \sum_{i,j} a_i^\dagger A_{ij} a_j$

$$\sum_{i \in G} f(r_i) \approx \int \mathrm{d}\mathcal{N}(r) f(r) = 14 \int \mathrm{d}\rho \; f(r)$$

$$= 14 \cdot 4 \int_0^L \frac{\mathrm{d}r \; r}{(1 - r^2)^2} f(r)$$

$$= \frac{28}{\pi} \int_{|z| \leq L} \frac{\mathrm{d}^2 z}{(1 - |z|^2)^2} f(|z|)$$

lattice sums $H = \sum_{i,j} a_i^\dagger A_{ij} a_j$

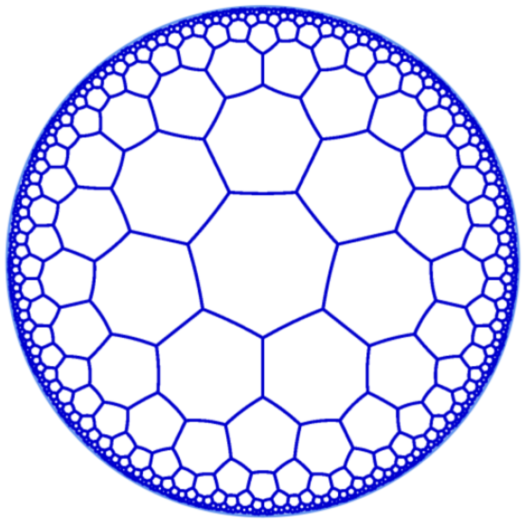
$$\sum_{i \in G} f(z_i) \approx \frac{28}{\pi} \int_{|z| \leq L} \frac{\mathrm{d}^2 z}{(1-|z|^2)^2} f(z)$$

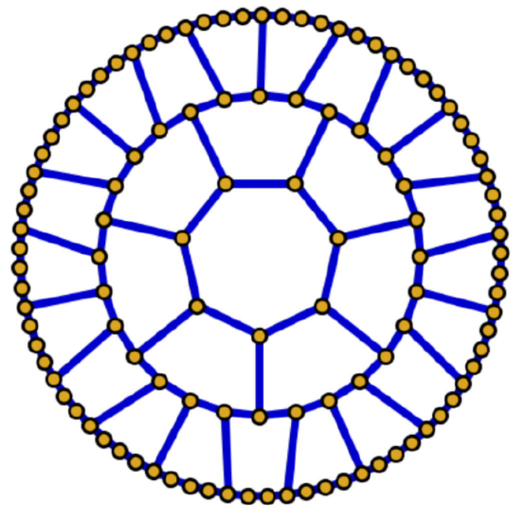
$$N \stackrel{!}{=} \frac{28}{\pi} \frac{\pi L^2}{(1-L^2)} \quad \Leftrightarrow \quad L = \sqrt{\frac{N}{N+28}}$$

number of rings ℓ	1	2	3	4	5	6	7	8	9	10
effective disk radius L	0.447	0.745	0.894	0.958	0.984	0.994	0.998	0.9990	0.9997	0.9999

Graph / Continuum Dictionary

finite graph with N sites	finite disk with radius $L < 1$
$i \in G$	$z_i \in \mathbb{D}$
f_i	$f(z_i)$
A_{ij}	$3 + \frac{3}{4} h^2 \Delta_g$
N	$L = \sqrt{\frac{N}{N+28}}$
$\sum_{i \in G}$	$\frac{28}{\pi} \int_{\lvert z \rvert \leq L} \frac{d^2 z}{(1-\lvert z \rvert^2)^2}$





number of rings ℓ	1	2	3	4	5	6	7	8	9	10
effective disk radius L	0.447	0.745	0.894	0.958	0.984	0.994	0.998	0.9990	0.9997	0.9999

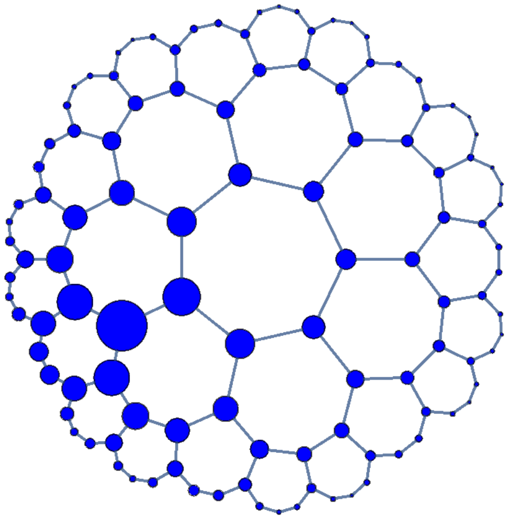
Example: Bose-Hubbard model

$$\hat{\mathcal{H}} = \sum_{i \in G} \left[-t \sum_{j \in G} \hat{a}_i^\dagger A_{ij} \hat{a}_j - \mu \hat{a}_i^\dagger \hat{a}_i + U(\hat{a}_i^\dagger \hat{a}_i)^2 \right]$$

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$$

$$\hat{\mathcal{H}}' = \int_{|z| \leq L} \frac{\mathrm{d}^2 z}{(1 - |z|^2)^2} \left[\hat{\alpha}_z^\dagger (-J\Delta_g - \mu') \hat{\alpha}_z + U' (\hat{\alpha}_z^\dagger \hat{\alpha}_z)^2 \right]$$

$$[\hat{\alpha}(z), \hat{\alpha}^\dagger(z')] = (1 - |z|^2)^2 \delta^{(2)}(z - z')$$



Applications

Application 1:

Ground state energy and spectral gap of

$$H = -\sum_{i,j} a_i^\dagger A_{ij} a_j$$

$$\mathrm{spec}(-A) \subset (-3,3), \;\; E_0 = \min\left[\mathrm{spec}(-A)\right]$$

What about $\ell \to \infty$?

number of rings ℓ	1	2	3	4	5	6	7	8	9	10
E_0 (graph)	-2	-2.636	-2.787	-2.847	-2.877	-2.894	-2.905	-2.91	-2.92	-2.92

need sparse matrix techniques

Application 1:

Ground state energy and spectral gap of

$$H = -\sum_{i,j} a_i^\dagger A_{ij} a_j$$

Mathematicians know: $\lim_{\ell \to \infty} E_0 \in [-2.966, -2.862]$

Higuchi and Shirai, Interdiscip. Inform. Sci. 9, 221 (2003)
Paschke, Math. Z., 225 (1992)
Kollar, Fitzpatrick, Sarnak, Houck, arXiv:1902.02794

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Mathematicians know: $\lim_{\ell\to\infty} E_0 \in [-2.966, -2.862]$

Continuum theory: $E_0 = -3 + \frac{3}{4} h^2 (1 + k_0^2)$

$$\xrightarrow{\ell\to\infty} -3 + \frac{3}{4} h^2 = -2.94295$$

number of rings ℓ	1	2	3	4	5	6	7	8	9	10
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Application 1:

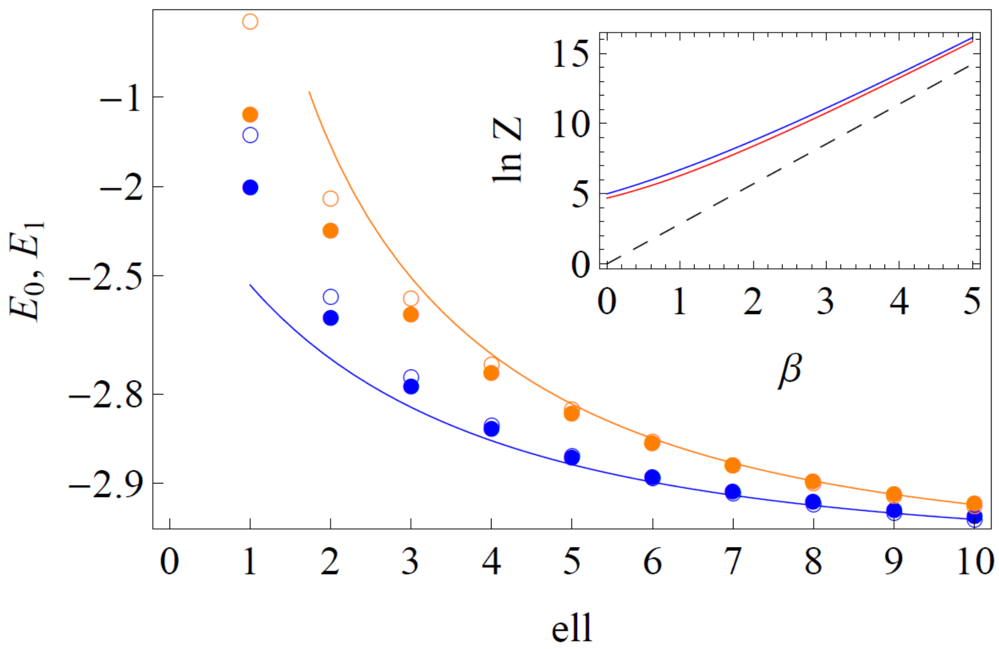
Ground state energy and spectral gap of

$$H = -\sum_{i,j} a_i^\dagger A_{ij} a_j$$

$$E_0^{(\mathrm{cont})} = -3 + \frac{3}{4} h^2 (k_0^2 + 1)$$

$$P_{\frac{1}{2}(-1+\mathrm{i}k_0)}\left(\frac{1+L^2}{1-L^2}\right) = 0$$

P_ν : Legendre function



number of rings ℓ	1	2	3	4	5	6	7	8	9	10
E_0 (graph)	-2	-2.636	-2.787	-2.847	-2.877	-2.894	-2.905	-2.91	-2.92	-2.92
E_0 (continuum)	-1.5	-2.570	-2.770	-2.842	-2.876	-2.895	-2.906	-2.914	-2.920	-2.924

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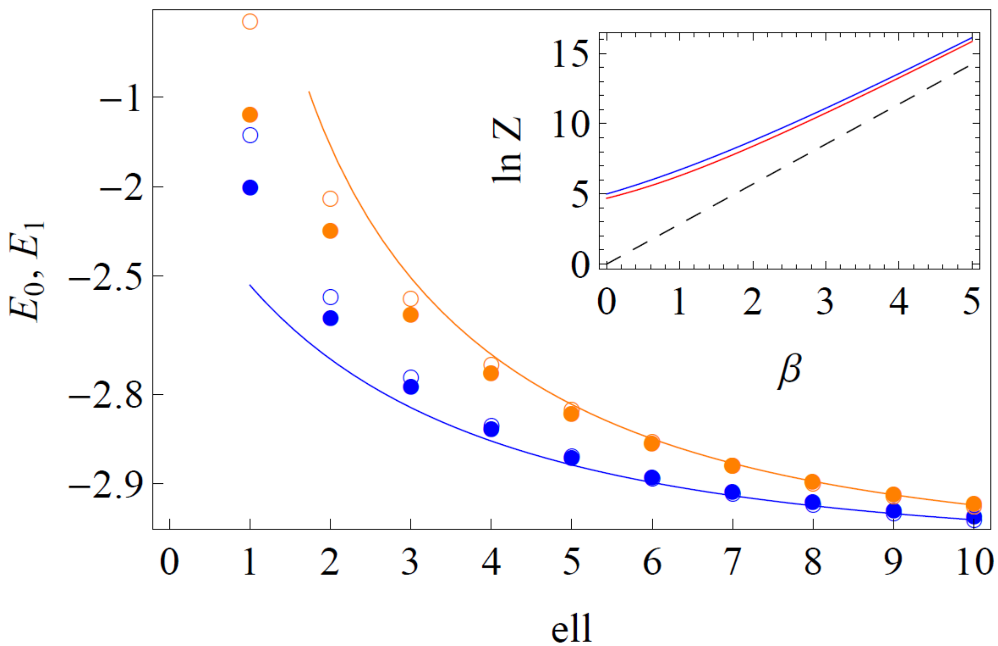

$$E_{0,1}^{(\mathrm{cont})} \sim E_\infty + \frac{3\pi^2 h^2}{4} \frac{1}{(\ln \varphi \cdot \ell + c_{0,1})^2}$$

$$\varphi = (1+\sqrt{5})/2, \; c_0 = \ln 2, \; c_1 = \ln 2 - 1$$

Application 2:

Correlation functions

$$G_{ij}(\omega) = \left(\frac{1}{\omega - H}\right)_{ij} = \langle a_i(\omega) a_j^{\dagger}(\omega) \rangle_0$$

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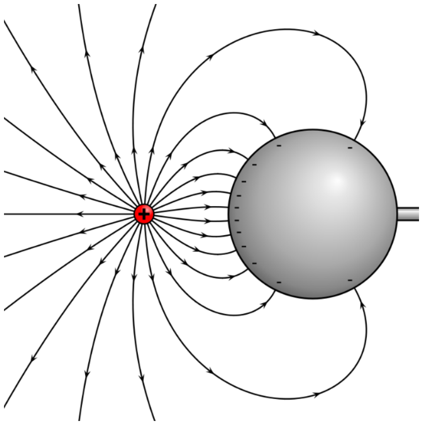
$$G_{ij}(\omega) \approx \frac{\pi}{21 h^2}\, G\left(z_i, z_j, \frac{4(\omega+3)}{3h^2}, L\right)$$

$$(\lambda + \Delta_g) G(z, z', \lambda, L) = -(1 - |z|^2)^2 \delta^{(2)}(z - z')$$



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Correlation functions

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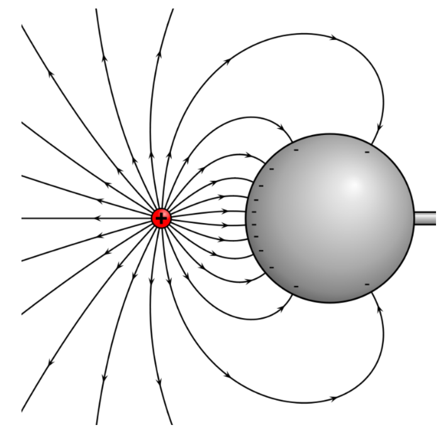
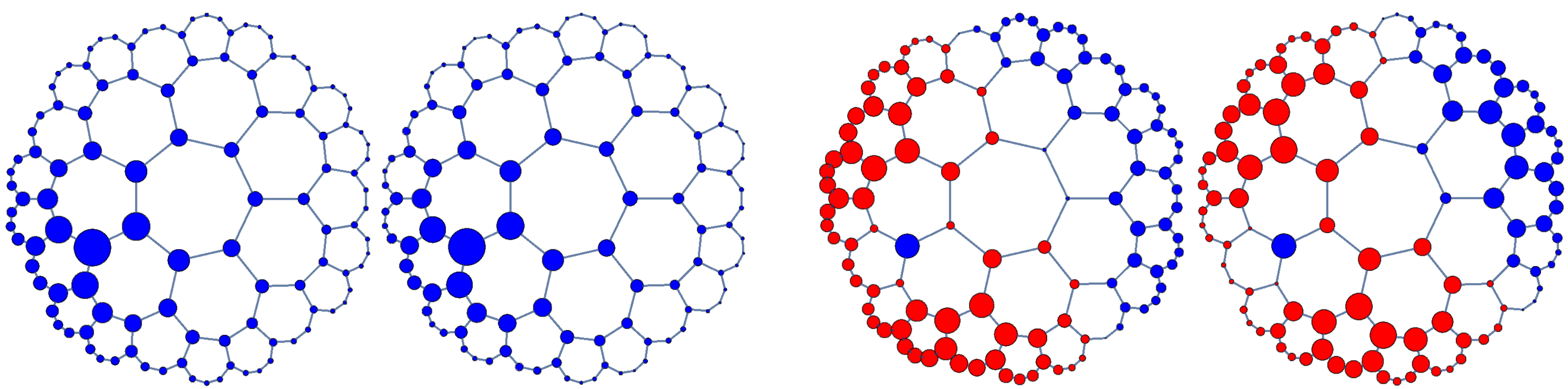


$\omega = -2.95$ $\qquad\qquad$ $\omega = -2.5 + 0.1\mathrm{i}$



G_{ij} $\qquad$ $G(z_i, z_j)$ $\qquad$ G_{ij} $\qquad$ $G(z_i, z_j)$

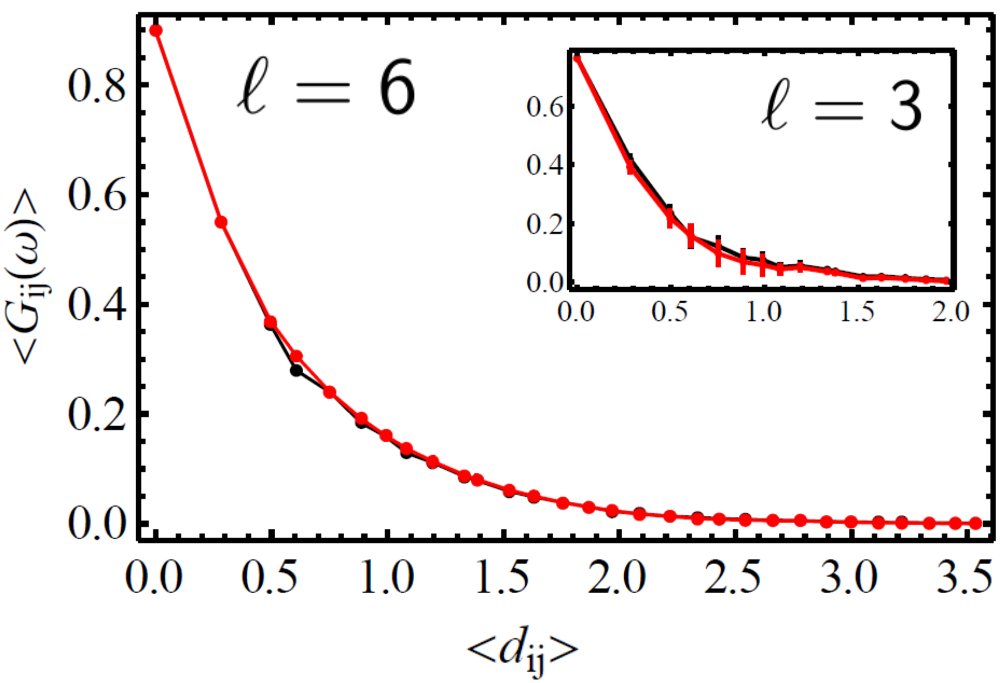
Application 2:

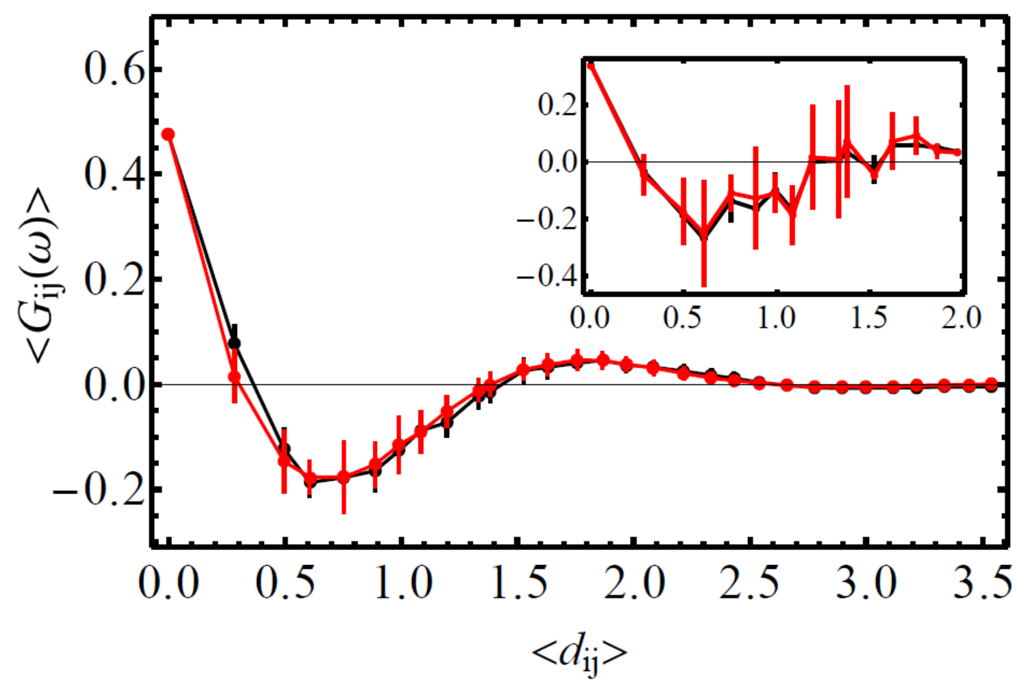
Correlation functions

$$G_{ij}(\omega) = \left(\frac{1}{\omega - H}\right)_{ij} = \langle a_i(\omega) a_j^\dagger(\omega) \rangle_0$$

emergent conformal symmetry:

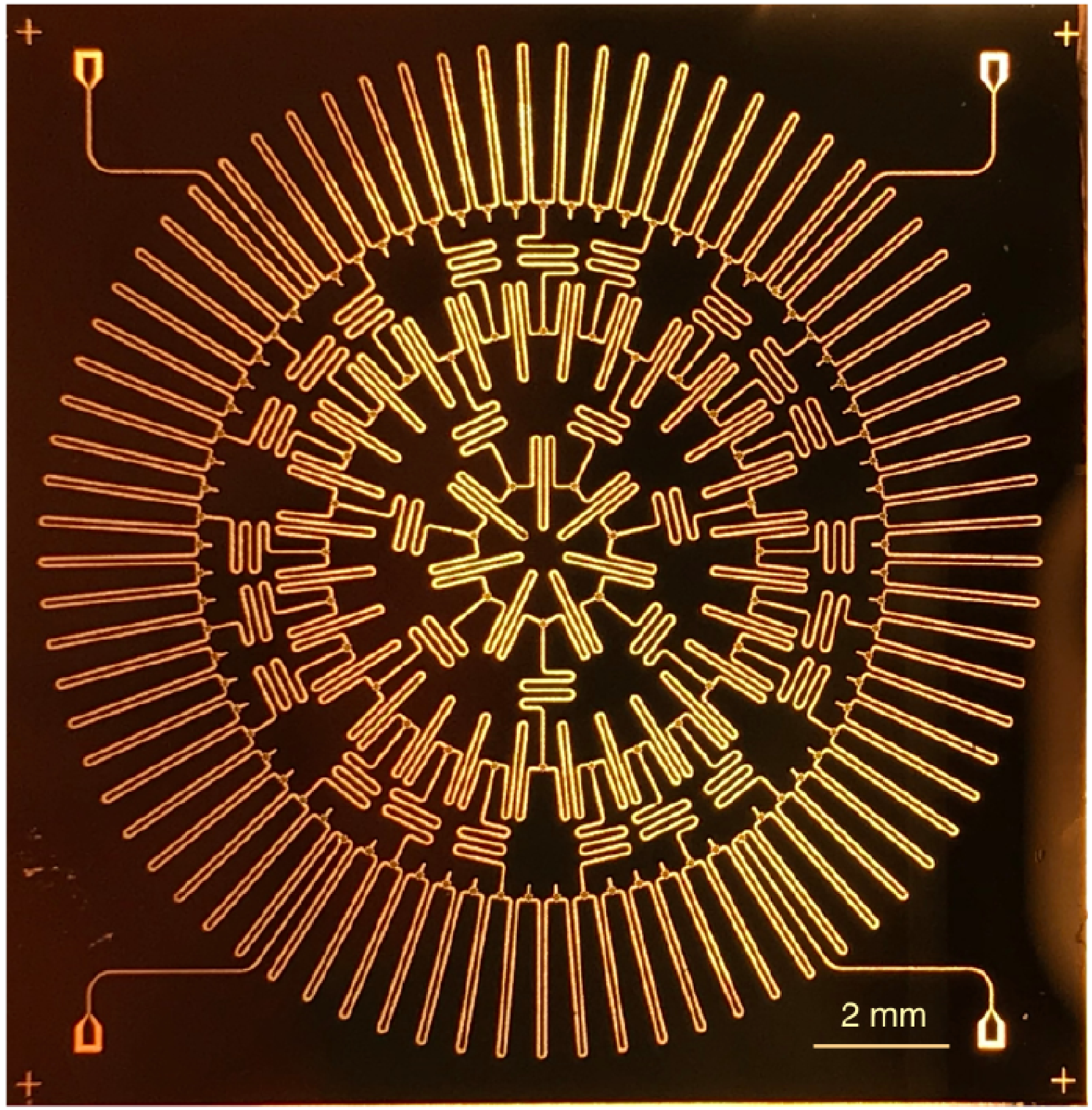
G_{ij} is (approximately) a function of $d_{ij} = d(z_i, z_j)$: $G_{ij} = f(d_{ij})$





Experimental outlook

$|\text{out}\rangle$



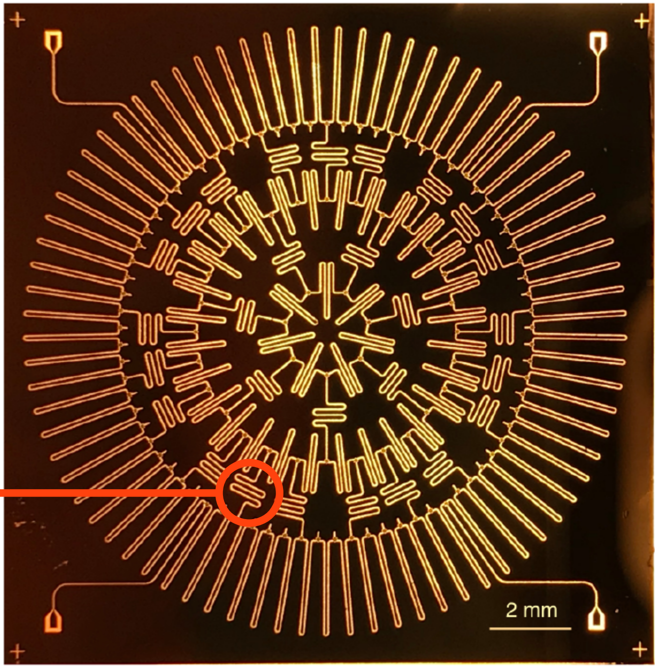
$|\text{in}\rangle$

scattering experiments, relation to quantum optics

Experimental outlook

$$\mathcal{H} = -\sum_{i,j} \underline{t_{ij}(B)} A_{ij} \hat{a}_i^\dagger \hat{a}_j$$

spatial / temporal variations in hopping



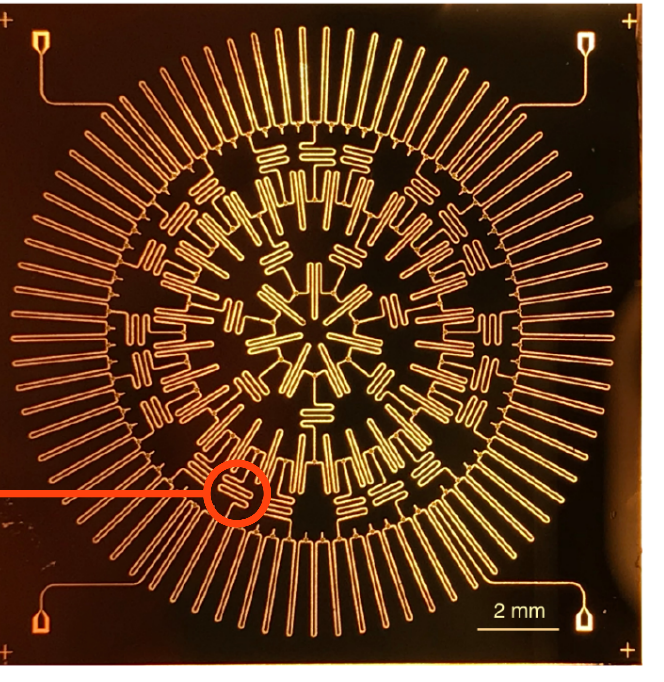
SQUID
loops (B)

Experimental outlook

$$\mathcal{H} = -\sum_{i,j} \underline{t_{ij}(B)} A_{ij} \hat{a}_i^\dagger \hat{a}_j + \sum_i \underline{\omega_i} \hat{a}_i^\dagger \hat{a}_i$$

spatial / temporal variations in hopping

disorder: localization transition?



SQUID
loops (B)

Experimental outlook

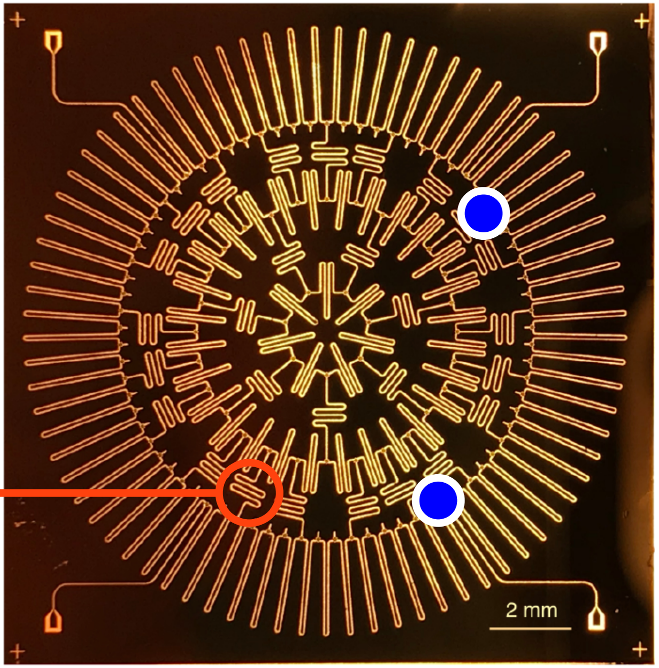
$$\mathcal{H} = -\sum_{i,j} \underline{t_{ij}(B)} A_{ij} \hat{a}_i^\dagger \hat{a}_j + \sum_i \underline{\omega_i} \hat{a}_i^\dagger \hat{a}_i$$

$$+ \underline{\Delta \sigma^z_{i_0} + g(\sigma^+_{i_0} \hat{a}_{i_0} + \sigma^-_{i_0} \hat{a}^\dagger_{i_0})}$$

spatial / temporal variations in hopping

disorder: localization transition?

tunable qubits: spin-boson model



SQUID
loops (B)

Experimental outlook

$$\langle S_i S_j \rangle \sim e^{-d(i,j)/\xi} \sim \frac{1}{|i-j|^{\xi}}$$

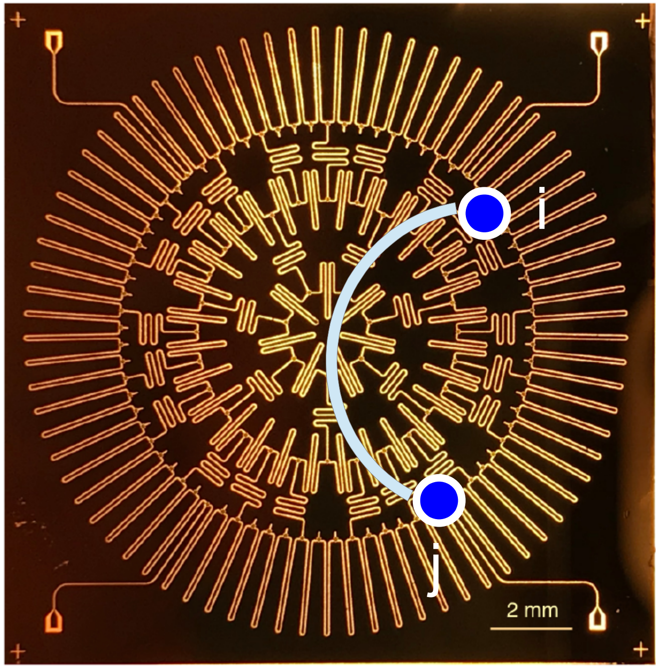
boundary critical correlations $d(i,j) \sim \log|i-j|$

$$\langle S_i S_j S_k \rangle, \ \langle S_i S_j S_k S_l \rangle$$??

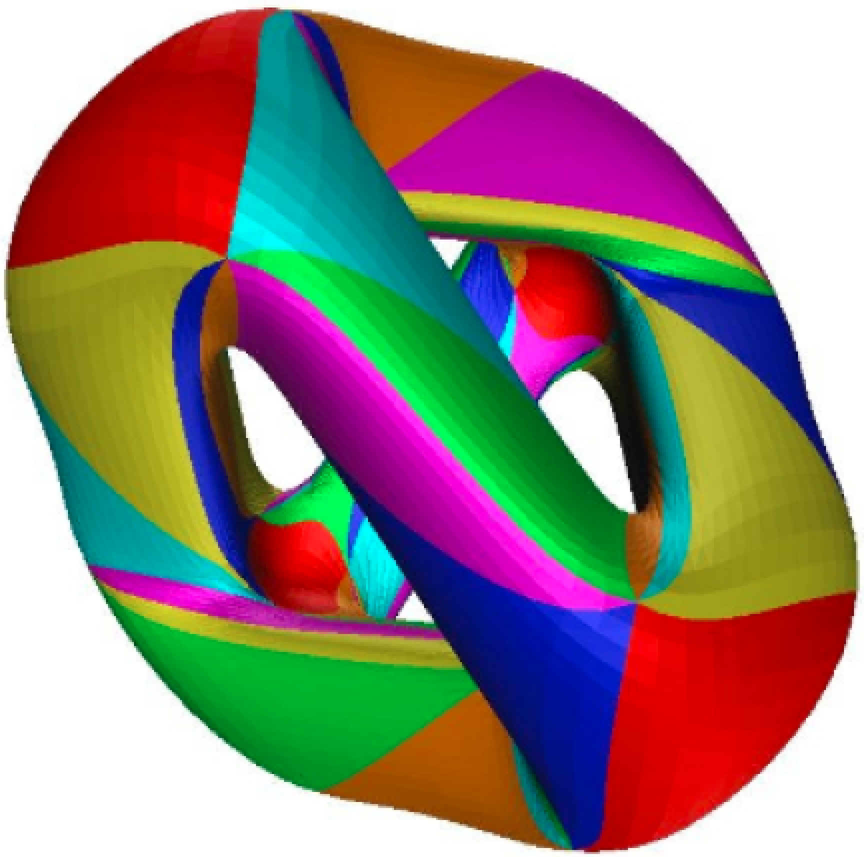
boundary fields

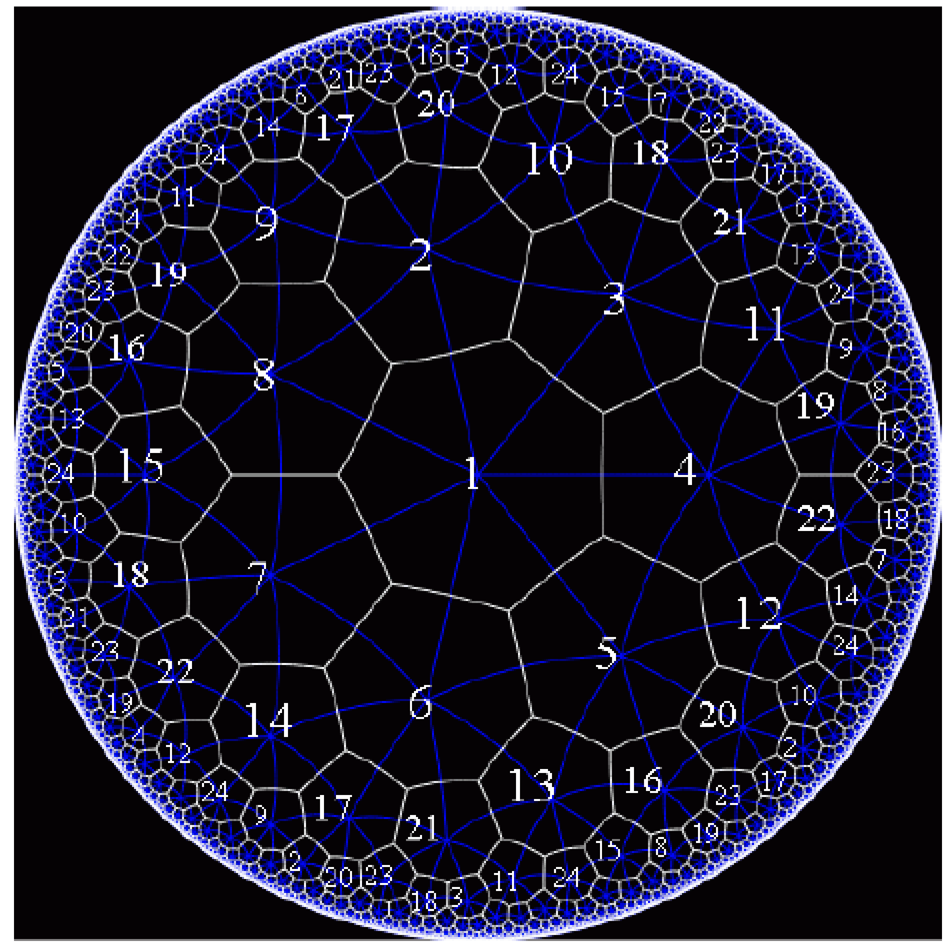
$$z \mapsto e^{\mathrm{i}\alpha} \frac{z-a}{1-z\bar{a}}, \ a \in \mathbb{D}$$

$$\hat{\psi}(\theta) = \hat{\alpha}(Le^{\mathrm{i}\theta})$$



Periodic boundary conditions





www.math.ucr.edu/home/baez/klein.html

Klein Quartic: hyperbolic surface of genus g=3