Holomorphic Gradient Flow and Lefschetz Thimbles in strongly correlated fermionic systems

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Outline

1. Non-relativistic Fermi gas in one-dimension
   - Continuum description
   - Path integral formulation
   - Non-interacting case ($g = 0$)
2. Monte-Carlo Simulation of the one-dimensional Fermi gas
3. Sign problem
   - One-dimensional Fermi gas
4. Holomorphic-Gradient-Flow (Generalized Thimble Method)
   - Lefschetz thimbles
   - HGF-algorithm
   - Example: One-Site Hubbard model
   - Non-realtivistic Fermi gas
5. Summary/Conclusion

Non-relativistic Fermi gas in one-dimension[1]

[1] L. Rammelmüller, W. J. Porter, J. E. Drut and J. Braun, Phys. Rev. D 96 (2017) 094506

Creation and Annihilation Operators

- fermion at position $x \in [-L/2, L/2]$ with spin $\sigma \in \{\uparrow, \downarrow\}$
- creation $\psi^\dagger_\sigma(x)$ and annihilation operators $\psi_\sigma(x)$:

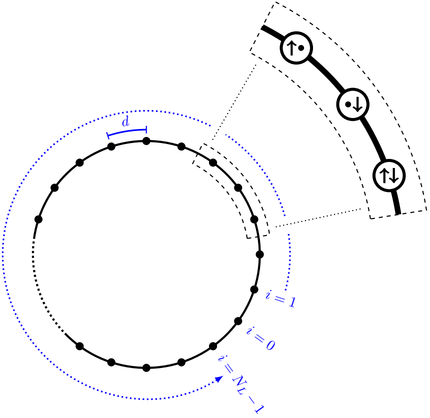
$$\left\{\psi_\sigma(x), \psi^\dagger_{\sigma'}(x')\right\} = \delta_{\sigma\sigma'}\delta(x - x')$$
$$\left\{\psi^\dagger_\sigma(x), \psi^\dagger_{\sigma'}(x')\right\} = \left\{\psi_\sigma(x), \psi_{\sigma'}(x')\right\} = 0$$

- periodic boundary conditions: $\psi_\sigma(x + L) = \psi_\sigma(x)$

Hamiltonian

$$\hat{H}_L := \sum_{\sigma\in\{\uparrow,\downarrow\}} \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}x\, \psi^\dagger_\sigma(x) \left(-\frac{1}{2m_\sigma}\partial_x^2\right) \psi_\sigma(x)$$
$$+ g \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}x\, \psi^\dagger_\uparrow(x)\psi_\uparrow(x)\psi^\dagger_\downarrow(x)\psi_\downarrow(x)$$

Discretization of Space



$$\hat{H}_L := \sum_{\sigma\in\{\uparrow,\downarrow\}} \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}x\, \psi_\sigma^\dagger(x) \left(-\frac{1}{2m_\sigma}\partial_x^2\right) \psi_\sigma(x) + g \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}x\, \psi_\uparrow^\dagger(x)\psi_\uparrow(x)\psi_\downarrow^\dagger(x)\psi_\downarrow(x)$$

$$\psi_\sigma^\dagger(x) \longrightarrow \psi_\sigma^\dagger(x_i) \quad \psi_\sigma(x) \longrightarrow \psi_\sigma(x_i) \quad \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}x \longrightarrow d\sum_{i=0}^{N_L-1}$$

$$\partial_x^2\psi_\sigma(x) \longrightarrow \frac{1}{d^2}\left(\psi_\sigma(x_{i+1}) - 2\psi_\sigma(x_i) + \psi_\sigma(x_{i-1})\right)$$

$$\hat{H}_{L,d} = \sum_{\sigma\in\{\uparrow,\downarrow\}} \sum_{i=0}^{N_L-1} \left(-\frac{1}{2m_\sigma d^2}\right) a_{i,\sigma}^\dagger \left(a_{i+1,\sigma} - 2a_{i,\sigma} + a_{i-1,\sigma}\right) + \frac{g}{d} \sum_{i=0}^{N_L-1} a_{i,\uparrow}^\dagger a_{i,\uparrow} a_{i,\downarrow}^\dagger a_{i,\downarrow}$$

Coupling to a Heat and Particle Reservoir

- partition function:

$$Z = \mathrm{tr}\left(\mathrm{e}^{-\beta\hat{h}}\right),\ \hat{h} := \hat{H}_{L,d} - \mu\hat{n}$$

Discretization of Euclidean Time

- separation between interacting and non-interacting part:

$$\hat{h}^{(I)} := \frac{g}{d}\sum_{l=0}^{N_L-1}\hat{n}_{l,\uparrow}\hat{n}_{l,\downarrow} \qquad \hat{h}^{(0)} := \hat{h} - \hat{h}^{(I)}$$

- Suzuki-Trotter decomposition:

$$\mathrm{e}^{-\beta\hat{h}} = \lim_{N_\beta\to\infty}\left(\mathrm{e}^{-\delta_\beta\hat{h}^{(0)}}\,\mathrm{e}^{-\delta_\beta\hat{h}^{(I)}}\right),\ \delta_\beta := \frac{\beta}{N_\beta}$$

Path Integral for $g < 0$

$$Z = \lim_{N_\beta \to \infty} \int \mathcal{D}\varphi \det\left(A(\varphi)\right) \mathrm{e}^{-\frac{\delta_\beta |g|}{2d} \sum_{i=0}^{N_\beta - 1} \sum_{l=0}^{N_L - 1} \varphi_{i,l}^2}$$

Path Integral for $g > 0$

$$Z = Z' \exp\left(-\beta N_L \left(\frac{1}{m_\downarrow d^2} - \mu\right)\right)$$

$$Z' = \lim_{N_\beta \to \infty} \int \mathcal{D}\varphi \det\left(A'(\varphi)\right) \mathrm{e}^{-\frac{\delta_\beta |g|}{2d} \sum_{i=0}^{N_\beta - 1} \sum_{l=0}^{N_L - 1} \varphi_{i,l}^2}$$

- From this point on, we shall use primed variables (e.g. $A', \ldots$) to denote the repulsive case.

Fermion Determinant $(g < 0)$

$$\det\left(A(\varphi)\right) = \prod_{\sigma} \det\left(\mathbb{1} + \Lambda_\sigma \Gamma_{\sigma,0} \Lambda_\sigma \Gamma_{\sigma,1} \Lambda_\sigma \ldots \Lambda_\sigma \Gamma_{\sigma,N_\beta - 1}\right)$$

Matrices

$$\Lambda_\sigma = \begin{pmatrix} \lambda_\sigma & \alpha_\sigma & 0 & \ldots & \ldots & 0 & \alpha_\sigma \\ \alpha_\sigma & \lambda_\sigma & \alpha_\sigma & 0 & \ldots & \ldots & 0 \\ 0 & \alpha_\sigma & \lambda_\sigma & \alpha_\sigma & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & \alpha_\sigma & \lambda_\sigma & \alpha_\sigma & 0 \\ 0 & \ldots & \ldots & 0 & \alpha_\sigma & \lambda_\sigma & \alpha_\sigma \\ \alpha_\sigma & 0 & \ldots & \ldots & 0 & \alpha_\sigma & \lambda_\sigma \end{pmatrix} \quad \Gamma_{\sigma,i} = \begin{pmatrix} \gamma^\sigma_{i,0} & 0 & 0 & \ldots & 0 \\ 0 & \gamma^\sigma_{i,1} & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & \gamma^\sigma_{i,N_L-2} & 0 \\ 0 & \ldots & 0 & 0 & \gamma^\sigma_{i,N_L-1} \end{pmatrix}$$

Elements

$$\lambda_\sigma = \frac{\delta_\beta}{m_\sigma d^2} - \delta_\beta \mu - 1,\ \alpha_\sigma = -\frac{\delta_\beta}{2 m_\sigma d^2},\ \gamma^\sigma_{i,l} = -\mathrm{e}^{-\mathrm{i}\frac{\delta_\beta |g|}{d}\left(\left(1 - 2\delta_{\downarrow,\sigma}\right)\varphi_{i,l} + \frac{\mathrm{i}}{2}\right)}$$

Momentum Space

- Fourier transform:

$$\psi_\sigma(p_j) = d \sum_{k=0}^{N_L-1} \mathrm{e}^{-\mathrm{i}p_j x_k} \psi_\sigma(x_k),\ p_j := \frac{2\pi}{dN_L} j$$

- creation and annihilation operators for fermions with momentum p_j:

$$\tilde{a}^\dagger_{j,\sigma} := \frac{1}{\sqrt{L}} \psi^\dagger_\sigma(p_j),\ \tilde{a}_{j,\sigma} := \frac{1}{\sqrt{L}} \psi_\sigma(p_j)$$

Diagonalization of the Hamiltonian

$$\hat{H}_{L,d} = \sum_\sigma \sum_{m=0}^{N_L-1} \tilde{a}^\dagger_{m,\sigma} \left(- \frac{\cos(p_m d) - 1}{m_\sigma d^2} \right) \tilde{a}_{m,\sigma}$$

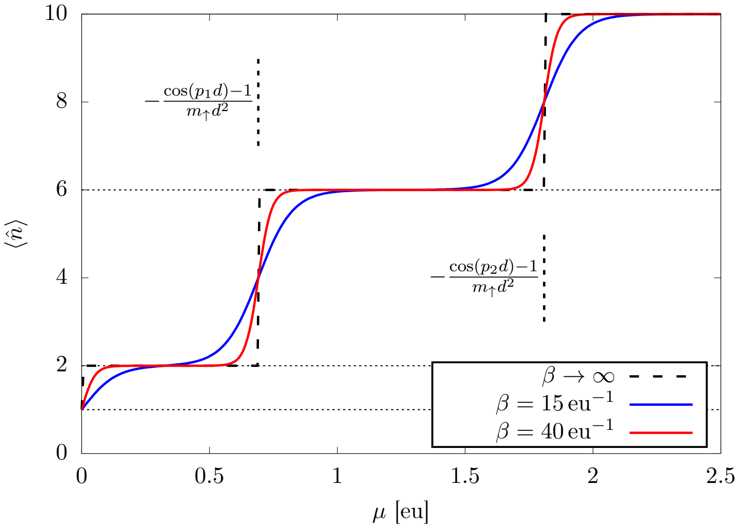


Figure: Non-interacting case with $m_\uparrow = m_\downarrow = 1\,\mathrm{eu}$, $L = 5\,\mathrm{eu}^{-1}$ and $N_L = 5$. The limit $\beta \to \infty$ is approximated by choosing $\beta = 10^4\,\mathrm{eu}^{-1}$.

Path Integral in the Limit $g \to 0^-$

$$\lim_{g \to 0^-} \langle \hat{n} \rangle = \lim_{N_\beta \to \infty} \mathcal{O}_0$$

$$\mathcal{O}_0 = (-1)^{N_\beta+1} \sum_\sigma \mathrm{tr} \left(\left(\mathbb{1} + (-1)^{N_\beta} \Lambda_\sigma^{N_\beta} \right)^{-1} \Lambda_\sigma^{N_\beta - 1} \right)$$

Path Integral in the Limit $g \to 0^+$

$$\lim_{g \to 0^+} \langle \hat{n} \rangle = N_L + \lim_{N_\beta \to \infty} \mathcal{O}_0'$$

$$\mathcal{O}_0' = \sum_\sigma (-1)^{N_\beta+1} (-1)^{\delta_{\downarrow,\sigma}} \mathrm{tr} \left(\left(\mathbb{1} + (-1)^{N_\beta} {\Lambda'}_{\sigma,g=0}^{N_\beta} \right)^{-1} {\Lambda'}_{\sigma,g=0}^{N_\beta - 1} \right)$$

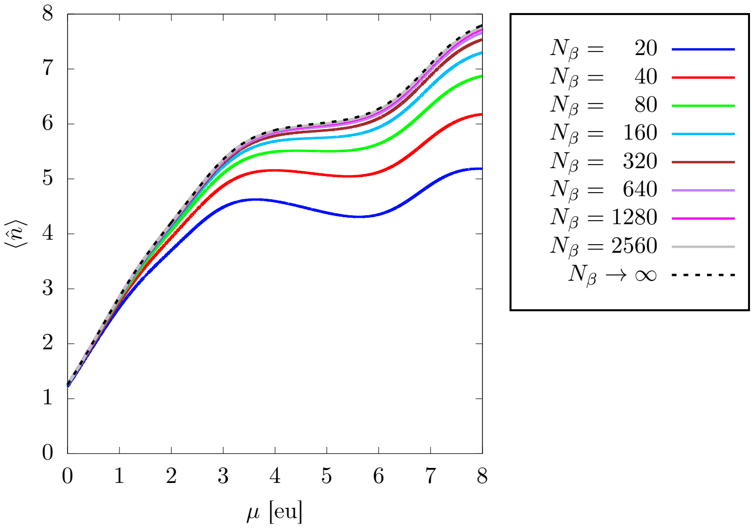


Figure: Estimation $\mathcal{O}_0$ of the expectation value $\langle \hat{n} \rangle_{L,d}$. Non-interacting case with $m_\uparrow = 0.1\,\mathrm{eu}$, $m_\downarrow = 0.7\,\mathrm{eu}$, $L = 5\,\mathrm{eu}^{-1}$, $N_L = 5$ and $\beta = 2\,\mathrm{eu}^{-1}$.

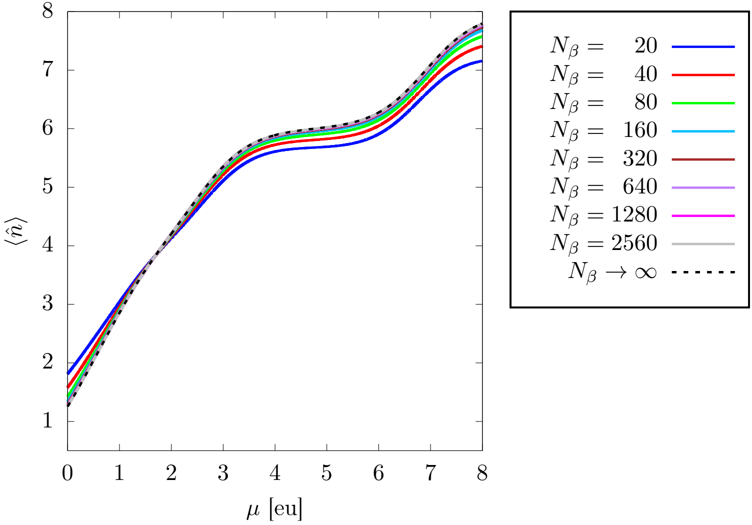


Figure: Estimation $\mathcal{O}_0' + N_L$ of the expectation value $\langle \hat{n} \rangle_{L,d}$. Non-interacting case with $m_\uparrow = 0.1\,\text{eu}$, $m_\downarrow = 0.7\,\text{eu}$, $L = 5\,\text{eu}^{-1}$, $N_L = 5$ and $\beta = 2\,\text{eu}^{-1}$.

Monte-Carlo Simulation of the one-dimensional Fermi gas

Action

- for finite N_β and $g < 0$:

$$Z_{N_\beta} = \int \mathcal{D}\varphi \, \mathrm{e}^{-S(\varphi)}$$

$$S(\varphi) \coloneqq \frac{\delta_\beta |g|}{2d} \sum_{i=0}^{N_\beta - 1} \sum_{l=0}^{N_L - 1} \varphi_{i,l}^2 - \ln \det \left(A(\varphi) \right)$$

- in the case of $g > 0$:

$$Z'_{N_\beta}, S'(\varphi), A'(\varphi), \dots$$

Probability Interpretation (Reweighting)

- We take the real part as a probability measure:

$$P(\varphi) \propto \mathrm{e}^{-\mathrm{Re}\, S(\varphi)}$$

- expectation value $\langle \hat{O} \rangle = \lim_{N_\beta \to \infty} \langle \hat{O} \rangle_{N_\beta}$ of an observable:

$$\langle \hat{O} \rangle_{N_\beta} = \frac{\left\langle \mathcal{O}(\varphi) \mathrm{e}^{-\mathrm{i}\, \mathrm{Im}\, S(\varphi)} \right\rangle_P}{\left\langle \mathrm{e}^{-\mathrm{i}\, \mathrm{Im}\, S(\varphi)} \right\rangle_P}$$

- $\left\langle \mathrm{Im}\, \mathrm{e}^{-\mathrm{i}\, \mathrm{Im}\, S(\varphi)} \right\rangle_P = 0$ because $S(-\varphi) = (S(\varphi))^*$

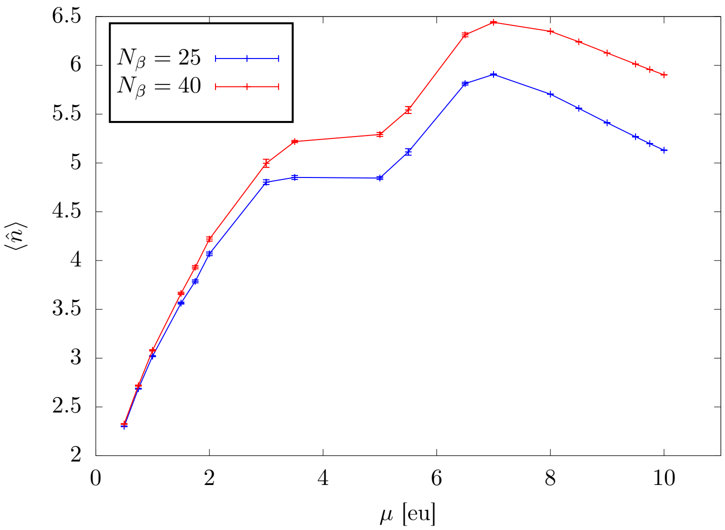


Figure: Estimation of $\langle \hat{n} \rangle$. The parameters are chosen as $m_\uparrow = 0.1\,\mathrm{eu}$, $m_\downarrow = 0.7\,\mathrm{eu}$, $\beta = 2\,\mathrm{eu}^{-1}$, $L = 5\,\mathrm{eu}^{-1}$, $g = -1$ and $N_L = 5$.

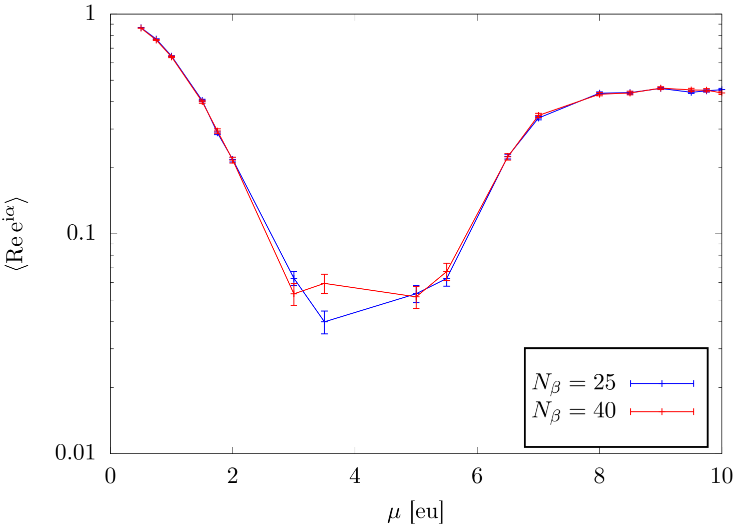


Figure: Estimation of $\langle \mathrm{Re}\, \mathrm{e}^{\mathrm{i}\alpha} \rangle$ with $\alpha \coloneqq -\mathrm{Im}\, S(\varphi)$. The parameters are chosen as $m_\uparrow = 0.1\,\mathrm{eu}$, $m_\downarrow = 0.7\,\mathrm{eu}$, $\beta = 2\,\mathrm{eu}^{-1}$, $L = 5\,\mathrm{eu}^{-1}$, $g = -1$ and $N_L = 5$.

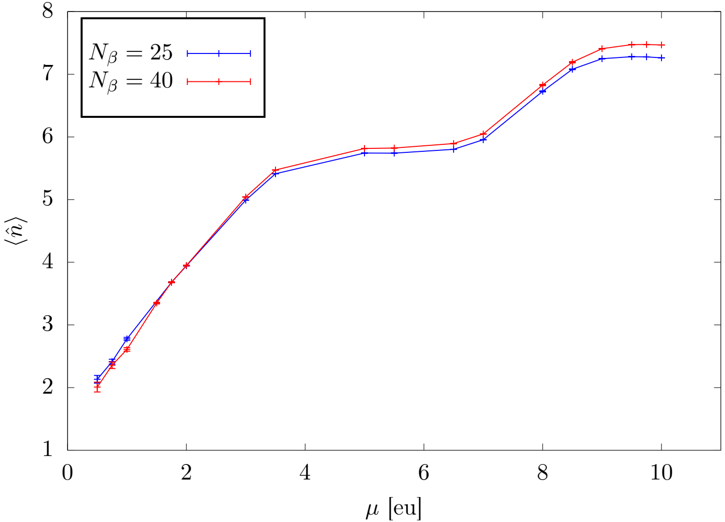


Figure: Estimation of $\langle \hat{n} \rangle$. The parameters are chosen as $m_\uparrow = 0.1\,\mathrm{eu}$, $m_\downarrow = 0.7\,\mathrm{eu}$, $\beta = 2\,\mathrm{eu}^{-1}$, $L = 5\,\mathrm{eu}^{-1}$, $g = 1$ and $N_L = 5$.

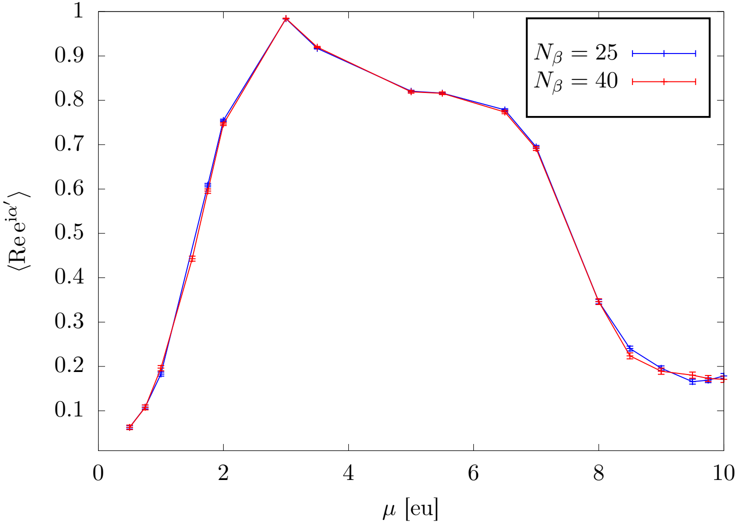


Figure: Estimation of $\langle \mathrm{Re}\, \mathrm{e}^{\mathrm{i}\alpha'} \rangle$ with $\alpha' := -\mathrm{Im}\, S'(\varphi)$. The parameters are chosen as $m_\uparrow = 0.1\,\mathrm{eu}$, $m_\downarrow = 0.7\,\mathrm{eu}$, $\beta = 2\,\mathrm{eu}^{-1}$, $L = 5\,\mathrm{eu}^{-1}$, $g = 1$ and $N_L = 5$.

Sign Problem

Full Theory

- finite $N_\beta < \infty$:

$$Z_{N_\beta} = \int \mathrm{d}^n x \, \mathrm{e}^{-S(x)}$$

- Euclidean time continuum ($N_\beta \to \infty$):

$$Z = \lim_{N_\beta \to \infty} Z_{N_\beta}$$

Phase Quenched

- finite $N_\beta < \infty$:

$$\tilde{Z}_{N_\beta} := \int \mathrm{d}^n x \, \left| \mathrm{e}^{-S(x)} \right| = \int \mathrm{d}^n x \, \mathrm{e}^{-\mathrm{Re}\, S(x)}$$

- Euclidean time continuum ($N_\beta \to \infty$):

$$\tilde{Z} = \lim_{N_\beta \to \infty} \tilde{Z}_{N_\beta}$$

Phase Expectation Value

- finite N_β:

$$\left\langle \mathrm{Re}\, \mathrm{e}^{-\mathrm{i}\, \mathrm{Im}\, S(x)} \right\rangle_P = \frac{Z_{N_\beta}}{\tilde{Z}_{N_\beta}}$$

- for $N_\beta \gg 1$ sufficiently large:

$$\left\langle \mathrm{Re}\, \mathrm{e}^{-\mathrm{i}\, \mathrm{Im}\, S(x)} \right\rangle_P \approx \frac{Z}{\tilde{Z}} = \mathrm{e}^{-\beta(\Phi - \tilde{\Phi})}$$

Upper Bound

- We notice that $0 < Z \leq \tilde{Z}$ because $Z_{N_\beta} \leq \tilde{Z}_{N_\beta}$.

$$\Longrightarrow \forall \varepsilon > 0 : \underbrace{\inf_{\beta \in [\varepsilon, \infty)} (\Phi - \tilde{\Phi})}_{\Delta\Phi_{\min} :=} \geq 0$$

- assumption:

$$\exists \varepsilon_0 > 0 : \Delta\Phi_{\min} > 0$$

- consequence:

$$\forall \beta \geq \varepsilon_0 : \frac{Z}{\tilde{Z}} \leq \mathrm{e}^{-\beta \Delta\Phi_{\min}}$$

Low Temperature Limit

- for $N_\beta \gg 1$ sufficiently large:

$$\left\langle \mathrm{Re}\, \mathrm{e}^{-\mathrm{i}\,\mathrm{Im}\, S(x)} \right\rangle_P \leq \mathrm{e}^{-\beta \Delta \Phi_{\min}} \longrightarrow 0 \quad (\beta \to \infty)$$

Relative Error

$$\frac{\sigma\left(\overline{\left(\mathrm{Re}\, \mathrm{e}^{-\mathrm{i}\,\mathrm{Im}\, S(x)}\right)}_N\right)}{\langle \mathrm{Re}\, \mathrm{e}^{-\mathrm{i}\,\mathrm{Im}\, S(x)} \rangle_P} = \frac{\tau}{\sqrt{N}} \frac{\sigma\left(\mathrm{Re}\, \mathrm{e}^{-\mathrm{i}\,\mathrm{Im}\, S(x)}\right)}{\langle \mathrm{Re}\, \mathrm{e}^{-\mathrm{i}\,\mathrm{Im}\, S(x)} \rangle_P}, \ \tau \geq 1$$

Relative Error (Lower Bound)

$$\frac{\sigma\left(\overline{\left(\mathrm{Re}\, \mathrm{e}^{-\mathrm{i}\,\mathrm{Im}\, S(x)}\right)}_N\right)}{\left\langle \mathrm{Re}\, \mathrm{e}^{-\mathrm{i}\,\mathrm{Im}\, S(x)} \right\rangle_P} \geq \frac{\sigma\left(\mathrm{Re}\, \mathrm{e}^{-\mathrm{i}\,\mathrm{Im}\, S(x)}\right)}{\sqrt{N}} \mathrm{e}^{+\beta \Delta \Phi_{\min}}$$

- minimal number of samples to reach a relative error of $100\,\%$:

$$N = \sigma^2 \left(\mathrm{Re}\, \mathrm{e}^{-\mathrm{i}\, \mathrm{Im}\, S(x)} \right) \mathrm{e}^{+2\beta \Delta \Phi_{\mathrm{min}}}$$

- Monte-Carlo method with an exponential complexity
- exponential complexity for other parameters ($\Delta \Phi_{\mathrm{min}} \propto V$, etc.)

One-Dimensional Fermi Gas

- no sign problem if $g < 0$, $m_\uparrow = m_\downarrow$
- otherwise $\det\left(A(\varphi)\right) \geq 0$ not guaranteed

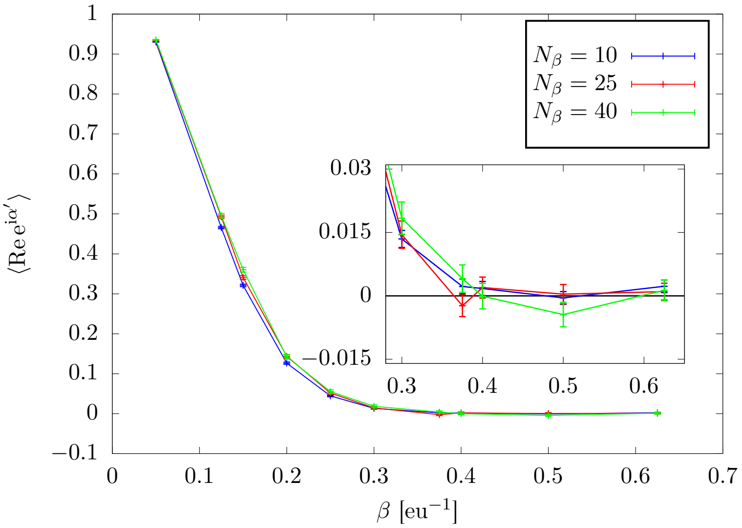
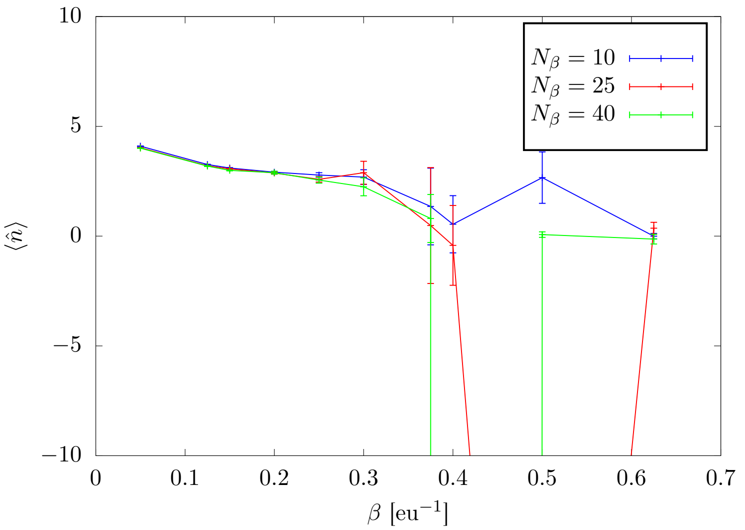


Figure: The parameters are chosen as $m_\uparrow = 0.1\,\mathrm{eu}$, $m_\downarrow = 0.7\,\mathrm{eu}$, $\mu = 1\,\mathrm{eu}$, $L = 5\,\mathrm{eu}^{-1}$, $g = 10$ and $N_L = 5$.



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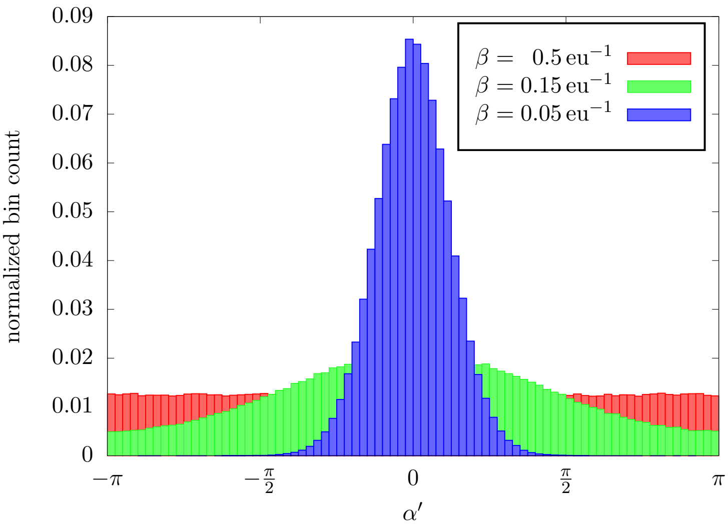


Figure: The parameters are chosen as $m_\uparrow = 0.1\,\mathrm{eu}$, $m_\downarrow = 0.7\,\mathrm{eu}$, $\mu = 1\,\mathrm{eu}$, $L = 5\,\mathrm{eu}^{-1}$, $g = 10$, $N_L = 5$ and $N_\beta = 10$.

Holomorphic-Gradient-Flow

Critical Points

$$\forall i \in \{1, \ldots, n\} : \frac{\partial \mathrm{Re}\, S(z_c)}{\partial z_i} = 0$$

Upward (+) and Downward (−) Flow Equations

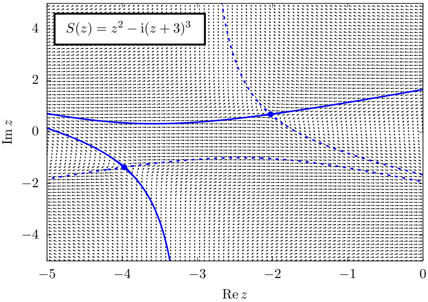
$$\frac{\mathrm{d}z_i}{\mathrm{d}t} = \pm \left(\frac{\partial S(z)}{\partial z_i} \right)^*, i \in \{1, \ldots, n\}$$

Lefschetz Thimble

$$J_{z_c} := \left\{ z_0 \in \mathbb{C}^n \middle| \lim_{t \to \infty} z_d(t, z_0) = z_c \right\}$$

Dual Thimble

$$K_{z_c} := \left\{ z_0 \in \mathbb{C}^n \middle| \lim_{t \to \infty} z_u(t, z_0) = z_c \right\}$$



Path Integral Decomposition[1] [2]

$$Z_{N_\beta} = \sum_{z_c} n_{z_c} \int_{J_{z_c}} \mathrm{d}^n z \, \mathrm{e}^{-S(z)}$$

- $\operatorname{Im} S(z)$ is constant on each Lefschetz thimble.
- no sign problem for integral over thimble J_{z_c}

Problem

- The decomposition is not feasible in general.

[1] see M. Cristoforetti, F. D. Renzo and L. Scorzato, Phys. Rev. D 86 (2012) 074506

[2] see E. Witten, 2010, arXiv: 1001.2933v4

Holomorphic-Gradient-Flow (Generalized Thimble Method)[1]

- deform integration domain $\mathbb{R}^n \to M_t := z_u(t, \mathbb{R}^n)$:

$$\int_{M_t} \mathrm{d}^n z \, \mathcal{O}(z) \mathrm{e}^{-S(z)} = \int \mathrm{d}^n x \, \mathcal{O}(z_u(t,x)) \det(J_t(x)) \mathrm{e}^{-S(z_u(t,x))}$$

- M_t approximates the contributing Lefschetz thimbles.

Sampling

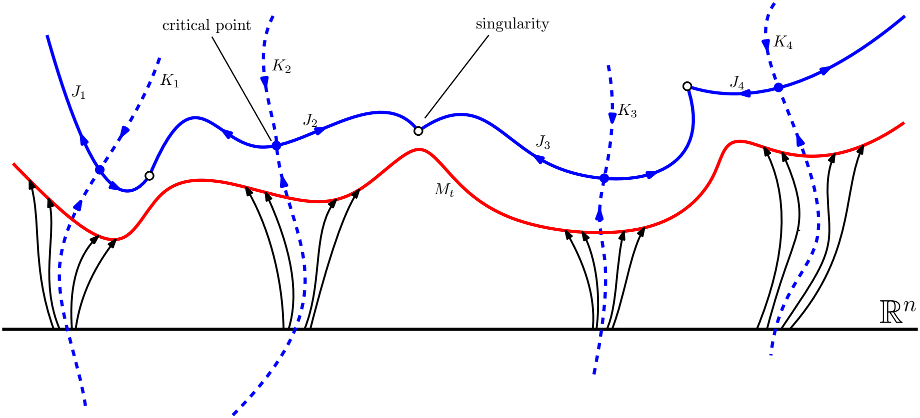
- use the reweighting method $\left(P \propto \mathrm{e}^{-\mathrm{Re}\, S(z_u(t,x)) + \ln |\det J_t(x)|}\right)$
- solve differential equations numerically:

$$\frac{\mathrm{d}z_i}{\mathrm{d}t} = \left(\frac{\partial S(z)}{\partial z_i}\right)^*, \quad \frac{\mathrm{d}(J_t)_{i,j}}{\mathrm{d}t} = \sum_{k=1}^{n} \left(\frac{\partial^2 S(z)}{\partial z_i \partial z_k} (J_t)_{k,j}\right)^*$$

[1] A. Alexandru, G. Başar, P. F. Bedaque, G. W. Ridgway and N. C. Warrington, J. High Energ. Phys. (2016) 2016: 53

Phase Expectation Value

- large t: small fluctuations of $\mathrm{e}^{\mathrm{i}\alpha_t}$ are expected
- $\langle \mathrm{Re}\, \mathrm{e}^{\mathrm{i}\alpha_t} \rangle_P$ in same order of magnitude as standard deviation



Hamiltonian

$$\hat{H}_{L,d} = U a^{\dagger}_{0,\uparrow} a_{0,\uparrow} a^{\dagger}_{0,\downarrow} a_{0,\downarrow} + \frac{1}{m_{\uparrow} d^2} \hat{n}_{\uparrow} + \frac{1}{m_{\downarrow} d^2} \hat{n}_{\downarrow}$$

Path Integral[1]

$$Z_{L,d} = \sqrt{\frac{\beta}{2\pi U}} \int \mathrm{d}x \, \mathrm{e}^{-S(x)}$$

$$S(x) := \frac{\beta}{2U} x^2 - 2 \ln \left(1 + \exp \left(\mathrm{i}x + \mu + \frac{U}{2} \right) \right)$$

- We choose: $\mu = 0$, $U = 1$ and $\beta = 30$.

[1] see Y. Tanizaki, Y. Hidaka and T. Hayata, New J. Phys. 18 (2016) 033002

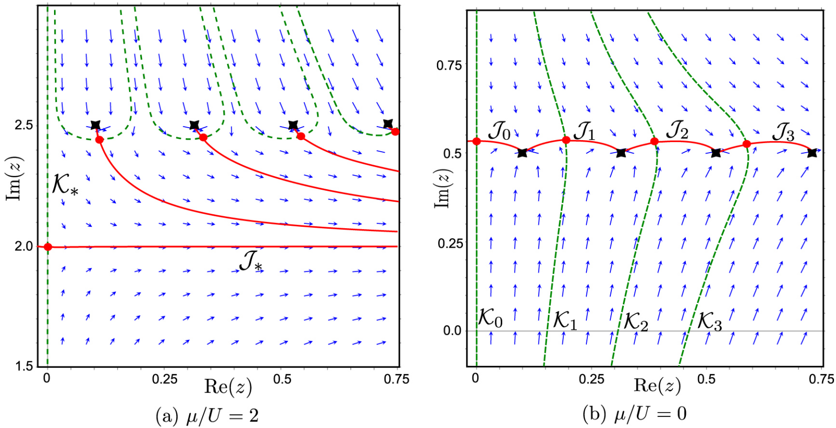
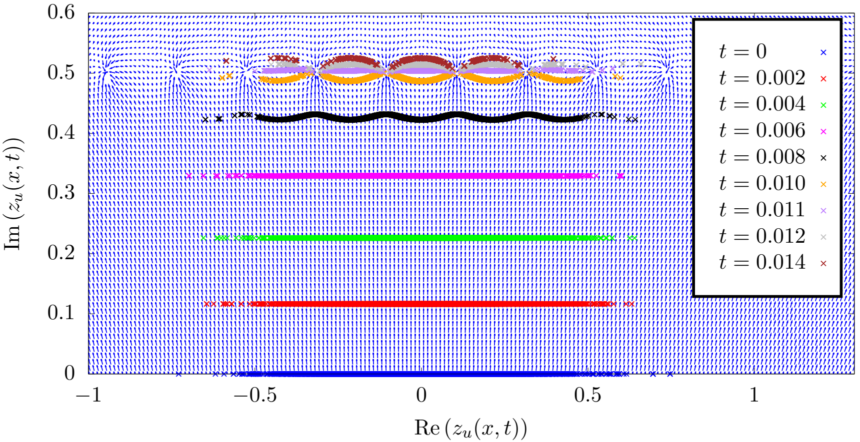


Figure: Illustration taken from New J. Phys. 18 (2016) 033002.[1]

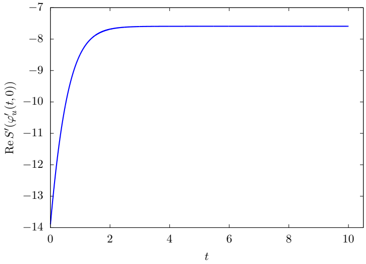

0.6
0.5
0.4
0.3
0.2
0.1
0
−1
−0.5
0
0.5
1
$\mathrm{Im}\,(z_u(x,t))$
$\mathrm{Re}\,(z_u(x,t))$
$t = 0$
$t = 0.002$
$t = 0.004$
$t = 0.006$
$t = 0.008$
$t = 0.010$
$t = 0.011$
$t = 0.012$
$t = 0.014$

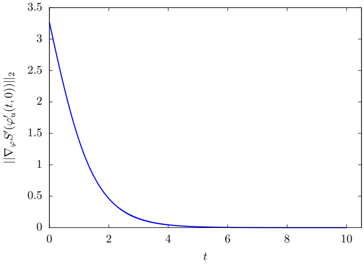
Parameters

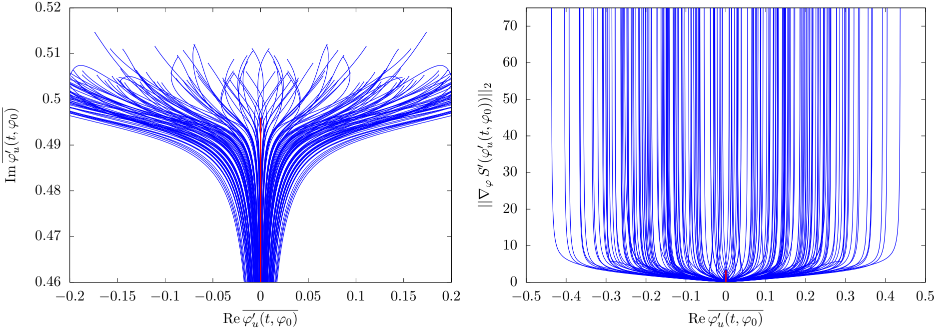
- We choose: $m_\uparrow = 0.1\,\mathrm{eu}$, $m_\downarrow = 0.7\,\mathrm{eu}$, $\mu = 1\,\mathrm{eu}$, $L = 5\,\mathrm{eu}^{-1}$, $g = 10$, $\beta = 0.5\,\mathrm{eu}^{-1}$, $N_L = 5$ and $N_\beta = 10$

Critical Point

- at least one critical point: $\varphi'_{c,0} = \lim_{t\to\infty} \varphi'_u(t,0)$

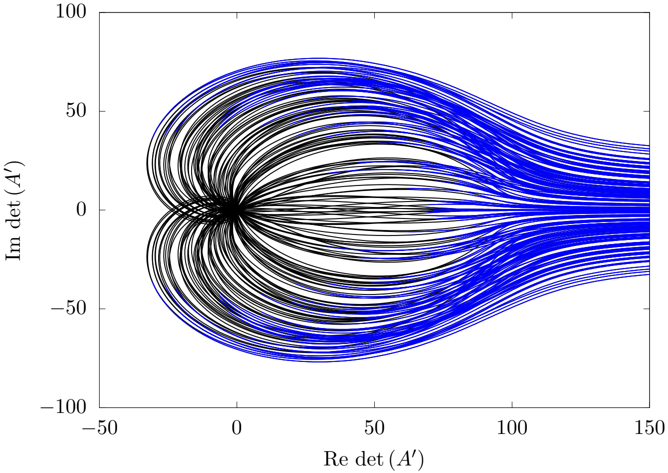


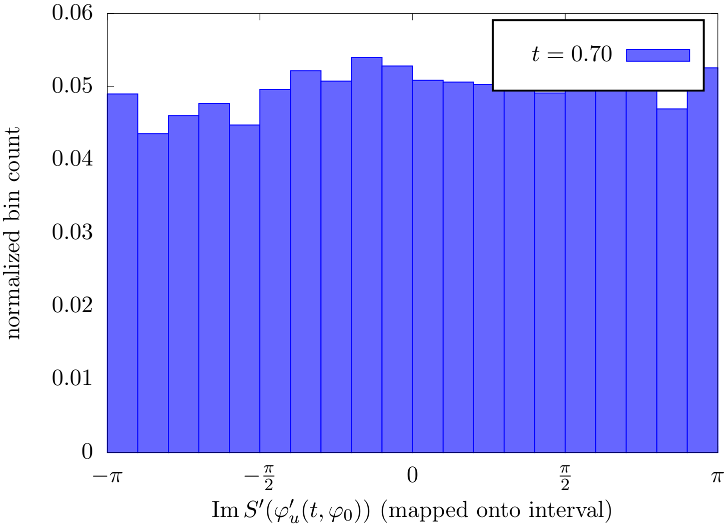


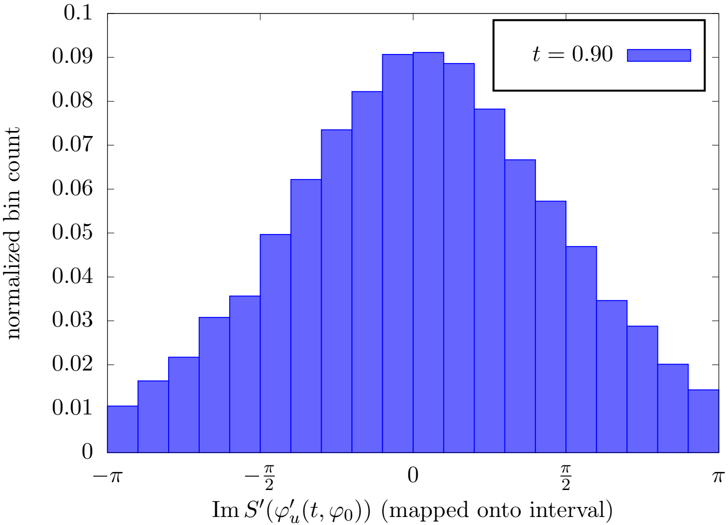

0.52
0.51
0.5
0.49
0.48
0.47
0.46
$\mathrm{Im}\,\overline{\varphi'_u(t,\varphi_0)}$
−0.2 −0.15 −0.1 −0.05 0 0.05 0.1 0.15 0.2
$\mathrm{Re}\,\overline{\varphi'_u(t,\varphi_0)}$
70
60
50
40
30
20
10
0
$\|\nabla_\varphi S'(\varphi'_u(t,\varphi_0))\|_2$
−0.5 −0.4 −0.3 −0.2 −0.1 0 0.1 0.2 0.3 0.4 0.5
$\mathrm{Re}\,\overline{\varphi'_u(t,\varphi_0)}$

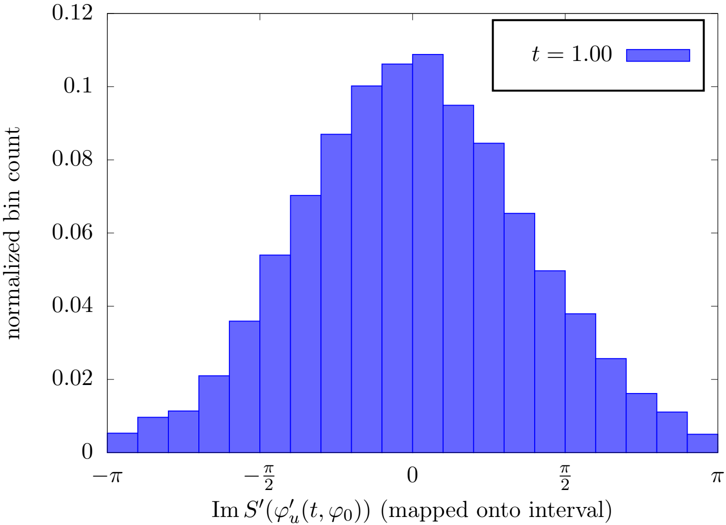
Problem

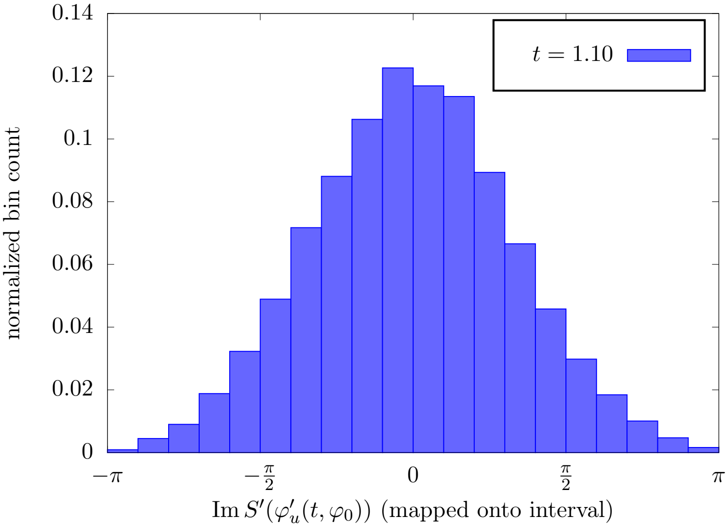
- Most points are mapped into singularities.
- difficult to solve the flow equations numerically for large t

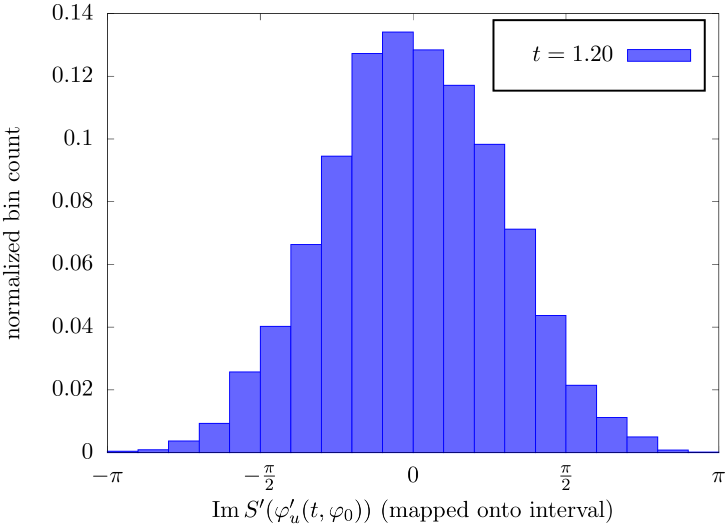


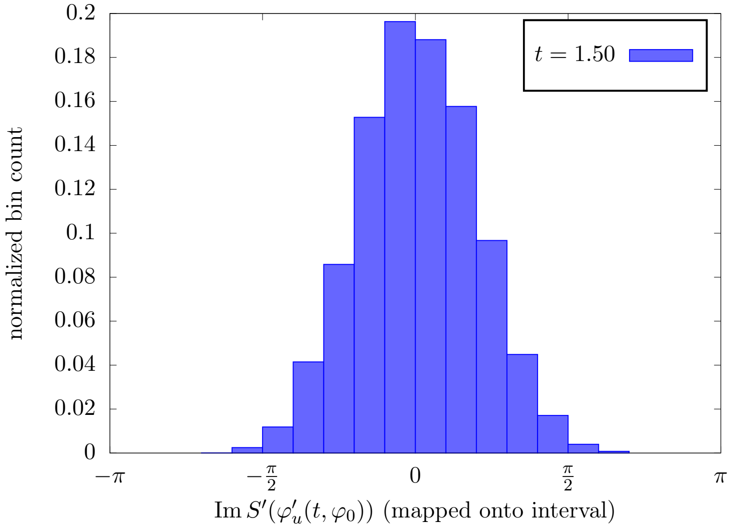

0.06
0.05
0.04
0.03
0.02
0.01
0
normalized bin count
$t = 0.70$
$-\pi$
$-\frac{\pi}{2}$
0
$\frac{\pi}{2}$
π
$\mathrm{Im}\, S'(\varphi'_u(t, \varphi_0))$ (mapped onto interval)

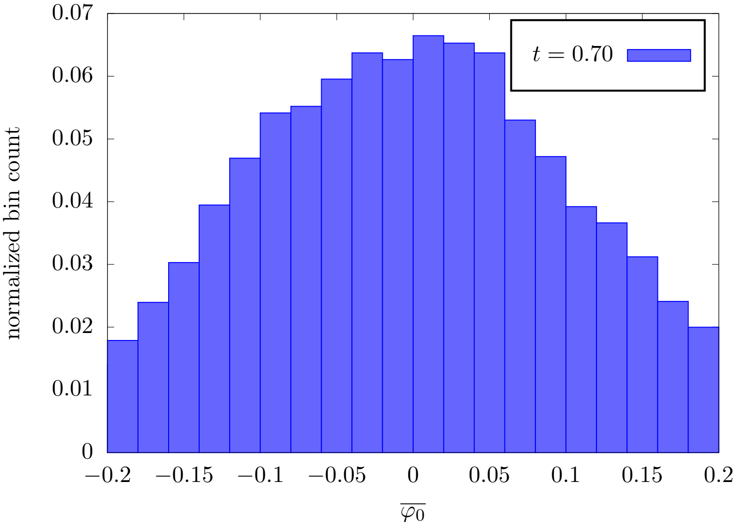

0.1
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
normalized bin count
$t = 0.90$
$-\pi$
$-\frac{\pi}{2}$
0
$\frac{\pi}{2}$
π
$\mathrm{Im}\, S'(\varphi'_u(t, \varphi_0))$ (mapped onto interval)


$t = 1.00$
normalized bin count
0.12
0.1
0.08
0.06
0.04
0.02
0
$-\pi$
$-\frac{\pi}{2}$
0
$\frac{\pi}{2}$
π
$\mathrm{Im}\, S'(\varphi'_u(t, \varphi_0))$ (mapped onto interval)

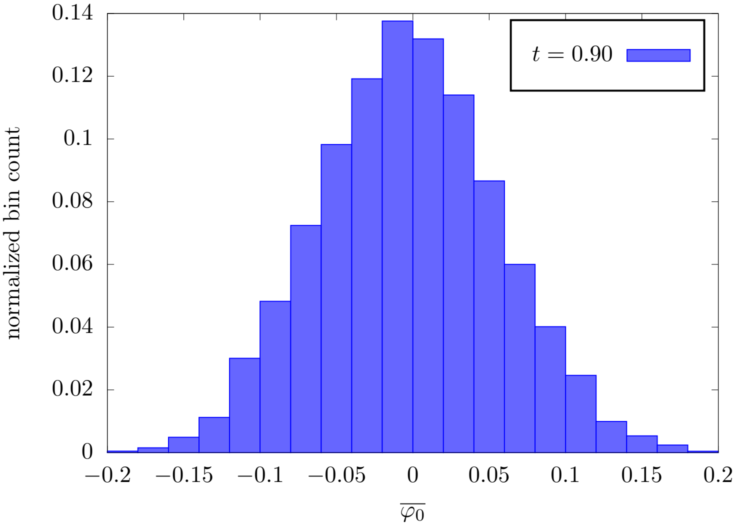

0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
normalized bin count
$t = 1.10$
$-\pi$
$-\frac{\pi}{2}$
0
$\frac{\pi}{2}$
π
$\mathrm{Im}\, S'(\varphi_u'(t, \varphi_0))$ (mapped onto interval)


$t = 1.20$
normalized bin count
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
$-\pi$
$-\frac{\pi}{2}$
0
$\frac{\pi}{2}$
π
$\mathrm{Im}\, S'(\varphi'_u(t, \varphi_0))$ (mapped onto interval)

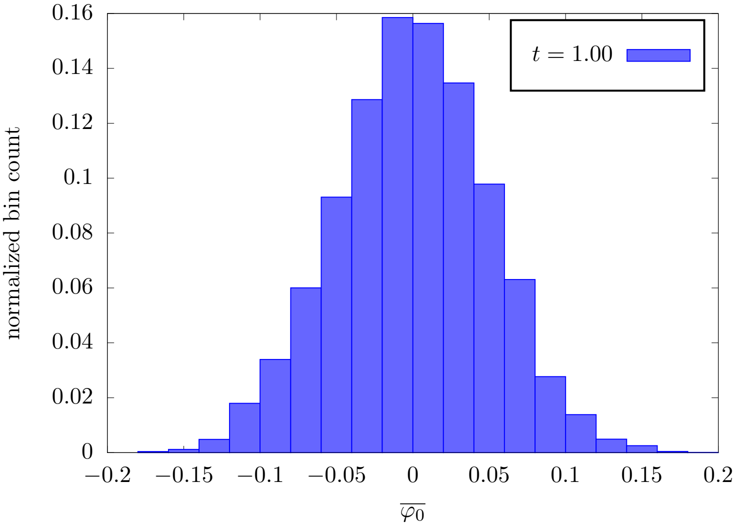

0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
normalized bin count
$t = 1.50$
$-\pi$
$-\frac{\pi}{2}$
0
$\frac{\pi}{2}$
π
$\mathrm{Im}\, S'(\varphi'_u(t, \varphi_0))$ (mapped onto interval)

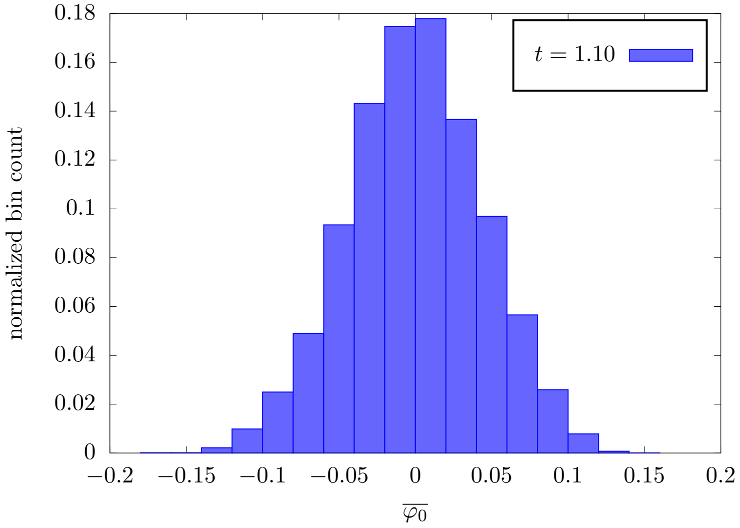


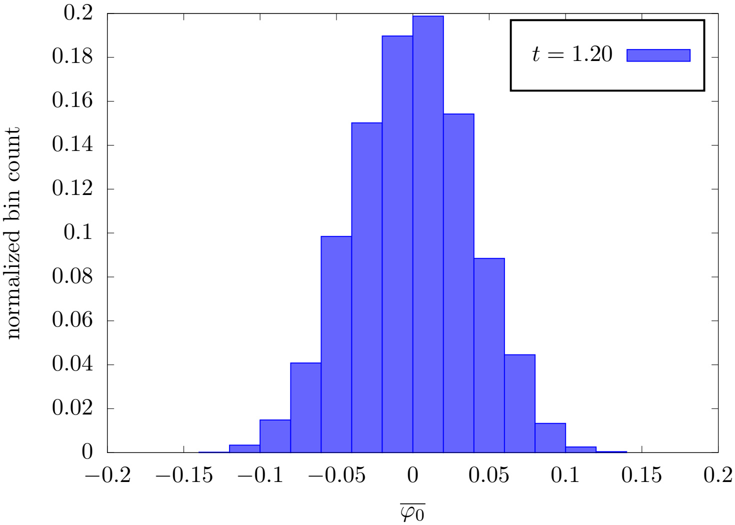
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
normalized bin count
−0.2
−0.15
−0.1
−0.05
0
0.05
0.1
0.15
0.2
$\overline{\varphi_0}$
$t = 0.70$



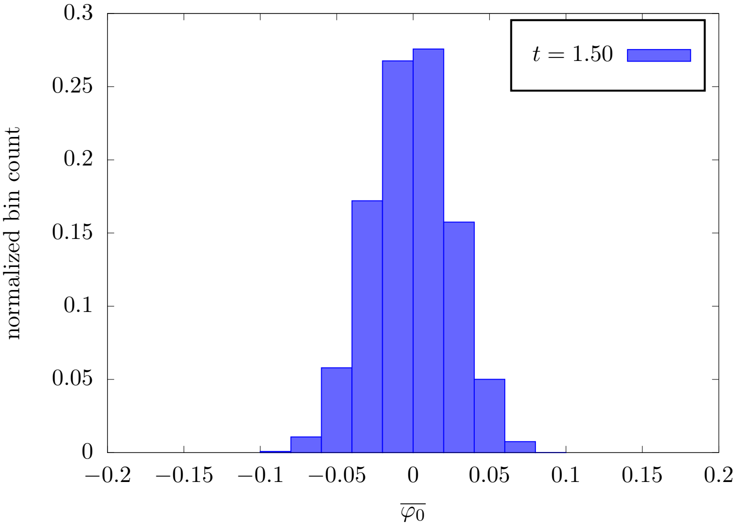
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
normalized bin count
$t = 0.90$
−0.2
−0.15
−0.1
−0.05
0
0.05
0.1
0.15
0.2
$\overline{\varphi_0}$


0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
normalized bin count
−0.2
−0.15
−0.1
−0.05
0
0.05
0.1
0.15
0.2
$\overline{\varphi_0}$
$t = 1.00$


0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
normalized bin count
−0.2
−0.15
−0.1
−0.05
0
0.05
0.1
0.15
0.2
$\overline{\varphi_0}$
$t = 1.10$

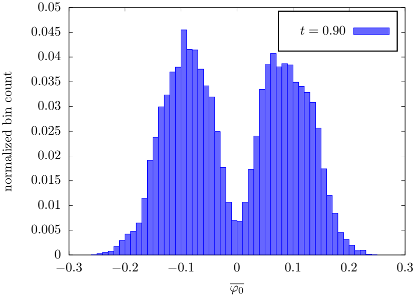


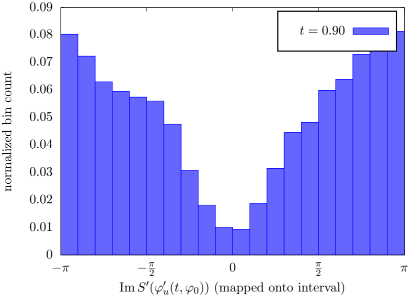
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
normalized bin count
−0.2
−0.15
−0.1
−0.05
0
0.05
0.1
0.15
0.2
$\overline{\varphi_0}$
$t = 1.20$


0.3
0.25
0.2
0.15
0.1
0.05
0
normalized bin count
−0.2
−0.15
−0.1
−0.05
0
0.05
0.1
0.15
0.2
$\overline{\varphi_0}$
$t = 1.50$

Inconsistency?

- Including the Jacobian J_t yields (qualitatively) different results.
- Reasons are unclear. Further investigations are required.





- HGF-algorithm is already successful in minimizing the impact of the sign problem, see one-site Hubbard model, A. Alexandru et. al.
- For the one-dimensional Fermi gas, further investigations are required.
- HGF-algorithm not feasible for large $N_\beta N_L$ (computation of J_t)
- It is questionable if useful for more complicated systems (QCD,...)
- New methods:
  - Find a new domain M of integration via an optimization process.[1] [2]
  - Machine Learning[3] (e.g. one-site Hubbard model)
  - Triangularization (see talk of Ziesché)
- All these methods do not solve the sign problem!

[1] Y. Mori, K. Kashiwa and A. Ohnishi, Phys. Rev. D 96 (2017) 111501

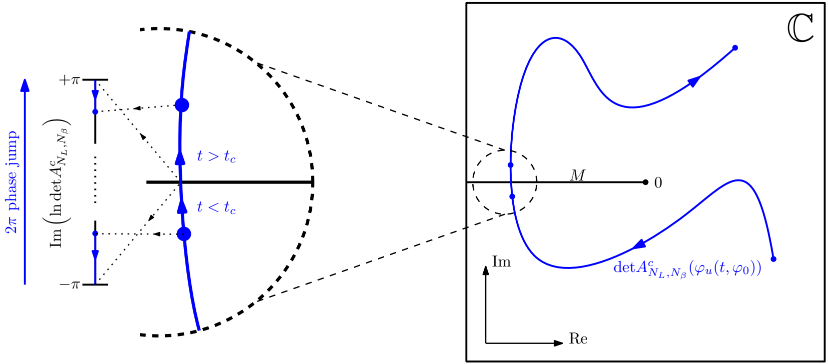
[2] A. Alexandru, P. F. Bedaque, H. Lamm and S. Lawrence, Phys. Rev. D 96 (2017) 094510

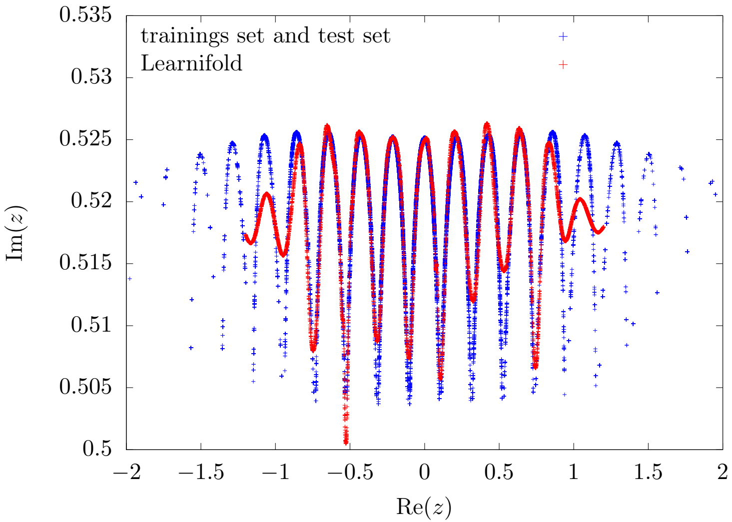
[3] A. Alexandru, P. F. Bedaque, H. Lamm and S. Lawrence, Phys. Rev. D 96 (2017) 094505

Thank You!

Action

$$S(\varphi) := \frac{\delta_\beta |g|}{2d} \sum_{i=0}^{N_\beta - 1} \sum_{l=0}^{N_L - 1} \varphi_{i,l}^2 - \ln\det\left(A(\varphi)\right)$$




trainings set and test set
Learnifold
Im(z)
Re(z)
0.535
0.53
0.525
0.52
0.515
0.51
0.505
0.5
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2