Dynamic Critical Behaviour and Spectral Functions of ϕ^4 Theory (A06)

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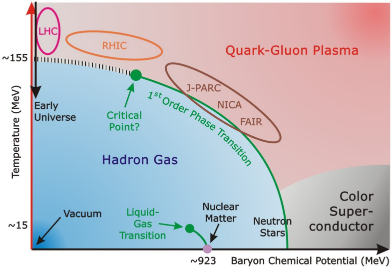
- Long-term objective: Find the critical point of QCD
- Experiments: heavy-ion collisions
  - Naturally very dynamic
  - Equilibrium not guaranteed

⇒ Need to know how critical point changes dynamics

- Accessible dynamic quantity: spectral functions



Second Order Phase Transitions

- *Continuous* phase transition
  - Infinite correlation length ξ
  - Scale invariance
  - $\Rightarrow$ Universality
- Power laws in observables ($\langle\phi\rangle$,χ,ξ,...):

$$\xi \sim |\tau|^{-\nu}$$

$$\langle O(T)\rangle \sim |\tau|^{\sigma}\,, \qquad \tau := \frac{T - T_c}{T_c}$$

- Examples: CO_2, ferromagnets, QCD, ϕ^4...

Fully described by classical physics!

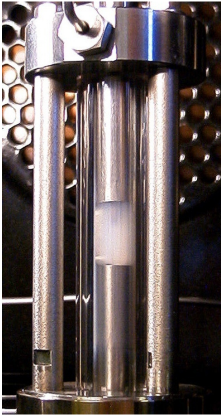


Figure: "Critical opalescence of ethane" by Dr. Sven Horstmann, used under CC BY-SA 3.0/cropped from original.

Example I

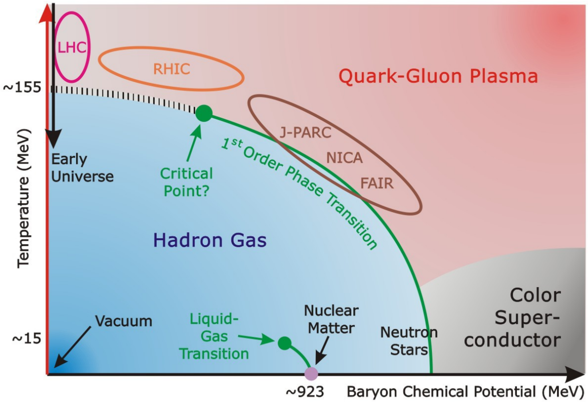


Figure: Semi-quantitative phase diagram of QCD

Example II

$$\mathcal{L}(\phi) = \frac{1}{2}\left(\partial^\mu\phi\partial_\mu\phi - m^2\phi^2\right) - \frac{\lambda}{4!}\phi^4 - J\phi$$

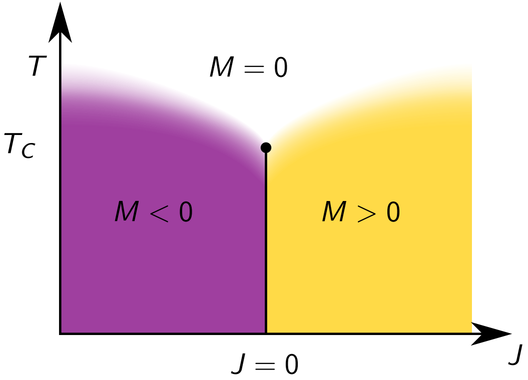


Figure: Phase diagram of a ϕ^4 theory. The line between between opposing magnetization phases ends in a critical point at the curie temperature.

Configurations

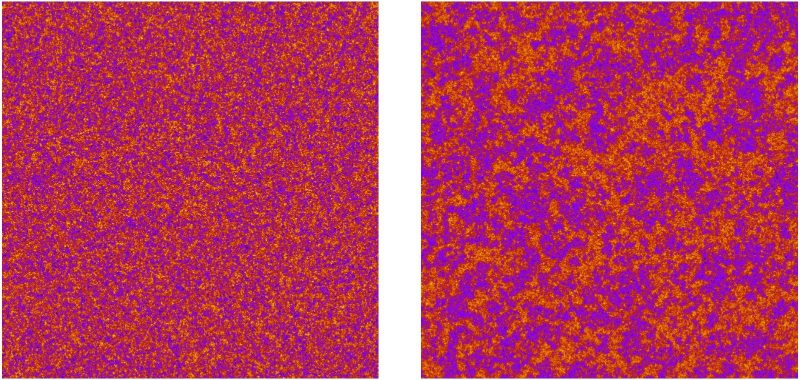


Figure: Configurations in the symmetric phase

Configurations

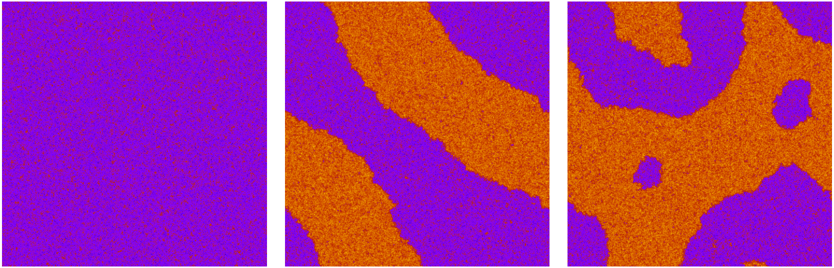


Figure: Configurations on line of phase coexistence

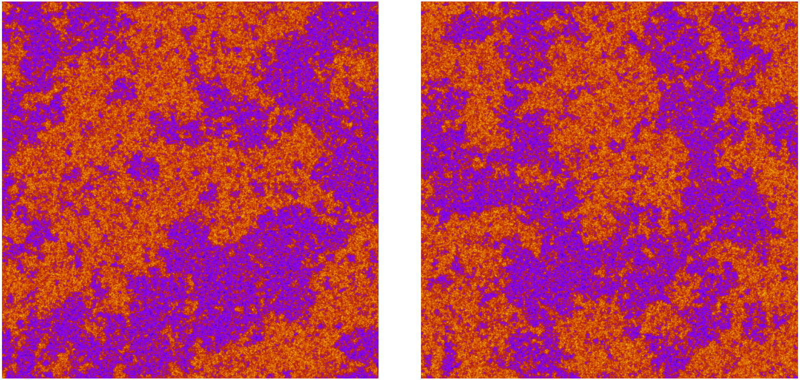


Figure: Configurations close to the critical point

- Time evolution of expectation values:

$$\left\langle \hat{O}(t) \right\rangle = Z^{-1}\,\mathrm{tr}\left(\hat{\rho}_0 e^{-i\hat{H}t}\hat{O}e^{i\hat{H}t}\right) \tag{1}$$

- Suzuki-Trotter decomposition ⇒ *Schwinger-Keldysh* technique

$$\left\langle \hat{O}(t) \right\rangle = Z^{-1}\int \mathcal{D}\phi_+(t)\mathcal{D}\phi_-(t)\rho_0[\phi_+(0),\phi_-(0)]$$
$$\exp\left[-i\int_0^t \mathrm{d}t_1 \hat{H}(\phi_+(t_1))\right] O\left[\phi_+(t),\phi_-(t)\right] \tag{2}$$
$$\exp\left[+i\int_t^0 \mathrm{d}t_2 \hat{H}(\phi_-(t_2))\right]$$
$$= \int \mathcal{D}\phi_+\mathcal{D}\phi_-\rho_0 e^{-i\int_0^t \mathrm{d}t' \frac{\dot{\phi}_+^2-\dot{\phi}_-^2}{2}-(V(\phi_+)-V(\phi_-))} O(\phi_+,\phi_-) \tag{3}$$

Real-time dynamics II

- Change of variables:
  - "classical" coordinate $\phi = \frac{\phi_+ + \phi_-}{2}$
  - "quantum fluctuation" $\tilde{\phi} = \phi_+ - \phi_-$
- Expand (3) to first order in $\tilde{\phi}$:

$$\left\langle \hat{O}(t) \right\rangle \approx \int \mathcal{D}\phi \mathcal{D}\tilde{\phi} e^{-i \int_0^t \mathrm{d}t' \tilde{\phi}(t') \left(-\ddot{\phi}(t') - \partial_\phi V(\phi) \right)} \rho_0(\phi, \dot{\phi}) O(\phi, \dot{\phi})$$

- $\tilde{x}$-integration enforces classical equations of motion: $\ddot{\phi} + \partial_\phi V(\phi) = 0$

Classical field theory - real-time lattice

$$H(\phi) = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$

$$V(\phi) = \frac{1}{2}\left((\nabla\phi)^2 + m^2\phi^2\right) + \frac{\lambda}{4!}\phi^4 - J\phi$$

$$\phi(t+\Delta t) - \phi(t) = \Delta t \cdot \dot{\phi}$$

$$\dot{\phi}(t+\Delta t) - \dot{\phi}(t) = -\Delta t \cdot \partial_\phi H$$

- Discretize $\phi(x_i), \dot{\phi}(x_i)$ on square/cubic lattice
- Solve equations of motion at every lattice site
- Hamiltonian dynamics: energy conserved $\rightarrow$ microcanonical ensembles
- Average over thermal initial conditions

$\Rightarrow$ Calculate real-time observables as functions of $\phi(t), \dot{\phi}(t)$

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$$\phi(t+\Delta t) - \phi(t) = \Delta t \cdot \dot{\phi}$$
$$\dot{\phi}(t+\Delta t) - \dot{\phi}(t) = -\Delta t \cdot \partial_\phi H - \Delta t \left(\gamma\dot{\phi} + \sqrt{2\gamma T}\eta(t)\right)$$

- Discretize $\phi(x_i), \dot{\phi}(x_i)$ on square/cubic lattice
- Solve equations of motion at every lattice site
- Langevin dynamics: coupling to heat bath $\rightarrow$ canonical ensembles
- Average over thermal initial conditions

$\Rightarrow$ Calculate real-time observables as functions of $\phi(t), \dot{\phi}(t)$

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WARNING

The following footage may potentially trigger seizures for people with photosensitive epilepsy. Viewer discretion is advised.

- Passing of critical point changes dynamics visibly
- Known phenomenon: Critical Slowing-Down (of MC algorithms)

$\Rightarrow$ Look at multi-time correlation functions

The Spectral Function I

- Spectral function defined via decomposition of Green's function:

$$\rho\left(t, t', x, x'\right) \equiv i \left\langle \left[\phi(t, x)\phi(t', x')\right] \right\rangle$$
$$F(t, t', x, x') \equiv \frac{1}{2} \left\langle \left\{\phi(t, x)\phi(t', x')\right\} \right\rangle - \langle\phi\rangle^2$$

- Use fluctuation-dissipation theorem to get ρ from F

$$F(\omega, \vec{p}) = -i \left(\frac{1}{2} + n_T(\omega)\right) \rho(\omega, \vec{p}) \tag{4}$$

The Spectral Function II

- Use classical limit of BE distribution $n_T(\omega) \approx \frac{T}{\omega}$ for large $\frac{T}{\omega}$:

$$F(\omega, \vec{p}, T) = -i\frac{T}{\omega}\rho(\omega, \vec{p}, T) \tag{5}$$

$$\Rightarrow \quad \rho(t, \vec{p}, T) = -\frac{1}{T}\frac{\partial}{\partial t}F(t, \vec{p}, T) \tag{6}$$

$$= -\frac{1}{T}\left\langle \dot{\phi}(t - t', p)\phi(0, -p)\right\rangle \tag{7}$$

Now calculate ρ from classical fields $\phi, \dot{\phi}$!

Spectral Functions

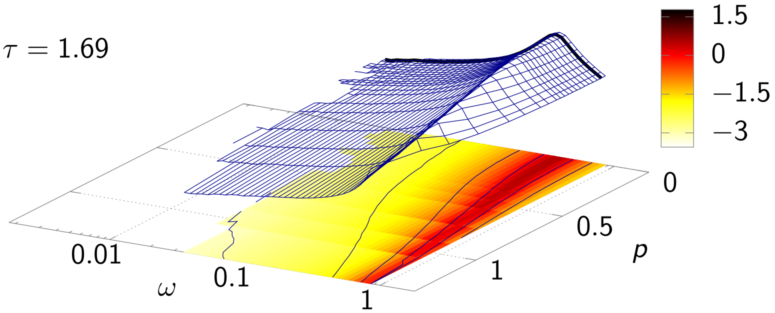
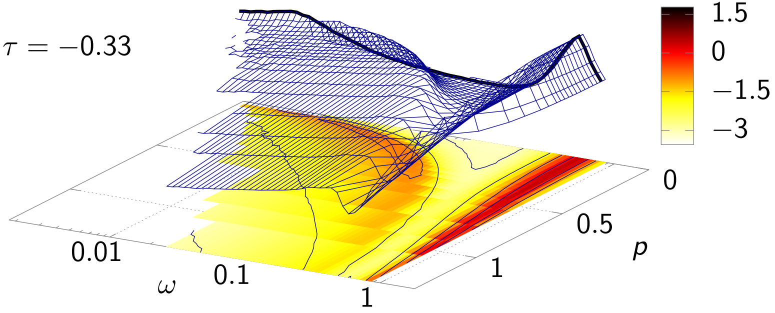


Figure: Spectral functions at different temperatures



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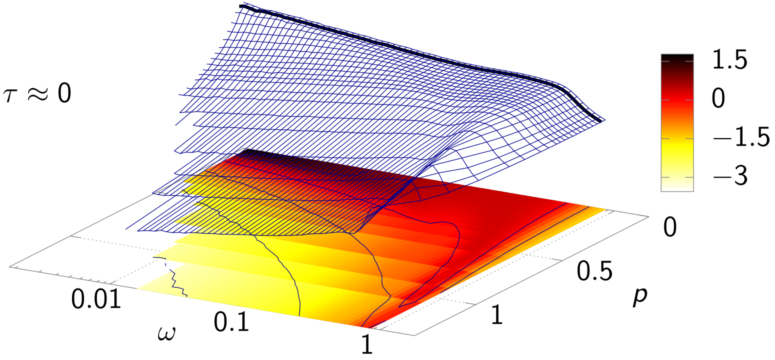

Figure: Spectral functions at different temperatures

Dynamic scaling hypothesis I

$$\tau \equiv \frac{T - T_c}{T_c}$$

- Time scale diverges with length scale: $\xi_t \sim \xi^z \sim \tau^{-\nu z}$
- Time/frequency variables only enter observables in rescaled form:

$$\rho\left(\omega, p, \tau\right) = s^{(2-\eta)} \rho\left(s^z \omega, sp, s^{\frac{1}{\nu}} \tau\right) \tag{8}$$

- Choose $s = \omega^{-1/z}$

$$\rho\left(\omega, p, \tau\right) = \omega^{-(2-\eta)/z} \tilde{\rho}\left(p/\omega^{1/z}, \tau/\omega^{1/\nu z}\right) \tag{9}$$

$\Rightarrow$ Universal scaling function $\tilde{\rho}$

$\Rightarrow$ IR power law at $p = 0, \tau \to 0$:

$$\rho\left(\omega, p, \tau\right) = \omega^{-(2-\eta)/z} \cdot \text{const.} \tag{10}$$

Dynamic scaling hypothesis II

- Fourier transform (9) to time domain:

$$\rho(t,p,\tau) = t^{-\frac{2-\eta}{z}-1}\tilde{\rho}_t\left(pt^{1/z},\tau/t^{1/\nu z}\right) \qquad (11)$$

- Ansatz for $\tilde{\rho}_t$ at vanishing spatial momentum:

$$\tilde{\rho}_t\left(0,\tau/t^{1/\nu z}\right) \propto \exp\left(-\frac{t}{\xi_t}\right) \qquad (12)$$

Additional quantities:

Dynamic critical exponent z, correlation time ξ_t

Dynamic "Universality Classes"

- Time/frequency scaling exponent z
  $\Rightarrow$ dynamic universality classes
- Additional influcences on z:
  - Conservation laws
  - Poisson brackets
- Classification scheme by Halperin/Hohenberg
  - "Models", ordered by conserved fields and non-vanishing Poisson brackets
  - φ^4 w. Langevin dynamics: Model A (3D: $z \approx 2.05$, 2D: $z \approx 2.17$)
  - φ^4 w. Hamiltonian dynamics: Model C (3D: $z = 2.17$, 2D: $z = 2$)
  - QCD: Model H $\rightarrow z = 3$ (Son, Stephanov 2004)

Find universal scaling functions, distinguish models A and C!

Spectral Functions near T_c, $p > 0$

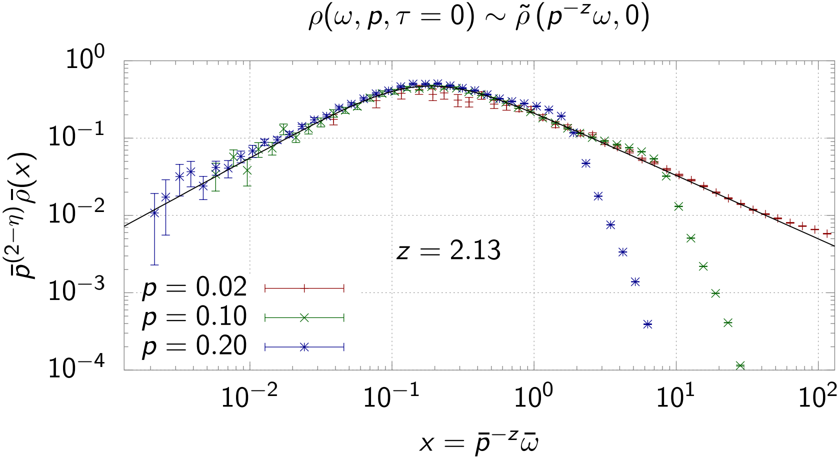

Figure: 2D Model A spectral functions at $\tau \approx 0, p > 0$

Spectral Functions near T_c, $p > 0$

$$\rho(\omega, p, \tau = 0) \sim \tilde{\rho}\left(p^{-z}\omega, 0\right)$$

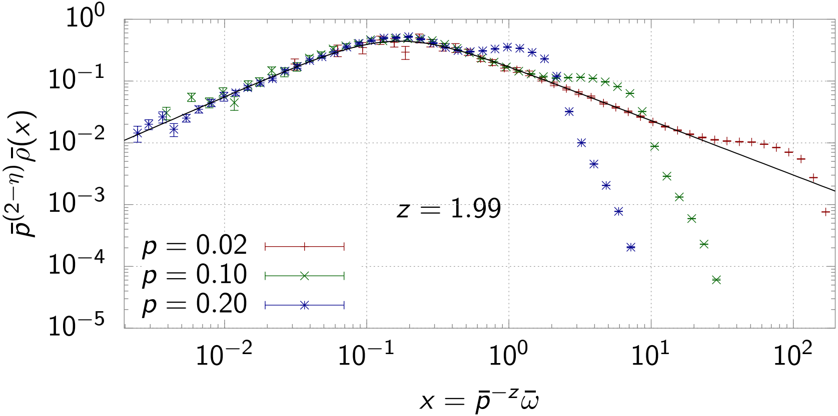

Figure: 2D Model C spectral functions at $\tau \approx 0, p > 0$

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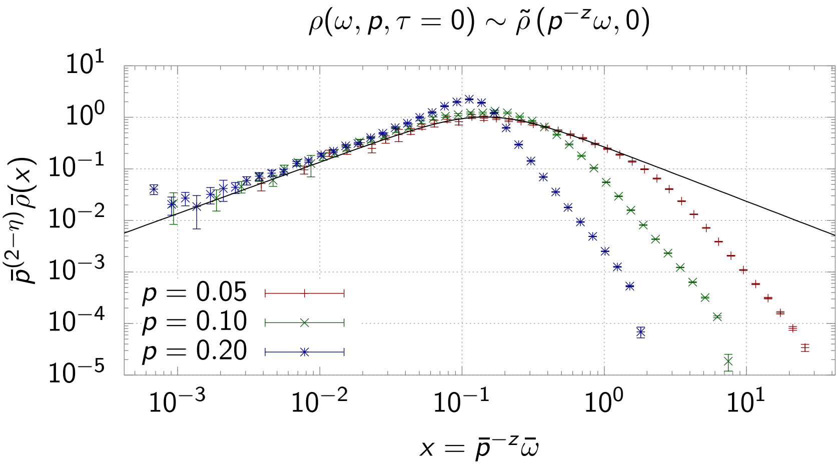

Figure: 3D Model A spectral functions at $\tau \approx 0, p > 0$

Spectral Functions near T_c, $p > 0$

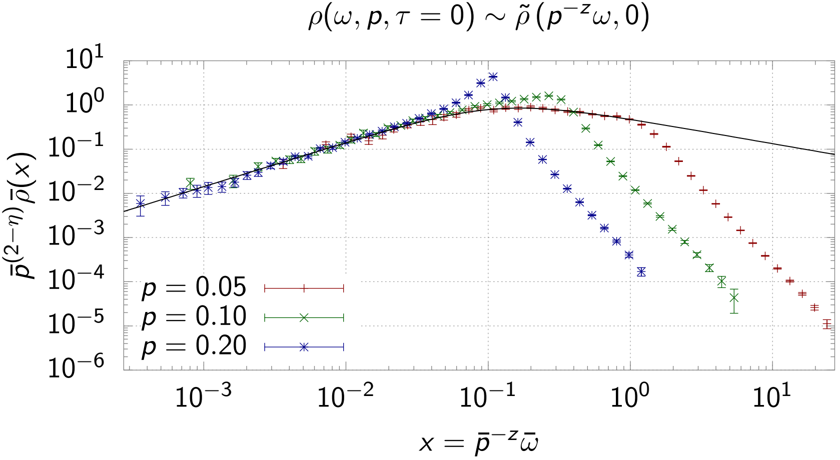

Figure: 3D Model C spectral functions at $\tau \approx 0, p > 0$

Spectral Functions at $\tau > 0$, $p = 0$

$$\rho(\omega, p = 0, \tau) \sim \tilde{\rho}\,(0, \omega/\tau^{\nu z})$$

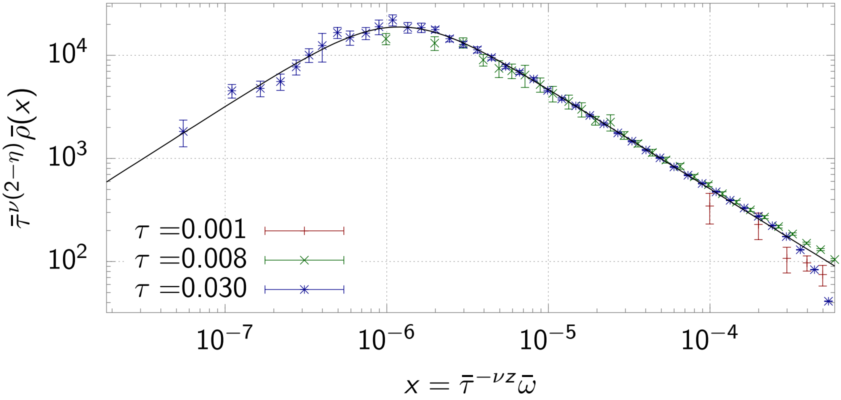

Figure: 2D Model A spectral functions at $\tau > 0, p = 0$

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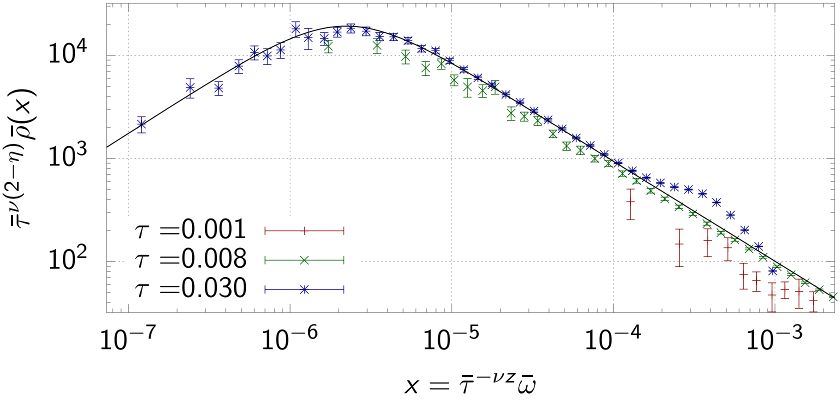

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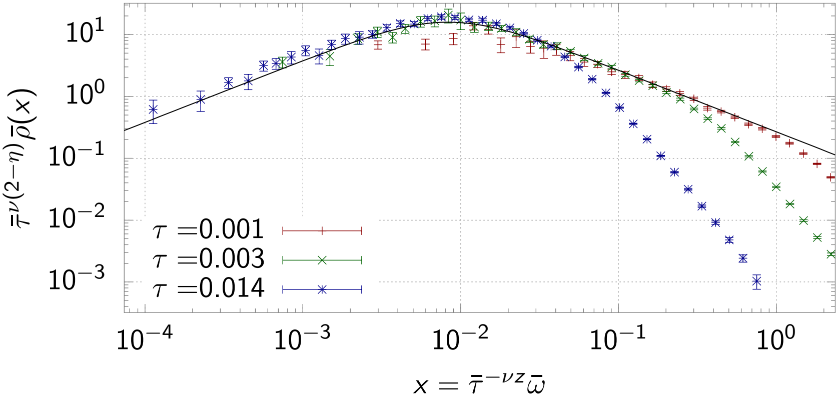

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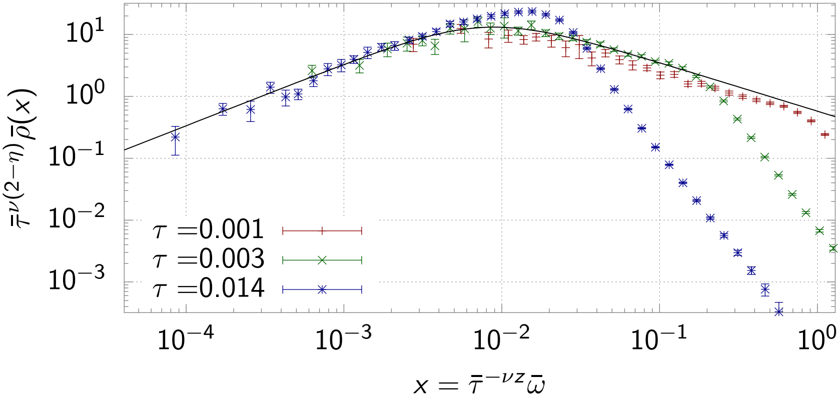


Figure: 3D Model C spectral functions at $\tau > 0, p = 0$

Correlation time ξ_t

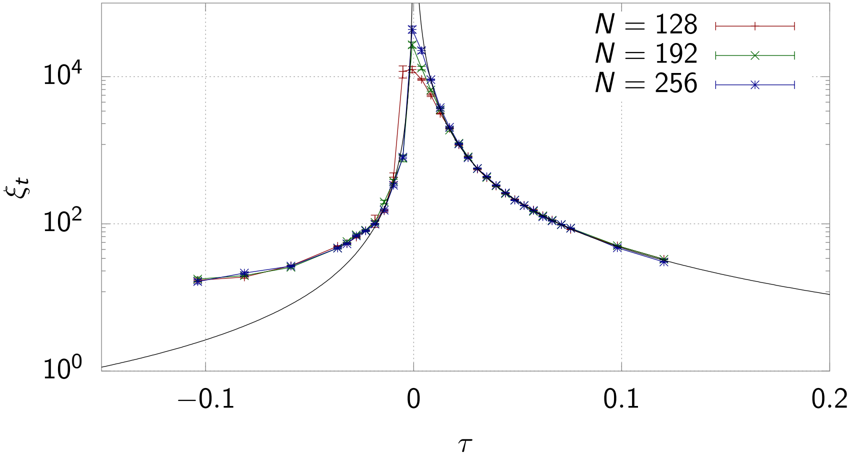
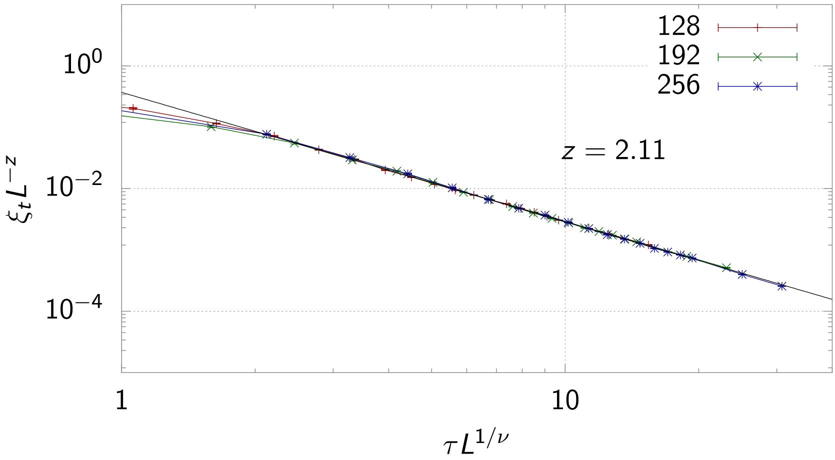


Figure: 2D Model A correlation times with fit function

Calculating z from ξ_t



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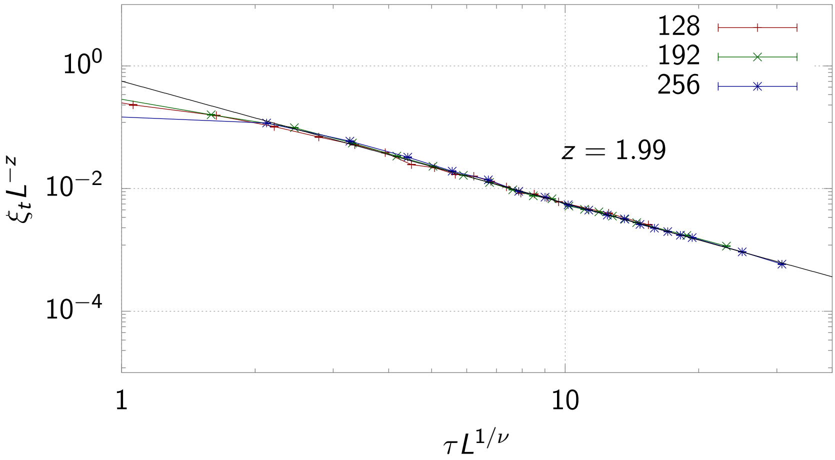


Figure: 2D Model C correlation times with fit function

Finding $z_C - z_A$

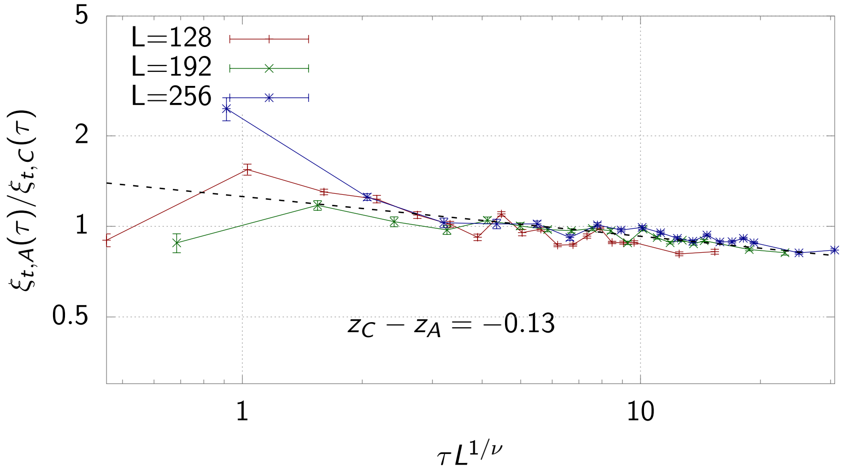


Figure: Ratios of correlation times in 2D. The exponent of the powerlaw corresponds to $\nu(z_C - z_A)$.

Critical dynamics – summary

- Plausible results for ξ_t, z in power law fits at $\tau = 0$
- Measureable difference $z_C - z_A$
- $d = 3$ has lower amplitudes on ξ_t than $d = 2$
- Model C has stronger non-critical contributions than model A

d	Model	$z_{meas.}$	Exp.	MC/scaling
2	A	2.12(1)	2.09(6)	2.1665(12)
2	C	1.98(1)	-	$2 + \frac{\alpha}{\nu} = 2$
3	A	2.02(3)	-	2.05(3)
3	C	2.18(7)	-	$2 + \frac{\alpha}{\nu} = 2.17$

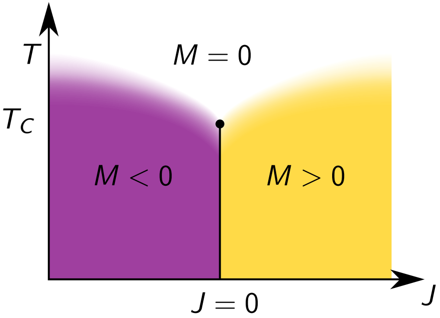
Figure: Comparison of our results to experiment and literature.

Trajectories in Phase Space

- Langevin dynamics couple to heat bath
  $\rightarrow$ handle on T
- Vary J over time

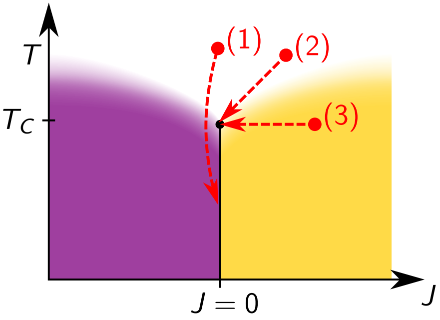
$\Rightarrow$ Trajectory in phase diagram

- Trajectory near critical point: ξ_t large, system falls out of equilibrium



Instant Quench

- Thermalize at $T = T_i, J = J_i$, instant quench to $T = T_q, J = J_q$
- Classify non-equilibrium phenomena
  - coarsening, phase ordering (1)
  - initial slip, exponent θ (2)
  - aging (3)



Instant Quench

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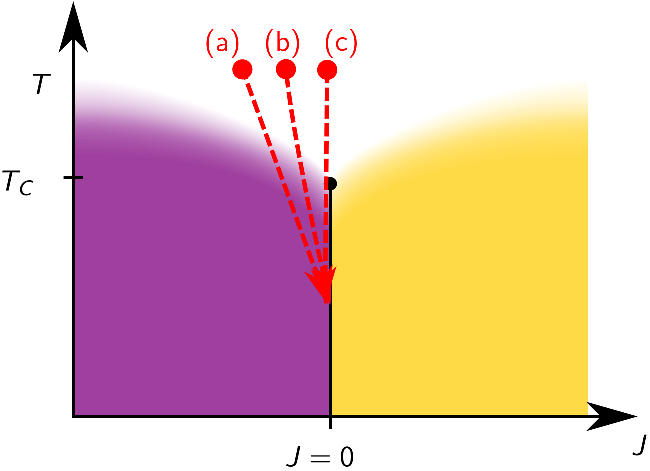

(a)
(b)
(c)
T
T_C
$J = 0$
J

Quench (1), version (a)

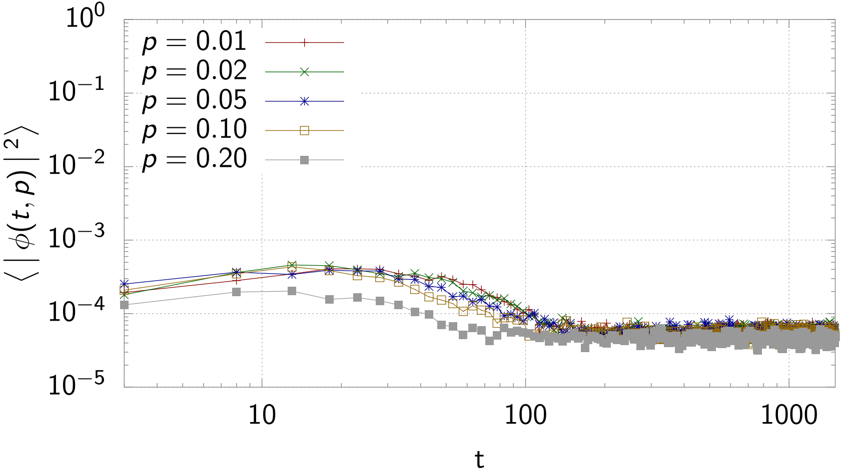
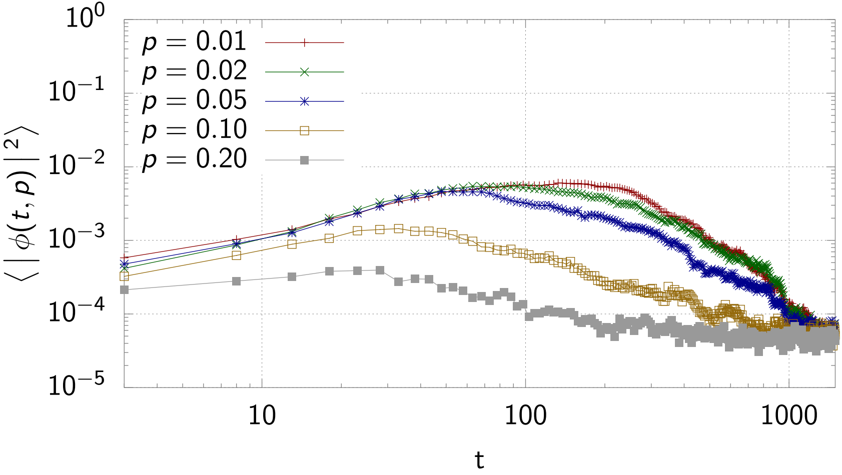


Figure: Order parameter modes fall out of equilibrium



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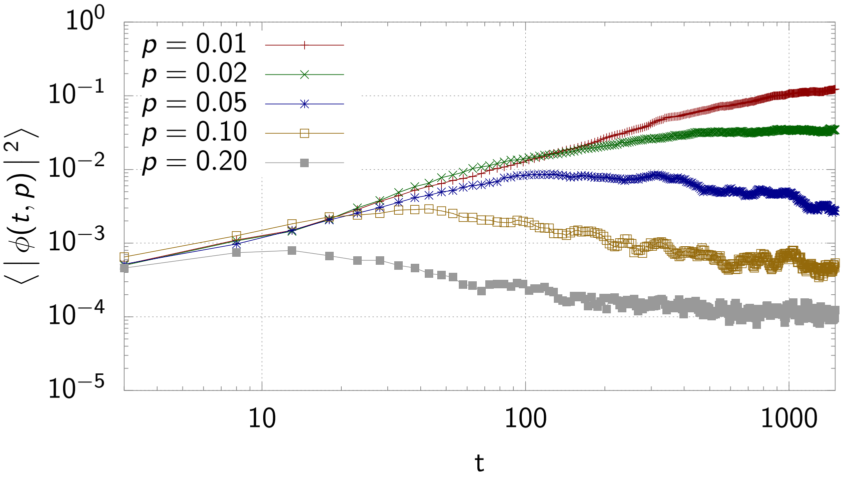
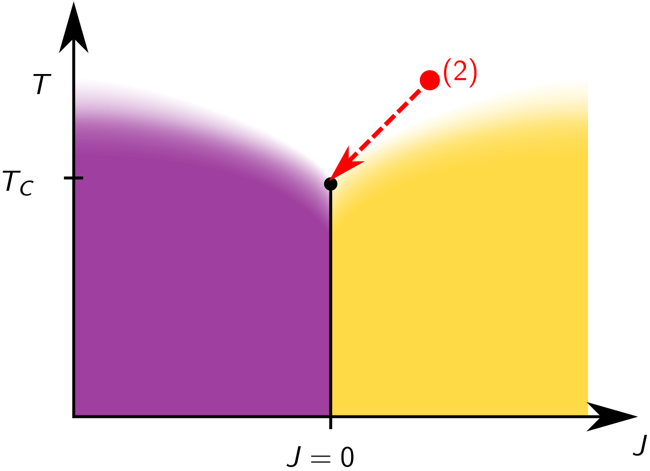


Figure: Order parameter modes fall out of equilibrium

Quench (2)



Relevant observables

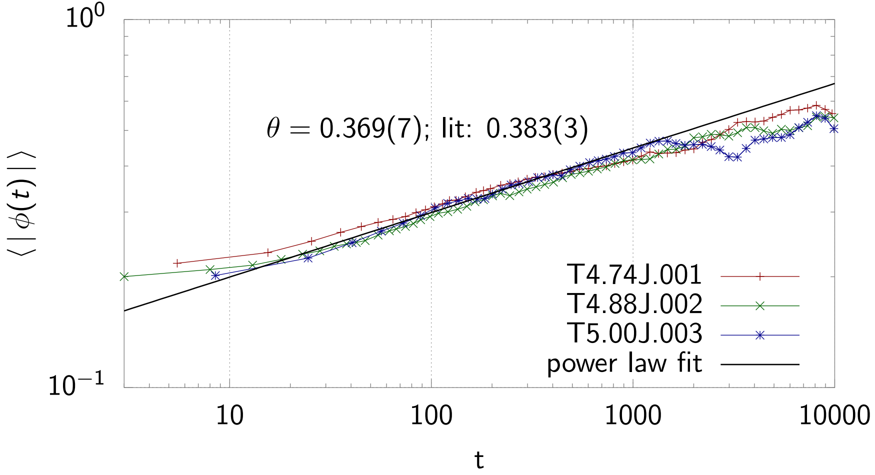
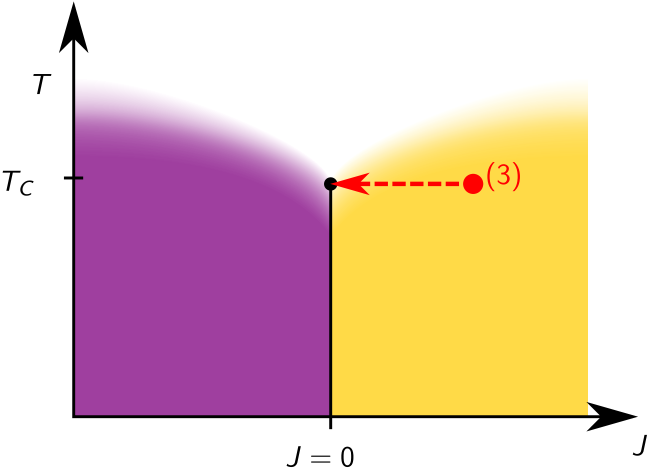


Figure: Algebraic growth, controlled by "initial slip" exponent θ


T
T_C
(3)
$J = 0$
J

Relevant observables

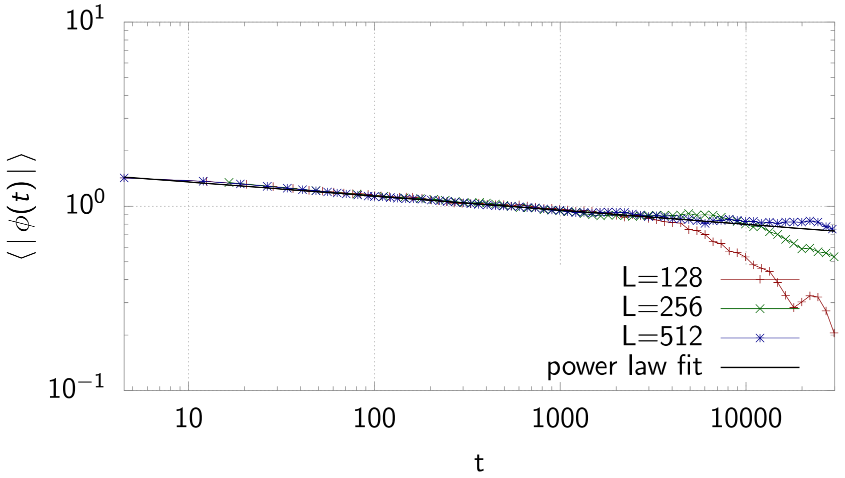
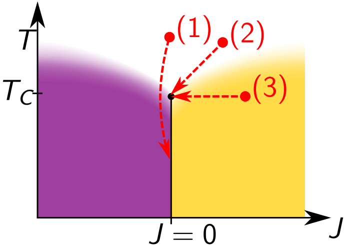


Figure: Algebraic decay, controlled by equilibrium exponents

- We can...
  - ... compute spectral functions from lattice data
  - ... extract scaling function of the spectral function
  - ... calculate dynamic critical exponent z, correlation time ξ_t

- Where to go from here:
  - Explore non-equilibrium phenomena
  - Apply framework to different models
  - Map results onto QCD



Backup

Spectral Functions near T_c, $p > 0$

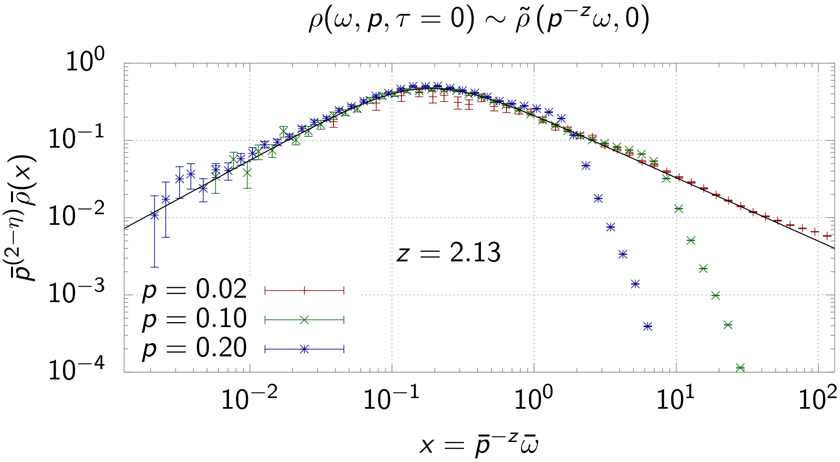


Figure: 2D Model A spectral functions at $\tau \approx 0$, $p > 0$

$$\rho(\omega, p, \tau = 0) \sim \tilde{\rho}\,(p^{-z}\omega, 0)$$

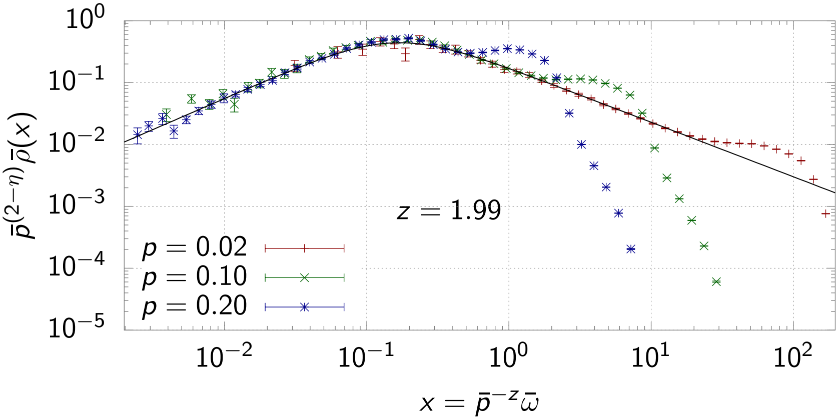


Figure: 2D Model C spectral functions at $\tau \approx 0, p > 0$

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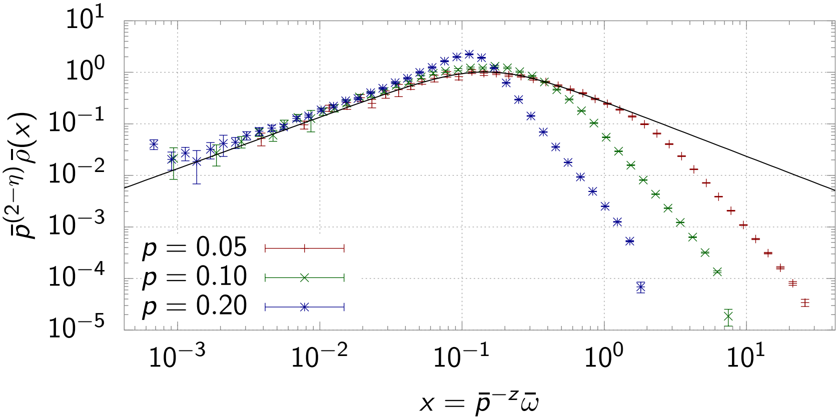

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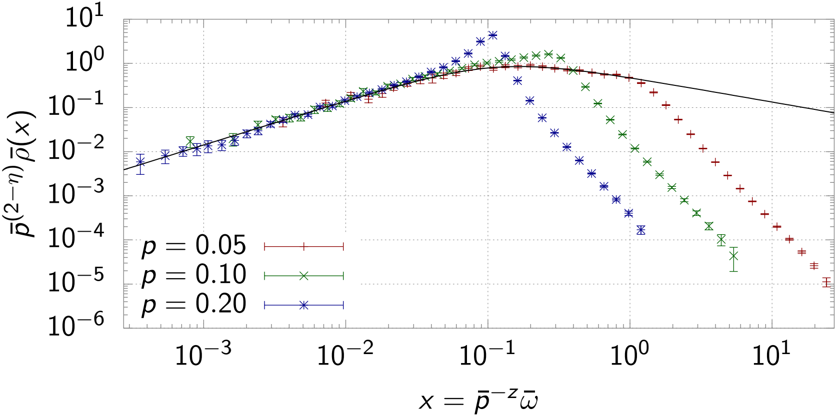


Figure: 3D Model C spectral functions at $\tau \approx 0, p > 0$

$$\rho(\omega, p = 0, \tau) \sim \tilde{\rho}\,(0, \omega/\tau^{\nu z})$$

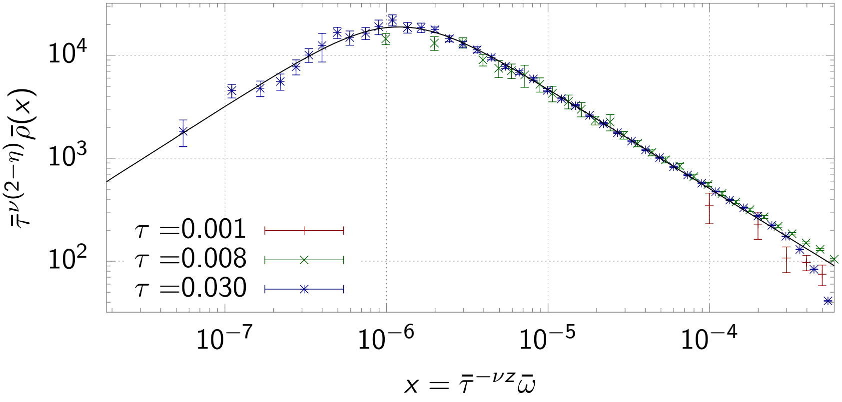


Figure: 2D Model A spectral functions at $\tau > 0, p = 0$

Spectral Functions at $\tau > 0$, $p = 0$

$$\rho(\omega, p=0, \tau) \sim \tilde{\rho}\left(0, \omega/\tau^{\nu z}\right)$$

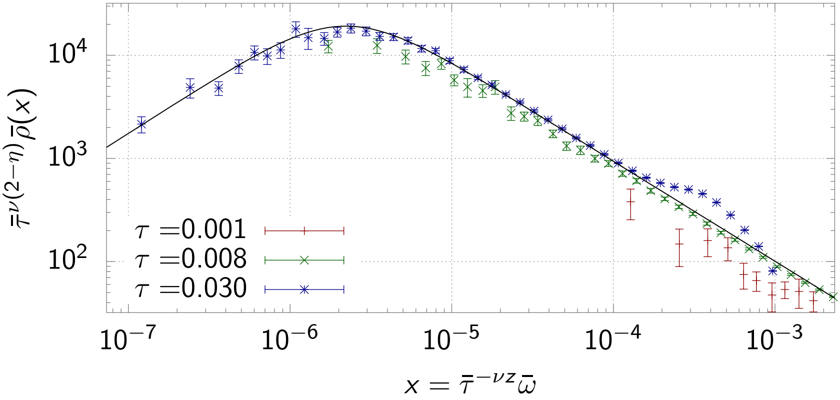


Figure: 2D Model C spectral functions at $\tau > 0, p = 0$

Spectral Functions at $\tau > 0$, $p = 0$

$$\rho(\omega, p = 0, \tau) \sim \tilde{\rho}\left(0, \omega/\tau^{\nu z}\right)$$

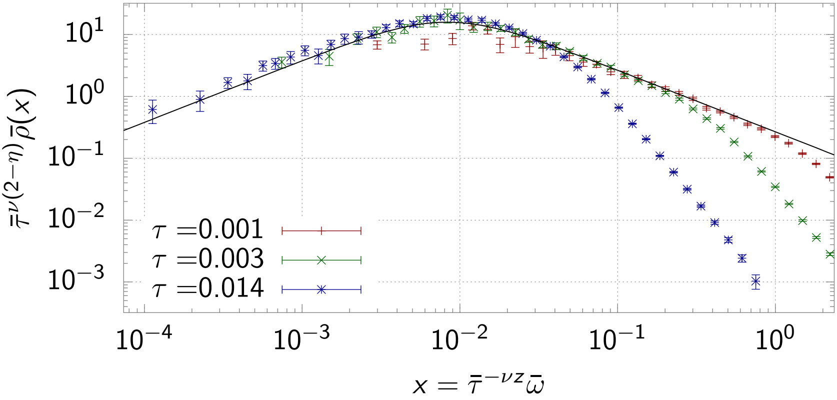


Figure: 3D Model A spectral functions at $\tau > 0, p = 0$

Spectral Functions at $\tau > 0$, $p = 0$

$$\rho(\omega, p = 0, \tau) \sim \tilde{\rho}\left(0, \omega/\tau^{\nu z}\right)$$

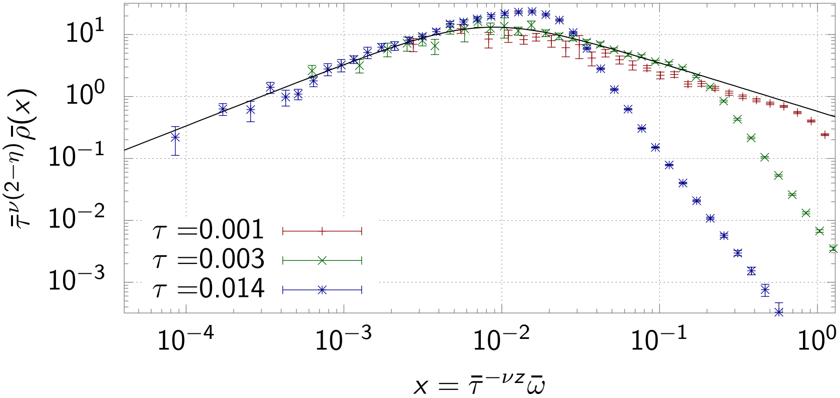


Figure: 3D Model C spectral functions at $\tau > 0, p = 0$

Correlation time ξ_t

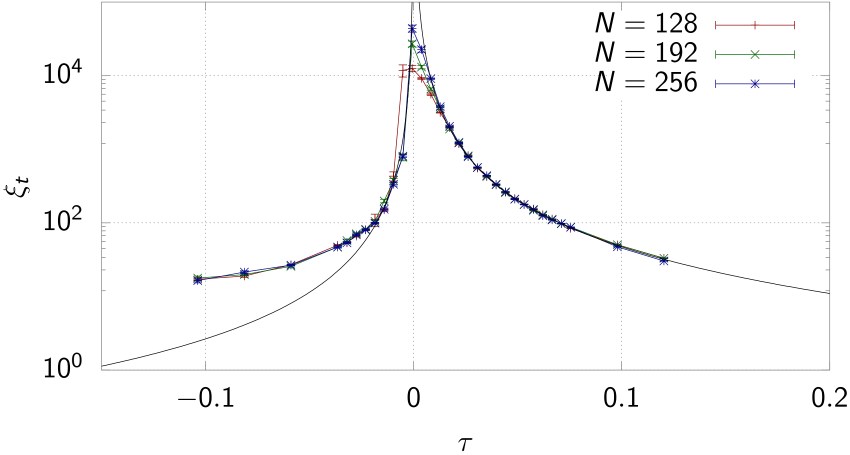
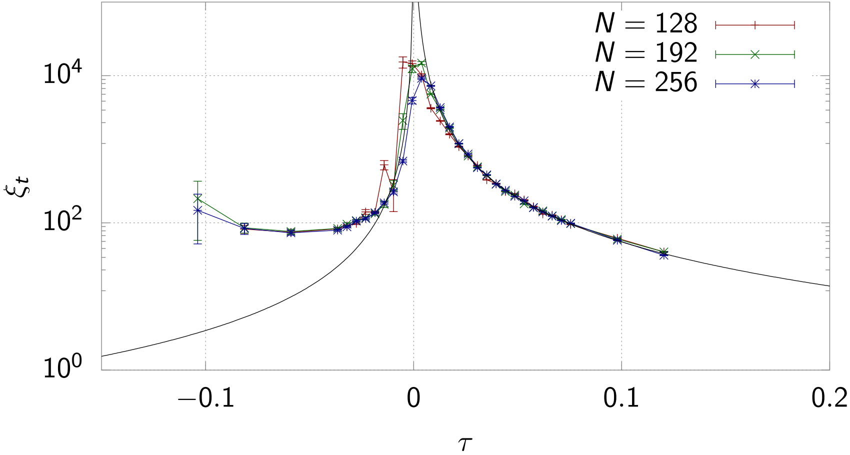


Figure: 2D Model A correlation times with fit function

Correlation time ξ_t



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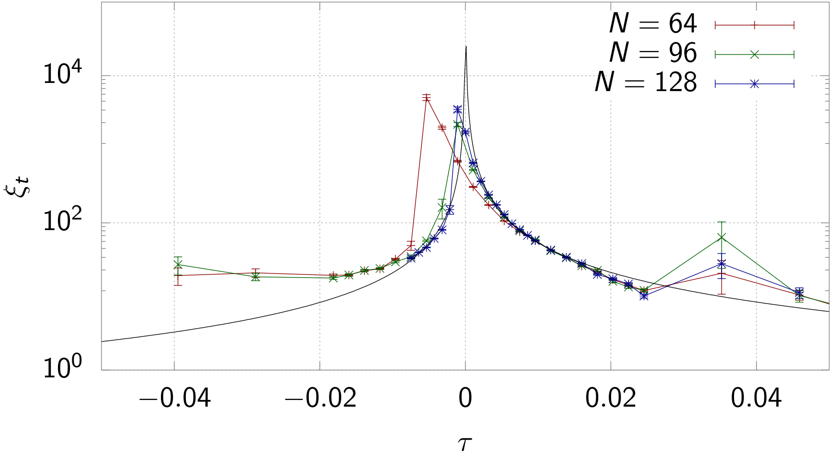
Correlation time ξ_t



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Correlation time ξ_t

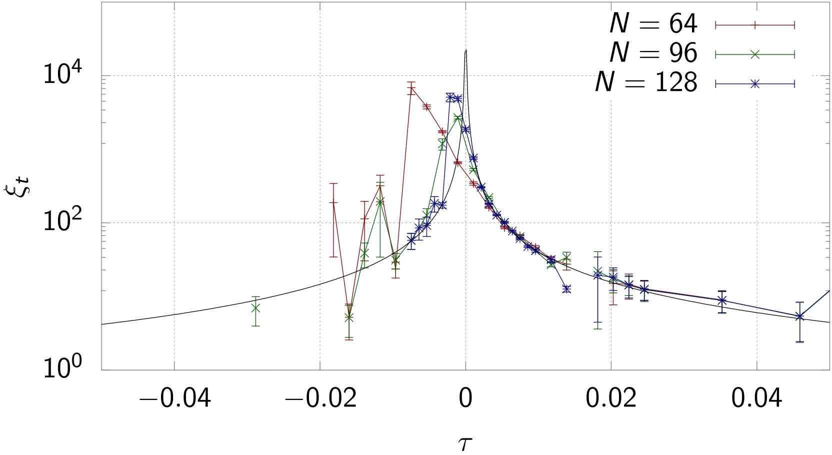

Figure: 3D Model C correlation times with fit function

Calculating z from ξ_t

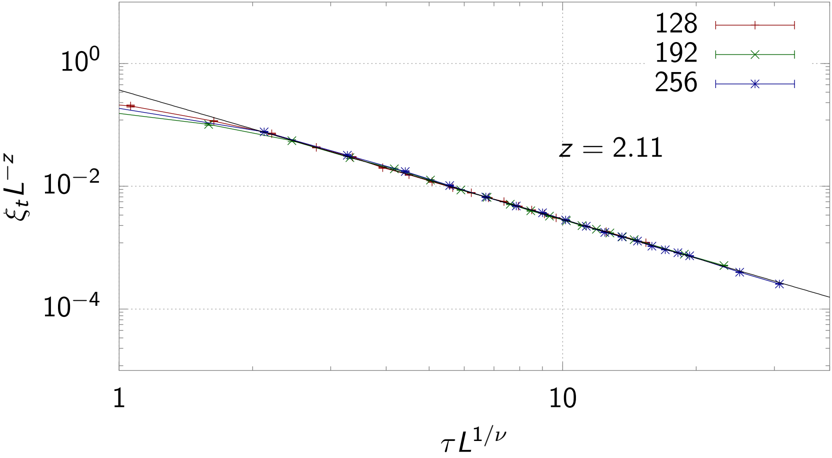

Figure: 2D Model A correlation times with fit function

Calculating z from ξ_t

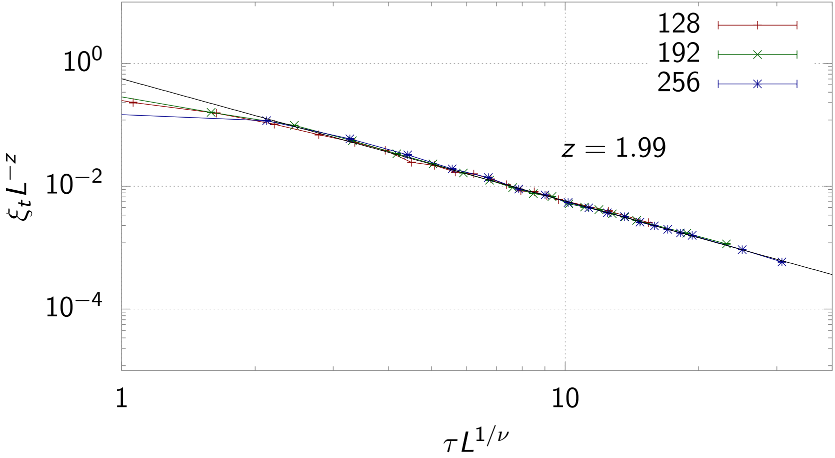

Figure: 2D Model C correlation times with fit function

Calculating z from ξ_t

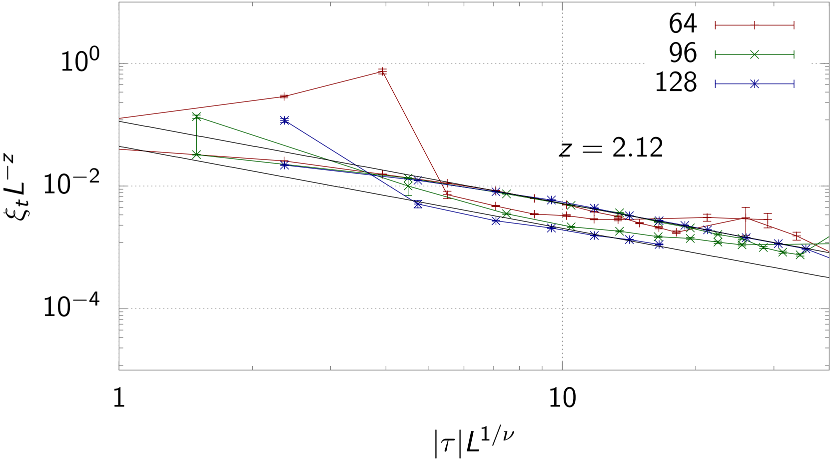

Figure: 3D Model A correlation times with fit function

Calculating z from ξ_t

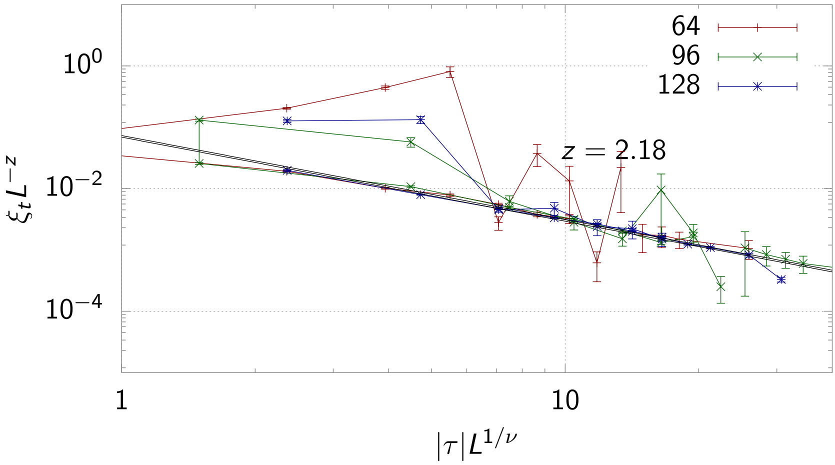

Figure: 3D Model C correlation times with fit function

Finding $z_C - z_A$

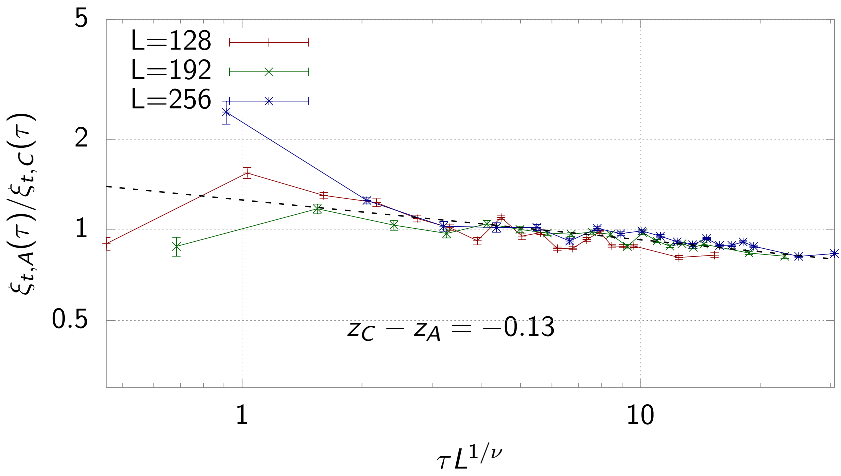

Figure: Ratios of correlation times in 2D. The exponent of the powerlaw corresponds to $\nu(z_C - z_A)$.

Finding $z_C - z_A$

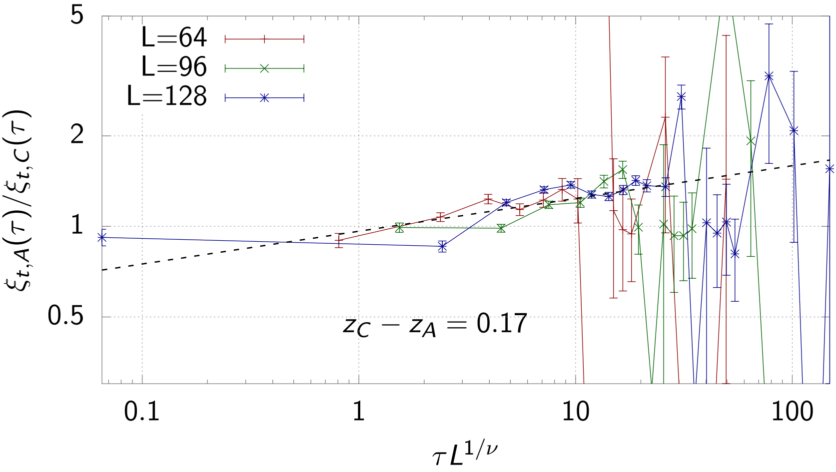

Figure: Ratios of correlation times in 3D. The exponent of the powerlaw corresponds to $\nu(z_C - z_A)$.

2D critical dynamics – summary

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- Plausible results for ξ_t, z in power law fits at $\tau = 0$
- Implausibly small results for z in fits at $\tau > 0$
- Measureable difference $z_C - z_A = -0.13(3) < 0$

Model	$z_{meas.}$	Exp.	MC/scaling
A	2.12(1)	2.09(6)	2.1665(12)
C	1.98(1)	-	$2 + \frac{\alpha}{\nu} = 2$

Figure: Comparison of our results to experiment and literature

- Much lower non-universal amplitudes than in 2D
  - power laws harder to fit
  - harder to get good results on z
- Still measureable difference $z_C - z_A = 0.17(9) > 0$

Model	$z_{meas.}$	MC/scaling
A	2.02(3)	2.05(3)
C	2.18(7)	$2 + \frac{\alpha}{\nu} = 2.17$

Figure: Comparison of our results to experiment and literature

Scaling in both τ and J

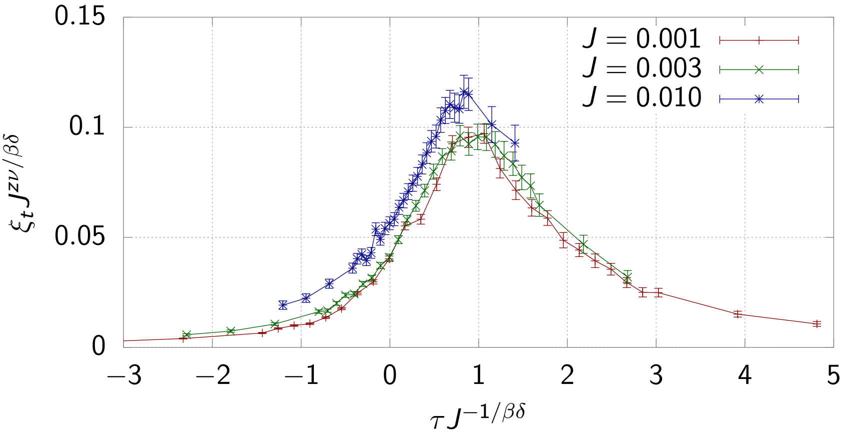

Figure: 2D Model A: Scaling of ξ_t at finite J.