Spectral functions and dynamic critical behaviour of relativistic Z_2 theories

arXiv:2007.03374

D. Schweitzer[1], S. Schlichting[2], L. v. Smekal[1]

[1]Institut für Theoretische Physik, Justus-Liebig-Universität Gießen
[2]Fakultät für Physik, Universität Bielefeld

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

JUSTUS-LIEBIG-UNIVERSITÄT GIESSEN

UNIVERSITÄT BIELEFELD

TECHNISCHE UNIVERSITÄT DARMSTADT

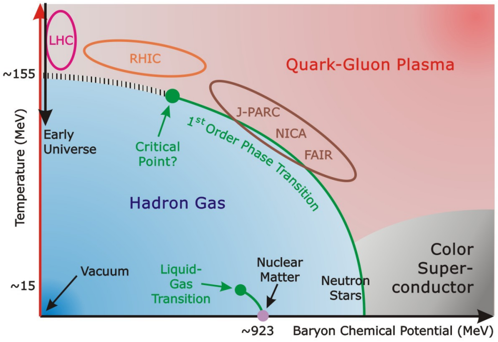
DFG

CRC-TR 211 Strong-interaction matter under extreme conditions

HGS-HIRe for FAIR Helmholtz Graduate School for Hadron and Ion Research

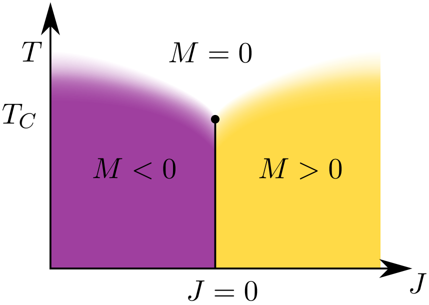
Motivation I

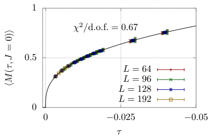
- Search for QCD critical endpoint is gaining attention
- HIC experiments ongoing → dynamic processes
- Theoretical understanding of dynamic CEP signatures needed
- First-principle lattice simulations operate in Euclidean framework
- Functional methods powerful, but require truncation

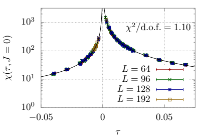


- First-principles method for real-time calculations required
- Make use of scale invariance at critical point:
  - Diverging correlation length $\to$ large clusters
  - Observables behave like $\langle O \rangle \sim |\tau|^\sigma$, $\tau \equiv \frac{T-T_c}{T_c}$
  - Universality
- Critical phenomena fully captured by classical description

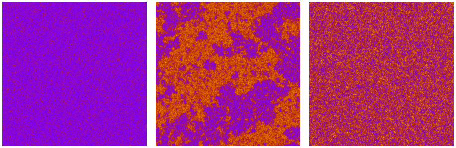
$\Rightarrow$ Study classical systems of the same universality class

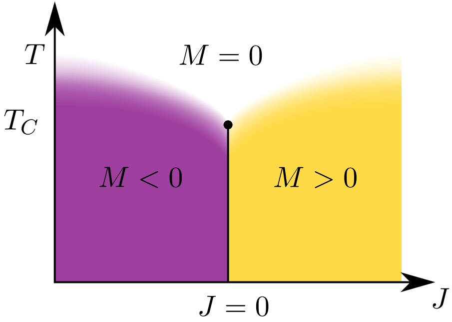






Z_2 theory





$$\mathcal{L}(\phi) = \frac{1}{2}\left(\partial^\mu \phi \partial_\mu \phi - m^2 \phi^2\right) - \frac{\lambda}{4!}\phi^4 - J\phi$$

- For $J \to 0$ invariant under change of sign $\phi \to -\phi$
- If $m^2 < 0$, $\tau = \frac{T - T_c}{T_c} < 0$: symmetry spontaneously broken, $M = \langle \phi \rangle \neq 0$
- CEP of phase border at $T = T_c, J = 0$, Ising universality class

Classical field theory - real-time lattice

$$H(\phi) = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$
$$V(\phi) = \frac{1}{2}\left((\nabla\phi)^2 + m^2\phi^2\right) + \frac{\lambda}{4!}\phi^4 - J\phi$$
$$\phi(t+\Delta t) - \phi(t) = \Delta t \cdot \dot{\phi}$$
$$\dot{\phi}(t+\Delta t) - \dot{\phi}(t) = -\Delta t \cdot \partial_\phi H$$

- Discretize $\phi(x_i), \dot{\phi}(x_i)$ on square/cubic lattice
- Solve equations of motion at every lattice site
- Average over thermal initial conditions
- Hamiltonian dynamics: energy conserved $\rightarrow$ microcanonical ensembles

$\Rightarrow$ Calculate real-time observables as functions of $\phi(t), \dot{\phi}(t)$

$$H(\phi) = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$

$$V(\phi) = \frac{1}{2}\left((\nabla\phi)^2 + m^2\phi^2\right) + \frac{\lambda}{4!}\phi^4 - J\phi$$

$$\phi(t+\Delta t) - \phi(t) = \Delta t \cdot \dot{\phi}$$

$$\dot{\phi}(t+\Delta t) - \dot{\phi}(t) = -\Delta t \cdot \partial_\phi H - \Delta t\left(\gamma\dot{\phi} + \sqrt{2\gamma T}\eta(t)\right)$$

- Discretize $\phi(x_i), \dot{\phi}(x_i)$ on square/cubic lattice
- Solve equations of motion at every lattice site
- Average over thermal initial conditions
- Langevin dynamics: coupling to heat bath $\rightarrow$ canonical ensembles

$\Rightarrow$ Calculate real-time observables as functions of $\phi(t), \dot{\phi}(t)$

Spectral functions

Why spectral functions?

CRC-TR 211

- Spectral function defined via decomposition of Green's function:

$$\rho\left(t,t',x,x'\right) \equiv i\left\langle\left[\phi(t,x),\phi(t',x')\right]\right\rangle$$
$$F(t,t',x,x') \equiv \frac{1}{2}\left\langle\left\{\phi(t,x),\phi(t',x')\right\}\right\rangle - \langle\phi\rangle^2$$

- Contains information about possible excitations of the system
  - Identify relevant d.o.f.
  - Basis for transport, hydrodynamic descriptions
- Accessible by scattering experiments
- Classical limit: commutator $\to$ Poisson bracket (tricky), anti-commutator $\to 2$

Why spectral functions?

CRC-TR 211

- Spectral function defined via decomposition of Green's function:

$$\rho\left(t,t',x,x'\right) \equiv i\left\langle\left[\phi(t,x),\phi(t',x')\right]\right\rangle$$
$$F(t,t',x,x') \equiv \frac{1}{2}\left\langle\left\{\phi(t,x),\phi(t',x')\right\}\right\rangle - \langle\phi\rangle^2$$

- Contains information about possible excitations of the system
  - Identify relevant d.o.f.
  - Basis for transport, hydrodynamic descriptions
- Accessible by scattering experiments
- Classical limit: commutator $\to$ Poisson bracket (tricky), anti-commutator $\to 2$ (easy)

Calculating the spectral function

- Use fluctuation-dissipation theorem to get ρ from F

$$F(\omega,\vec{p}) = -i\left(\frac{1}{2} + n_T(\omega)\right)\rho(\omega,\vec{p}) \tag{1}$$

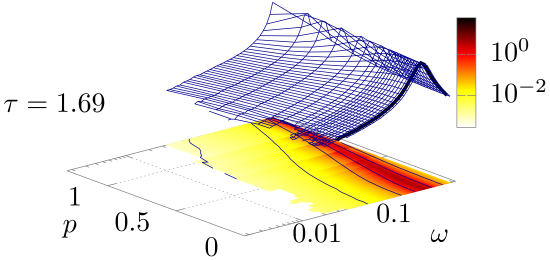
- Classical limit (large $\frac{T}{\omega}$) of thermal distribution $\frac{1}{2} + n_T(\omega) \rightarrow \frac{T}{\omega}$:

$$F(\omega,\vec{p},T) = -i\frac{T}{\omega}\rho(\omega,\vec{p},T) \tag{2}$$

$$\Rightarrow \quad \rho(t-t',\vec{p},T) = -\frac{1}{T}\frac{\partial}{\partial t}F(t-t',\vec{p},T) \tag{3}$$

$$= -\frac{1}{T}\left\langle \dot{\phi}(t,\vec{p})\phi(t',-\vec{p})\right\rangle \tag{4}$$

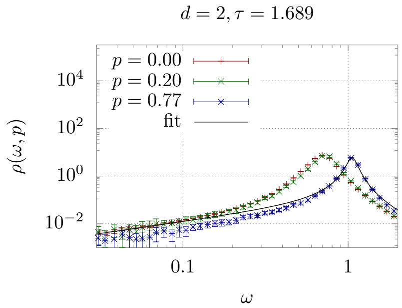
Now calculate ρ from classical fields $\phi, \dot{\phi}$!

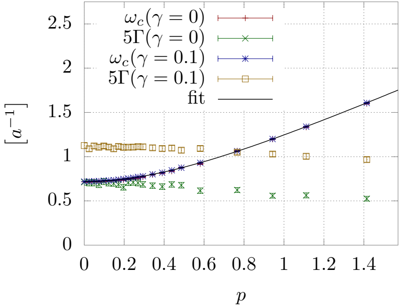


- Symmetric phase: single quasi-particle excitation(?)
- Fit to Breit-Wigner shape $\rho_{BW}(\omega, p)$

$$\rho_{BW}(\omega, p) = \frac{\Gamma(p)\omega}{(\omega^2 - \omega_c^2(p))^2 + \omega^2\Gamma^2(p)} \tag{5}$$

Spectral Functions II



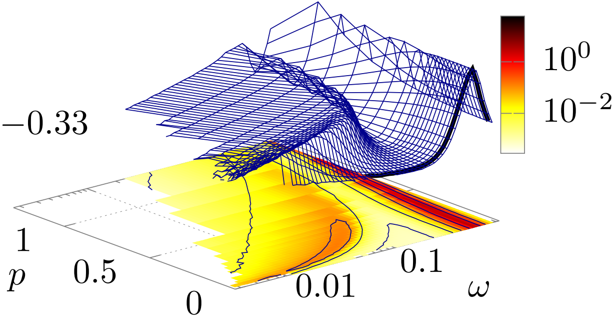


- Breit-Wigner shape fits data points around peak
- Resulting central frequency fits $\omega_c^2(p) = m^2 + p^2$ perfectly
- No p-dependent corrections to damping rate:

$$\Gamma(p, \gamma) \approx \gamma + \text{const.} \tag{6}$$

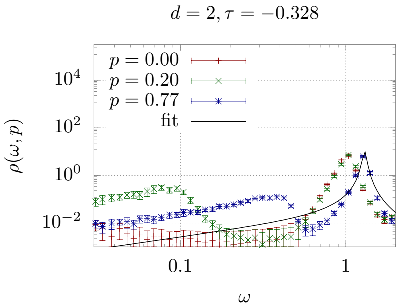
Spectral Functions III

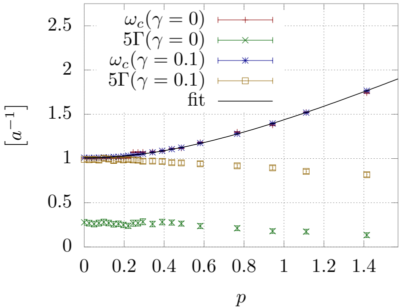
$\tau = -0.33$



- Ordered phase: quasi-particle excitation + X
- X $\equiv$ capillary wave, "ripples"
- Isolate large frequencies for fit to $\rho_{BW}(\omega, p)$

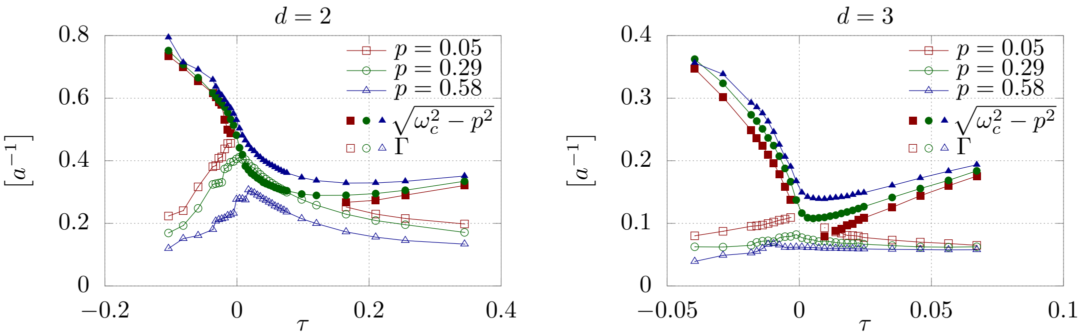
Spectral Functions IV





- Breit-Wigner shape fits narrow region around peak
- Resulting central frequency still fits $\omega_c^2(p) = m^2 + p^2$ perfectly
- Also no p-dependent corrections to damping rate:

$$\Gamma(p, \gamma) \approx \gamma + \text{const.} \tag{7}$$

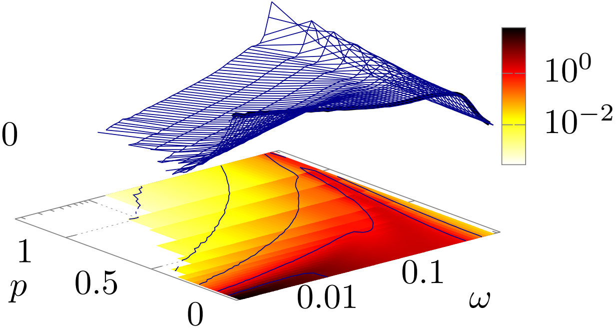


- $\Gamma \to 0$, $m^2 \to 2$ as $T \to 0$ (mean-field limit)
- Quasi-particle mass drops, damping rate increases as $T \to T_c$
- Continuous process at large spatial momentum

Dynamic critical behaviour

Critical spectral function

$\tau \approx 0$



- Strong IR divergence building up at low p
- Critical contribution suppressed at high p
- Quasi-particle mass $m^2 \to 0$

Dynamic scaling hypothesis I

- Time scale diverges with length scale: $\xi_t \sim \xi^z \sim \tau^{-\nu z}$
- Time/frequency variables only enter observables in rescaled form:

$$\tau \equiv \frac{T - T_c}{T_c}$$

$$\rho\left(\omega, p, \tau\right) = s^{(2-\eta)} \rho\left(s^z \omega, sp, s^{\frac{1}{\nu}} \tau\right) \quad (8)$$

- Choose $s = \omega^{-1/z}$

$$\rho\left(\omega, p, \tau\right) = \omega^{-(2-\eta)/z} f_\omega\left(p/\omega^{1/z}, \tau/\omega^{1/\nu z}\right) \quad (9)$$

⇒ Universal scaling function f_ω

⇒ If $f_\omega(0,0) > 0$, find IR power law at $p = 0, \tau \to 0$:

$$\rho\left(\omega, 0, 0\right) = \omega^{-(2-\eta)/z} f_\omega(0,0) \quad (10)$$

Dynamic “Universality Classes”

- Time/frequency scaling exponent z
  $\Rightarrow$ dynamic universality classes
- Additional influcences on z:
  - Conserved charges
  - Coupled modes
- Classification scheme by Halperin/Hohenberg
  - “Models”, ordered by conserved fields and non-vanishing Poisson brackets
  - φ^4 w. Langevin dynamics: Model A (3D: $z \approx 2.05$, 2D: $z \approx 2.17$)
  - φ^4 w. Hamiltonian dynamics: Model C (3D: $z = 2.17$, 2D: $z = 2$)
  - QCD: Model H $\rightarrow z = 3$ (Son, Stephanov 2004)

Challenge:

Find universal scaling functions, distinguish models A and C!

Universal scaling function

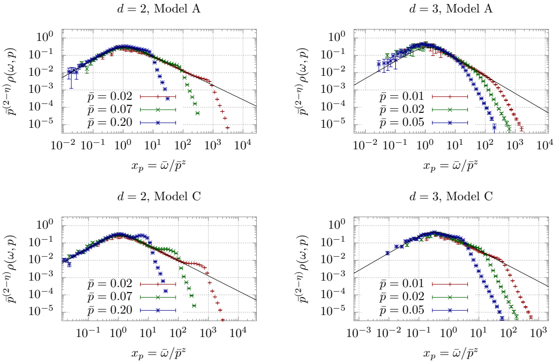


Figure: Rescaled spectral functions at $\tau = 0$. Overlapping data points reveal the scaling function. The black line is the inverse sum of the limits.

- Scale-invariant spectral function:

$$\rho(\omega, p, \tau) = s^{(2-\eta)}\rho\left(s^z\omega, sp, s^{\frac{1}{\nu}}\tau\right)$$

- Let $s = p^{-1}$:

$$\rho(\omega, p, \tau = 0) = p^{-(2-\eta)} f_p\left(p^{-z}\omega, 0\right)$$

- Limiting behaviour:

$$\lim_{x\to\infty} f_p(x, 0) \sim x^{-(2-\eta)/z}$$

$$\lim_{x\to 0} f_p(x, 0) \sim x$$

Universal scaling function

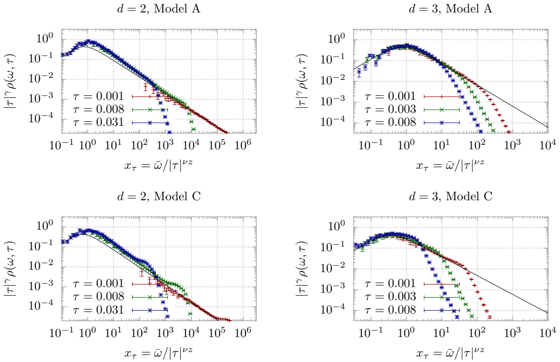


Figure: Rescaled spectral functions at $p = 0, \tau > 0$. Overlapping data points reveal the scaling function. The black line is the inverse sum of the limits.

- Scale-invariant spectral function:

$$\rho\left(\omega, p, \tau\right) = s^{(2-\eta)} \rho\left(s^z \omega, sp, s^{\frac{1}{\nu}} \tau\right)$$

- Let $s = |\tau|^{-\nu}$ where $\tau > 0$:

$$\rho\left(\omega, p = 0, \tau\right) = \tau^{-(2-\eta)/\nu} f_\tau^+ \left(\tau^{-\nu z} \omega, 0\right)$$

- Limiting behaviour:

$$\lim_{x \to \infty} f_\tau^+(x, 0) \sim x^{-(2-\eta)/z}$$

$$\lim_{x \to 0} f_\tau^+(x, 0) \sim x$$

Universal scaling function

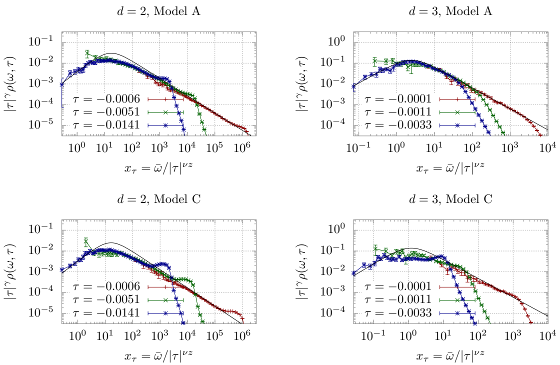


Figure: Rescaled spectral functions at $p = 0, \tau < 0$. Overlapping data points reveal the scaling function. The black line is the inverse sum of the limits.

- Scale-invariant spectral function:

$$\rho\left(\omega, p, \tau\right) = s^{(2-\eta)} \rho\left(s^z \omega, sp, s^{\frac{1}{\nu}} \tau\right)$$

- Let $s = |\tau|^{-\nu}$ where $\tau < 0$:

$$\rho\left(\omega, p = 0, \tau\right) = \tau^{-(2-\eta)/\nu} f_\tau^- \left(\tau^{-\nu z} \omega, 0\right)$$

- Limiting behaviour:

$$\lim_{x \to \infty} f_\tau^- \left(x, 0\right) \sim x^{-(2-\eta)/z}$$

$$\lim_{x \to 0} f_\tau^- \left(x, 0\right) \sim x$$

Universal scaling function

- Scaling behaviour accesible in all cases
  - Critical scaling has wider range for $d = 2$ than for $d = 3$
  - Stronger non-critical corrections for Model C than for Model A
- Scaling function at $\tau = 0, p > 0$ can be parametrized by

$$f_p\left(x_p, 0\right) = \frac{1}{\left(a_p x_p\right)^{-1} + f_\omega^{-1} x_p^{(2-\eta)/z}} \text{ for } x_p = \bar{\omega}/\bar{p}^z \tag{11}$$

- Scaling functions at $\tau \neq 0, p = 0$ similar

- Fourier-transform of scale-invariant $\rho(\omega, p, \tau)$ to time domain yields

$$\rho(t, p, \tau) \sim t^{-(2-\eta)/z-1} f_t\left(p^z t, \tau t^{1/\nu z}\right) \tag{12}$$

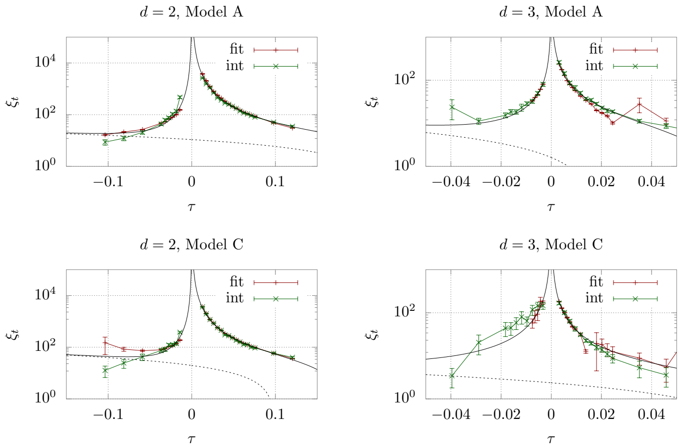
$$\Rightarrow \quad \rho(t, p=0, \tau=0) \sim t^{-(2-\eta)/z-1} \tag{13}$$

- For finite τ, p, one expects

$$\lim_{t\to\infty} \rho(t, p, \tau) \sim \exp -\frac{t}{\xi_t}, \tag{14}$$

⇒ Autocorrelation time $\boxed{\xi_t \sim \xi^z \sim \tau^{-\nu z}}$

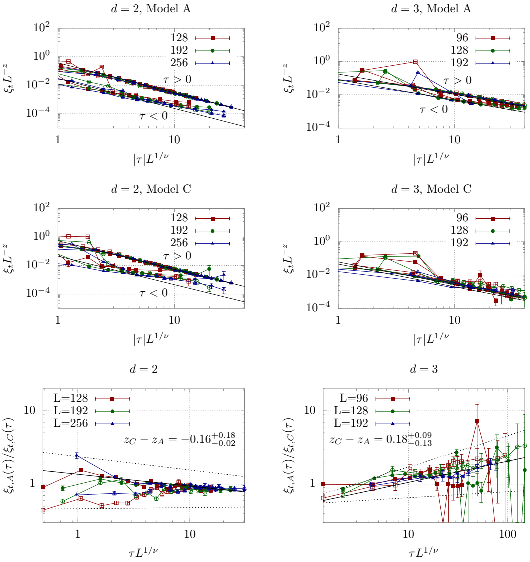
Autocorrelation time ξ_t II



- Find power-law divergence around $\tau = 0$
- Strong divergence for $\tau > 0$, smaller amplitudes at $\tau < 0$
- Relatively large regular part

Extracting z

Divergence of ξ_t

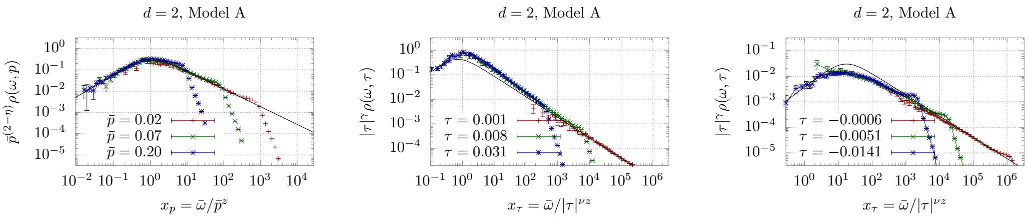


- Fit the exponent of the power-law divergence
  - strong correlation of exponent and amplitude
  - corrections from regular part

⇒ Use ratio of ξ_t for different models to estimate $z_A - z_C$

$$\frac{\xi_{t,A}}{\xi_{t,C}} \sim \frac{|\tau|^{-\nu z_A}}{|\tau|^{-\nu z_C}} \sim |\tau|^{\nu(z_C - z_A)} \tag{15}$$

- larger error, jitter
+ better to visualize difference

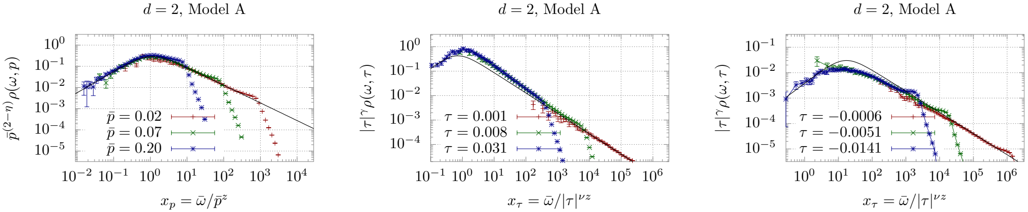


- Minimize 2-norm of pair-wise distance between interpolated spectral functions

$$\Delta^2(z) = \sum_{p_i} \sum_{p_i<p_j} \int_{\omega_l}^{\omega_h} \mathrm{d}\omega \frac{\left| p_i^{2-\eta}\rho(p_i^z\omega, p_i, 0) - p_j^{2-\eta}\rho(p_j^z\omega, p_j, 0)\right|^2}{\left(p_i^{2-\eta}\Delta\rho(p_i^z\omega, p_i, 0)\right)^2 + \left(p_j^{2-\eta}\Delta\rho(p_j^z\omega, p_j, 0)\right)^2}, \tag{16}$$

  + No a priori information necessary, only specify ω interval
  - Limited precision

Limiting behaviour of universal function



- Large-x behaviour of the universal function is controlled by z

$$f_p\left(x_p,0\right) = \frac{1}{\left(a_p x_p\right)^{-1} + f_\omega^{-1} x_p^{(2-\eta)/z}} \quad \text{for } x_p = \bar{\omega}/\bar{p}^z \tag{17}$$

+ Higher precision
- Dependence on borders of fit interval

Critical dynamics – summary

- Plausible results for ξ_t, z in power law fits at $\tau = 0$
- Measureable difference $z_C - z_A$

d	Model	$z_{meas.}$	Exp.	MC/scaling
2	A	2.12(1)	2.09(6)	2.1665(12)
2	C	1.98(1)	-	$2 + \frac{\alpha}{\nu} = 2$
3	A	2.02(3)	-	2.05(3)
3	C	2.18(7)	-	$2 + \frac{\alpha}{\nu} = 2.17$

Table: Comparison of our results to experiment and literature.

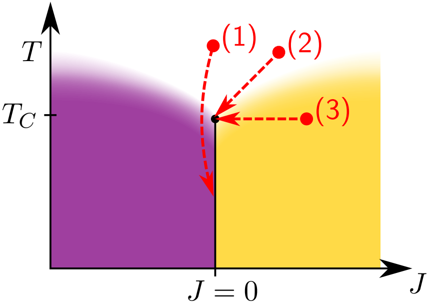
Conclusion

CRC-TR 211

- First-principles calculation of spectral functions using classical-statistical simulations
- Identified relevant excitations in all regions of the phase diagram
- Explicitely verified dynamic scaling hypothesis
  - Observed scaling behaviour in frequency, temperature and spatial momentum
  - Extracted universal scaling functions, developed parametrization
  - Analyzed divergence of autocorrelation time ξ_t
  - Extracted the dynamic scaling exponent z
- Compared dynamic critical behaviour of Models A,C in $d = 2$ and $d = 3$
  - Very similar up to amplitudes
  - Stronger non-critical contributions in Model C

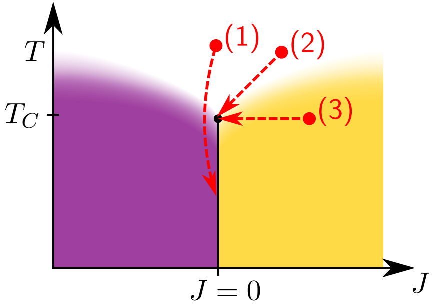
Outlook

- Currently underway: Study on Models B, D (conserved order parameter)
- Analysis of multi-time correlations of $T^{\mu\nu}$
  - Effects of non-continuous symmetry on conservation laws
  - Transport coefficients
- Non-equilibrium critical phenomena (Thesis of C. Kummer)



Outlook

- Currently underway: Study on Models B, D (conserved order parameter)
- Analysis of multi-time correlations of $T^{\mu\nu}$
  - Effects of non-continuous symmetry on conservation laws
  - Transport coefficients
- Non-equilibrium critical phenomena (Thesis of C. Kummer)



Happy holidays!

Thank you for your attention.