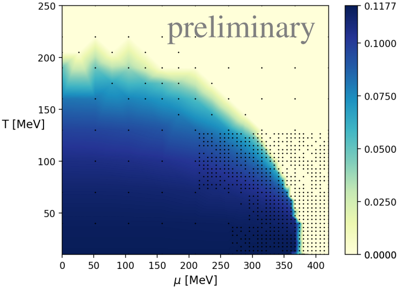
Towards resolving the QCD phase structure with Discontinuous Galerkin Methods

Friederike Ihssen

24.11.2021 - Lunch Club Seminar @ JLU Gießen



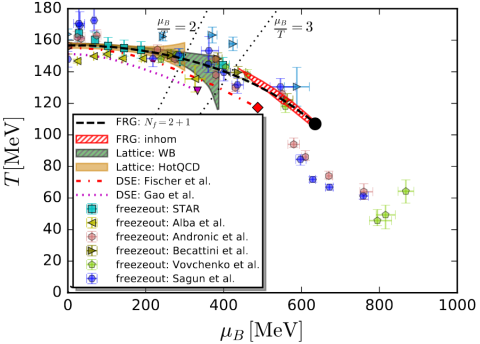
Work in collaboration with: E. Grossi, J. M. Pawlowski, F. Sattler, N. Wink

UNIVERSITÄT HEIDELBERG ZUKUNFT SEIT 1386

Studienstiftung des deutschen Volkes

HGS-HIRe for FAIR
Helmholtz Graduate School for Hadron and Ion Research

The QCD-Phase Structure, so far...



W. F. Fu, J. M. Pawlowski, F. Rennecke, (Phys.Rev.D 101, 054032)

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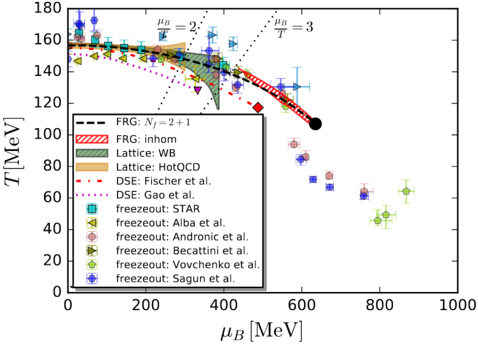

W. Fu, J. M. Pawlowski, F. Rennecke, (Phys.Rev.D 101, 054032)

1. The LEFTs of QCD in the Functional Renormalisation Group
2. Discontinuous Galerkin Methods
3. Results: Quark Meson Model
4. Improvements: Local Discontinuous Galerkin
5. Conclusion and Outlook

A Functional Approach

The infrared regularized path-integral / generating functional:

$$\mathcal{Z}_k[J] = \int [d\varphi]_{\mathrm{ren},p^2 \geq k^2} \exp\Big\{ - S[\varphi] + J \cdot \varphi \Big\} ,$$

$$\int [d\varphi]_{\mathrm{ren},p^2 \geq k^2} = \int [d\varphi]_{\mathrm{ren}} \exp\Big\{ - \frac{1}{2} \varphi \cdot R_k \cdot \varphi \Big\}$$

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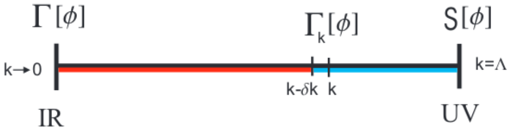
$$\mathcal{W}_k[J] = \ln \mathcal{Z}_k[J]$$

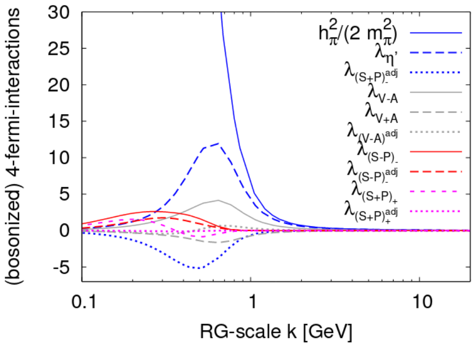
$$\Gamma_k[\phi] = \sup \left\{ \int_x J(x)\phi(x) - \mathcal{W}_k[J] - \Delta S_k[\phi] \right\}$$



Successive integration of momentum shells:

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2}\mathrm{Tr}\left[\left(\frac{1}{\Gamma_k^{(2)}[\Phi] + R_k}\right)_{ab} \partial_t R_k^{ab}\right], \qquad t = \ln\left(\frac{\Lambda}{k}\right).$$

QCD at low Energies



(M. Mitter, J. M. Pawlowski, N. Strodthoff, Phys.Rev.D 91, 054035)

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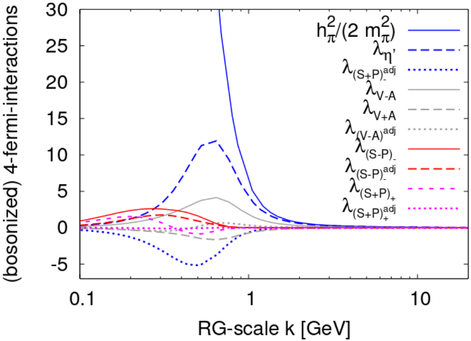

- Complete basis consistent with symmetry.

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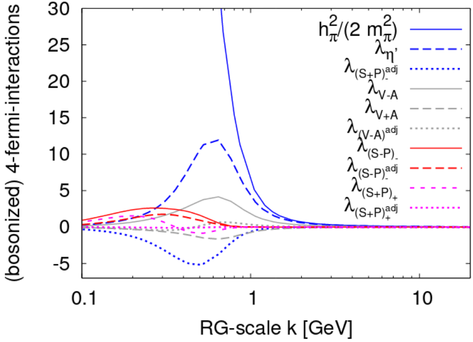

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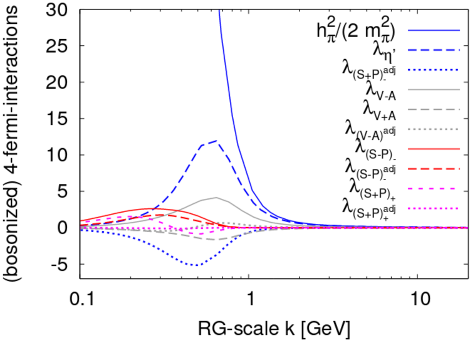

(M. Mitter, J. M. Pawlowski, N. Strodthoff, Phys.Rev.D 91, 054035)

- Complete basis consistent with symmetry.
- We have one dominant resonant channel.
- Introduction of a quark-condensate: $\bar{q}q$.

A Simple Low Energy Effective Model of QCD:

$$\Gamma_k = \int_x \left\{ \imath\bar{q}(\gamma_\mu\partial_\mu + \gamma_0\mu)q + \frac{1}{2}(\partial_\mu\phi)^2 + \bar{h}_k\bar{q}(\tau\cdot\phi)q + V_k \right\}$$

- Kinetic quark term
- Kinetic meson term
- Quark-meson coupling
- Higher-order mesonic scatterings

A Simple Low Energy Effective Model of QCD:

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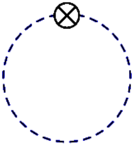
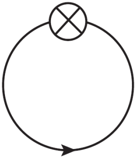
Use the condensate as an order parameter:

$$\langle \bar{q} q \rangle = \sigma = \begin{cases} 0 & \text{symmetric phase} \\ \sigma_0 & \text{broken phase} \end{cases}$$

Solving FRG-Equations

$$\partial_t V + f(\partial_\rho V) + g(\partial_\rho V, \partial_\rho^2 V) = s(h(\rho), \rho) \ , \qquad \rho = \frac{\phi^2}{2}$$

$+\frac{1}{2}$ 

$-$ 

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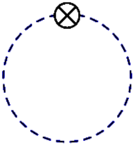
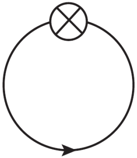
$+\frac{1}{2}$ 

$-$ 

PDE with second order derivatives:

$\Rightarrow$ Analogy to Hydrodynamics

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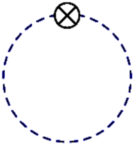
$+\frac{1}{2}$



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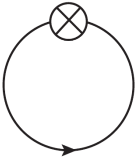

PDE with second order derivatives:

$\Rightarrow$ Analogy to Hydrodynamics

PDE with discontinuities:

$\Rightarrow$ Requires discontinuous methods

Solving Conservation Laws

Scalar conservation law:

$$\partial_t u + \partial_x f(u) = 0$$

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With numerical approximations:

$$u_h(t, x_k),\ f_h(u_h(t, x_k))$$

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The scheme converges if the residual vanishes:

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Convergence on Nodes:

$$\mathcal{R}_h(t, x_k) = 0 \quad \forall x_k \in \Omega$$

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Convergence on Nodes:

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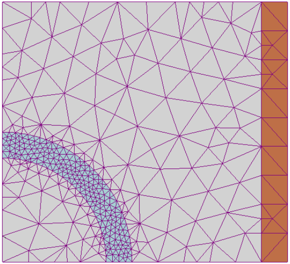
Orthogonal to test-functions:

$$\int_\Omega \mathcal{R}_h(t, x)\psi(x) = 0$$

Discontinuous Galerkin

Computational domain:

$$\Omega \simeq \Omega_h = \bigcup_{k=1}^{K} D^k$$



https://en.wikipedia.org/wiki/Finite_element_method

Solution in each cell:

$$u_h^k(t,x) = \sum_{n=1}^{N+1} \hat{u}_n^k(t)\psi_n(x)$$

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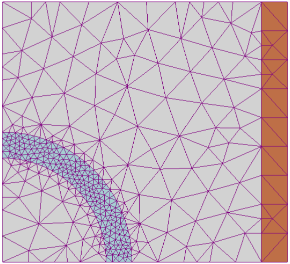

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Best of Finite Volume

- Cell average vanishes.
- geometrically flexible
- Inherently discontinuous
- Higher order accuracy problems

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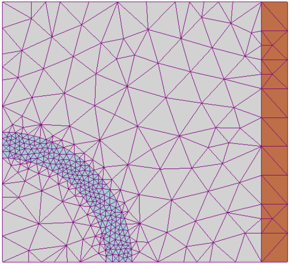

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Best of Finite Volume

- Cell average vanishes.
- geometrically flexible
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Best of Finite Element

- Residue vanishes weakly.
- Higher order accuracy
- Method is global

Numerical Flux and Weak Convergence

$$\int_{D^k} \Big((\partial_t u_{i,h} + a_{i,h} \partial_x u_{i,h} + s_{i,h}) \psi_n + f_{i,h} \partial_x \psi_n \Big) dx$$

$$= - \int_{\partial D^k} \psi_n \Big(f_i^* \, \hat{\mathbf{n}} + \mathbf{D}(u_{i,h}^+, u_{i,h}^-, \hat{\mathbf{n}}) \Big) dx$$

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- Conservative numerical flux:

$$f^*(u_h^+, u_h^-) = \frac{1}{2} \big(f_h(u_h^+) + f_h(u_h^-) \big) + \frac{C}{2} [[\mathbf{u_h}]]$$

- Non-Conservative numerical flux:

$$\mathbf{D}(u^+, u^-, \hat{\mathbf{n}}) = \frac{1}{2} \int_0^1 |a| \hat{\mathbf{n}} \frac{\partial \phi}{\partial s} ds$$

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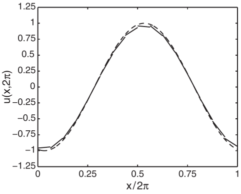
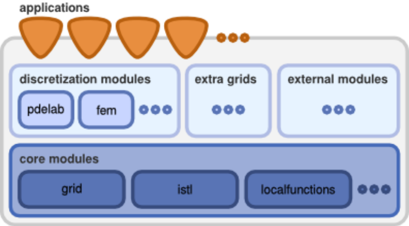


Hesthaven, Warburton, Nodal Discontinuous Galerkin Methods

Implementation

Distributed and Unified Numerics Environment

- Modular toolbox (C++)
  - Variety of grids (1D, 2D ...).
  - Implementations of FEM, FVM, DG.
  - Various time-stepping modules
- Large-scale computing



Applying DG to FRG

Solve explicitly for $m_\pi^2 = \partial_\rho V(\rho) = u(\rho)$:

$\Rightarrow$ Take a derivative w.r.t. $\rho = x$

$$\partial_t u - \partial_x \big(F(u) + G(u, \partial_x u) - s(x) \big) = 0$$

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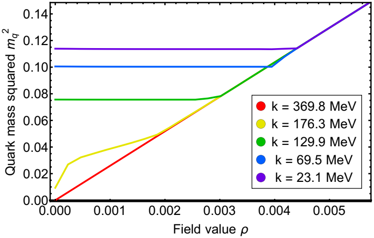
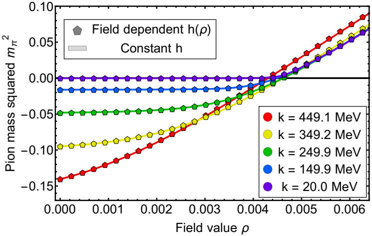
Large-N limit ($N_f \to \infty$), solve additionally for $m_q^2 = 2\rho h(\rho)^2 = v(\rho)$:

$$\partial_t u - \partial_x \big(F(u) - s(v, x) \big) = 0$$

$$\partial_t v - a(u)\partial_x v + L(u, v, x) = 0$$

RG-Time Evolution of the Masses





Initial conditions:

$$m_\pi^2 = 0\,\rho + \lambda\rho^2$$

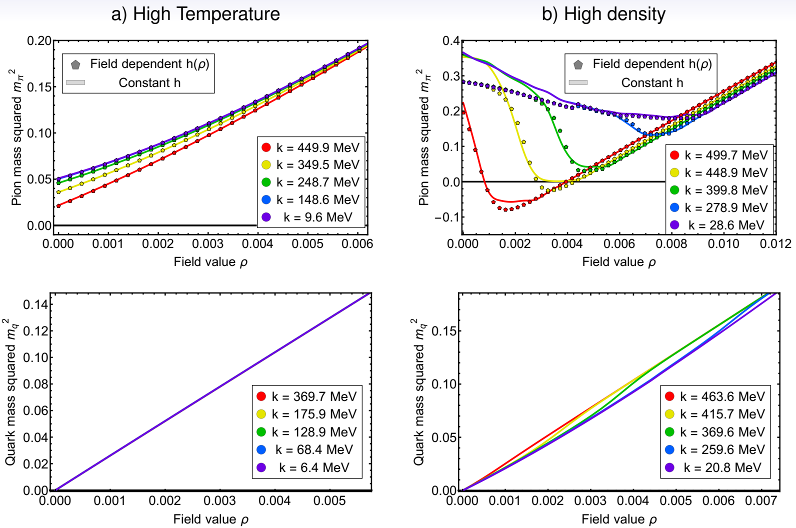
$$m_q^2 = 2h^2\rho$$

Mean field expectation value:

$$\sigma_{0,\pi} = 87.4(17) \quad \mathrm{MeV}$$

$$\sigma_{0,q} = 86.0(17) \quad \mathrm{MeV}$$

E. Grossi, F. Ihssen, J.M. Pawlowski, N. Wink, Phys.Rev.D.104.016028



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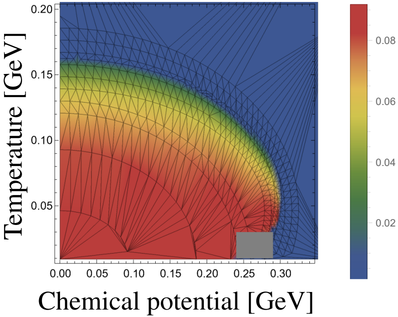
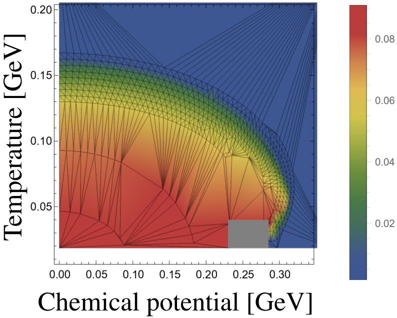
The Chiral Phase Structure



Phase diagram for constant quark mass, finite N_f.



Phase diagram with running field dependent quark mass, large N_f limit.

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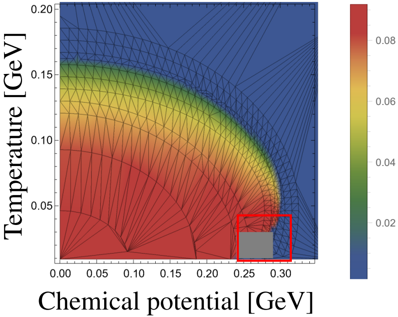

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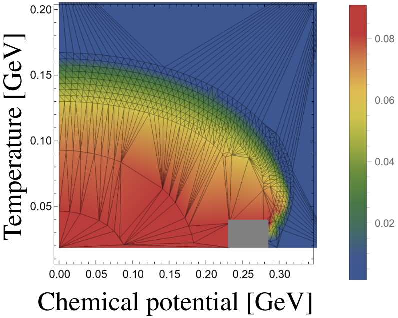

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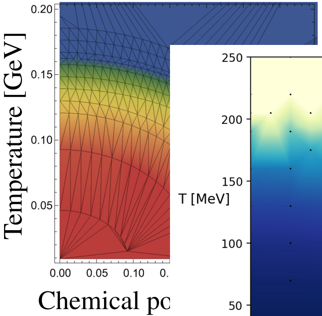

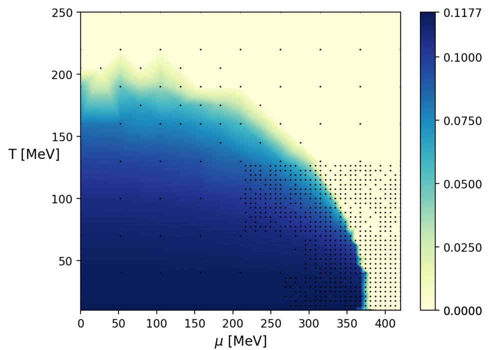

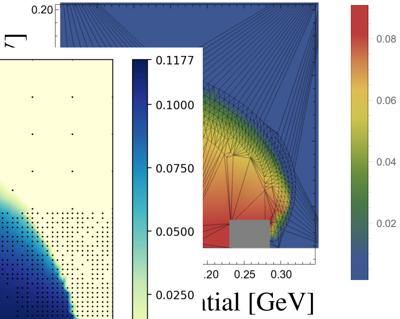

Phase diagram
mass,

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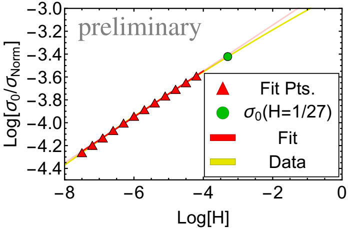
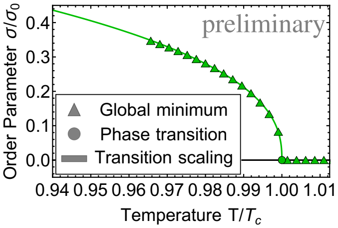
E. Grossi, F. Ihssen, J.M. Pawlowski, N. Wink, Phys.Rev.D.104.016028

Chiral Crossover Transition

F. Ihssen, J.M. Pawlowski in preparation

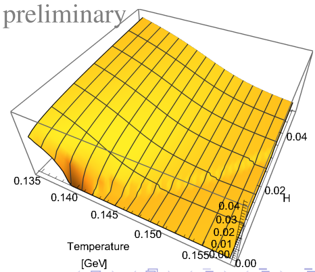




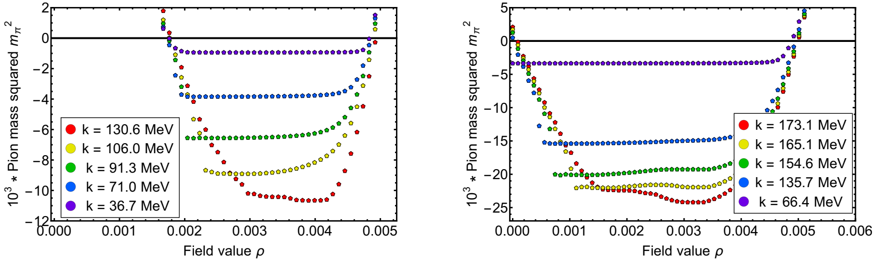
$$\beta = 0.4028(33)$$

$$\delta = 4.978(35)$$

$$T_c \approx 0.139 \text{ GeV}$$



Shock Development for $N_f \to \infty$



Derivative of the potential in the symmetric phase.

Derivative of the potential in the broken phase.

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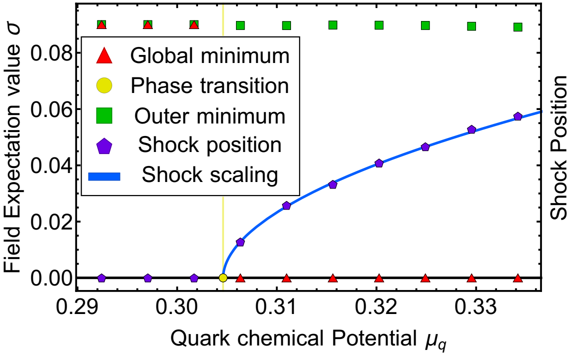


Parameter	Critical chemical potential
χ^2-Fit	0.30460(33)

Transition driven by particle density μ_q.

- $\langle \bar{q}q \rangle$ undergoes a first order phase transition.
- Hidden second order phase transition of the shock position.

E. Grossi, F. Ihssen, J.M. Pawlowski, N. Wink, Phys.Rev.D.104.016028

Dealing with Diffusion: LDG-Methods

$\Rightarrow$ Solve explicitly for $m_\sigma^2 = \partial_\rho V(\rho) + 2\rho\partial_\rho^2 V(\rho) = w(\rho)$

$$\partial_t u - \partial_x\big(F(u) + G(w)\big) = 0$$
$$\partial_t w - \partial_x\big(H(u,w) + x a(w)\partial_x w\big) = 0$$

$\Rightarrow$ Introduce a stationary equation:

$$0 = v - \partial_x j(w)\,, \quad j(w) = \int_v \sqrt{a(s)}\, ds$$

B. Cockburn, C. Shu, SIAM J. Numer. Anal., 35(6), 2440-2463.

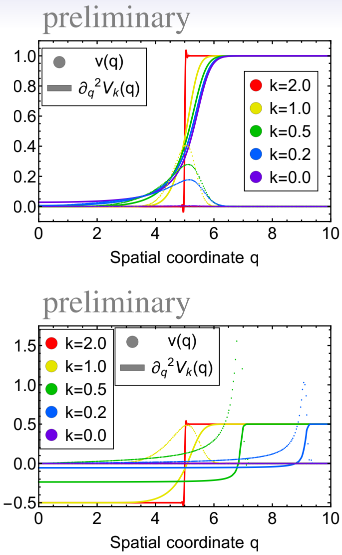
Riemann problem for an $O(1)$ model in 0+0 dim (in the 2^{nd} derivative):

$$\partial_t u = \partial_x \left(\sqrt{a(t,v)}\, v \right)$$

$$v = \sqrt{a(t)}\partial_x u$$

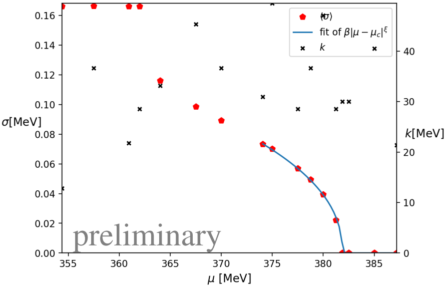
- Slopes travel with convection.
- Slopes smooth out with diffusion.

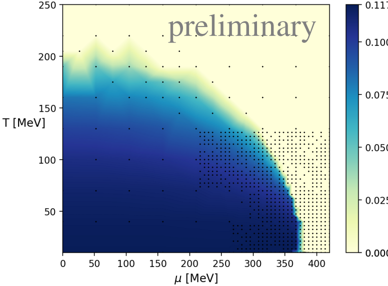
$\Rightarrow$ What happens to Shocks?



F. Ihssen, F. Sattler, N. Wink, in preparation

A first look at the Quark-Meson model





- Open technical issue: shock development & convexity
- Diffusion smooths phase transition
  $\Rightarrow$ Strong dynamics

F. Ihssen, J. M. Pawlowski, F. Sattler, N. Wink, in preparation

Conclusion

- Shock development within the 1st order regime in the large-N_f limit.
- Diffusion shifts the shock to second derivative at finite N_f.

- Discontinuities may heavily influence the position of the QCD phase transition at high densities.

Discontinuous methods are necessary to capture full physics!

Conclusion

- Shock development within the 1st order regime in the large-N_f limit.
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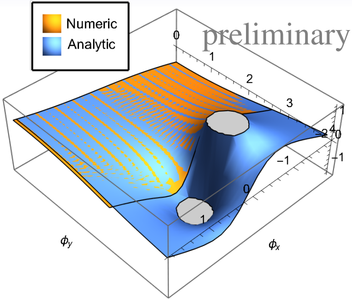
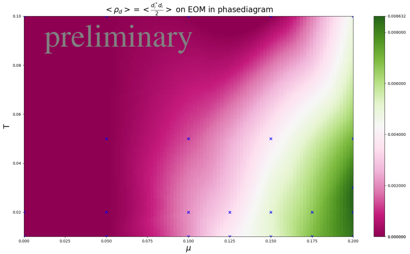
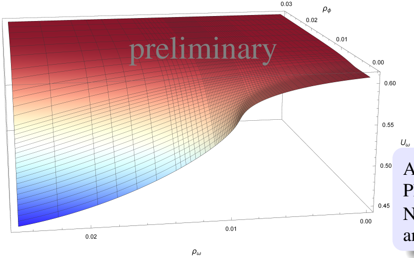
Results are right around the corner!







A. Connelly, G. Johnson, F. Rennecke, V. Skokov, Phys.Rev.Lett.125, 191602
N. Khan, J. M. Pawlowski, F. Rennecke, M. Scherer, arXiv:1512.03673

currently developing the DUNE-FRGDG module: F. Ihssen, W. Kritsin, F. Sattler, V. Wüst

Thank you for
your attention!

- Evaluate at constant fields:

$$\phi = (\sigma, \vec{\pi}) = (\sigma_0, \vec{0})$$

$$q = \bar{q} = 0$$

$\Rightarrow$ LHS simplifies to: $\quad \partial_t \Gamma[\phi] = \int \mathrm{d}^d x \, \partial_t V_k(\rho) \,,$

with $\rho = \phi^2/2$.

- Compute the second derivatives $\Gamma_k^{(2)}$ for the RHS:

$$\Gamma^{(2)}_{\sigma\sigma,k}[\phi](p) = p^2 + V_k'(\rho) + 2\rho V''(\rho)$$

$$\Gamma^{(2)}_{\pi_i\pi_i,k}[\phi](p) = p^2 + V_k'(\rho)$$

$$\Gamma^{(2)}_{\bar{q}q,k}[\phi](p) = \gamma_\mu p_\mu + h_k(\rho)\sigma_0$$

$$\partial_t V_k(\rho) = \frac{1}{2}\int_p (\Gamma^{(2)} + R_k)^{-1}(p)\,\partial_t R_k(p)\ \left[\frac{1}{\mathrm{Vol}_d}(2\pi)^d\delta(0)\right]$$

$$= \frac{1}{2}\int_p \frac{1}{p^2 + V_k'(\rho) + 2\rho V_k''(\rho) + R_{B,k}(p^2)}\,\partial_t R_{B,k}(p^2)$$

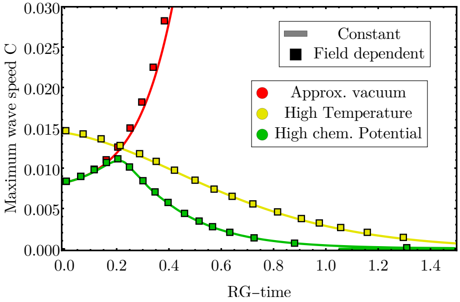
$$+ \frac{1}{2}\int_p \frac{N_f - 1}{p^2 + V_k'(\rho) + R_{B,k}(p^2)}\,\partial_t R_{B,k}(p^2)$$

$$- \int_p \frac{4N_f N_c}{p^2 + 2\rho h_k^2(\rho) + R_{F,k}^2(p^2)}\,\partial_t R_{F,k}(p^2)$$

⇒ Choose regulator

⇒ Similar derivation for $\partial_t h(\rho)$

Time-stepping



Wavespeed f_{max}: Biggest contribution to the conservative flux.

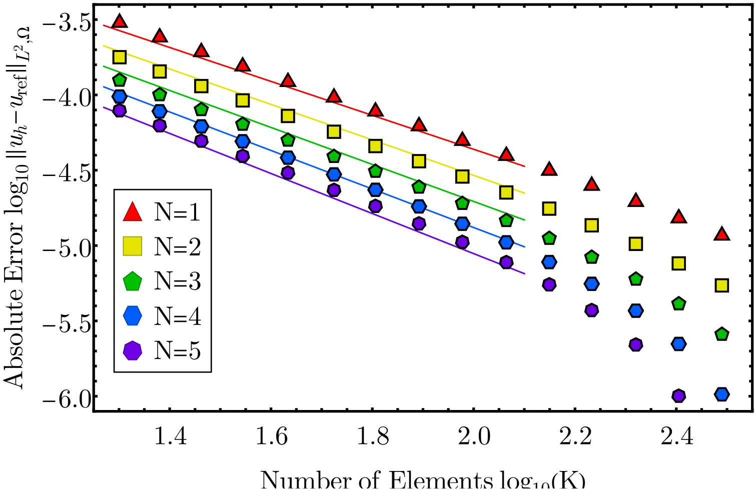
CFL-Conditions:

$$\Delta t \leq \frac{\Delta x}{(2N+1)} \frac{1}{f_{max}}$$

With maximum wavespeed:

$$f_{max} \geq \max_{D^{\{i,i+1\}}} |\partial_u f(u)|$$

Convergence



Error in comparision to a solution with high approximation order

Large-N Flow Equations

Flow equation of the pion mass:

$$\partial_t m^2_{\pi,k} = \partial_\rho \left[\frac{k^5}{12\pi^2} \left\{ \frac{N_\pi}{\epsilon^\pi_k}(1+2n_B(\epsilon^\pi_k)) - \frac{4\times 2\times 3}{\epsilon^q_k}(1-n_f(\epsilon^q_k+\mu)-n_f(\epsilon^q_k-\mu)) \right\} \right].$$

Flow of the quark mass:

$$\partial_t m^2_{q,k} = A(m^2_{\pi,k})\partial_\rho m^2_{q,k} + \frac{m^2_{q,k}}{\rho}\left[m^2_{q,k} B(m^2_{q,k}, m^2_{\pi,k}) - A(m^2_{\pi,k}) \right],$$

with

$$A(m^2_{\pi,k};T,\mu) = -2N_\pi v_3 k^2 l_1^{(B,4)}(m^2_{\pi,k};T)\,,$$

$$B(m^2_{q,k}, m^2_{\pi,k};T,\mu) = -4N_\pi v_3 L^{(4)}_{(1,1)}(m^2_{q,k}, m^2_{\pi,k};T,\mu)\,,$$