A new hamiltonian approach to a few exactly solvable models

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ABSTRACT: Simple two-dimensional models with massless and massive fermions are studied in the hamiltonian framework. One of the motivations is to understand better the relationship between the usual (space-like) and light front forms of field theory. The models include the derivative coupling model (Schroer, Rothe-Stamatescu), the Thirring, Thirring-Wess and Schwinger model. The correct quantum Hamiltonians that incorporate the knowledge of the operator solutions, are derived. While the derivative-coupling model is found to be almost equivalent to a free theory, the physical vacuum states of the (massless) Thirring and Thirring-Wess can be obtained by means of a Bogoliubov transformation in the form of a coherent state quadratic in composite boson operators. The hermitian point-splitting is used to derive the interacting currents from the known operator solutions in the TW and Schwinger models. The axial anomaly is derived in a simple manner, the truly gauge-invariant currents are found and the Schwinger mechanism is elucidated.

INTRODUCTION

Quantum Field Theory (QFT) – language of elementary particle physics

Perturbative solutions/approximations

Non-perturbative (analytic) treatment difficult (cf. F. Strocchi's recent book)

rigorous treatments (axiomatic/constructive/algebraic FT) use complicated mathematics and have gained a limited physical insight

a more physical but still sufficiently rigorous and reliable scheme would be desirable

area of solvable models is a very good testing ground for NP approaches

Soluble models: simple relativistic field theories in two-dimenional space-time in which operator solutions of the field equations can be found (versions of the model with derivative coupling, Thirring, Federbush, Thirring-Wess, Schwinger...)

sometimes taken as prototype for more realistic theories (Schwinger m. for QCD)

Original goal: soluble models – suitable for studying structure of and relationship between spacelike (SL) and light front (LF) formulations of QFT

explicit non-approximativ solutions of the Heisenberg field equations can be obtained at the quantum level in both schemes – full physical content can be extracted

advantages of the LF noticed long time ago (”infinite momentum frame”)

useful for analysis of processes at high energies

no systematic formulation of LF QFT available, partial approaches although structure (properties) of the two schemes differ, physical results (correlation functions) should agree – not always the case

Hamiltonians have different structure in some models

front-form Hamiltonian formulation (Dirac 1949)

LF variables:

$$
\begin{aligned}
& x^\mu = (x^+, x^-), \;\; p.x = \frac{1}{2}p^+x^- + \frac{1}{2}p^-x^+, \;\; p.p = m^2 \Rightarrow \hat{p}^- = m^2/p^+, \\
& \partial_+ = \frac{\partial}{\partial x^+}, \; \partial_- = \frac{\partial}{\partial x^-}, \quad \partial_\mu \partial^\mu = 4\partial_+\partial_- \\
& \psi^\dagger(x) = ({\psi_1}^\dagger, {\psi_2}^\dagger).
\end{aligned} \tag{1}
$$

x^+ – LF time, P^+ – LF momentum, P^- – LF Hamiltonian, etc.

Striking differences between the conventional SL and LF theories: mathematical structure as well as some physical aspects

- nature of field variables: dynamical vs. constrained
- minimal number (3) of dynamical Poincaré generators
- status of the vacuum state: by kinematical reasons, it is an eigenstate of the FULL Hamiltonian, not just of the free part H_0 (due to positivity and conservation of the LF momentum p^+
- consistent Fock expansion of bound states, amplitudes with direct probabilistic interpretation a la QM

SL form: physical vacuum (the lowest-energy eigenstate of the full

Hamiltonian) has to be obtained from dynamical calculations, very difficult, often neglected

possible by a Bogoliubov transformation in some very simple models (quadratic interaction Hamiltonian after the current bosonization)

important aspect: work with the correct Hamiltonian!

Novel strategy: knowledge of the operator solution taken into account at the Lagrangian level – Lagrangiansi re-expressed in terms of true degrees of freedom - free fields – similar to elimination of constraints

OUTLINE

1. SL and LF quantization of the derivative coupling model

2. Klaiber's formulation of the massless Thirring model, Hamiltonian treatment, physical vacuum state

3. Thirring-Wess model – operator solution of the coupled field equations, Hamiltonian and its physical ground state

4. Schwinger model in the Landau gauge, truly gauge-invariant currents, axial anomaly, generation of the gauge-boson mass...

5. Summary and conclusions

SL AND LF DERIVATIVE COUPLING MODEL

The simplest model – illustration of the derivation of the correct Hamiltonians and SL-LF comparison

The classical Lagrangian density

$$\mathcal{L} = \frac{i}{2}\overline{\Psi}\gamma^\mu \overset{\leftrightarrow}{\partial_\mu} \Psi - m\overline{\Psi}\Psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2 - g\partial_\mu\phi J^\mu, \quad J^\mu = \overline{\Psi}\gamma^\mu\Psi. \quad (2)$$

For $\mu = 0$ known as the Schroer's model, for axial vector current interaction as Rothe-Stamatescu model ($m = 0, \mu \neq 0$).

Euler-Lagrange eqs.

$$\begin{aligned} i\gamma^\mu\partial_\mu\Psi &= m\Psi + g\partial_\mu\phi\gamma^\mu\Psi, \\ \partial_\mu\partial^\mu\phi + \mu^2\phi &= g\partial_\mu J^\mu. \end{aligned} \quad (3)$$

Convention: capital Greek letters - interacting Heisenberg fields, small - free fields

Classically, the vector current is conserved, $\partial_\mu J^\mu(x) = 0 \Rightarrow$ free scalar field (not guaranteed at the quantum level)

classical solution of the Dirac eq.

$$\Psi(x) = e^{ig\phi(x)}\psi(x), \quad i\gamma^\mu\partial_\mu\psi(x) = m\psi(x). \tag{4}$$

irrespectively if scalar field is free or interacting

$\phi(x)$ quantized by $\left[a(k^1), a^\dagger(l^1)\right] = \delta(k^1 - l^1)$,
using notation $\hat{p}.x \equiv \omega(p^1)t - p^1x^1, \omega(p^1) = \sqrt{p_1^2 + \mu^2}$

$$
\begin{aligned}
\phi(x) &= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\frac{dk^1}{\sqrt{2\omega(k^1)}}\left[a(k^1)e^{-i\hat{k}.x}+a^\dagger(k^1)e^{i\hat{k}.x}\right] \\
&\equiv \phi^{(+)}(x)+\phi^{(-)}(x).
\end{aligned} \tag{5}
$$

The free massive fermion field quantized as

$$
\psi(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} dp^1\left[u(p^1)b(p^1)e^{-i\hat{p}.x}+v(p^1)d^\dagger(p^1)e^{i\hat{p}.x}\right],
$$

$$
u^\dagger(p^1) = (\sqrt{p^+},\sqrt{p^-}),\quad v(p^1)^\dagger = (\sqrt{p^+},-\sqrt{p^-}),
$$

$$
p^\pm = E(p^1)\pm p^1, E(p^1) = \sqrt{p_1^2+m^2}
$$

$$\{b(p^1), b^\dagger(q^1\} = \{d(p^1), d^\dagger(q^1)\} = \delta(p^1 - q^1). \tag{6}$$

Remark Approach here a bit heuristic, operators have to be regularized, finite-volume treatment.

At the quantum level, the solution has to be regularized:

$$\Psi(x) = Z^{1/2}(\epsilon) e^{-ig\phi^{(-)}(x)} e^{-ig\phi^{(+)}(x)} \psi(x), \tag{7}$$

where $Z(\epsilon) = \exp\left\{g^2\left[\phi^{(+)}(x - \frac{\epsilon}{2}), \phi^{(-)}(x + \frac{\epsilon}{2})\right]\right\} = \exp\left\{-ig^2 D^{(+)}(\epsilon)\right\}$.

Apply the point-splitting regularization to the interacting currents:

$$J^\mu(x) = s \lim_{\epsilon \to 0} \frac{1}{2}\{Z(\epsilon)\overline{\psi}(x + \frac{\epsilon}{2}) e^{ig\phi^{(-)}(x+\frac{\epsilon}{2})} e^{ig\phi^{(+)}(x+\frac{\epsilon}{2})} \gamma^\mu$$
$$\times e^{-ig\phi^{(-)}(x-\frac{\epsilon}{2})} e^{-ig\phi^{(+)}(x-\frac{\epsilon}{2})} \psi(x - \frac{\epsilon}{2}) + H.c.\} =$$

$$=: \overline{\psi}(x)\gamma^\mu\psi(x) : +\frac{g}{2\pi}\partial^\mu\phi(x). \tag{8}$$

Symmetric limit ($s\lim_{\epsilon\to 0}\frac{\epsilon^\mu\epsilon^\nu}{\epsilon^2} = \frac{1}{2}g^{\mu\nu}$), free field relation

$$\overline{\psi}(x+\frac{\epsilon}{2})\gamma^\mu\psi(x-\frac{\epsilon}{2}) =: \overline{\psi}(x)\gamma^\mu\psi(x) : -\frac{i}{\pi}\frac{\epsilon^\mu}{\epsilon^2} \tag{9}$$

No need to subtract the VEV part by hand if one defines the (free) current as a hermitian sum

$$j^\mu(x) = \frac{1}{2}\Big[\overline{\psi}(x+\frac{\epsilon}{2})\gamma^\mu\psi(x-\frac{\epsilon}{2}) + \overline{\psi}(x-\frac{\epsilon}{2})\gamma^\mu\psi(x+\frac{\epsilon}{2})\Big] \tag{10}$$

$Z(\epsilon)$ cancelled by the opposite factor from commuting two middle terms

The quantum vector current received a correction (”anomaly”, $\partial_\mu j^\mu = 0$),

$$\partial_\mu J^\mu(x) = \frac{g}{2\pi}\partial_\mu\partial^\mu\phi(x) \tag{11}$$

The only effect: finite mass ”renormalization”:

$$\partial_\mu\partial^\mu\phi(x) + \tilde{\mu}^2 = 0, \quad \tilde{\mu}^2 = \frac{\mu^2}{1-\frac{g^2}{2\pi}}. \tag{12}$$

Axial-vector current is conserved (if $m = 0$):

$$J_5^\mu(x) =: \overline{\psi}(x)\gamma^\mu\gamma^5\psi(x) : -\frac{g}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi(x). \tag{13}$$

Conjugate momenta directly

$$\Pi_\phi = \partial_0\phi(x) - gJ^0, \quad \Pi_\Psi = \frac{i}{2}\Psi^\dagger(x), \quad \Pi_{\Psi^\dagger} = -\frac{i}{2}\Psi(x). \tag{14}$$

The Hamiltonian

$$
\begin{aligned}
H &= H_{0B} + H^{'}, \\
H_{0B} &= \int\limits_{-\infty}^{+\infty} \mathrm{d}x^1 \Big[\frac{1}{2}\Pi_\phi^2 + \frac{1}{2}(\partial_1\phi)^2 + \frac{1}{2}\mu^2\phi^2\Big], \\
H^{'} &= \int\limits_{-\infty}^{+\infty} \mathrm{d}x^1 \big[- i\Psi^\dagger\alpha^1\partial_1\Psi + m\Psi^\dagger\gamma^0\Psi + g\partial_1\phi J^1\big].
\end{aligned} \tag{15}
$$

In the kinetic term the free field $\psi(x)$ taken,

$$
H_{0F} = \int\limits_{-\infty}^{+\infty} \mathrm{d}x^1 \big[- i\psi^\dagger\alpha^1\partial_1\psi + m\psi^\dagger\gamma^0\psi\big],
$$

$$H_{0F} = \int\limits_{-\infty}^{+\infty} \mathrm{d}p^1 E(p^1) \Big[b^\dagger(p^1) b(p^1) + d^\dagger(p^1) d(p^1) \Big],$$

$$H_{0B} = \int\limits_{-\infty}^{+\infty} \mathrm{d}p^1 \omega(p^1) a^\dagger(p^1) a(p^1), \;\; \omega(p^1) = \sqrt{p_1^2 + \mu^2}, \tag{16}$$

The interacting Hamiltonian becomes

$$H_{int} = \frac{g}{2\sqrt{\pi}} \int\limits_{-\infty}^{+\infty} \mathrm{d}k^1 \big[c^\dagger(k^1) a(k^1) + a^\dagger(k^1) c(k^1) + a^\dagger(k^1) c^\dagger(k^1) + a(k^1) c(k^1) \big]. \tag{17}$$

where the composite boson operators satisfying $[c(k^1), c^\dagger(l^1)] = \delta(k^1 - l^1)$

correspond to the vector current

$$j^\mu(x) = -\frac{i}{\sqrt{2\pi}} \int \frac{dk^1}{\sqrt{2k^0}} k^\mu \{c(k^1) e^{-i\hat{k}.x} - c^\dagger(k^1) e^{i\hat{k}.x}\}, \tag{18}$$

The Hamiltonian non-diagonal, a Bogoliubov transformation necessary for $m = 0$ implemented by means of a unitary operator $U = \exp(iS)$ with

$$S(\gamma) = -i \int_{-\infty}^{+\infty} \mathrm{d}k^1 \gamma(k) \left[c^\dagger(k^1) a^\dagger(-k^1) - c(k^1) a(-k^1)\right]. \tag{19}$$

The physical vacuum found as

$$|\Omega\rangle = N \exp\left[\int_{-\infty}^{+\infty} \mathrm{d}k^1 \gamma(g) c^\dagger(-k^1) a^\dagger(k^1)\right] |0\rangle. \tag{20}$$

nontrivial vacuum structure
Also: momentum operator contains interaction!

THE LF TREATMENT

Covariant Lagrangian in terms of LF space-time and field variables

$$\mathcal{L}_{lf} = i{\Psi_2}^\dagger \overleftrightarrow{\partial_+} \Psi_2 + i{\Psi_1}^\dagger \overleftrightarrow{\partial_-} \Psi_1 - m({\Psi_1}^\dagger\Psi_2 + {\Psi_2}^\dagger\Psi_1) + \\ + 2\partial_+\phi\partial_-\Phi - \frac{1}{2}\mu^2\phi^2 - g\partial_+\phi J^+ - g\partial_-\phi J^-, \tag{21}$$

Euler-Lagrange equations in the component form read

$$2i\partial_+\Psi_2 = m\Psi_1 + 2g\partial_+\phi\Psi_2, \qquad 2i\partial_-\Psi_1 = m\Psi_2 + 2g\partial_-\phi\Psi_1 \tag{22}$$

Inserting the constraint into the Lagrangian leads to the free LF

Hamiltonian!

$$P^- = \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}x^-}{2} \Big[m\big({\psi_1}^\dagger \psi_2 + {\psi_2}^\dagger \psi_1\big) + \mu^2 \phi^2 \Big] \tag{23}$$

CLEAR CONTRADICTION BETWEEN THE SL and LF FORMALISMS!

WAY OUT:

The solution of the field equations not taken into account!

The solution tells us that there is no ”independent” interacting field – it is composed out of free fields. The free fields are the true physical degrees of freedom and the Lagrangian has to be re-expressed in terms of them first (analogously to inserting a constraint into Lagrangian), then calculate conjugate momenta and derive the Hamiltonian.

NOTE: this is not the same as inserting the field equation (Dirac eq., Klein-Gordon eq.) into $\mathcal{L}$ – the latter leads to vanishing Lagrangian (extremum of the action)

Dirac eq. implies knowledge of $\gamma^\mu \partial_\mu \Psi$, knowing the solution implies knowing $\partial_\mu \Psi$

Inserting the solution of the Dirac eq. of the DCM in the form

$$\partial_\mu \Psi(x) = -ig\partial_\mu \phi(x)\Psi(x) + e^{-ig\phi(x)}\partial_\mu \psi(x) \tag{24}$$

into $\mathcal{L}$ yields

$$\mathcal{L} = \frac{i}{2}\overline{\psi}\gamma^\mu \overleftrightarrow{\partial_\mu} \psi - m\overline{\psi}\psi + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}\mu^2 \phi^2, \tag{25}$$

i.e. the interaction part got cancelled! Free form of the Lagrangian, free fields and conjugate momenta ($\Pi_\psi = i\psi^\dagger, \Pi_\phi = \partial_0\phi$) and the Hamiltonian

$$H = \int\limits_{-\infty}^{+\infty} \mathrm{d}x^1 \big[- i\psi^\dagger\alpha^1\partial_1\psi + m\psi^\dagger\gamma^0\psi + \frac{1}{2}\Pi_\phi^2 + \frac{1}{2}(\partial_1\phi)^2 + \frac{1}{2}\mu^2\phi^2\big], \tag{26}$$

which is just the sum of free Hamiltonians of the massive scalar and fermion fields. Correct Heisenberg equations generated with this Hamiltonian:

$$-i\partial_0\Psi(x) = [H, \Psi(x)] \tag{27}$$

Physical vacuum coincides with the Fock vacuum. The only trace of the interacting theory is the non-canonical form of the anticommutation relation of the interacting fermion field and the form of the correlation functions. The latter expressed in terms of the correlation functions of free fields,

$$\langle vac|\Psi_\alpha(x)\overline{\Psi}_\beta(y)|vac\rangle = \langle 0| : e^{-ig\phi(x)} : \psi_\alpha(x)\overline{\psi}_\beta(y) : e^{ig\phi(y)} : |0\rangle =$$

$$= e^{g^2 D^{(+)}(x-y)} S^{(+)}_{\alpha\beta}(x-y), \tag{28}$$

where

$$\begin{aligned} D^{(+)}(x-y) &= \langle 0|\phi(x)\phi(y)|0\rangle, \\ D^{(+)}(z) &= -\frac{1}{4}\theta(z^2)\Big[N_0(\mu\sqrt{z^2}) + \\ &\qquad + i\mathrm{sgn}(z^0)J_0(\mu\sqrt{z^2})\Big] + \frac{1}{2\pi}\theta(-z^2)K_0(\mu\sqrt{-z^2}). \end{aligned} \tag{29}$$

The fermionic two-point function is

$$\begin{aligned} S^{(+)}_{\alpha\beta}(x-y) &= \langle 0|\psi_\alpha(x)\overline{\psi}_\beta(y)|0\rangle, \\ S^{(+)}_{\alpha\beta}(z) &= \Big(i\gamma^\mu\partial_\mu + m\Big)_{\alpha\beta} D^{(+)}(z). \end{aligned} \tag{30}$$

Explictly,

$$S^{(\pm)}(z) = \frac{i}{2\pi}(i\gamma^\mu \partial^x_\mu + m) \int d^2p \, \delta(p^2 - m^2)\theta(\pm p^0) e^{\pm ip.z} =$$
$$= \frac{i}{4\pi} \int \frac{dp}{E(p)} \begin{pmatrix} m & p^- \\ p^+ & m \end{pmatrix} e^{\pm i\hat{p}.z} \tag{31}$$

Remark: B. Schroer used this model in 1961 (Fort. Physik 1) to illustrate the concept of ”infraparticle”. His results are ok if one considers his interacting Lagrangian – which however is not the true Lagrangian of the model

The LF analysis proceeds analogously:

$$\mathcal{L}_{lf} = i\Psi_2{}^\dagger \overset{\leftrightarrow}{\partial}_+ \Psi_2 + i\Psi_1{}^\dagger \overset{\leftrightarrow}{\partial}_- \Psi_1 - m(\Psi_1{}^\dagger\Psi_2 + \Psi_2{}^\dagger\Psi_1) + \\ + 2\partial_+\phi\partial_-\phi - \frac{1}{2}\mu^2\phi^2 - g\partial_+\phi J^+ - g\partial_-\phi J^-, \tag{32}$$

Field equations

$$2i\partial_+\Psi_2 = m\Psi_1 + 2g\partial_+\phi\Psi_2, \\ 2i\partial_-\Psi_1 = m\Psi_2 + 2g\partial_-\phi\Psi_1 \tag{33}$$

solved by

$$\Psi_2(x) = e^{-ig\phi(x)}\psi_2(x), \quad 2i\partial_+\psi_2 = m\psi_1 \\ \Psi_1(x) = e^{-ig\phi(x)}\psi_1(x), \quad 2i\partial_-\psi_1 = m\psi_2. \tag{34}$$

Inserting these solutions into the LF Lagrangian yields the free one:

$$\mathcal{L}_{lf} = i{\psi_2}^\dagger \overleftrightarrow{\partial_+} \psi_2 + i{\psi_1}^\dagger \overleftrightarrow{\partial_-} \psi_1 - m({\psi_1}^\dagger\psi_2 + {\psi_2}^\dagger\psi_1) + 2\partial_+\phi\partial_-\phi - \frac{1}{2}\mu^2\phi^2. \quad (35)$$

Free Hamiltonian follows:

$$P^- = \int_{-\infty}^{+\infty} \frac{\mathrm{d}x^-}{2}\Big[m\big({\psi_1}^\dagger\psi_2 + {\psi_2}^\dagger\psi_1\big) + \mu^2\phi^2\Big]. \quad (36)$$

The same as before. Reason: no kinetic term in LF Hamiltonian present by construction

Correlation functions coincide with those from the space-like treatment.

$$\langle 0|\Psi(x)\overline{\Psi}(y)|0\rangle = e^{-\frac{g^2}{\pi}D^{(+)}(x-y)}S^{(+)}(x-y). \quad (37)$$

$$S_{22}(x-y) = \langle 0|\psi_2(x)\psi_2^\dagger(y)|0\rangle = \int\limits_0^\infty \frac{dp^+}{8\pi} e^{-\frac{i}{2}p^+(x^- - y^- - i\epsilon) - \frac{i}{2}\frac{m^2}{p^+}(x^+ - y^+ - i\epsilon)},$$

$$S_{11}(x-y) = \langle 0|\psi_1(x)\psi_1^\dagger(y)|0\rangle = \int\limits_0^\infty \frac{dp^+}{8\pi} \frac{m^2}{p^{+2}} e^{-\frac{i}{2}p^+(x^- - y^- - i\epsilon) - \frac{i}{2}\frac{m^2}{p^+}(x^+ - y^+ - i\epsilon)},$$

$$S_{12}(x-y) = \langle 0|\psi_1(x)\psi_2^\dagger(y)|0\rangle = \int\limits_0^\infty \frac{dp^+}{8\pi} \frac{m}{p^+} e^{-\frac{i}{2}p^+(x^- - y^- - i\epsilon) - \frac{i}{2}\frac{m^2}{p^+}(x^+ - y^+ - i\epsilon)}. \quad (38)$$

Note that we have introduced the small imaginary parts in time and space coordinates. This step dictated by the mathematical consistency. Without the damping factors the integrals would not exist as mathematical objects [Gradshteyn and Ryzhik]. The scalar-field function is

$$D^{(+)}(z) = mS^{(+)}_{12}(z). \tag{39}$$

The fermion-field functions are

$$\begin{aligned} S^{(+)}_{22}(z) = & - & \theta\big(z^2\big)\frac{m}{8}\sqrt{\frac{z^+}{z^-}}\Big[J_1\big(m\sqrt{z^2}\big) - i\,\mathrm{sgn}(z^+)N_1\big(m\sqrt{z^2}\big)\Big] + \\ & + & \theta\big(-z^2\big)\mathrm{sgn}(z^+)\frac{im}{4\pi}\sqrt{-\frac{z^+}{z^-}}K_1\big(m\sqrt{-z^2}\big), \\ S^{(+)}_{11}(z) = & & \theta\big(z^2\big)\frac{m}{8}\sqrt{\frac{z^-}{z^+}}\Big[J_1\big(m\sqrt{z^2}\big) - i\,\mathrm{sgn}(z^+)N_1\big(m\sqrt{z^2}\big)\Big] - \end{aligned}$$

$$
\begin{aligned}
& - \quad \theta\big(-z^2\big)\mathrm{sgn}(z^+)\frac{im}{4\pi}\sqrt{-\frac{z^-}{z^+}}K_1\big(m\sqrt{-z^2}\big), \\
S_{12}^{(+)}(z) = & - \quad \theta\big(z^2\big)\frac{m}{8}\Big[N_0\big(m\sqrt{z^2}\big) + i\ \mathrm{sgn}(z^+)J_0(m\sqrt{z^2})\Big] + \\
& + \quad \theta\big(-z^2\big)\frac{m}{4\pi}K_0\big(m\sqrt{-z^2}\big)
\end{aligned}
\tag{40}
$$

Small imaginary parts in the arguments with appropriate sign are understood. Calculation of the analogous correlation functions in the conventional theory is more complicated and requires a clever change of variables [Bogoliubov and Shirkov].

The scalar-field correlation function diverges for $\mu = 0$ in both schemes. LF calculation with the massless fermion field inconsistent (yields vanishing $S_{11}^{(+)}(z)$.) The $m = 0$ limit of the LF fermion correlation function coincides with the SL case.

MASSIVE ROTHE-STAMATESCU MODEL – A FEW REMARKS

$$\mathcal{L} = \frac{i}{2}\overline{\Psi}\gamma^{\mu} \overleftrightarrow{\partial}_{\mu} \Psi - m\overline{\Psi}\Psi + \frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi - \frac{1}{2}\mu^2\phi^2 - g\partial_{\mu}\phi J_5^{\mu}, \quad J_5^{\mu} = \overline{\Psi}\gamma^{\mu}\gamma^5\Psi. \quad (41)$$

Non-trivial physics found in literature (Belvedere and Rodrigues in a series of papers) : axial anomaly, anomalous dimension of the fermion field, relation to the massive Thirring and sine-Gordon models...

Based on the definition of the vector current:

$$j^{\mu}_{\epsilon}(x) = \overline{\Psi}(x+\epsilon)\gamma^{\mu}\Psi(x)\exp\left(ig\int_{x}^{x+\epsilon} dy_{\lambda}\epsilon^{\lambda\nu}\partial_{\nu}\phi(y)\right) - VEV. \qquad (42)$$

”an extended treatment”

”conservative approach”:

Field equations:

$$
\begin{aligned}
i\gamma^\mu\partial_\mu\Psi &= m\Psi + g\partial_\mu\phi\gamma^\mu\gamma^5\Psi, \\
\partial_\mu\partial^\mu\phi + \mu^2\phi^2 &= g\partial_\mu J_5^\mu = 2img\overline{\Psi}\gamma^5\Psi.
\end{aligned} \tag{43}
$$

Scalar field is no longer free, Dirac eq. seems to have an operator solution similar to the one from the DCM:

$$
\Psi(x) = e^{-ig\gamma^5\phi(x)}\psi(x). \tag{44}
$$

Check:

$$
\begin{aligned}
i\gamma^\mu\partial_\mu\Psi(x) &= i\gamma^\mu\big[-ig\gamma^5\partial_\mu\phi(x)\Psi(x) + e^{-ig\gamma^5\phi(x)}\partial_\mu\psi(x)\big] = \\
&= g\partial_\mu\phi(x)\gamma^\mu\gamma^5\Psi(x) + e^{+ig\gamma^5\phi(x)}i\gamma^\mu\partial_\mu\psi(x),
\end{aligned} \tag{45}
$$

where $i\gamma^\mu \partial_\mu \psi = m\psi$. The sign in the last exponential is opposite due to $\gamma^\mu \gamma^5 = -\gamma^5 \gamma^\mu$.

Thus, the massive RS model is not exactly solvable. The original massless RS model (Rothe and Stamatescu, Annals of Physics 1977): The massless axial current is conserved, hence scalar field is free. Dirac eq. is exactly solvable but inserting the solution to the Lagrangian generates the free Hamiltonian. Similar to the massive derivative-coupling model.

iterative (perturbative) approach in the Heisenberg picture?

MASSLESS THIRRING MODEL

Thirring model played important role in history of QFT (see Wightman's Cargese lectures and Klaiber's paper)

operator solution due to B. Klaiber (Boulder 1967), n-point correlation functions constructed, basis of the Coleman's (perturbative) bosonization

all aspects clarified?

not quite true: a series of papers by Faber and Ivanov (discovery of a broken phase claimed based on Nambu – Jona-Lasinio BCS-like Ansatz for the ground state)

similar conclusions done by Fujita et al. using the Bethe Ansatz solution

systematic Hamiltonian study based on the model's solvability not given so far (however some ideas and methods by S. Korenblit are close to ours

– LM and P. Grange, PLB (2013))

Classical Lagrangian density

$$\mathcal{L} = \frac{i}{2}\overline{\Psi}\gamma^\mu \overleftrightarrow{\partial_\mu} \Psi - \frac{1}{2} g J_\mu J^\mu, \quad J^\mu = \overline{\Psi}\gamma^\mu \Psi. \tag{46}$$

Field equations and current conservation

$$\begin{aligned} i\gamma^\mu \partial_\mu \Psi(x) &= g J^\mu(x) \gamma_\mu \Psi(x), \\ \partial_\mu J^\mu(x) &= 0. \end{aligned} \tag{47}$$

The general solution is

$$\begin{aligned} \Psi(x) &= e^{-i(g/\sqrt{\pi})\left(\alpha j(x) - \beta\gamma^5 \tilde{j}(x)\right)} \psi(x), \\ \gamma^\mu \partial_\mu \psi(x) &= 0 \end{aligned} \tag{48}$$

with $\alpha + \beta = 1$. "Potentials" $j(x)$ and $\tilde{j}(x)$ connected to the free vector current by $\partial_\mu j(x) = -\sqrt{\pi} j_\mu(x)$, $\partial_\mu \tilde{j}(x) = \sqrt{\pi}\epsilon_{\mu\nu} j^\nu(x)$. This corresponds to replacing $J^\mu(x)$ by $j^\mu(x)$ in the field equation – rather restrictive, does not represent the most general quantum solution. The latter can be obtained as follows. Consider the $\beta = 0$ case for simplicity

$$\Psi(x) = e^{i(g/\sqrt{\pi})J(x)}\psi(x) \tag{49}$$

with the unknown potential $J(x)$ of the interacting current $J^\mu(x)$, i.e. defining $\partial_\mu J(x) = -\sqrt{\pi} J_\mu(x)$. Regularizing (49) like in the DCM model and calculating the corresponding current using the point-split product of the above $\Psi^\dagger$ and Ψ, we find

$$J^\mu(x) =: \overline{\psi}(x)\gamma^\mu\psi(x) : + \frac{g}{2\pi} J^\mu(x) \Rightarrow$$

$$J^\mu(x) = G(g) j^\mu(x),\ G(g) = (1 - \frac{g}{2\pi})^{-1}. \tag{50}$$

Interacting current = rescaled free current. Potential consequences for Coleman's bosonization.

Fourier representation

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int dp^1 \{b(p^1)u(p^1)e^{-ip.x} + d^\dagger(p^1)v(p^1)e^{ip.x}\}, \;\; p^0 = |p^1|$$
$$\{b(p^1), b^\dagger(q^1)\} = \{d(p^1), d^\dagger(q^1)\} = \delta(p^1 - q^1),$$
$$b(k^1)|0\rangle = d(k^1)|0\rangle = 0. \quad (51)$$

The spinors $u(p^1), v(p^1)$ are $m = 0$ limits of the massive spinors,

$$u^\dagger(p^1) = \big(\theta(-p^1), \theta(p^1)\big), \;\; v^\dagger(p^1) = \big(- \theta(-p^1), \theta(p^1)\big). \quad (52)$$

Vector current $j^\mu = (:\psi^\dagger\psi:, :\psi^\dagger\alpha^1\psi:)$:

$$j^0(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\mathrm{d}p^1\int\limits_{-\infty}^{+\infty}\mathrm{d}q^1\Big\{f_0(p^1,q^1)\Big[\Big[b^\dagger(p^1)b(q^1) - d^\dagger(p^1)d(q^1)\Big]e^{i(\hat{p}-\hat{q}).x}\Big] +$$

$$+ g_0(p^1,q^1)\Big[b^\dagger(p^1)d^\dagger(q^1)e^{i(\hat{p}+\hat{q}).x} + d(p^1)b(q^1)e^{-i(\hat{p}+\hat{q}).x}\Big]\Big\},$$

$$j^1(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\mathrm{d}p^1\int\limits_{-\infty}^{+\infty}\mathrm{d}q^1\Big\{g_0(p^1,q^1)\Big[\Big[b^\dagger(p^1)b(q^1) - d^\dagger(p^1)d(q^1)\Big]e^{i(\hat{p}-\hat{q}).x}\Big] +$$

$$+ f_0(p^1,q^1)\Big[b^\dagger(p^1)d^\dagger(q^1)e^{i(\hat{p}+\hat{q}).x} + d(p^1)b(q^1)e^{-i(\hat{p}+\hat{q}).x}\Big]\Big\},$$

$$f_0(p^1,q^1) = \theta(p^1)\theta(q^1) + \theta(-p^1)\theta(-q^1),$$

$$g_0(p^1,q^1) = \theta(p^1)\theta(q^1) - \theta(-p^1)\theta(-q^1). \qquad (53)$$

can be represented in terms of composite fermion operators

$$j^{\mu}(x) = -\frac{i}{\sqrt{2}\pi}\int \frac{dk^1}{\sqrt{2k^0}} k^{\mu}\{c(k^1)e^{-i\hat{k}.x} - c^{\dagger}(k^1)e^{i\hat{k}.x}\}, \tag{54}$$

where (Fourier transform)

$$c(k^1) = \frac{i}{\sqrt{k^0}}\int dp^1\{\theta\big(p^1k^1\big)\big)\big[b^{\dagger}(p^1)b(p^1+k^1) - d^{\dagger}(p^1)d(p^1+k^1)\big] +$$
$$+\epsilon(p^1)\theta\big(p^1(p^1-k^1)\big)d(k^1-p^1)b(p^1)\big\}. \tag{55}$$

Canonical Fock commutation relation follow

$$\big[c(p^1), c^{\dagger}(q^1)\big] = \delta(p^1 - q^1), \;\; c(k^1)|0\rangle = 0. \tag{56}$$

Problem: infrared divergence – the two-point correlation function of a massless scalar field in D=1+1 is divergent,

$$D^{(+)}(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle = \frac{1}{4\pi}\int \frac{dk^1}{|k^1|} e^{-i\hat{k}.x}. \tag{57}$$

True ground state of the massless Thirring model:

Hamiltonian in the usual treatment (kinetic term taken as built from free field) is

$$H = \int_{-\infty}^{+\infty} \mathrm{d}x^1 \Big[-i\psi^\dagger \alpha^1 \partial_1 \psi + \frac{1}{2} g \big(j^0 j^0 - j^1 j^1 \big) \Big] \tag{58}$$

Not correct. Insert the operator solution to the Lagrangian first:

$$\mathcal{L} = i\overline{\Psi}\gamma^\mu \Big[-\frac{ig}{\sqrt{\pi}} \partial_\mu j \Psi + e^{-\frac{ig}{\sqrt{\pi}} j} \partial_\mu \psi \Big] - \frac{g}{2} j_\mu j^\mu. \tag{59}$$

The first term in the bracket combines with the interaction term reversing its sign. The correct Hamiltonian is

$$H = \int_{-\infty}^{+\infty} \mathrm{d}x^1 \Big[- i\psi^\dagger \alpha^1 \partial_1 \psi - \frac{1}{2} g \big(J^0 J^0 - J^1 J^1 \big) \Big]. \tag{60}$$

Fock representation: the free Hamiltonian is

$$H_0 = \int_{-\infty}^{+\infty} \mathrm{d}p^1 |p^1| \Big[b^\dagger(p^1) b(p^1) + d^\dagger(p^1) d(p^1) \Big]. \tag{61}$$

The interacting Hamiltonian greatly simplifies in terms of composite operators $c(k^1), c^\dagger(k^1)$:

$$H_g = G^2(g)\frac{g}{\pi}\int_{-\infty}^{+\infty} dk^1|k^1|\Big[c^\dagger(k^1)c^\dagger(-k^1) + c(k^1)c(-k^1)\Big]. \tag{62}$$

Obviously $H = H_0 + H_g$ is not diagonal and $|0\rangle$ is not its eigenstate.

DETAILS:

H_0 satisfies

$$\big[H_0, c(k^1)\big] = -|k^1|c(k^1), \quad \big[H_0, c^\dagger(k^1)\big] = |k^1|c^\dagger(k^1). \tag{63}$$

Remark: mathematically correct treatment requires cut-offs or test functions to have well defined quantities, here the approach a little heuristic (but checked in a finite volume)

To diagonalize H, define the operator T with the same commutation property:

$$T = \int\limits_{-\infty}^{+\infty} \mathrm{d}k^1 |k^1| c^\dagger(k^1) c(k^1),$$

$$\big[T, c(k^1)\big] = -|k^1| c(k^1), \quad \big[T, c^\dagger(k^1)\big] = |k^1| c^\dagger(k^1). \tag{64}$$

Consider now the unitary operator U,

$$U = e^{iS}, \;\; S = -\frac{i}{2} \int\limits_{-\infty}^{+\infty} \mathrm{d}p^1 \gamma(p^1) \big[c^\dagger(p^1) c^\dagger(-p^1) - c(p^1) c(-p^1)\big]. \tag{65}$$

Form new free and interacting Hamiltonians

$$\hat{H}_0 = H_0 - T, \quad \hat{H}_g = H_g + T. \tag{66}$$

By construction, due to $\left[S, \hat{H}_0\right] = 0$, $\hat{H}_0$ is invariant with respect to U:

$$\hat{H}_0 \rightarrow e^{iS}\hat{H}_0 e^{-iS} = \hat{H}_0 + i\left[S, \hat{H}_0\right] + \ldots = \hat{H}_0. \tag{67}$$

On the other hand, $\hat{H}_{int}$ transforms non-trivially due to

$$\left[S, c(k^1)\right] = i\gamma(k^1)c^\dagger(-k^1), \quad \left[S, c^\dagger(k^1)\right] = i\gamma(k^1)c(-k^1), \;\; \gamma(-k^1) = \gamma(k^1). \tag{68}$$

Using the operator identity $e^A B e^{-A} = B + [A, B] + \frac{1}{2}[A, [A, B]] + + \frac{1}{3!}[A, [A, [A, B]]] + \ldots$:

$$e^{iS}c(k^1)e^{-iS} = c(k^1) + i\big(i\gamma(k^1)\big)c^\dagger(-k^1) + \frac{i^2}{2}\big(i\gamma(k^1)\big)^2 c(k^1) + \frac{i^3}{3!}\big(i\gamma(k^1)\big)^3 c^\dagger(-k^1) +$$

$$+\ldots. \qquad (69)$$

Thus

$$c(k^1) \to e^{iS}c(k^1)e^{-iS} = c(k)\cosh\gamma(k^1) - c^\dagger(-k^1)\sinh\gamma(k^1),$$
$$c^\dagger(k^1) \to e^{iS}c^\dagger(k^1)e^{-iS} = c^\dagger(k^1)\cosh\gamma(k^1) - c(-k^1)\sinh\gamma(k^1). \qquad (70)$$

It follows

$$\hat{H}_g \to e^{iS}\hat{H}_g e^{-iS} =$$
$$\int\limits_{-\infty}^{+\infty} \mathrm{d}k^1|k^1|\Big\{\Big[c^\dagger(k^1)c^\dagger(-k^1) + c(k^1)c(-k^1)\Big]\Big[\frac{g}{2\pi}\Big(\cosh^2\gamma(k^1) + \sinh^2\gamma(k^1)\Big) -$$
$$-\cosh\gamma(k^1)\sinh\gamma(k^1)\Big] -$$

$$-c^{\dagger}(k^1)c(k)\Big[4\frac{g}{2\pi}\sinh\gamma(k^1)\cosh\gamma(k^1)-\Big(\cosh^2\gamma(k^1)+\sinh^2\gamma(k^1)\Big)\Big]-$$
$$-\delta(0)\Big[2\sinh\gamma(k^1)\cosh\gamma(k^1)+\sinh^2\gamma(k^1)\Big]\Big\}. \tag{71}$$

The last (divergent) term removed by normal ordering. Diagonal form if $\gamma(k^1)=\gamma_d=\frac{1}{2}\mathrm{arctanh}(2G(g)\frac{g}{\pi})$.

Thus we have achieved

$$e^{iS}\big(\hat{H}_0+\hat{H}_g\big)e^{-iS}|0\rangle=0 \tag{72}$$

and $|\Omega\rightarrow=e^{-iS}|0\rangle$ is the new vacuum state. or

$$|\Omega\rangle=\exp\Big[-\frac{1}{2}\gamma_D\int_{-\infty}^{+\infty}\mathrm{d}p^1\big[c^{\dagger}(p^1)c^{\dagger}(-p^1)-c(p^1)c(-p^1)\big]\Big]|0\rangle. \tag{73}$$

Simplification due to the operator identity (Kirzhnits)

$$e^{\tau[A+B]} = e^{\alpha(\tau)B} e^{\beta(\tau)C} e^{\gamma(\tau)C} \tag{74}$$

valid if

$$[A, B] = C, \quad [A, C] = -\lambda A, \quad [B, C] = \lambda B,$$
$$\alpha(\tau) = \gamma(\tau) = \sqrt{\frac{2}{\lambda}} \tanh\left(\sqrt{\frac{\lambda}{2}}\tau\right), \quad \beta(\tau) = \frac{2}{\lambda} \ln\cosh\left(\sqrt{\frac{\lambda}{2}}\tau\right). \tag{75}$$

For our case,

$$A \equiv \int_{-\infty}^{+\infty} \mathrm{d}q^1 c(q^1) c(-q^1), \quad B \equiv \int_{-\infty}^{+\infty} \mathrm{d}q^1 c^\dagger(q^1) c^\dagger(-q^1), \quad C = -4 \int_{-\infty}^{+\infty} \mathrm{d}q^1 c^\dagger(q^1) c(q^1),$$

$$\tau = \frac{1}{2}\gamma_D, \quad \lambda = 8, \quad \alpha = \frac{1}{2}\tanh\left(\frac{1}{2}\mathrm{arctanh}\frac{g}{\pi}\right) \equiv \kappa. \tag{76}$$

and the vacuum state becomes

$$|\Omega\rangle = N e^{-\kappa \int\limits_{-\infty}^{+\infty} \mathrm{d}p^1 c^\dagger(p^1) c^\dagger(-p^1)} |0\rangle. \tag{77}$$

A coherent state of pairs of effective bosons (bilinear in fermion Fock operators) with zero total momentum:

$$P^1|\Omega\rangle = 0, \quad P^1 = \int\limits_{-\infty}^{+\infty} \mathrm{d}p^1 p^1 \left[b^\dagger(p^1)b(p^1) + d^\dagger(p^1)d(p^1)\right]. \tag{78}$$

The vacuum $|\Omega\rangle$ is invariant under $U(1)$ and $U_A(1)$ transformations (i.e.

carries vanishing charge and axial charge):

$$U(\alpha)|\Omega\rangle = |\Omega\rangle, \;\; U(\alpha) = e^{i\alpha Q}, \;\; Q = \int\limits_{-\infty}^{+\infty} \mathrm{d}q^1 \big[b^{\dagger}(q^1)b(q^1) - d^{\dagger}(q^1)d(q^1)\big],$$

$$V(\beta)|\Omega\rangle = |\Omega\rangle, \;\; V(\beta) = e^{i\beta Q_5}, \;\; Q_5 = \int\limits_{-\infty}^{+\infty} \mathrm{d}q^1 \epsilon(q^1)\big[b^{\dagger}(q^1)b(q^1) - d^{\dagger}(q^1)d(q^1)\big].$$

The vacuum state $|\Omega\rangle$ corresponds to the symmetric phase (is invariant with respect to axial-vector transformations, i.e. no chiral symmetry breaking) – in contradiction with the results of Faber and Ivanov true vacuum should be an eigenstate of the full Hamiltonian! $|\Omega\rangle$ is such a state

TWO-POINT FUNCTION

Correlation functions calculated from the known operator solution

$$\Psi(x) = e^{(-ig/\sqrt{\pi})j^{+}(x)}\psi(x)e^{(-ig/\sqrt{\pi})j^{-}(x)}. \tag{79}$$

$\psi(x)$ is the free massless fermion field and $j^{\pm}(x)$ are the positive and negative-frequency parts of the integrated current $j(x) = j^{(+)}(x) + j^{(-)}(x)$:

$$j^{(+)}(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \mathrm{d}q^1 \frac{c^{\dagger}(q^1)}{\sqrt{2|q^1|}} \big[e^{i\hat{q}.x} - \theta\big(\lambda - |q^1|\big)\big],$$

$$j^{(-)}(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \mathrm{d}q^1 \frac{c(q^1)}{\sqrt{2|q^1|}} \big[e^{-i\hat{q}.x} - \theta\big(\lambda - |q^1|\big)\big]. \tag{80}$$

The infared regularization necessary to have meaningful objects. The scale

λ introduced. The two-point function defined as

$$C_2(x-y) = \langle vac|\Psi(x)\overline{\Psi}(y)|vac\rangle. \tag{81}$$

What is $|vac\rangle$? As a rule, the perturbative vacuum state taken. Commuting the fermion operators through the exponentials and the exponentials themselves, one arrives at

$$C_2(x-y) = e^{\frac{g^2}{\pi}D^{(+)}(x-y)}e^{-2g\left[D^{(+)}(y-x)+\gamma^5\tilde{D}^{(+)}(y-x)\right]}\langle 0|\psi(x)\overline{\psi}(y)|0\rangle. \tag{82}$$

Here, with $\mu = e^{\gamma_E}\lambda$,

$$D^{(+)}(x) = \frac{1}{2\pi}\int\frac{dk}{2|k|}\theta(|k|-\lambda)e^{-ik.x} = -\frac{1}{4\pi}\ln\big(-\mu^2x^2+ix^0\epsilon\big) \tag{83}$$

Calculation with $|\Omega\rangle$ more complicated:

$$\langle\Omega|\Psi(x)\overline{\Psi}(y)|\Omega\rangle = F_2(x-y;\kappa)C_2(x-y). \tag{84}$$

The function $F_2(x-y;\kappa) \to 1$ for $\kappa \to 0$.

The other aspects:

canonical quantization may not always be valid for interacting fields:

$$\left\{\Psi(x), \Psi^\dagger(y)\right\} = Z^{-1}\delta(x-y), \quad Z^{-1} = \exp\left(g^2 D^{(+)}(0)\right)$$

Calculation of the spectrum possible using the discrete plane-wave basis and the Fock expansion. Not trivial since $c(k^1)$ composed from $b(p^1), d(p^1)$ so that $[c(k^1), b(p^1)]$ is non-zero.

FEDERBUSH MODEL

non-trivial *massive* solvable model

permits us to generalize the Klaiber's bosonization to the massive case and search for the true physical ground state generalizing the treatment tested for the massless Thirring model

The Lagrangian of the Federbush model

$$\mathcal{L} = \frac{i}{2}\overline{\Psi}\gamma^{\mu} \overset{\leftrightarrow}{\partial}_{\mu} \Psi - m\overline{\Psi}\Psi + \frac{i}{2}\overline{\Phi}\gamma^{\mu} \overset{\leftrightarrow}{\partial}_{\mu} \Phi - \mu\overline{\Phi}\Phi - g\epsilon_{\mu\nu}J^{\mu}H^{\nu}, \tag{85}$$

describes two species of the fermion field interacting via specific current–current coupling, where $J^{\mu} = \overline{\Psi}\gamma^{\mu}\Psi$, $H^{\mu} = \overline{\Phi}\gamma^{\mu}\Phi$. Unlike the closely related massive Thirring model, Federbush model is exactly solvable.

Field equations:

$$i\gamma^\mu \partial_\mu \Psi(x) = m\Psi(x) + g\epsilon_{\mu\nu}\gamma^\mu H^\nu(x)\Psi(x)$$
$$i\gamma^\mu \partial_\mu \Phi(x) = \mu\Phi(x) - g\epsilon_{\mu\nu}\gamma^\mu J^\nu(x)\Phi(x). \tag{86}$$

The "integrated currents" $j(x)$ and $h(x)$

$$J^\mu(x) = \frac{\epsilon^{\mu\nu}}{\sqrt{\pi}}\partial_\nu j(x), \quad H^\mu(x) = \frac{\epsilon^{\mu\nu}}{\sqrt{\pi}}\partial_\nu h(x), \tag{87}$$

enter into the solutions in an "off-diagonal" way:

$$\Psi(x) = e^{-i\frac{g}{\sqrt{\pi}}h(x)}\psi(x), \quad \Phi(x) = e^{i\frac{g}{\sqrt{\pi}}j(x)}\phi(x). \tag{88}$$

The exponentials of the composite fields are more singular than in the massless case and have to be defined using the "triple-dot ordering"

(Wightman, Schroer) which generalizes the normal ordering (subtractions of the vacuum expactation values order by order). We avoid this by bosonization of the massive current.

$\psi(x)$ and $\phi(x)$ are free fields:

$$i\gamma^{\mu}\partial_{\mu}\psi(x) = m\psi(x), \;\; i\gamma^{\mu}\partial_{\mu}\phi(x) = \mu\phi(x). \tag{89}$$

The usual treatment yields

$$H = \int_{-\infty}^{+\infty} \mathrm{d}x^1 \Big[-\frac{i}{2}\psi^{\dagger}\alpha^1 \overleftrightarrow{\partial_1} \psi + m\psi^{\dagger}\gamma^0\psi - \frac{i}{2}\phi^{\dagger}\alpha^1 \overleftrightarrow{\partial_1} \phi + \mu\phi^{\dagger}\gamma^0\phi - gj^0h^1 + gj^1h^0 \Big]. \tag{90}$$

again in contrast with the LF Hamiltonian (obtained after inserting two fermion constraints)

$$P^- = \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}x^-}{2} \Big[m\Big({\psi_1}^\dagger \psi_2 + {\psi_2}^\dagger \psi_1 \Big) + \mu \Big({\phi_1}^\dagger \phi_2 + {\phi_2}^\dagger \phi_1 \Big) \Big]. \tag{91}$$

The free one!

The new approach – inserting the solutions into the Lagrangian:

$$\mathcal{L} = \frac{i}{2} \psi^\dagger \gamma^0 \gamma^\mu \overleftrightarrow{\partial_\mu} \psi - m\overline{\psi}\psi + \frac{i}{2} \phi^\dagger \gamma^0 \gamma^\mu \overleftrightarrow{\partial_\mu} \phi - \mu \overline{\phi}\phi + g \epsilon_{\mu\nu} j^\mu h^\nu. \tag{92}$$

Free fields only, opposite sign in the interaction piece in comparison with the conventional derivation (like in Thirring model).

The discrepancy removed: the SL and LF Hamiltonians acquire the same interacting structure

$g(j^0 h^1 - j^1 h^0)$ vs. $g(j^+ h^- - j^- h^+)$.

LF massive bosonization simple (like massless SL), the SL complicated

THIRRING – WESS MODEL

$$\mathcal{L} = \frac{i}{2}\overline{\Psi}\gamma^{\mu} \overleftrightarrow{\partial_{\mu}} \Psi - \frac{1}{4}\tilde{G}_{\mu\nu}\tilde{G}^{\mu\nu} + \mu_0^2\tilde{B}_{\mu}\tilde{B}^{\mu} - eJ_{\mu}\tilde{B}^{\mu}, \quad \tilde{G}_{\mu\nu} = \partial_{\mu}\tilde{B}_{\nu} - \partial_{\nu}\tilde{B}_{\mu}. \quad (93)$$

L. Brown, Nuovo Cim. (1962), W. Thirring & J. Wess, Ann. Phys. (1964)

TW paper – Ansaetze, Brown: point-splitting with the ”gauge-field” exponential, not correct and necessary;

Coupled field equations (massless Dirac + Proca (massive B^{μ})):

$$i\gamma^{\mu}\partial_{\mu}\Psi(x) = e\gamma^{\mu}\tilde{B}_{\mu}(x)\Psi(x), \quad (94)$$

$$\partial_{\mu}\tilde{G}^{\mu\nu} + \mu_0^2\tilde{B}^{\nu} = eJ^{\nu}, \quad (95)$$

QUANTIZATION: free fields will be useful

Massless fermion field

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int dp^1 \{ b(p^1) u(p^1) e^{-ip.x} + d^\dagger(p^1) v(p^1) e^{ip.x} \}, \;\; p^0 = |p^1| \quad (96)$$

$$\{b(p^1), b^\dagger(q^1)\} = \{d(p^1), d^\dagger(q^1)\} = \delta(p^1 - q^1), \; b(k^1)|0\rangle = d(k^1)|0\rangle = 0.$$

The spinors $u(p^1), v(p^1)$ are $m = 0$ limits of the massive spinors,

$$u^\dagger(p^1) = \big(\theta(-p^1), \theta(p^1)\big), \;\; v^\dagger(p^1) = \big(-\theta(-p^1), \theta(p^1)\big). \quad (97)$$

Massive vector field $B^0(x)$ ($B^1(x)$ from $\partial_0 B^0 + \partial_1 B^1 = 0$ – see below):

$$B^0(x) = \frac{1}{\sqrt{2\pi}} \int \frac{dk^1}{\sqrt{2E(k^1)}} \frac{k^1}{\mu_0} \big[a(k^1) e^{-i\hat{k}.x} + a^\dagger(k^1) e^{i\hat{k}.x} \big],$$

$$B^1(x) = \frac{1}{\sqrt{2\pi}} \int \frac{dk^1}{\sqrt{2E(k^1)}} \frac{E(k^1)}{\mu_0} \big[a(k^1) e^{-i\hat{k}.x} + a^\dagger(k^1) e^{i\hat{k}.x} \big],$$

$$\left[a(p^1), a^\dagger(q^1)\right] = \delta(p^1 - q^1). \tag{98}$$

Taking ∂_ν of the Proca eq. yields $\partial_\mu \tilde{B}^\mu = 0$. With this condition, the Dirac eq. is solved in terms of $\tilde{B}^0(x)$ and the free fermion field $\psi(x), \gamma^\mu \partial_\mu \psi = 0$:

$$\Psi(x) = \exp\left\{ -\frac{ie}{2}\gamma^5 \int_{-\infty}^{+\infty} \mathrm{d}y^1 \epsilon(x^1 - y^1)\tilde{B}^0(y^1, t)\right\}\psi(x),$$
$$\epsilon(x) = \theta(x) - \theta(-x),\ \partial_x \epsilon(x) = 2\delta(x). \tag{99}$$

Normal-ordering of the exponential understood. Product of two fermion operators has to be regularized by the point-splitting, $x^\mu \pm \frac{\epsilon^\mu}{2}$:

$$\Psi(x \pm \frac{\epsilon}{2}) = \exp\Big\{-\frac{ie}{2}\gamma^5 \int_{-\infty}^{+\infty} \mathrm{d}y^1 \epsilon(x^1 \pm \frac{\epsilon^1}{2} - y^1)\tilde{B}^0(y^1, t \pm \frac{\epsilon^0}{2})\Big\}\psi(x \pm \frac{\epsilon}{2}). \quad (100)$$

$$J^\mu(x) = \frac{1}{2}\Big[\Psi^\dagger(x + \frac{\epsilon}{2})\gamma^0\gamma^\mu\Psi(x - \frac{\epsilon}{2}) + H.c.\Big],$$
$$J_5^\mu(x) = \frac{1}{2}\Big[\Psi^\dagger(x + \frac{\epsilon}{2})\gamma^0\gamma^\mu\gamma^5\Psi(x - \frac{\epsilon}{2}) + H.c.\Big]. \quad (101)$$

We find, by means of

$$\psi^\dagger(x + \frac{\epsilon}{2})\gamma^0\gamma^\mu\psi(x - \frac{\epsilon}{2}) =: \psi(x)^\dagger\gamma^0\gamma^\mu\psi(x) : -\frac{i}{2\pi}Tr\Big(\frac{\gamma^\alpha\epsilon_\alpha\gamma^\mu}{\epsilon^2}\Big) \quad (102)$$

(analogously for $J_5^\mu(x)$) and using the definition of the symmetric limit:

$$J^\mu(x) = j^\mu(x) + \frac{e}{\pi}\tilde{B}^\mu(x), \quad J_5^\mu(x) = j_5^\mu(x) + \frac{e}{\pi}\epsilon^{\mu\nu}\tilde{B}_\nu(x). \tag{103}$$

$j^\mu(x)$ and $j_5^\mu(x)$ are free currents (normal-ordered products).

crucial: the expression in the exponential contains a term of order $O(\epsilon)$ which cancels a singularity in the free-field contraction $\Rightarrow$ a finite term, representing contributions due to the interaction on the quantum level

vector current is obviously conserved, axial anomaly $a(x)$ equal to

$$a(x) = \frac{g}{2\pi}\epsilon^{\mu\nu}\tilde{G}_{\mu\nu}(x). \tag{104}$$

Similar to the conventional result in the Schwinger model although no exponential of the integral over gauge field inserted ! (??)

Derivation of the Hamiltonian requires solution of the field eqs. for $\tilde{B}^{\mu}(x)$

The Proca equations become, due to the operator relation $\partial_{\mu}\tilde{B}^{\mu} = 0$ and the form of the interacting current, SOLUBLE. Indeed, with an appropriate choice of the Green's function, the equation

$$\partial_{\mu}\partial^{\mu}\tilde{B}^{\nu} + \mu^{2}\tilde{B}^{\nu} = ej^{\nu}, \quad \mu^{2} = \mu_{0}^{2} + \frac{e^{2}}{\pi} \tag{105}$$

can be inverted. Define

$$\left(\partial_{\mu}\partial^{\mu} + \mu^{2}\right)D_{R}(x-y) = \delta^{(2)}(x-y), \quad \left(\partial_{\mu}\partial^{\mu} + \mu^{2}\right)B^{\mu}(x) = 0, \tag{106}$$

then

$$\tilde{B}^{\nu}(x) = B^{\nu}(x) + e\int_{-\infty}^{\infty} d^{2}y D_{R}(x-y)j^{\nu}(y). \tag{107}$$

Then the Hamiltonian can be expressed in terms of the independent field variables (just free massless fermion field and its vector current, plus free $B^0(x)$ (and a zero mode $b^1(t)$ in a finite-volume treatment with periodic boundary conditions) Explicitly,

$$D_R(x-y) = -\frac{1}{2\pi^2}\int_{-\infty}^{+\infty} d^2l \frac{e^{-il\dot{(}x-y)}}{l^2-\mu_0^2+il^0\eta}, \tag{108}$$

or, after l^0 integration,

$$D_R(x-y) = -\frac{i}{2\pi}\int_{-\infty}^{+\infty} dl^1 \frac{e^{il^1(x^1-y^1)}}{2E(l^1)}\left[e^{-i(E(l^1)-i\eta/2)(x^0-y^0)} - H.c.\right]\theta(x^0-y^0), \tag{109}$$

The interacting Hamiltonian found as

$$H_{int} = -\frac{e^2}{\pi} \int\limits_{-\infty}^{+\infty} \frac{dk^1}{E(k^1)} \Big[a^\dagger(k^1)a(k^1) + \Big(\frac{E^2(k^1)}{\mu^2} - \frac{1}{2} \Big) \times$$

$$\times \left[a^\dagger(k^1)a^\dagger(-k^1) + a(k^1)a(-k^1) \right] \Big] +$$

$$+ \frac{e^2}{2\pi\mu^2} \int\limits_{-\infty}^{+\infty} dk^1 |k^1| \left[c^\dagger(k^1)c^\dagger(-k^1) + c(k^1)c(-k^1) \right]$$

$$- \frac{ie}{\sqrt{\pi}} \int\limits_{-\infty}^{+\infty} \frac{dk^1}{\sqrt{2E(k^1)}} \frac{k^1}{\sqrt{2\mu |k^1|}} \times$$

$$\times \Big[\left(E(k^1) + |k^1| \right) \left[a^\dagger(k^1)c^\dagger(-k^1) - a(k^1)c(-k^1) \right] +$$

$$+\big(E(k^1)-|k^1|\big)\left[a^\dagger(k^1)c(k^1)-c^\dagger(k^1)a(k^1)\right]\Big]. \tag{110}$$

Non-diagonal, Bogoliubov transformation needed

$$\begin{aligned}
&-L\le x^1\le L,\quad \psi(t,-L)=-\psi(t,L),\\
&B^\mu(t,-L)=B^\mu(t,L)\Rightarrow B^\mu(x)=B^\mu_N(x)+b^\mu(t),\\
&\text{periodic BC}: k^1_n=\frac{2\pi}{L}n,\ n=0,\pm1,\pm2\ldots,\\
&\text{antiperiodic BC}: p^1_n=\frac{2\pi}{L}n,\ n=\pm\frac{1}{2},\pm\frac{3}{2},\ldots.
\end{aligned} \tag{111}$$

SCHWINGER MODEL IN THE LANDAU GAUGE

$$\mathcal{L} = \frac{i}{2}\overline{\Psi}\gamma^{\mu} \overleftrightarrow{\partial_{\mu}} \Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eJ_{\mu}A^{\mu} - G(x)\partial_{\mu}A^{\mu} + \frac{1}{2}(1-\gamma)G^2(x),$$

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}, \;\; J^{\mu}(x) = \overline{\Psi}(x)\gamma^{\mu}\Psi(x). \tag{112}$$

Two terms with the auxiliary field $G(x)$ instead of usual $-\frac{\lambda}{2}\big(\partial_{\mu}A^{\mu}(x)\big)^2$.

The gauge-fixing term in the Lagrangian guarantees restriction to an arbitrary covariant gauge in which neither the condition $\partial_{\mu}A^{\mu}(x) = 0$ nor the Maxwell equations can be satisfied at the operator level:

$$\partial_{\mu}F^{\mu\nu}(x) = eJ^{\nu}(x) - \partial^{\nu}G(x), \tag{113}$$

$$\partial_{\mu}A^{\mu}(x) = (1-\gamma)G(x), \quad \partial_{\mu}\partial^{\mu}G(x) = 0. \tag{114}$$

Choose $\gamma = 1$: the gauge condition is satisfied at the operator level and the solution of the Dirac equation

$$i\gamma^\mu \partial_\mu \Psi(x) = e\gamma^\mu A_\mu(x)\Psi(x) \tag{115}$$

is completely analogous to the Thirring-Wess model case:

$$\Psi(x) = \exp\Big\{ -\frac{ie}{2}\gamma^5 \int\limits_{-\infty}^{+\infty} \mathrm{d}y^1 \epsilon(x^1 - y^1) A^0(y^1, t)\Big\}\psi(x), \quad \gamma^\mu \partial_\mu \psi = 0. \tag{116}$$

Again, the vector and axial-vector currents calculated via point-splitting.

IMPORTANT: the gauge freedom has been restricted (fixed) only partially, the above Lagrangian is still invariant with respect to gauge transformations parametrized by the gauge function obeying

$$\partial_\mu \partial^\mu \Lambda(x) = 0 \Rightarrow \partial_0^2 \Lambda = \partial_1^2 \Lambda \quad \Rightarrow \frac{\partial_0}{\partial_1}\Lambda = \frac{\partial_1}{\partial_0}\Lambda. \tag{117}$$

In order to work with the original theory, we will have to impose a condition on physical states

$$G^{(+)}(x)|phys\rangle = 0, \tag{118}$$

which generalizes the Gupta-Bleuler condition $\partial_\mu A^{(+)\mu}|phys\rangle = 0$.

The massive vector field in a gauge theory á la Lowenstein and Swieca:

The Ansatz ($\tilde{\partial}_\mu = \epsilon_{\mu\nu}\partial^\nu$)

$$A_\mu = -\frac{\sqrt{\pi}}{e}\big(\tilde{\partial}_\mu \Sigma + \partial_\mu \tilde{\eta}\big). \tag{119}$$

In the $\partial_\mu A^\mu = 0$ gauge, one gets $\partial_\mu \partial^\mu \tilde{\eta} = 0$

$$F_{\mu\nu} = \frac{\sqrt{\pi}}{e} \epsilon_{\mu\nu} \partial_\rho \partial^\rho \Sigma. \tag{120}$$

Assuming $J^\mu = \overline{\Psi} \gamma^\mu \Psi = -\frac{1}{\sqrt{\pi}} \tilde{\partial}^\mu \Phi$, $J_5^\mu = \overline{\Psi} \gamma^\mu \gamma^5 \Psi = -\frac{1}{\sqrt{\pi}} \partial^\mu \Phi$, we get from the (assumed) anomalous divergence of the axial current

$$\partial_\mu J_5^\mu = \frac{e}{2\pi} \epsilon_{\mu\nu} F^{\mu\nu} \tag{121}$$

that $\partial_\mu \partial^\mu \Phi = \partial_\mu \partial^\mu \Sigma$ or $\Phi = \Sigma + h$ with $\partial_\mu \partial^\mu h = 0$ and the vector current is

$$J_\mu = -\frac{1}{\sqrt{\pi}} \tilde{\partial}_\mu \Sigma + L_\mu, \quad L_\mu = -\frac{1}{\sqrt{\pi}} \tilde{\partial}_\mu h. \tag{122}$$

From the Maxwell eqs.

$$\tilde{\partial}^{\nu}(\partial_{\rho}\partial^{\rho}+\frac{e^2}{\pi})\Sigma-\frac{e^2}{\sqrt{\pi}}L^{\nu}=0 \quad (123)$$

$\tilde{\partial}^{\mu}L_{\mu}=0$ or

$$(\partial_{\rho}\partial^{\rho}+\frac{e^2}{\pi})\Sigma=0. \quad (124)$$

Gauge field became massive – the Schwinger mechanism of the mass generation for the gauge field.

The above solution of LS crucially depends on the axial anomaly

How is it calculated?

The usual ”gauge-invariant” definition of the currents leads to (Peskin&Schroeder, Strocchi,. . .)

$J^\mu(x) = j^\mu(x) + \frac{e}{\pi}A^\mu$, $J_5^\mu(x) = j^\mu(x) + \frac{e}{\pi}\epsilon^{\mu\nu}A_\nu$ – really GI?

Peskin and Schroeder calculate $\partial_\mu j_5^\mu(x) = \frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}$ inserting the gauge-field exponential without any gauge fixing

repeat their calculation directly for the vector and axial-vector current:

$$J^\mu_{(5)}(x) = \Psi^\dagger(x + \frac{\epsilon}{2})\gamma^0\gamma^\mu(\gamma^5)\exp\{-ie\int_{x-\epsilon/2}^{x+\epsilon/2} dz_\mu A^\mu(z)\}\Psi(x - \frac{\epsilon}{2}). \quad (125)$$

Both currents are formally gauge invariant under

$$\Psi(x) \to e^{ie\Lambda(x)}\Psi(x), \quad A^\mu(x) \to A^\mu(x) - \partial^\mu\Lambda(x). \quad (126)$$

NOTE: the fields are considered to be independent

The vector current becomes

$$J^\mu(x) = \Big[:\Psi^\dagger(x)\gamma^0\gamma^\mu\Psi(x): + \overbrace{\Psi^\dagger(x+\frac{\epsilon}{2})\gamma^0\gamma^\mu\Psi(x-\frac{\epsilon}{2})}\Big]\Big[1 - ie\epsilon_\nu A^\nu(x)\Big]. \tag{127}$$

CRUCIAL STEP: contraction of the FREE current taken

The result is precisely

$$J^\mu(x) = j^\mu(x) + \frac{e}{\pi}A^\mu(x), \quad J_5^\mu(x) = j_5^\mu(x) + \frac{e}{\pi}\epsilon^{\mu\nu}A_\nu(x), \tag{128}$$

i.e. NOT gauge-invariant (GI). This fact is hidden since one usualy calculates directly the divergence which gives the "familiar" anomaly

$a(x) = \frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}(x)$.

What is the result in the Landau gauge? - BACK TO OUR

OPERATOR SOLUTION

$$\Psi(x) = \exp\Big\{ -\frac{ie}{2}\gamma^5 \int_{-\infty}^{+\infty} \mathrm{d}y^1 \epsilon(x^1 - y^1) A^0(y^1, t) \Big\} \psi(x), \quad \gamma^\mu \partial_\mu \psi = 0. \quad (129)$$

OBSERVATION: we have the transformation law $A^\mu \to A^\mu - \partial^\mu \Lambda$, i.e.

$$A^0(x) \to A^0(x) - \partial_0 \Lambda(x), \quad \partial_\mu \partial^\mu \Lambda = 0,$$

THIS COMPLETELY DETERMINES THE TRANSFORMATION LAW FOR THE INTERACTING FERMION FIELD since the free fermion field

$\psi(x)$ does not transform!

$$\Psi(x) \to \exp\Big\{\frac{ie}{2}\gamma^5 \int\limits_{-\infty}^{+\infty} \mathrm{d}y^1 \epsilon(x^1 - y^1)\partial_0\Lambda(y^1, t)\Big\}\Psi(x) = \exp\big\{\frac{ie}{2}\gamma^5 \frac{\partial_0}{\partial_1}\Lambda\big\}\Psi(x). \tag{130}$$

IN OTHER WORDS, knowledge of the exact operator solution tells us how the full (interacting) fermion field transforms ($\Psi(x)$ and $A^\mu(x)$ are not independent!) and we should modify the "gauge exponential" in such a way that the currents given by the regularized (point-split) products of fermion fields are invariant under the specific transformations (130)

Check: Lagrangian is invariant w. r. to this particular gauge freedom Direct calculation also: the correct gauge-invariant form of the current is

$$J^\mu_{(5)}(x) = \Psi^\dagger(x + \frac{\epsilon}{2})\gamma^0\gamma^\mu(\gamma^5)\exp\big\{-ie\gamma^5\epsilon_{\mu\nu}A^\mu(x)\epsilon^\nu\big\}\Psi(x - \frac{\epsilon}{2}), \tag{131}$$

so that the gauge variations in the exponential cancel (due to the presence of γ^5, one has to calculate free-field contraction in a component form, but the result is the usual one)

RESULT: the interacting currents contain no gauge-field terms, they coincide with the free ones!

no axial anomaly $\Rightarrow$ no Schwinger mechanism

weird, strange, stupid, wrong???

FIND OUT IN A REGULARIZED TREATMENT – FINITE VOLUME

restrict $-L \leq x^1 \leq L$ and impose antiperiodic boundary conditions for the (free) fermion field and periodic ones for the gauge field:

$$\begin{aligned} &\psi(t,-L) = -\psi(t,L), \quad A^\mu(t,-L) = A^\mu(t,L) \\ &\Rightarrow A^\mu(x) = A^\mu_N(x) + A^\mu_0(t). \end{aligned} \tag{132}$$

The zero (Fourier) mode or x^1-independent part of the gauge field present

Rewrite Lagrangian, field equations... in terms of (anti)periodic fields

Dirac eq. and its solution becomes

$$i\gamma^0\partial_0\Psi + i\gamma^1\partial_1\Psi = e\big(\gamma^0 A_N^0 - \gamma^1 A_N^1\big)\Psi + e\Big(\gamma^0 a_0^0 - \gamma^1 A_0^1(t)\Big)\Psi \quad (133)$$

$$\Psi(x) = \exp\Big\{ie\gamma^5 \int_{t_0}^{t} d\tau A_0^1(\tau) - ie\gamma^5 x^1 a_0^0\Big\} \times$$

$$\times \exp\Big\{-\frac{ie}{2}\gamma^5 \int_{-L}^{+L} dy^1 \epsilon_N(x^1 - y^1) A_N^0(x^1 - y^1)\Big\}\psi(x). \quad (134)$$

The gauge condition becomes $\partial_0 A_N^0(x) + \partial_1 A_N^1(x) = 0$ (from the term

$-G_N(x)\partial_\mu A_N^\mu(x)$) and $A_0^0(t) = 0$. The gauge transformations act also in the sector of zero modes (ZM):

$$\begin{aligned}
&A_N^\mu(x) \to A_N^\mu(x) - \partial^\mu \Lambda_N(x), \\
&A_0^0(t) \to A_0^0(t) - \partial_0 \Lambda_0(t), \quad A_0^1(t) \to A_0^1(t) + \partial_1 \Lambda_0(t) = A_0^1(t).
\end{aligned} \tag{135}$$

The ZM $A_0^1(t)$ is gauge invariant! No need to add a term in the exponential to compensate for its non-invariance. The GI currents have the form

$$J^\mu_{(5)}(x) = \exp\big\{ - ie\gamma^5 \epsilon^0 A_0^1(t) \big\} \psi(x + \frac{\epsilon}{2}) \gamma^0 \gamma^\mu (\gamma^5) \psi(x - \frac{\epsilon}{2}). \tag{136}$$

Contraction in the discrete basis has the same singular structure $\Rightarrow$

$$J^\mu(x) = j^\mu(x) + \frac{e}{\pi}(0, A_0^1(t)), \quad J_5^\mu(x) = j_5^\mu(x) + \frac{e}{\pi}(A_0^1(t), 0). \tag{137}$$

Both currents are gauge invariant since $A_0^1(t)$ component is GI by itself. The divergences are

$$\partial_\mu J^\mu(x) = \partial_\mu j^\mu(x) + \frac{e}{\pi}(0, \partial_x A_0^1(t)) = 0, \tag{138}$$

$$\partial_\mu J_5^\mu(x) = \partial_\mu j_5^\mu(x) + \frac{e}{\pi}(\partial_0 A_0^1(t), 0) = \frac{e}{\pi}\partial_0 A_0^1(t) \neq 0. \tag{139}$$

From the ZM part of the Maxwell eq. one directly has

$$\partial_0^2 A_0^1(t) = -\frac{e^2}{\pi} A_0^1(t). \tag{140}$$

The Schwinger mechanism works in the zero-mode sector only. The massive Schwinger boson with $\mu^2 = \frac{e^2}{\pi}$ exists, at least in a finite volume. The continuum limit has to be studied.

Further steps:

- indefinite-metric space
- solution of the Maxwell eqs. possible
- Hamiltonian in terms of independent field variables and its symmetries

IN DETAIL:

There should be no dynamical gauge degrees of freedom in two dimensions (Coulomb gauge tells that) except for a zero mode (cf. Hetrick and Hosotani, Phys. Rev. D (1988) and other works), so we may expect that the $A^{\mu}_{N}(x)$ field are unphysical; we will treat them as ghost with zero norm acting in an indefinite metric space (K. Haller)

Covariant gauge: all gauge degrees of freedom kept, quantized as independent, the physical picture restored by a condition on allowed physical states. This condition guarantees validity of the Gauss law for expectation values. In the usual covariant-gauge formulation, in the Feynman version, the free two-dimensional gauge field satisfies the simple equation $\partial_\mu \partial^\mu A^\nu(x) = 0$. In the continuum formulation, it can be expanded as

$$A^\mu(x) = \int \frac{dk^1}{\sqrt{4\pi E(k^1)}} \left[a^{(\mu)} e^{-i\hat{k}.x} + a^{(\mu)\dagger} e^{i\hat{k}.x} \right]. \tag{141}$$

The covariant form of the canonical quantization rule

$$\left[A^\mu(x^0, x^1), \Pi^\nu(x^0, y^1) \right] = i g^{\mu\nu} \delta(x^1 - y^1), \tag{142}$$

$$\Pi^1(x) = F^{10}(x), \;\; \Pi^0(x) = -G(x) \tag{143}$$

leads to

$$[a^{(0)}(k^1), a^{(0)\dagger}(l^1)] = -\delta(k^1 - l^1), \;\; [a^{(1)}(k^1), a^{(1)\dagger}(l^1)] = \delta(k^1 - l^1). \quad (144)$$

Opposite sign in the first commutator causes well known problems with the usual probabilistic interpretation of the theory: norm of the states with odd number of time-like photons is negative (number operator contains additional minus sign to satisfy $N|1_0\rangle = |1_0\rangle \Rightarrow$ Hamiltonian has negative expactation values).

We have $\partial_\mu A^\mu(x) = G(x)$ (in Feynman gauge $\gamma = 0$) in our formulation. From this we obtain

$$\partial_\mu A^\mu(x) = -i \int \frac{dk^1}{\sqrt{4\pi}} \big[\big(a^{(0)}(k^1) - \epsilon(k^1) a^{(1)}(k^1) \big) e^{-i\hat{k}.x} - $$
$$- \big(a^{(0)\dagger}(k^1) - \epsilon(k^1) a^{(1)\dagger}(k^1) \big) e^{i\hat{k}.x} \big]. \quad (145)$$

Defining the linear combination $a_Q^\dagger(k^1) = -\frac{1}{\sqrt{2}}\big(a^{(0)} - \epsilon(k^1)a^{(1)}\big)$ with the property $\big[a_Q(k^1), a_Q^\dagger(l^1)\big] = 0$, we have

$$G(x) = -i \int \frac{dk^1}{\sqrt{4\pi}} \big[a_Q(k^1)e^{-i\hat{k}.x} - a_Q^\dagger(k^1)e^{i\hat{k}.x}\big]. \tag{146}$$

For a meaningful theory, we also have to define another linear combination $a_R^\dagger(k^1) = \frac{1}{\sqrt{2}}\big(a^{(0)} + \epsilon(k^1)a^{(1)}\big)$,
$\big[a_R(k^1), a_R^\dagger(l^1)\big] = 0$, $\big[a_Q(k^1), a_R^\dagger(l^1)\big] = \delta(k^1 - l^1)$.
These commutator rules follow from the CR Eq.(144) and ensure that the "ghost" states created by $a_Q^\dagger, a_R^\dagger$ have zero norm. This is a necessary condition for unphysical degrees of freedom.

The expansion $A^\mu(x)$ now takes the form (Feynman g.)

$$A^0(x) = \int \frac{dk^1}{\sqrt{8\pi k^0}} \Big[\big(a_R(k^1) - a_Q(k^1)\big)e^{-i\hat{k}.x} + \big(a_R^\dagger(k^1) - a_Q^\dagger(k^1)\big)e^{i\hat{k}.x}\Big], \tag{147}$$

$$A^1(x) = \int \frac{dk^1}{\sqrt{8\pi k^0}} \frac{k^1}{|k^1|} \Big[\big(a_R(k^1) + a_Q(k^1)\big)e^{-i\hat{k}.x} + \big(a_R^\dagger(k^1) + a_Q^\dagger(k^1)\big)e^{i\hat{k}.x}\Big].$$

For odd number of the "time-like" photons there is a negative norm – indefinite-metric space – corrected by the metric operator η that anticommutes with A^0 – the self-adjointness dictates conjugation $a_Q^*(k^1) = a_R(k^1)$, etc.

The (modified) Maxwell eqs.

$$\partial_\mu \partial^\mu \tilde{A}\nu = ej^\nu - \partial^\nu G \tag{148}$$

can now be inverted:

$$\tilde{A}^\mu(x) = A^\mu(x)+e\int\limits_{-\infty}^{+\infty} dy^2 D_{R0}(x-y)j^\mu(y)-\int\limits_{-\infty}^{+\infty} dy^2 D_{R0}(x-y)\partial^\mu G(y). \quad (149)$$

NEXT STEPS:

- Insert the solutions $\Psi(x)$ and $\tilde{A}^\mu$ to the Lagrangian and derive the Hamiltonian in terms of $\psi(x),\ j^\mu(x),\ A^\mu(x)$ and $G(x)$.

- remove the residual gauge freedom, related to spurious (unphysical) variables – a unitary transformation to the Coulomb gauge representation ("gauge fixing" by unitary transformation – cf. F. Lenz et al., Ann. Phys. (1994)) – all operators, Hamiltonian,... have to be transformed

- find a mechanism for the vacuum degeneracy (cf. spurion operators of Lowenstein and Swieca) – the gauge zero mode may play a role (cf. the light-front case, L. Martinovic, Phys. Lett. B (2001) - role of large gauge transformations – the covariant (Landau) gauge admits transformations of the form cx^1)

- analyze these residual large gauge transformations and also chiral symmetry of the resultant Hamiltonian, if there is a problem with the would-be Goldstone boson ($U(1)$ problem), how to construct the θ-vacuum and how to calculate the fermion condensate

SUMMARY AND CONCLUSIONS

- Dynamics of solvable models reformulated in terms of free fields which are the true dofs – structure of the LF and SL Hamiltonians unifies

- careful definition of interacting quantum currents - point-splitting regularization

- Derivative-coupling model is almost a free theory – free SL and LF Hamiltonians, correlation functions composed from free two-point functions

- True ground state of the Thirring model found by a Bogoliubov transformation – a coherent state effectively quartic in fermion Fock operators, no chiral symmetry breaking

- In terms of free fields, LF and SL Hamiltonians of the Federbush model have the same structure, but the SL one has to be diagonalized $\Rightarrow$ massive version of Klaiber's bosonization required – complicated;
  LF Hamiltonian diagonal, massive bosonization as simple as the SL massless case

model very suitable for a comparison of the non-perturbative structure of the two quantization schemes (LF vs. SL)

- axial anomaly in the Thirring-Wess model follows directly from the explicit form of the solution of the Dirac equation when currents regularized by a (hermitian) point-splitting, non-diagonal Hamiltonian, a Bogoliubov transformation required to find its physical ground state

- Schwinger model in the (reformulated) covariant gauge (Landau) revisited, solution of the Dirac eq. in terms of the original fields (no Ansaetze)
  usual definition of GI currents problematic, the invariance with respect to the residual gauge freedom in the covariant gauge dictates the form of the truly GI currents $\Rightarrow$ axial anomaly present only in the ZM sector, as well as the dynamical-mass generation
  gauge field quantized in terms of ghost modes, condition on physical

states

WORK OUT THE OTHER PHYSICAL PROPERTIES OF THE SCHWINGER MODEL IN THE FINITE-VOLUME FOMULATION:

large gauge transformations, vacuum degeneracy, chiral symmetry, fermion condensate