Thermodynamics of an exactly solvable nonlocal quark model

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Outlook

Outlook

Motivation

- QCD is the theory of strong interactions.
- Perturbative (high-energy) QCD: ok in some situations.
- However, much of the physics of the strong interaction is nonperturbative.
- Nonperturbative methods and models are necessary tools to develop an understanding of the theory.

Motivation

- Each model/method encompasses different (partial) aspects of the strong interaction.
- Two very important features: chiral symmetry breaking and confinement.
- One possible approach: “cousin theories” to QCD.
- We shall consider a certain limit of a QFT that is just like QCD in the ultraviolet but somewhat different in the infrared.
- NB: Ongoing work!

Outlook

A renormalizable infrared extension to QCD

- The gauge-fixed euclidean QCD lagrangian (for one quark flavor) reads

$$\mathcal{L}_{QCD} = \frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}+\bar{\psi}^i_\alpha[(\gamma_\mu)_{\alpha\beta}D^{ij}_\mu-\eta\delta_{\alpha\beta}\delta^{ij}]\psi^j_\beta+ib^a\partial_\mu A_\mu+\bar{c}^a\partial_\mu D^{ab}_\mu c^b,$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu,$$
$$D^{ij}_\mu = \delta^{ij}\partial_\mu - ig(T^a)^{ij}A^a_\mu$$

and the indexes

$$(i,j,\dots) = 1,\dots,N_c$$
$$(a,b,\dots) = 1,\dots,N_c^2-1$$
$$(\mu,\nu,\dots) = 1,2,3,4$$
$$(\alpha,\beta,\dots) = 1,2,3,4$$

A renormalizable infrared extension to QCD

- The QCD action is invariant under the BRST transformations

$$
\begin{aligned}
sA^a_\mu &= -D^{ab}_\mu c^b \\
s\psi^i_\alpha &= -igc^a(T^a)^{ij}\psi^j_\alpha \\
s\bar{\psi}^i_\alpha &= -ig\bar{\psi}^j_\alpha c^a(T^a)^{ji} \\
sc^a &= \frac{1}{2}gf^{abc}c^bc^c \\
s\bar{c}^a &= ib^a \\
sb^a &= 0
\end{aligned}
$$

- The BRST is nilpotent, i.e., $s^2 = 0$.
- BRST invariance is crucial for the renormalizability of the theory.

A renormalizable infrared extension to QCD

- Following [Balieu *et al.*, 2009], we introduce two BRST doublet of spinor fields (ξ^i, θ^i) and (η^i, λ^i):

$$s\xi^i_\alpha = \theta^i_\alpha \qquad s\theta^i_\alpha = 0$$
$$s\eta^i_\alpha = \lambda^i_\alpha \qquad s\lambda^i_\alpha = 0$$

and notice that

$$1 = \int [D\xi][D\theta][D\eta][D\lambda]e^{-S_{\xi\lambda}}$$

where

$$S_{\xi\lambda} = s\int d^4x \left[-\bar{\eta}^i_\alpha \partial^2 \xi^i_\alpha + \xi^i_\alpha \partial^2 \eta^i_\alpha + m^2(\bar{\eta}^i_\alpha \xi^i_\alpha - \bar{\xi}^i_\alpha \eta^i_\alpha)\right]$$

- In other words, the mere introduction of $S_{\xi\lambda}$ doesn't change the theory, but...

A renormalizable infrared extension to QCD

- ... we can add a coupling between the auxiliary fields and the fermions

$$S_M = \int d^4x \left[M_1^2(\bar{\xi}^i_\alpha \psi^i_\alpha + \bar{\psi}^i_\alpha \xi^i_\alpha) - M_2(\bar{\lambda}^i_\alpha \psi^i_\alpha + \bar{\psi}^i_\alpha \lambda^i_\alpha) \right]$$

  which does violate BRST symmetry because $sS_M \neq 0$.
- S_M corresponds to a *soft breaking* of the BRST symmetry.
- Therefore, the ultraviolet behaviour of the theory is not affected by the BRST violating term S_M!
- Indeed, the action $S = S_{YM} + S_{\xi\lambda} + S_M$ was shown to be multiplicatively renormalizable in [Balieu *et al.*, 2009].

A renormalizable infrared extension to QCD

- Integrating the auxiliary fields (ξ^i, θ^i) and (η^i, λ^i) in $S = S_{YM} + S_{\xi\lambda} + S_M$, the fermion sector reads

$$S_\psi = \int d^4x \left[\bar{\psi}^i_\alpha (\gamma_\mu)_{\alpha\beta} D^{ij}_\mu \psi^j_\beta - \bar{\psi}^i_\alpha \left(\frac{2M_1^2 M_2}{\partial^2 - m^2} + \eta \right) \psi^i_\alpha \right]$$

- Although this corresponds to a non-local action, it is equivalent to the fermion sector of the QCD action in the UV region.
- The infrared behavior of the theory is deeply changed. For example, the tree-level quark propagator reads

$$\langle \bar{\psi}(k)\psi(-k) \rangle = \frac{i\gamma_\mu k_\mu + \mathcal{A}(k)}{k^2 + \mathcal{A}^2(k)},$$

where the momentum dependent mass $\mathcal{A}(k)$ is

$$\mathcal{A}(k) = \frac{2M_1^2 M_2}{k^2 + m^2} + \eta.$$

Good.

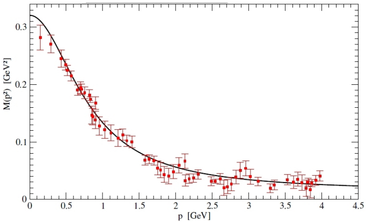
But does it have anything to do with actual QCD?

A renormalizable infrared extension to QCD

- A mass function compatible with $(2M_1^2 M_2 \equiv M_3)$

$$\mathcal{A}(k) = \frac{M_3}{k^2 + m^2} + \eta.$$

can be well fitted to lattice QCD results [Parappily *et al.*, PRD 2006].



- The mass parameters can be fitted [Dudal *et al.*, arXiv:1303.7134]

$$M_3 = 0.1960\, GeV^3 \qquad m^2 = 0.639\, GeV^2$$
$$\eta = 0.014\, GeV \qquad [\chi^2/(d.o.f.) = 1.18]$$

A renormalizable infrared extension to QCD

- Quark propagator

$$\langle\bar{\psi}(k)\psi(-k)\rangle = \frac{i\gamma_\mu k_\mu + \mathcal{A}(k)}{k^2 + \mathcal{A}^2(k)},$$

with

$$\mathcal{A}(k) = \frac{M_3}{k^2 + m^2} + \eta.$$

- Notice that the poles of the propagator (masses) are complex.
- Physical interpretation: quarks as dressed "quasi-particles" with
  - Positivity violation (complex mass): a sign of confinement.
  - Dynamically generated mass: chiral symmetry restoration as $p \to \infty$.
- Composite quark operators may have real masses, i.e., they may enter the physical spectrum. E.g.: $\rho^- = \bar{u}\gamma_\mu d$.
- This form of the propagator has been known for much time from other NP methods.

A renormalizable infrared extension to QCD

- In this preliminary study, we shall now calculate the partition function of the quark sector with $g = 0$, i.e.,

$$Z = \int [D\bar{\psi}][D\psi] e^{-S_\psi[g=0]},$$

where

$$S_\psi[g=0] = \int d^4x \bar{\psi} \left[\gamma_\mu \partial_\mu - \left(\frac{M_3}{\partial^2 - m^2} + \eta \right) \right] \psi$$

- Although this is a quadratic action, it emcompasses nonperturbative physics through the non-local term.
- The partition function can be calculated exactly!

Outlook

The partition function

- The grand canonical partition for a system of $N_f = 2$ quarks at temperature T and baryon chemical potential μ is

$$Z(T,\mu) = \int [D\bar{\psi}][D\psi] e^{-S_\psi[\bar{\psi},\psi] + \int_0^\beta d\tau \int d^3x \mu \bar{\psi}\gamma_0\psi}$$

- Being a quadratic functional of the fermion fields, one can readily integrate Z as a determinant in the imaginary time formalism

$$Z(T,\mu) = \det_{D,f,c,p} \left[\beta(-i\omega_n + \mu)\gamma^0 - \beta\vec{\gamma}\cdot\vec{p} - \beta M_{n,p}\right],$$

over spinor, flavor, color and momentum space.

- $M_{n,p}$ is the mass function

$$M_{n,p} = \frac{M_3}{\omega_n^2 + \mathbf{p}^2 + m^2} + \eta$$

The partition function

- The determinant in spinor space can be straightforwardly calculated, as well as in color and flavor spaces.

$$Z(T,\mu) = \det_p \left\{ \beta^4 \left[\mathbf{p}^2 + M_{n,p}^2 + (\omega_n + i\mu)^2 \right] \right\}^{N_c N_f}$$

- The remaining momentum space determinant is most easily computed by considering

$$\begin{aligned} \log Z(T,\mu) &= \log \det_p \left\{ \beta^4 \left[\mathbf{p}^2 + M_{n,p}^2 + (\omega_n + i\mu)^2 \right] \right\}^{N_c N_f} \\ &= \mathrm{Tr}_p \log \left\{ \beta^4 \left[\mathbf{p}^2 + M_{n,p}^2 + (\omega_n + i\mu)^2 \right] \right\}^{N_c N_f} \\ &= 2 N_c N_f \, \beta V \sum\!\!\!\!\!\!\int \log \left[\beta^2 (\mathbf{p}^2 + M_{n,p}^2 + (\omega_n + i\mu)^2) \right] \end{aligned}$$

The partition function

- For $\mu = 0$, this expression simplifies to

$$\log Z(T,0) = N_c N_f \, \beta V \sum\!\!\!\!\!\!\!\int \log\left[\beta^2(\mathbf{p}^2 + M_{n,p}^2 + \omega_n^2)\right]$$

- After substituting the expression for $M_{n,p}$, the argument of the logarithm is written as

$$L = \frac{(\omega_n^2 + \mathbf{p}^2 + m^2)^2(\mathbf{p}^2 + \omega_n^2 + \eta^2) + 2M_3\eta(\mathbf{p}^2 + \omega_n^2 + m^2) + M_3^2}{(\omega_n^2 + \mathbf{p}^2 + m^2)^2}$$
$$\equiv \frac{P_3(\omega_n^2)}{(\omega_n^2 + \mathbf{p}^2 + m^2)^2} = \frac{(\omega_n^2 - A_1)(\omega_n^2 - A_2)(\omega_n^2 - A_3)}{(\omega_n^2 + \mathbf{p}^2 + m^2)^2}$$

where $P_3(\omega_n^2)$ is a polynomial of the third degree in ω_n^2 and A_i are its (p-dependent) roots.

The partition function

- Once the argument of the log is a product of four terms, the $\mu = 0$ partition function can therefore be split into four terms:

$$\log Z(T,0) = N_c N_f \, \beta V \sum_{i=1}^{3} \sum\!\!\!\!\!\!\!\int \log(\omega_n^2 - A_i) - 2\log[\beta^2(\mathbf{p}^2 + \omega_n^2 + m^2)],$$

where A_1, A_2 and A_3 are the three roots of the polynomial $P_3(\omega_n^2)$.

- This expression formally corresponds to a sum of four of the usual expression found in any FTFT book (for $M_{n,p} = \eta$)

$$\log Z(T,0) = N_c N_f \, \beta V \sum\!\!\!\!\!\!\!\int \log\left[\beta^2(\omega_n^2 + \mathbf{p}^2 + \eta^2)\right],$$

The partition function

- A reminder of the FTFT 1.0.1 lectures...

After subtraction of (infinite) numerical constants,

$$\log Z(T,0) = N_c N_f \, \beta V \sum\!\!\!\!\!\!\int \log\left[\beta^2(\omega_n^2 + \mathbf{p}^2 + \eta^2)\right],$$

is found to give the well-known expression

$$\begin{aligned}\log Z(T,0) = \; & V \int \frac{d^3p}{(2\pi)^3} \sqrt{\mathbf{p}^2 + \eta^2} + \\ & + 2TV \int \frac{d^3p}{(2\pi)^3} \left[\log\left(1 + e^{-\beta(\sqrt{\mathbf{p}^2+\eta^2})}\right)\right].\end{aligned}$$

Partition function

- The pressure is related to the partition function as

$$P(T,\mu) = \frac{T}{V}\log Z(T,\mu)$$

- Applying the exact same reasoning, we find for the pressure at $\mu = 0$

$$P(T,0) = P_0 + 4TN_cN_f \int_P \log\left[\frac{\left(1+e^{-\beta\omega_1}\right)\left(1+e^{-\beta\omega_2}\right)\left(1+e^{-\beta\omega_3}\right)}{\left(1+e^{-\beta\omega_0}\right)^2}\right],$$

  where $\omega_i = [-A_i]^{1/2}$ and A_i are the three roots of the third degree polynomial $P(\omega_n^2)$.

- Note that two of the ω_i are complex conjugates and the other is real (due to complex conjugate masses).
- Therefore, $P(T,0)$ is a real function!

The partition function

- And what about $\mu \neq 0$? Let's remember the general expression

$$\log Z = 2N_cN_f\,\beta V \sum\!\!\!\!\!\!\!\int \log\left[\beta^2(\mathbf{p}^2 + M_{n,p}^2 + (\omega_n + i\mu)^2)\right]$$

- It can be written as

$$\log Z(T,\mu) = \log Z(T,0) + VN_cN_f \sum\!\!\!\!\!\!\!\int \log\left\{1 + \frac{2(\omega_n^2 - \Omega_{n,p}^2)\mu^2 + \mu^4}{(\Omega_{n,p}^2 + \omega_n^2)^2}\right\}$$

  where $\Omega_{n,p} \equiv \mathbf{p}^2 + M_{n,p}^2$.
- This is an exact solution for the model!
- The $\mu \neq 0$ term contains no divergencies and can be treated numerically. (Analytical expression? We don't know yet...)

Outlook

Thermodynamics of the model

- What to expect for the thermodynamics?
- Given the dispersion relation

$$\Omega_{n,p}^2 = \mathbf{p}^2 + \left[\frac{M_3}{\omega_n^2 + \mathbf{p}^2 + m^2} + \eta \right]^2$$

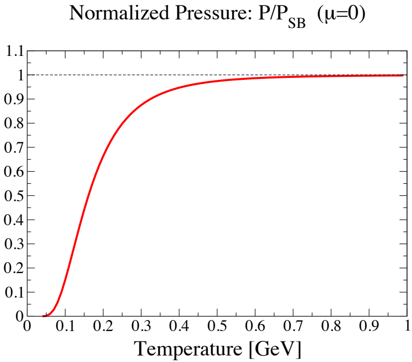
one could expect a free quark behavior as $p \to \infty$.

- Thermodynamically, this would correspond to the pressure of a relativistic gas of massless quarks

$$P_{SB} = N_c N_f \left(\frac{7\pi^2 T^4}{180} + \frac{\mu^2 T^2}{6} + \frac{\mu^4}{12\pi^2} \right)$$

- NB: We only have three free parameters in the model: M_3, m^2 and η from the $T = 0$ quark mass function of [Parapilly *et al.*, 2006].

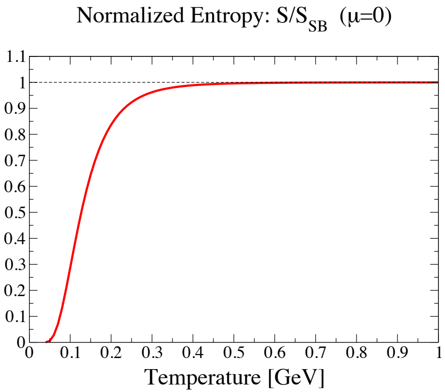
Thermodynamics of the model



- Nontrivial behavior at low T and the approach of the Stefan-Boltzmann limit as T grows.

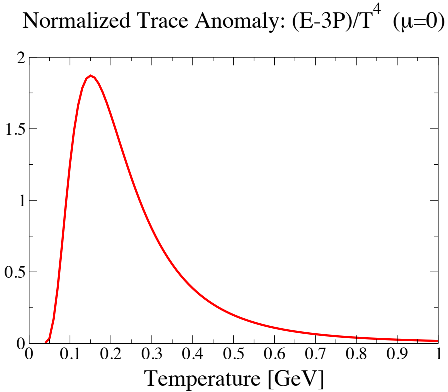
Thermodynamics of the model

- Normalized entropy density at $\mu = 0$



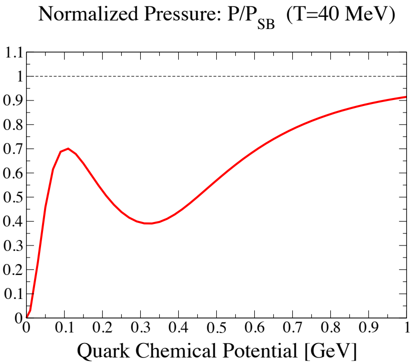
Thermodynamics of the model

- Trace anomaly $(E-3P)/T^4$ at $\mu=0$



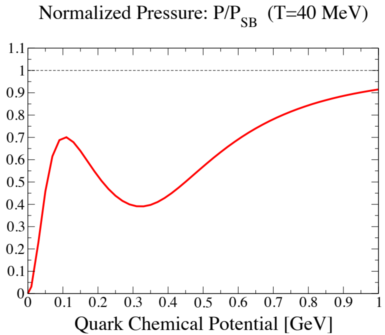
Thermodynamics of the model

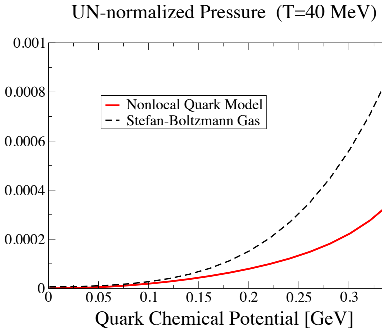
- Pressure at low temperature $(T = 40MeV)$ for varying μ



- Notice the approach of the Stefan-Boltzmann limit as μ grows.

Thermodynamics of the model





- NB: The pressure doesn't decrease! It just grows faster than SB, than slower and then reaches SB again.

Outlook

Summary and future prospects

- We considered the quark sector of an infrared extension of QCD.
- This model naturally incorporates nonperturbative information about the infrared behaviour of quark masses (compatible with lattice calculations).
- Its partition function can be calculated exactly for any (T, μ).
- The thermodynamics has a nontrivial behavior for low (T, μ) and reproduces the Stefan-Boltzmann limit as $(T, \mu) \to \infty$.
- Thermodynamical observables *somewhat resembles* lattice QCD results: up to which point is it a coincidence? We don't know!

Summary and future prospects

- Include dynamical gluons!
- The partition function (without gluons) can be calculated exactly: starting point for perturbation theory?
- Reminder: the *local* model (with gluons and auxiliary fields) is renormalizable.
- Closed expression for the $T \to 0$, $\mu \neq 0$ limit.
- Formal relation to other nonperturbative approaches (FRG, DSE, GZ,...)?