Extracting scattering and resonance properties from the lattice

Maxwell T. Hansen

Institut für Kernphysik and Helmholtz-Institut Mainz

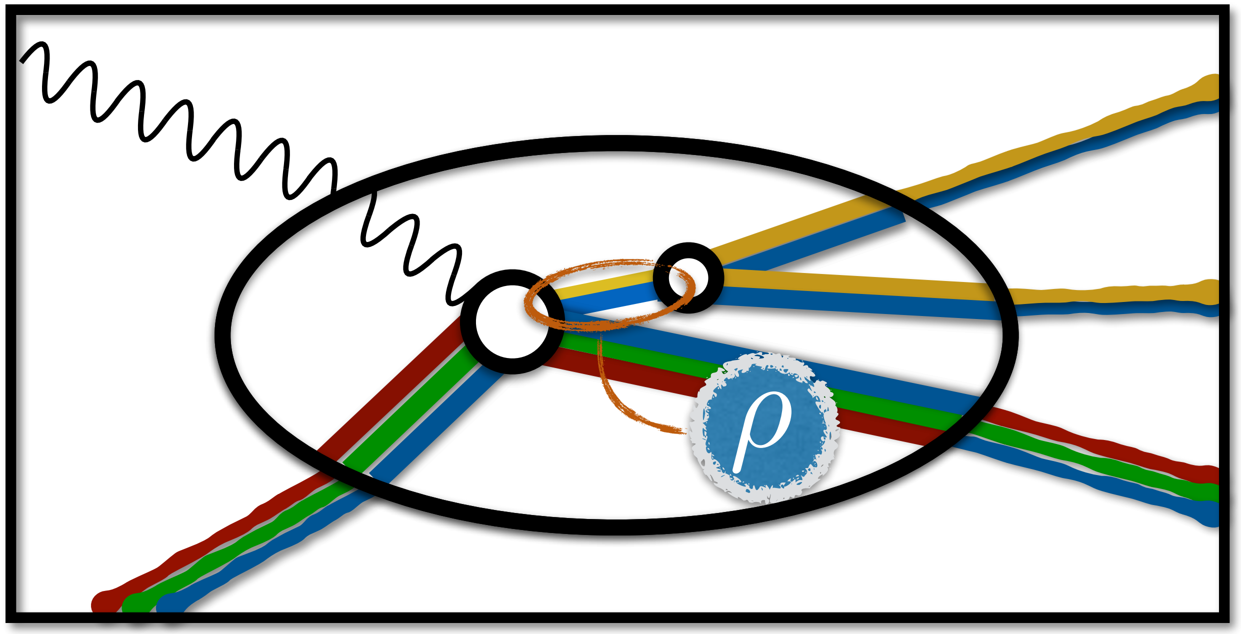
Johannes Gutenberg Universität

Mainz, Germany

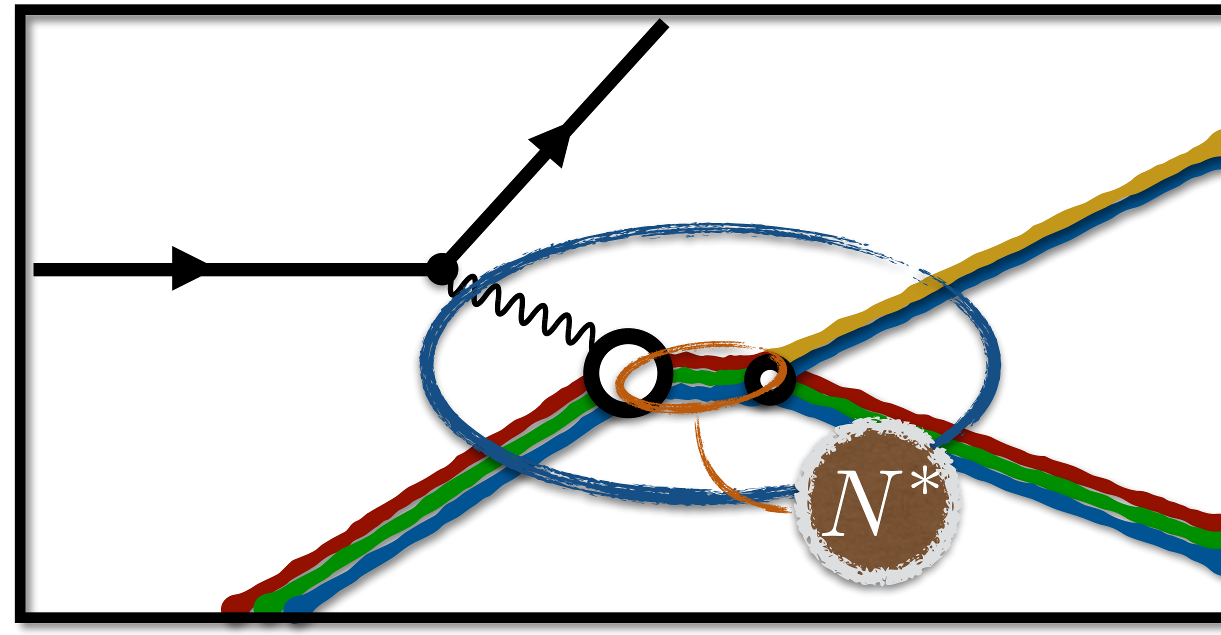
July 6th, 2016

HIM Helmholtz-Institut Mainz

JG|U



$p\gamma \to N\rho \to N\pi\pi$



$p\gamma^* \to N^* \to N\pi, N\eta$

Resonances are not directly detected.

Outgoing hadrons are used to reconstruct resonance properties.

It is thus highly valuable to predict these transition amplitudes from the underlying theory of QCD

Combining accurate, model-independent predictions with experiment will lead to a deeper understanding of QCD's rich resonance structure

What can we extract from the lattice?

We are trying to evaluate a difficult integral numerically

$$\textbf{observable} = \int \mathcal{D}\phi \; e^{iS} \left[\begin{array}{c} \textbf{interpolator} \\ \textbf{for observable} \end{array} \right]$$

What can we extract from the lattice?

We are trying to evaluate a difficult integral numerically

$$\textbf{observable} = \int \prod_{i}^{N} d\phi_i \; e^{-S} \left[\begin{array}{c} \textbf{interpolator} \\ \textbf{for observable} \end{array} \right]$$

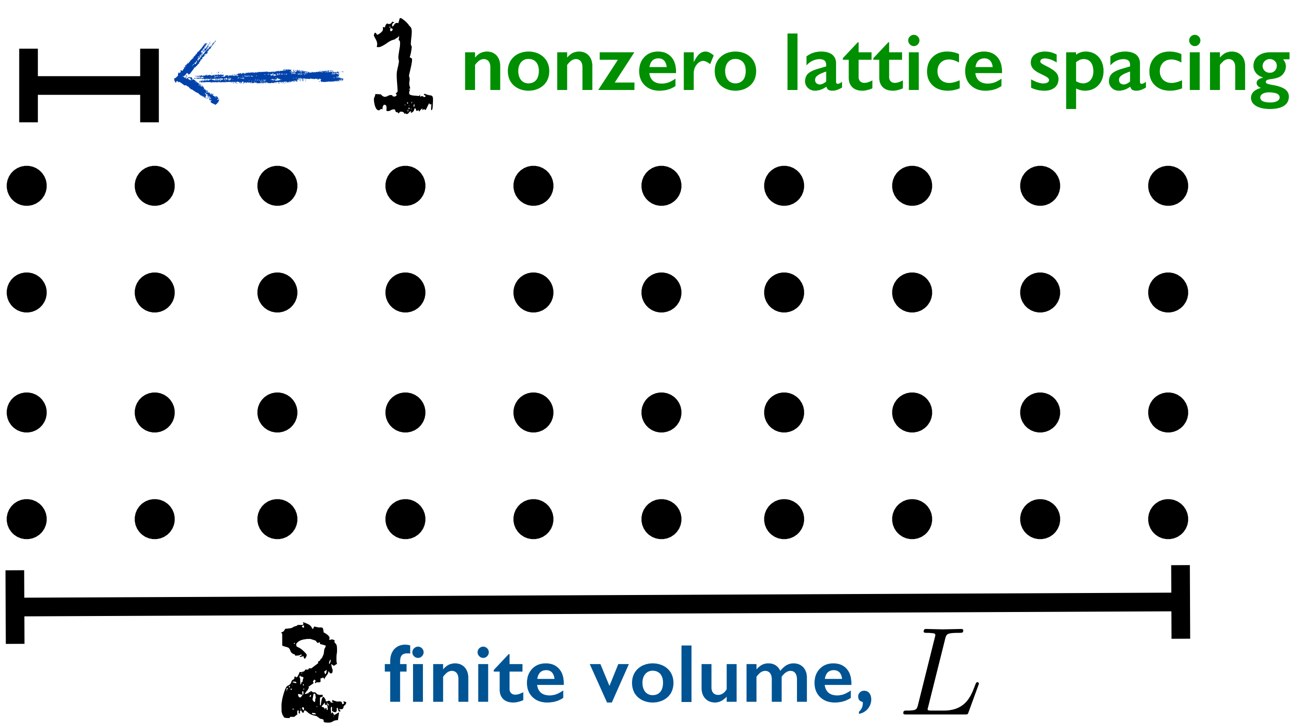
To do so we have to make four compromises

What can we extract from the lattice?

We are trying to evaluate a difficult integral numerically

$$\text{observable} = \int \prod_{i}^{N} d\phi_i \; e^{-S} \left[\begin{array}{c} \text{interpolator} \\ \text{for observable} \end{array} \right]$$

To do so we have to make four compromises

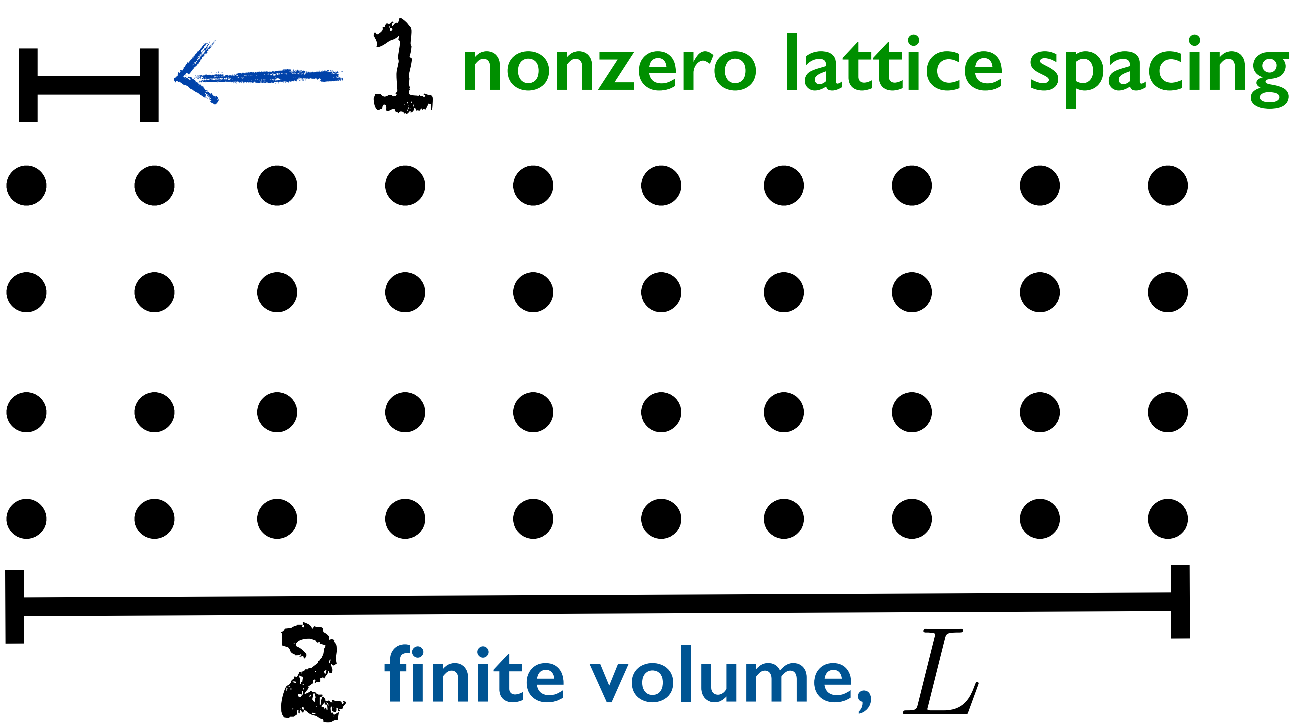


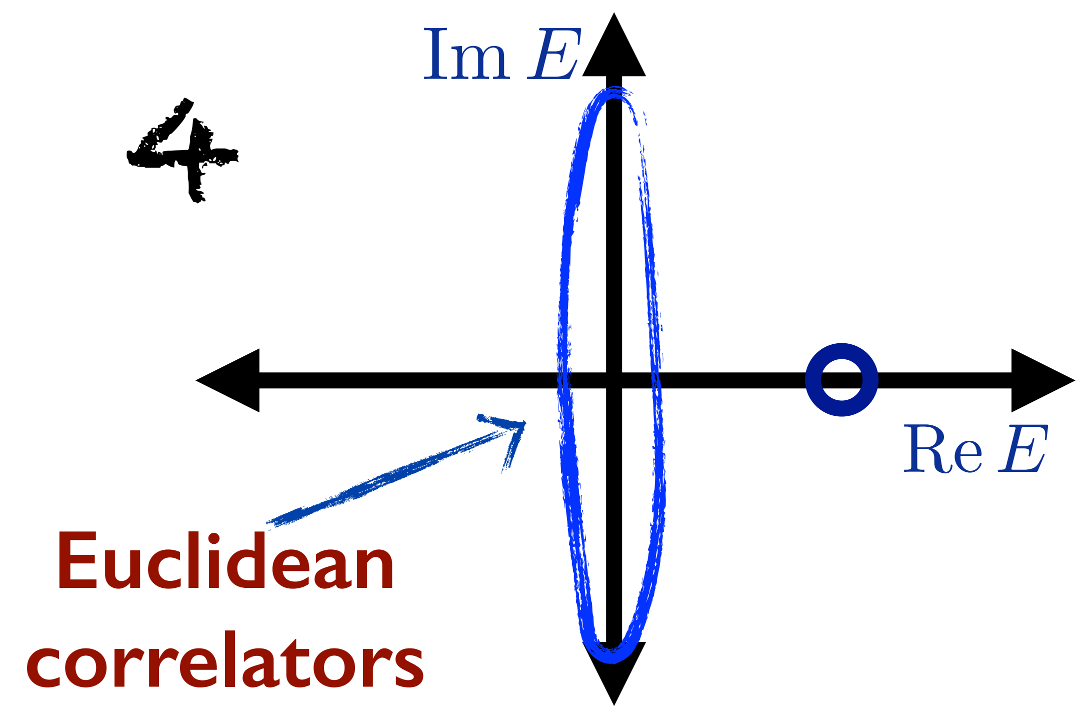
What can we extract from the lattice?

We are trying to evaluate a difficult integral numerically

$$\text{observable} = \int \prod_i^N d\phi_i \, e^{-S} \left[\begin{array}{c} \text{interpolator} \\ \text{for observable} \end{array} \right]$$

To do so we have to make four compromises





3 Unphysical pion masses $M_{\pi,\text{lattice}} > M_{\pi,\text{our universe}}$

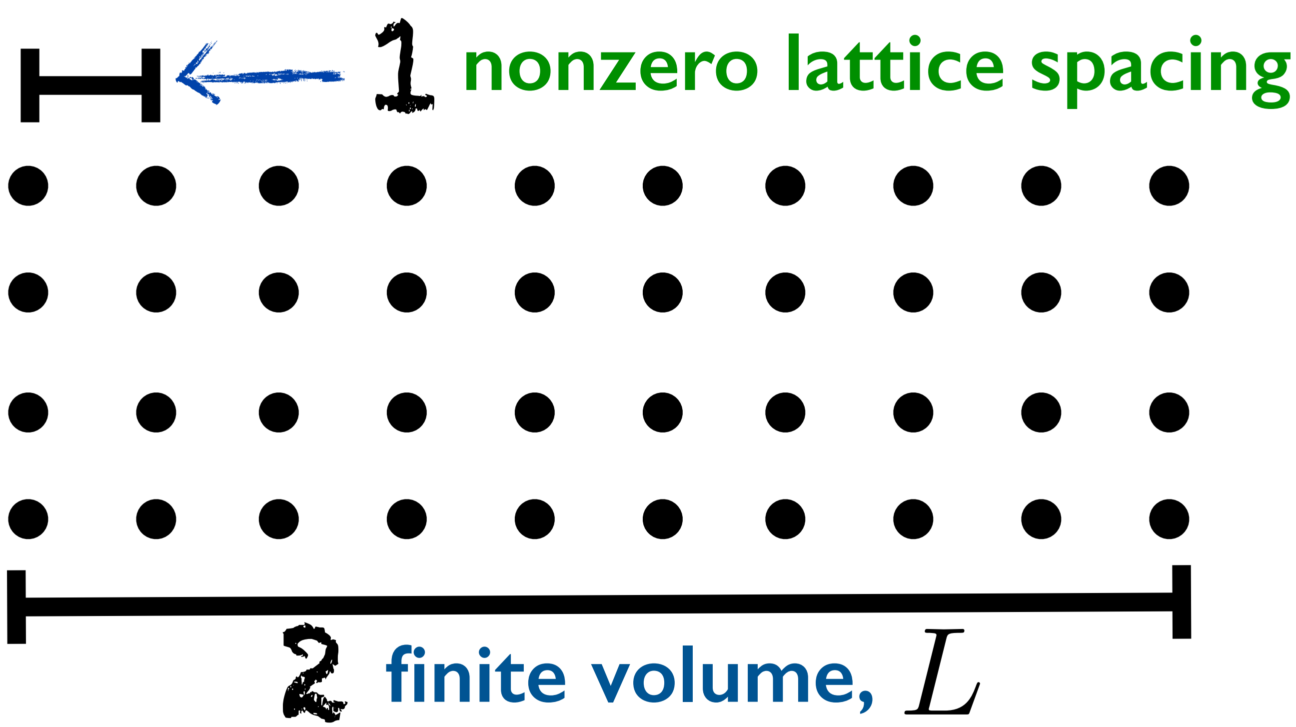
But calculations at the physical pion mass do now exist

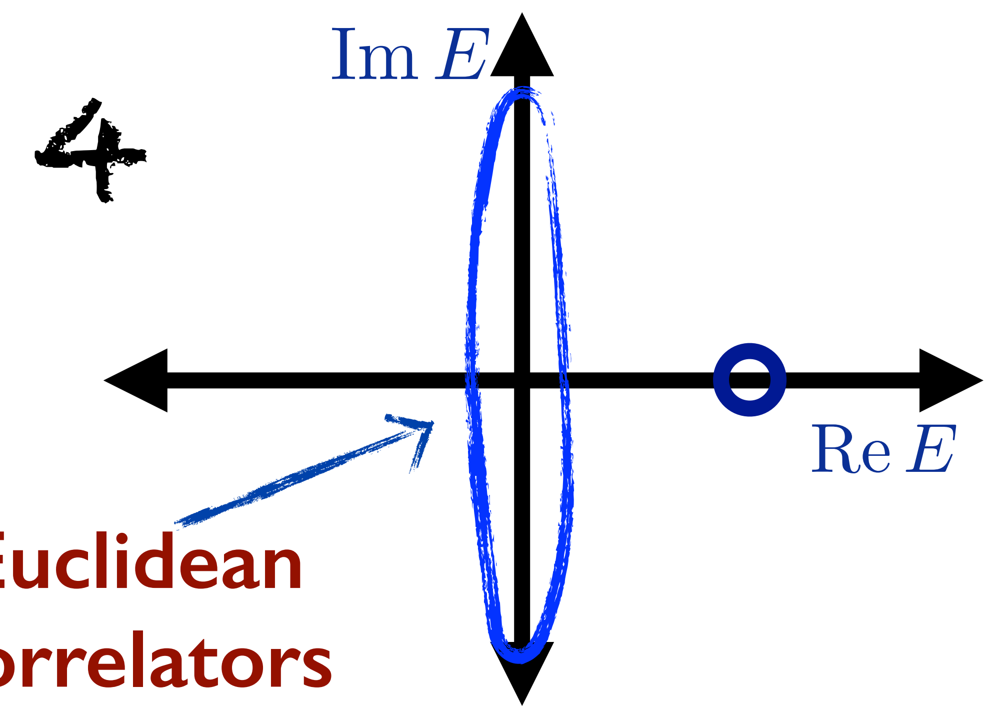
What can we extract from the lattice?

We are trying to evaluate a difficult integral numerically

$$\left(\begin{array}{c}\textbf{observable?} \\ \text{discretized, finite volume,} \\ \text{Euclidean, heavy pions}\end{array}\right) = \int \prod_i^N d\phi_i \; e^{-S} \left[\begin{array}{c}\textbf{interpolator} \\ \textbf{for observable}\end{array}\right]$$

To do so we have to make four compromises





3 **Unphysical pion masses** $M_{\pi,\mathrm{lattice}} > M_{\pi,\mathrm{our\ universe}}$

But calculations at the physical pion mass do now exist

Stable particle masses

$$C(\tau) \equiv \langle 0|\mathcal{O}(\tau)\mathcal{O}^\dagger(0)|0\rangle = \int \prod d\phi e^{-S}\mathcal{O}(\tau)\mathcal{O}^\dagger(0)$$

Stable particle masses

$$C(\tau) \equiv \langle 0|\mathcal{O}(\tau)\mathcal{O}^\dagger(0)|0\rangle = \int \prod d\phi e^{-S}\mathcal{O}(\tau)\mathcal{O}^\dagger(0)$$

The correlator is equal to a sum of decaying exponentials

$$C(\tau) = \sum_n \langle 0|e^{H\tau}\mathcal{O}(0)e^{-H\tau}|E_n\rangle\langle E_n|\mathcal{O}^\dagger(0)|0\rangle$$

$$= \sum_n |\langle 0|\mathcal{O}(0)|E_n\rangle|^2 e^{-E_n\tau} \underset{\tau\to\infty}{\longrightarrow} |\langle 0|\mathcal{O}(0)|E_1\rangle|^2 e^{-E_1\tau}$$

Stable particle masses

$$C(\tau) \equiv \langle 0|\mathcal{O}(\tau)\mathcal{O}^\dagger(0)|0\rangle = \int \prod d\phi e^{-S}\mathcal{O}(\tau)\mathcal{O}^\dagger(0)$$

The correlator is equal to a sum of decaying exponentials

$$C(\tau) = \sum_n \langle 0|e^{H\tau}\mathcal{O}(0)e^{-H\tau}|E_n\rangle\langle E_n|\mathcal{O}^\dagger(0)|0\rangle$$

$$= \sum_n |\langle 0|\mathcal{O}(0)|E_n\rangle|^2 e^{-E_n\tau} \xrightarrow[\tau\to\infty]{} |\langle 0|\mathcal{O}(0)|E_1\rangle|^2 e^{-E_1\tau}$$

One can fit to the long time behavior to extract the ground state

Stable particle masses

$$C(\tau) \equiv \langle 0|\mathcal{O}(\tau)\mathcal{O}^\dagger(0)|0\rangle = \int \prod d\phi e^{-S}\mathcal{O}(\tau)\mathcal{O}^\dagger(0)$$

The correlator is equal to a sum of decaying exponentials

$$C(\tau) = \sum_n \langle 0|e^{H\tau}\mathcal{O}(0)e^{-H\tau}|E_n\rangle\langle E_n|\mathcal{O}^\dagger(0)|0\rangle$$

$$= \sum_n |\langle 0|\mathcal{O}(0)|E_n\rangle|^2 e^{-E_n\tau} \xrightarrow[\tau\to\infty]{} |\langle 0|\mathcal{O}(0)|E_1\rangle|^2 e^{-E_1\tau}$$

One can fit to the long time behavior to extract the ground state

If we choose the operator…

$$\mathcal{O} = \bar{u}\gamma_5 d$$

Then in the infinite-volume, continuum limit we recover…

$$E_1(a, L, m_q) \longrightarrow M_\pi(m_q)$$

Stable particle masses

$$C(\tau) \equiv \langle 0|\mathcal{O}(\tau)\mathcal{O}^\dagger(0)|0\rangle = \int \prod d\phi e^{-S}\mathcal{O}(\tau)\mathcal{O}^\dagger(0)$$

The correlator is equal to a sum of decaying exponentials

$$C(\tau) = \sum_n \langle 0|e^{H\tau}\mathcal{O}(0)e^{-H\tau}|E_n\rangle\langle E_n|\mathcal{O}^\dagger(0)|0\rangle$$

$$= \sum_n |\langle 0|\mathcal{O}(0)|E_n\rangle|^2 e^{-E_n\tau} \underset{\tau\to\infty}{\longrightarrow} |\langle 0|\mathcal{O}(0)|E_1\rangle|^2 e^{-E_1\tau}$$

One can fit to the long time behavior to extract the ground state

If we choose the operator…

$$\mathcal{O} = \bar{u}\gamma_5 d$$

$$\mathcal{O} = \bar{u}\gamma_5 s$$

Then in the infinite-volume, continuum limit we recover…

$$E_1(a, L, m_q) \longrightarrow M_\pi(m_q)$$

$$E_1(a, L, m_q) \longrightarrow M_K(m_q)$$

Stable particle masses

$$C(\tau) \equiv \langle 0|\mathcal{O}(\tau)\mathcal{O}^\dagger(0)|0\rangle = \int \prod d\phi e^{-S}\mathcal{O}(\tau)\mathcal{O}^\dagger(0)$$

The correlator is equal to a sum of decaying exponentials

$$C(\tau) = \sum_n \langle 0|e^{H\tau}\mathcal{O}(0)e^{-H\tau}|E_n\rangle\langle E_n|\mathcal{O}^\dagger(0)|0\rangle$$

$$= \sum_n |\langle 0|\mathcal{O}(0)|E_n\rangle|^2 e^{-E_n\tau} \xrightarrow[\tau\to\infty]{} |\langle 0|\mathcal{O}(0)|E_1\rangle|^2 e^{-E_1\tau}$$

One can fit to the long time behavior to extract the ground state

If we choose the operator…	Then in the infinite-volume, continuum limit we recover…
$\mathcal{O} = \bar{u}\gamma_5 d$	$E_1(a, L, m_q) \longrightarrow M_\pi(m_q)$
$\mathcal{O} = \bar{u}\gamma_5 s$	$E_1(a, L, m_q) \longrightarrow M_K(m_q)$
$\mathcal{O} = u(u^T\Gamma d)$	$E_1(a, L, m_q) \longrightarrow M_N(m_q)$

Stable particle masses

Full error budget calculation of isospin splittings

Four lattice spacings

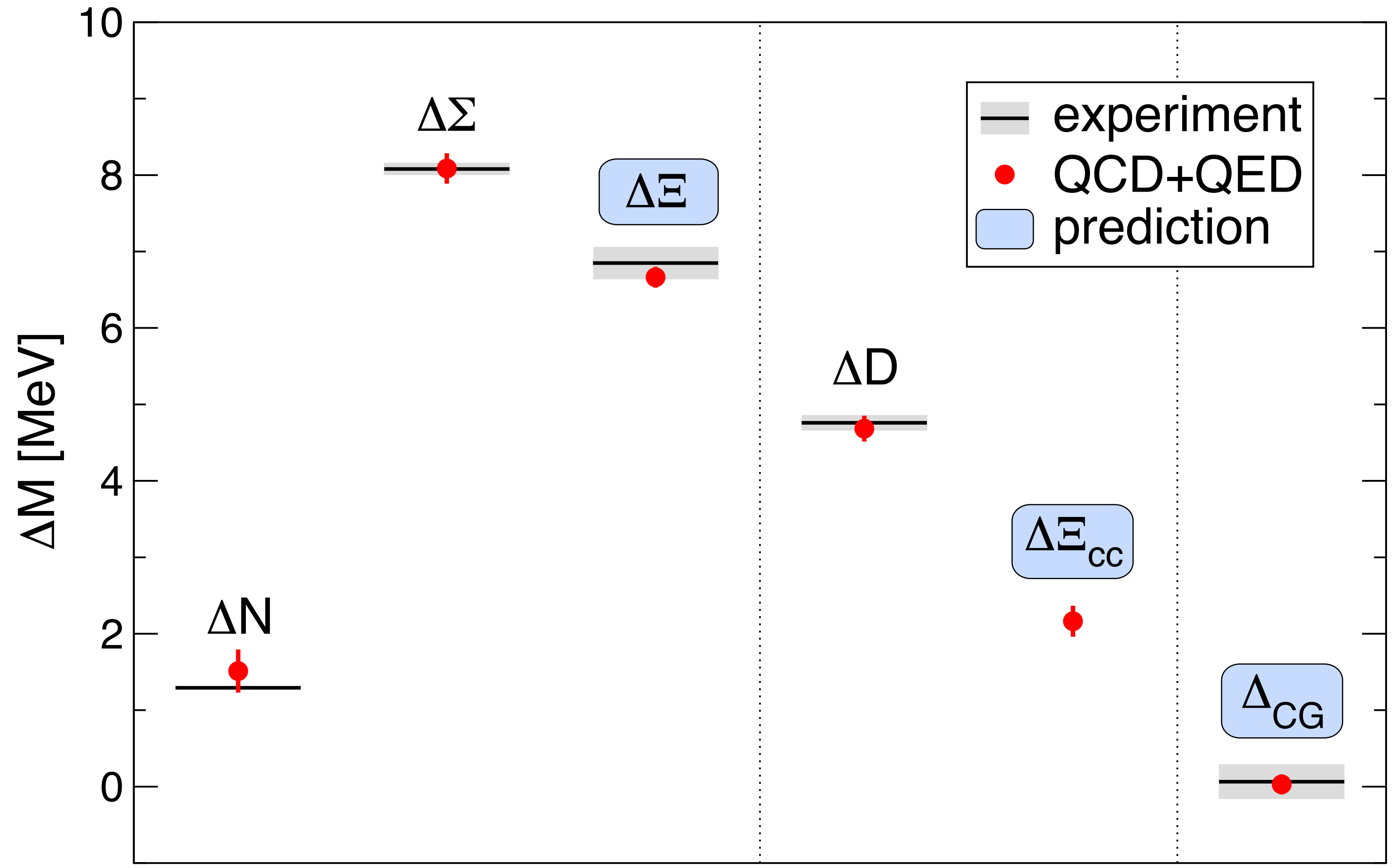
0.06fm **to** 0.10fm

Four volumes, ranging up to

8fm

Many pion masses, ranging down to

197MeV



Dynamical up, down, strange and charm quarks + QED

Borsanyi et.al. (BMW Collaboration) *Science* 347, 1452-1455 (2015)

Decay constants

$$\langle 0|A_\mu(0)\mathcal{O}^\dagger(-\tau)|0\rangle = \int \prod d\phi e^{-S} A_\mu(0)\mathcal{O}^\dagger(-\tau)$$

$$\langle 0|A_\mu(0)\mathcal{O}^\dagger(-\tau)|0\rangle \underset{\tau\to\infty}{\longrightarrow} ip_\mu f$$

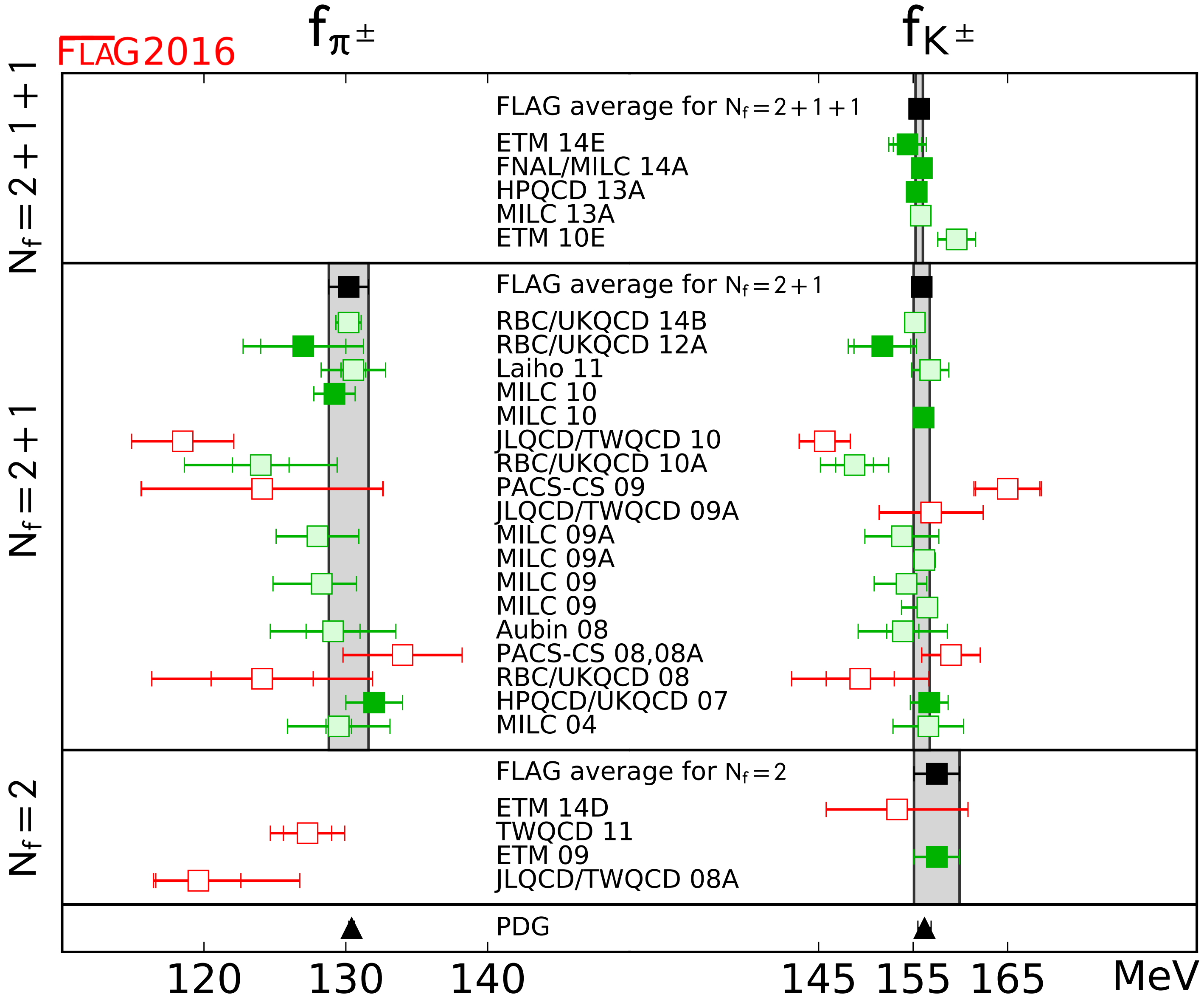
requires renormalization of the axial-vector current

Decay constants

$$\langle 0|A_\mu(0)\mathcal{O}^\dagger(-\tau)|0\rangle = \int \prod d\phi e^{-S} A_\mu(0)\mathcal{O}^\dagger(-\tau)$$

$$\langle 0|A_\mu(0)\mathcal{O}^\dagger(-\tau)|0\rangle \underset{\tau\to\infty}{\longrightarrow} ip_\mu f$$

requires renormalization of the axial-vector current



FLAG

(Flavor Lattice Averaging Group)

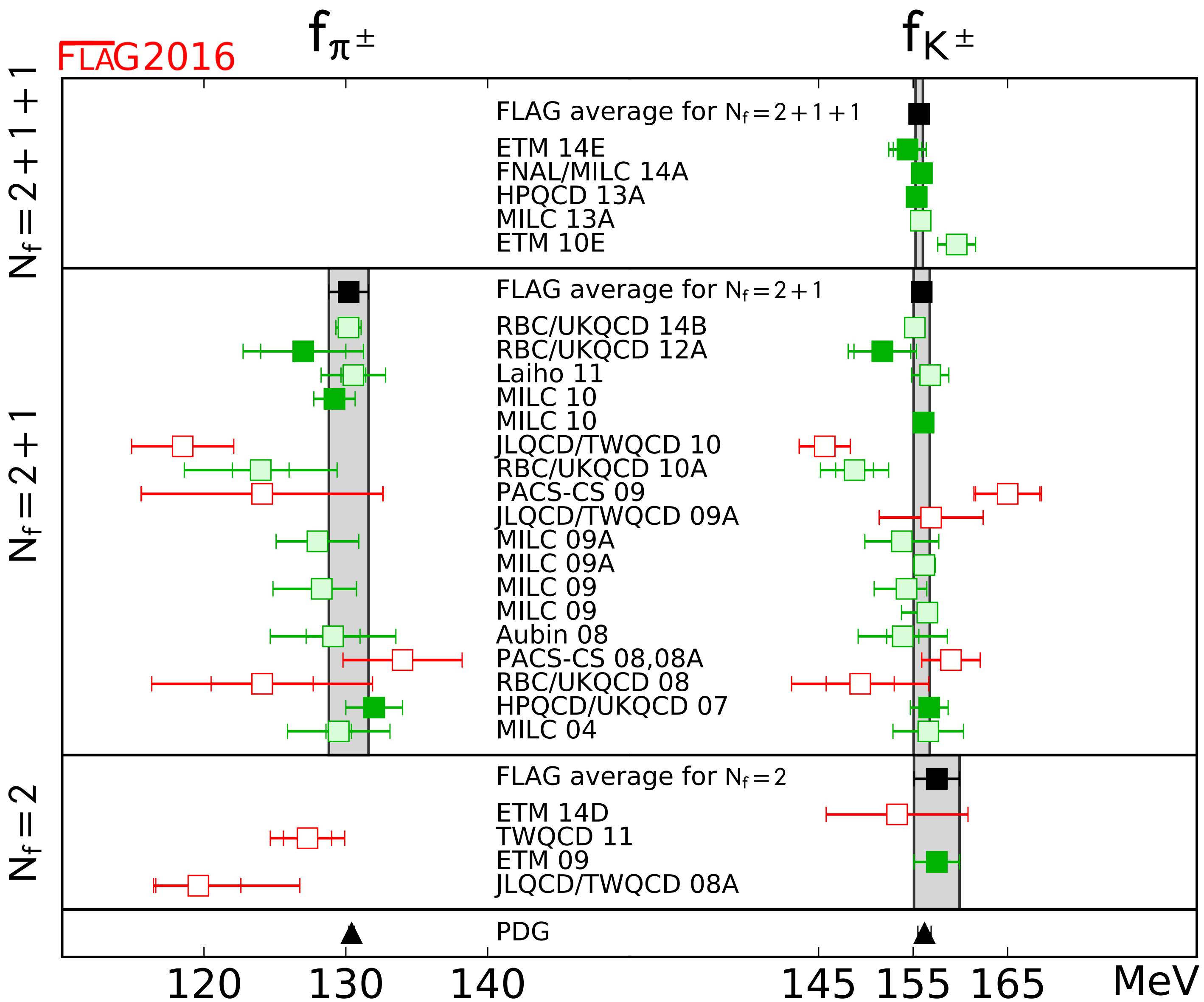
Collaboration for systematically averaging results from different collaborations

Decay constants

$$\langle 0|A_\mu(0)\mathcal{O}^\dagger(-\tau)|0\rangle = \int \prod d\phi e^{-S} A_\mu(0)\mathcal{O}^\dagger(-\tau)$$

$$\langle 0|A_\mu(0)\mathcal{O}^\dagger(-\tau)|0\rangle \underset{\tau\to\infty}{\longrightarrow} ip_\mu f$$

requires renormalization of the axial-vector current



FLAG

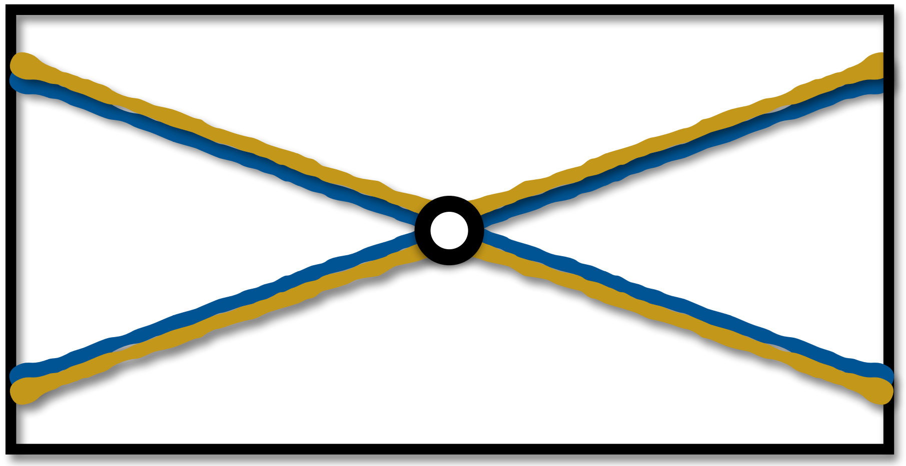
(Flavor Lattice Averaging Group)

Collaboration for systematically averaging results from different collaborations

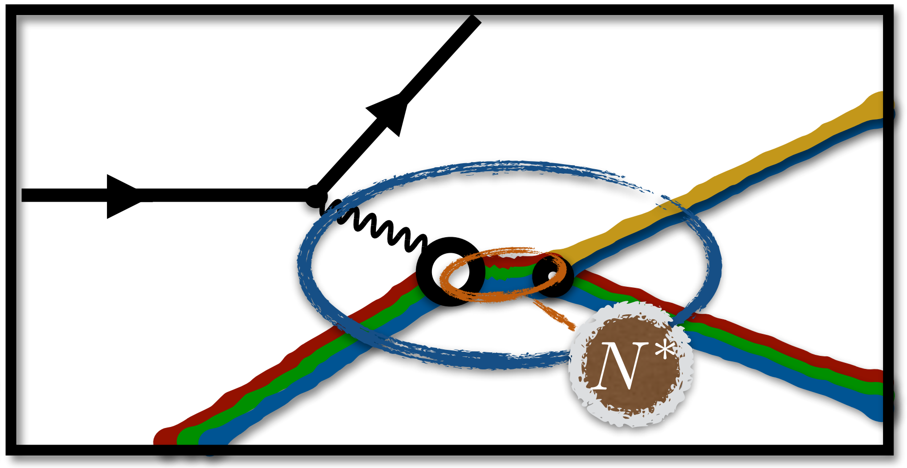
Results are only included if they meet certain standards:

1. Chiral extrapolation, pion below 400MeV
2. Continuum extrapolation, minimum two lattices (below 0.1 fm, sufficiently different)
3. Two volumes or demonstrably small effect
4. Non-perturbative renormalization

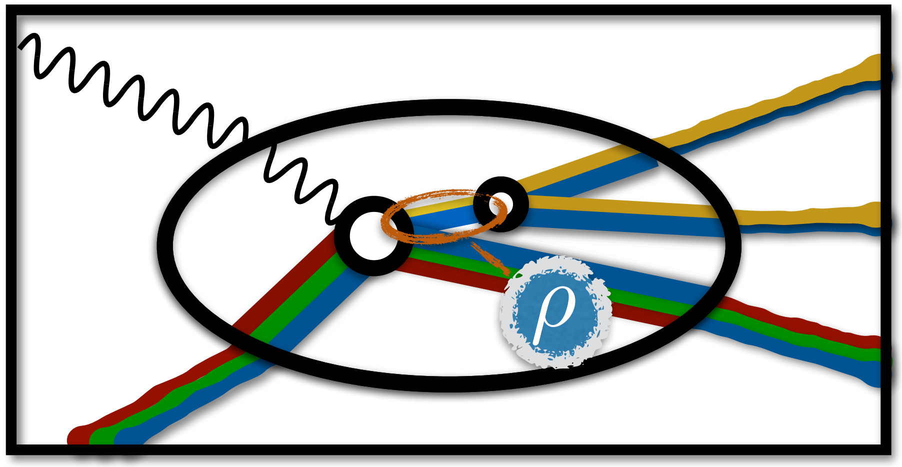
What can we extract from the lattice?



$\langle \pi\pi | \pi\pi \rangle$



$\langle N\pi | \mathcal{J}_\mu | N \rangle$



$\langle N\pi\pi | \mathcal{J}_\mu | N \rangle$

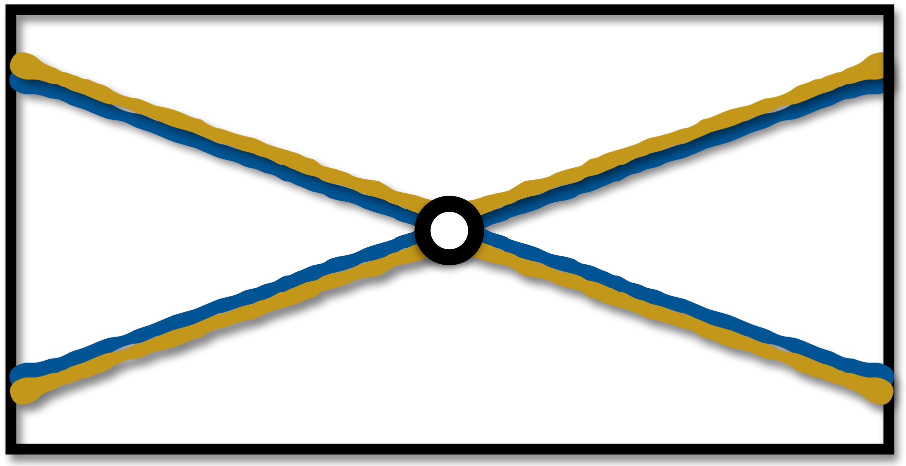
Resonances are fundamentally different from stable particles

$$C(\tau) = \langle 0 | \mathcal{O}_\rho(\tau) \mathcal{O}_\rho^\dagger(0) | 0 \rangle = \sum_n |\langle 0 | \mathcal{O}_\rho(0) | E_n \rangle|^2 e^{-E_n \tau}$$

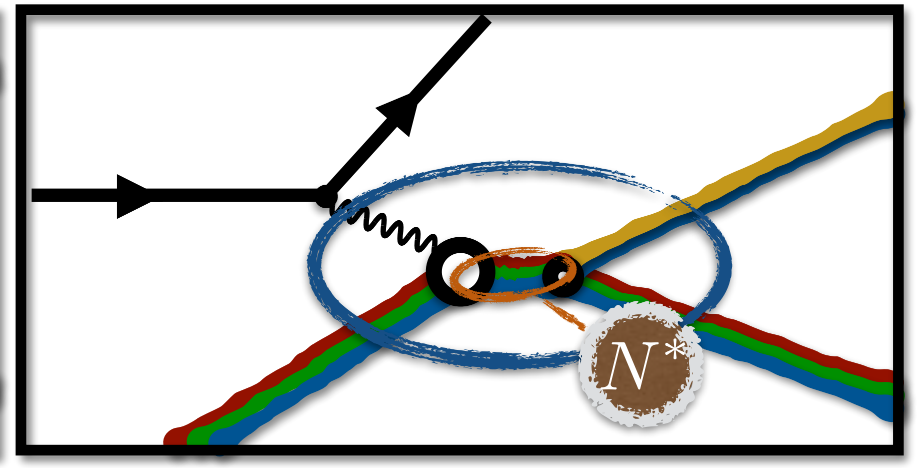
$$\lim_{L \to \infty} E_n(L) \neq M_\rho$$

What can we extract from the lattice?

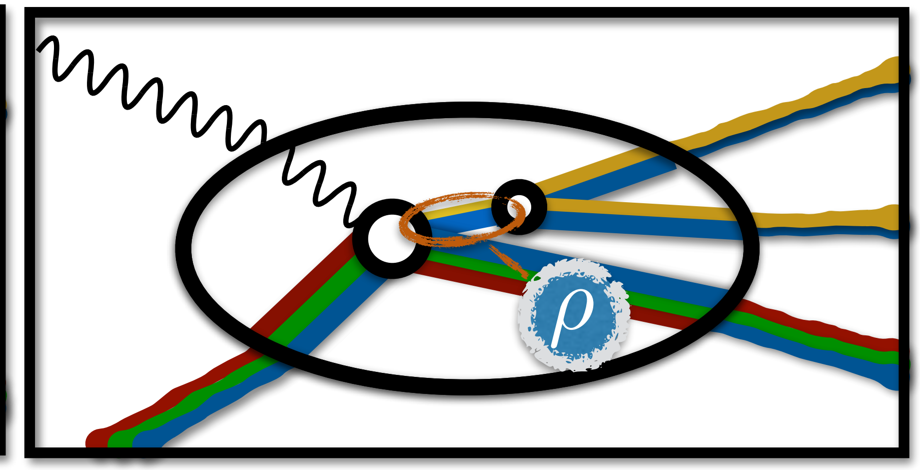
Not possible to directly calculate



$\langle \pi\pi | \pi\pi \rangle$



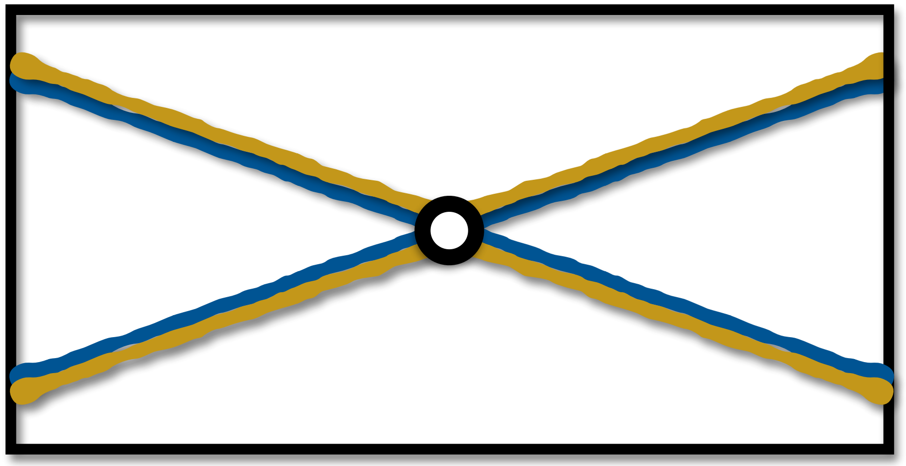
$\langle N\pi | \mathcal{J}_\mu | N \rangle$



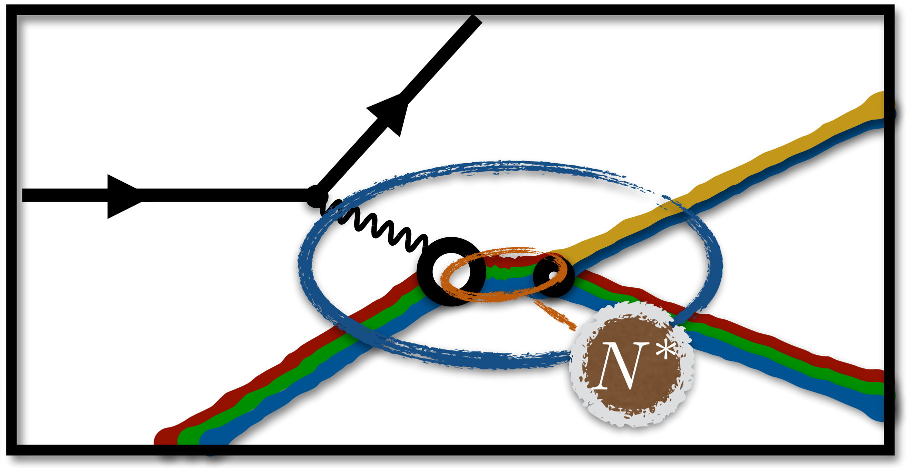
$\langle N\pi\pi | \mathcal{J}_\mu | N \rangle$

What can we extract from the lattice?

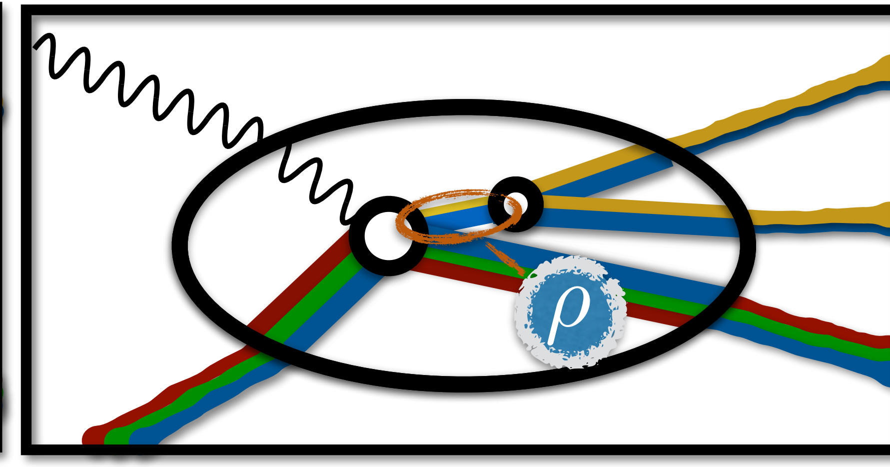
Not possible to directly calculate



$\langle \pi\pi | \pi\pi \rangle$



$\langle N\pi | \mathcal{J}_\mu | N \rangle$

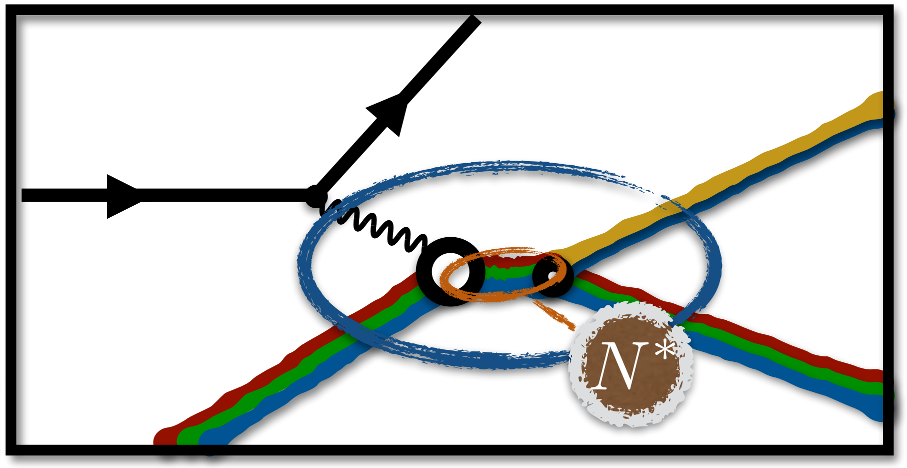


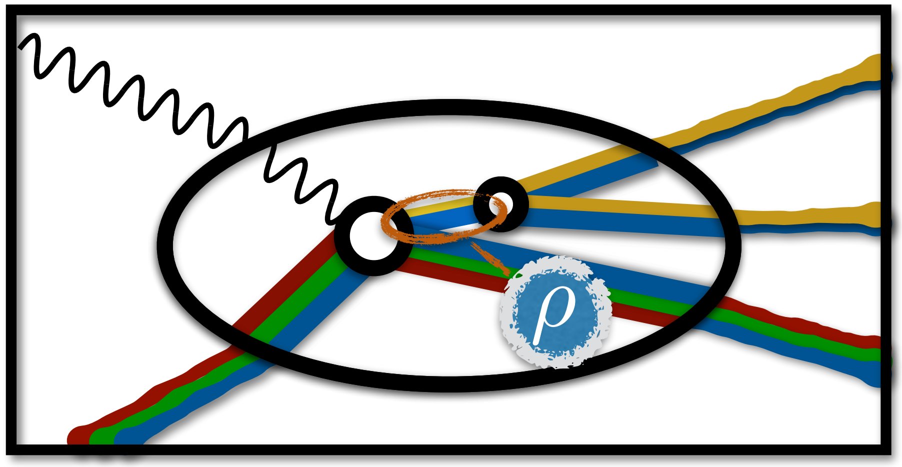
$\langle N\pi\pi | \mathcal{J}_\mu | N \rangle$

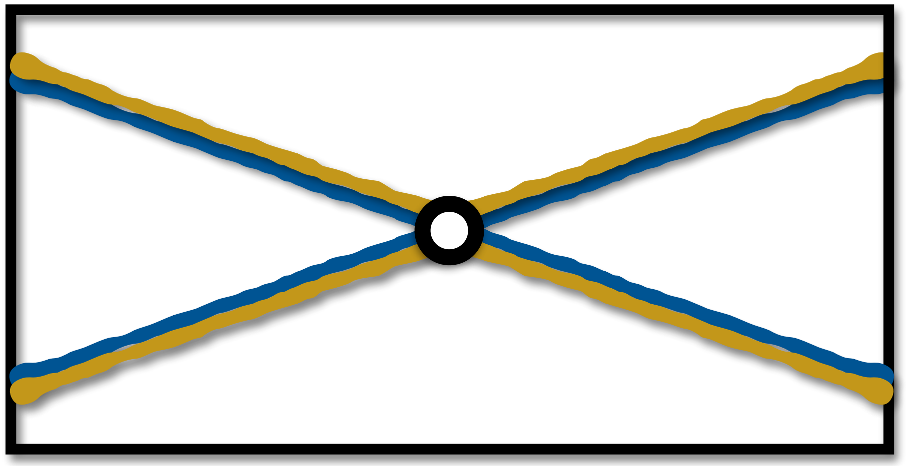
multi-particle in- and outstates

What can we extract from the lattice?

Not possible to directly calculate







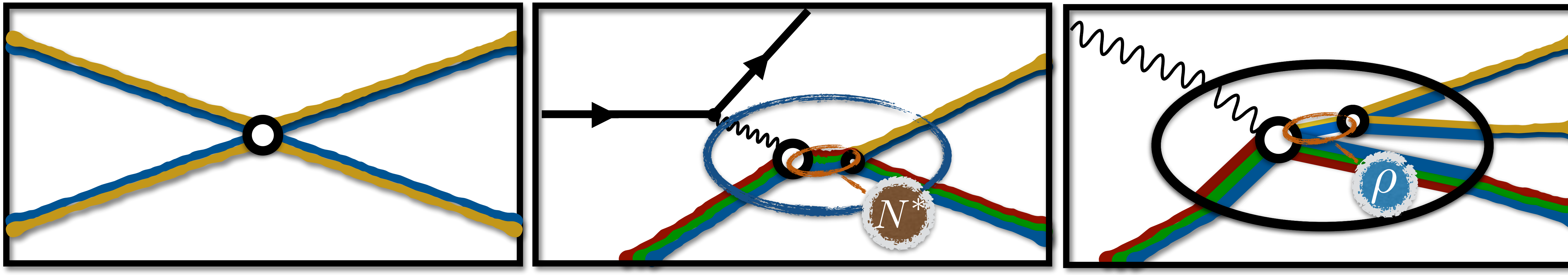
$\langle \pi\pi | \pi\pi \rangle$ $\langle N\pi | \mathcal{J}_\mu | N \rangle$ $\langle N\pi\pi | \mathcal{J}_\mu | N \rangle$

multi-particle in- and outstates

$$\langle \pi\pi, \mathrm{out} | \pi\pi, \mathrm{in} \rangle = \underset{\text{amputate and put on-shell}}{\langle 0 | \tilde{\pi}(p')\tilde{\pi}(k')\ \tilde{\pi}(p)\tilde{\pi}(k) | 0 \rangle}$$

What can we extract from the lattice?

Not possible to directly calculate



$\langle \pi\pi | \pi\pi \rangle$ $\qquad$ $\langle N\pi | \mathcal{J}_\mu | N \rangle$ $\qquad$ $\langle N\pi\pi | \mathcal{J}_\mu | N \rangle$

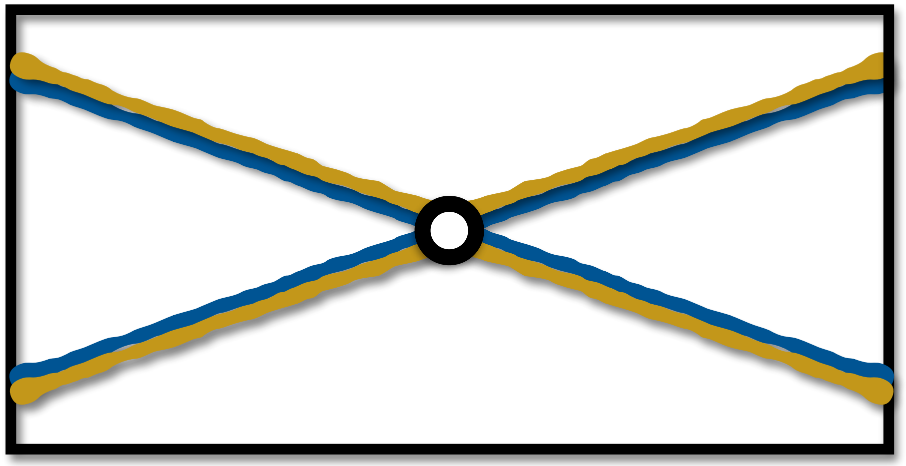
multi-particle in- and outstates

$$\langle \pi\pi, \mathrm{out} | \pi\pi, \mathrm{in} \rangle = \underset{\textbf{amputate and put on-shell}}{} \langle 0 | \tilde{\pi}(p')\tilde{\pi}(k') \; \tilde{\pi}(p)\tilde{\pi}(k) | 0 \rangle$$

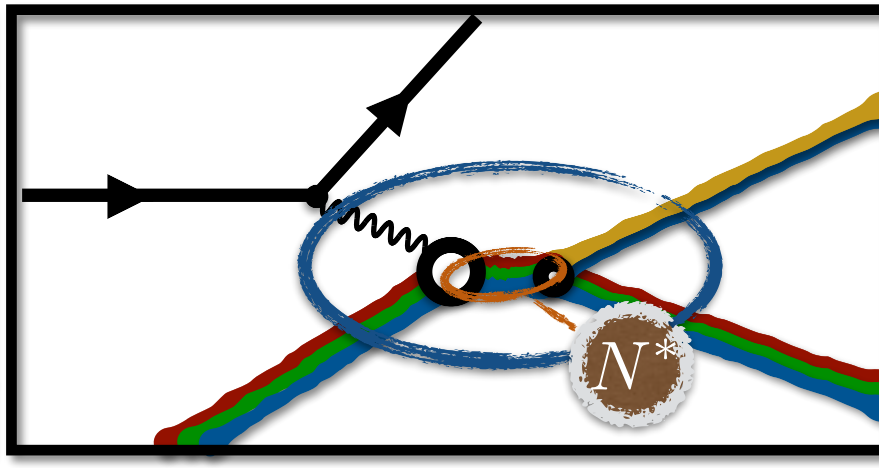
$$\langle N\pi\pi, \mathrm{out} | \mathcal{J}_\mu(x) | N \rangle = \underset{\textbf{amputate and put on-shell}}{} \langle 0 | \tilde{N}(p_1')\tilde{\pi}(p_2')\tilde{\pi}(p_3') \; \mathcal{J}_\mu(x) \; \tilde{N}(P) | 0 \rangle$$

What can we extract from the lattice?

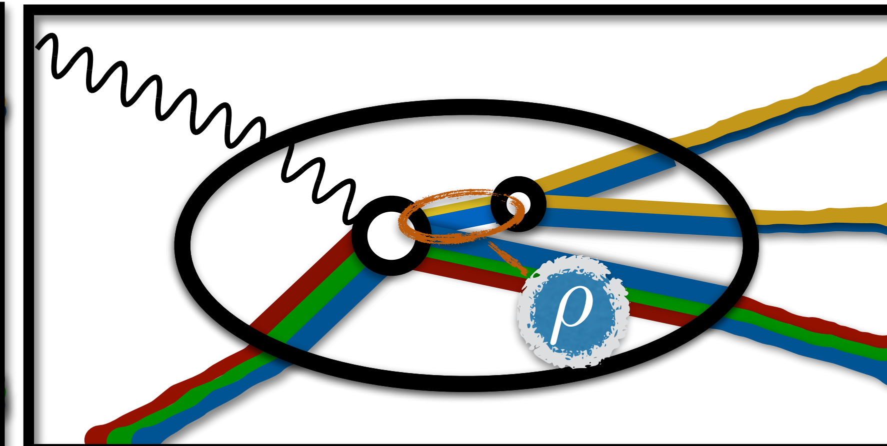
Not possible to directly calculate



$\langle \pi\pi | \pi\pi \rangle$



$\langle N\pi | \mathcal{J}_\mu | N \rangle$



$\langle N\pi\pi | \mathcal{J}_\mu | N \rangle$

multi-particle in- and outstates

$$\langle \pi\pi, \mathrm{out} | \pi\pi, \mathrm{in} \rangle = \underset{\textbf{amputate and put on-shell}}{} \langle 0 | \tilde{\pi}(p')\tilde{\pi}(k') \; \tilde{\pi}(p)\tilde{\pi}(k) | 0 \rangle$$

$$\langle N\pi\pi, \mathrm{out} | \mathcal{J}_\mu(x) | N \rangle = \underset{\textbf{amputate and put on-shell}}{} \langle 0 | \tilde{N}(p_1')\tilde{\pi}(p_2')\tilde{\pi}(p_3') \; \mathcal{J}_\mu(x) \; \tilde{N}(P) | 0 \rangle$$

Requires Minkowski momenta and infinite volume

What can we extract from the lattice?

Instead we can only access

$$H_{\rm QCD}|n,L\rangle = |n,L\rangle E_n(L) \qquad \langle n, L, \text{“}N\pi\pi\text{”}|\mathcal{J}_\mu(x)|\text{“}N\text{”}, L\rangle$$

finite-volume energies and matrix elements

labels in quotes indicate quantum numbers

What can we extract from the lattice?

Instead we can only access

$$H_{\rm QCD}|n,L\rangle = |n,L\rangle E_n(L) \qquad \langle n, L, \text{“}N\pi\pi\text{”}|\mathcal{J}_\mu(x)|\text{“}N\text{”}, L\rangle$$

finite-volume energies and matrix elements

labels in quotes indicate quantum numbers

How can we determine

$\langle \pi\pi, \text{out}|\pi\pi, \text{in}\rangle$ and $\langle N\pi\pi, \text{out}|\mathcal{J}_\mu(x)|N\rangle$

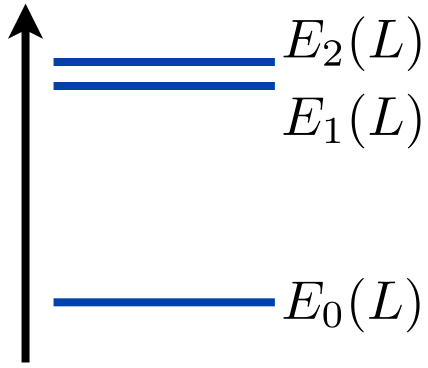
from

$E_n(L)$ and $\langle n, L, \text{“}N\pi\pi\text{”}|\mathcal{J}_\mu(x)|\text{“}N\text{”}, L\rangle$?

It is possible to derive relations between finite- and infinite-volume physics

It is possible to derive relations between finite- and infinite-volume physics

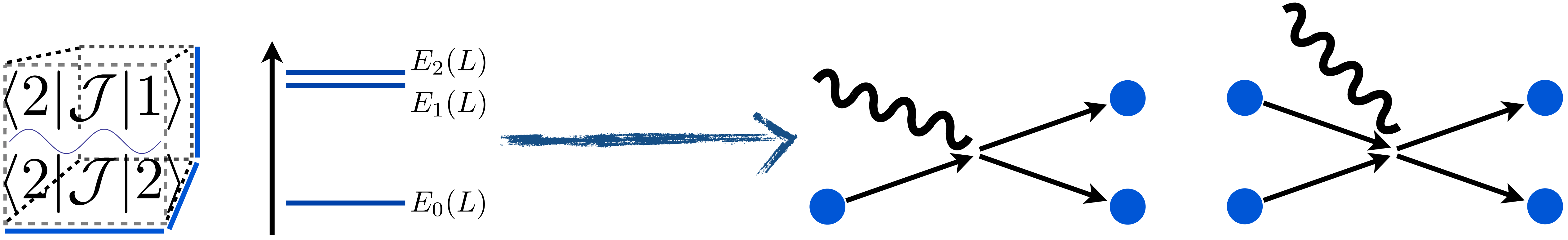
Two-particle scattering



It is possible to derive relations between finite- and infinite-volume physics

Two-particle scattering

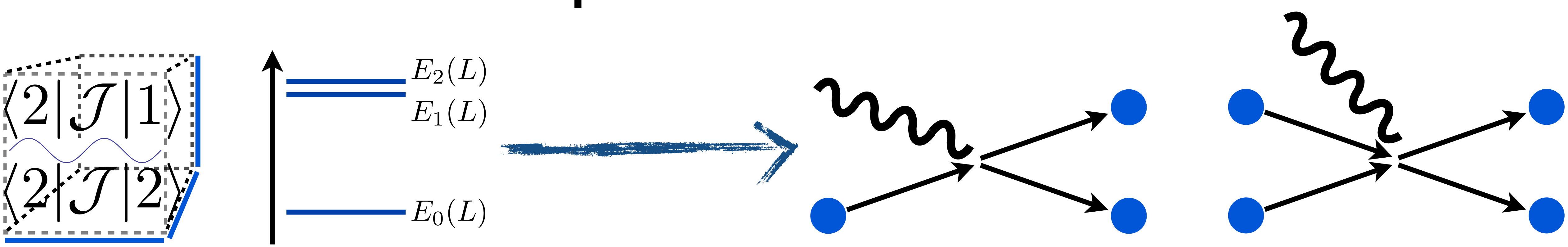
Photo- and electroproduction



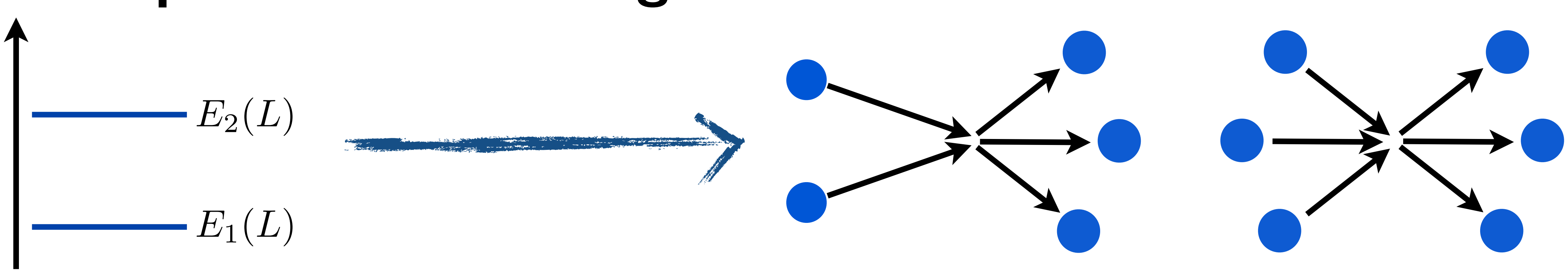
It is possible to derive relations between finite- and infinite-volume physics

Two-particle scattering

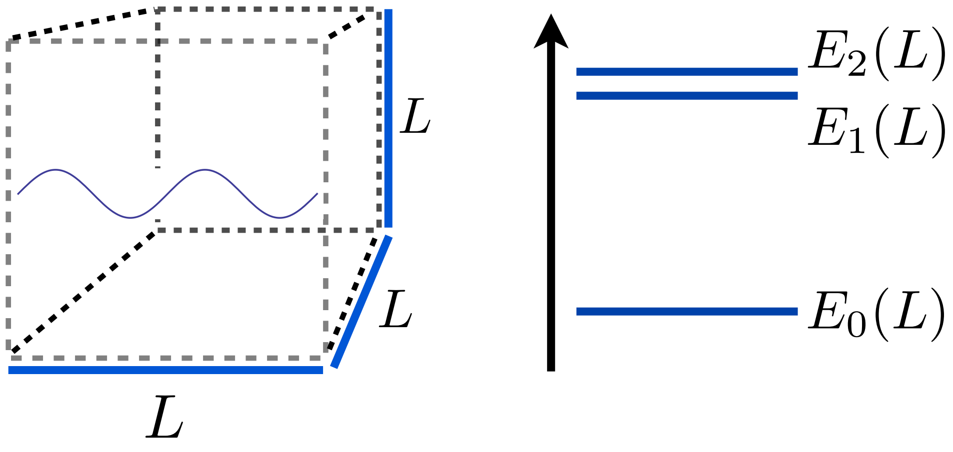
Photo- and electroproduction



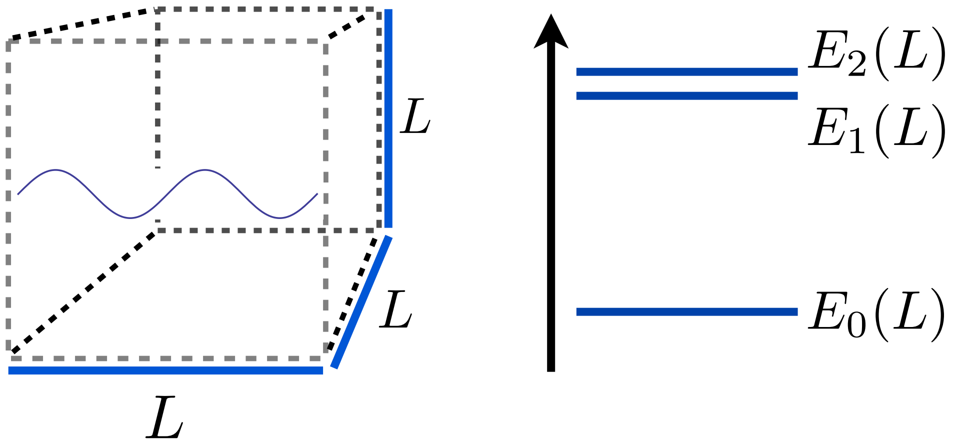
Three-particle scattering



Finite volume



Finite volume



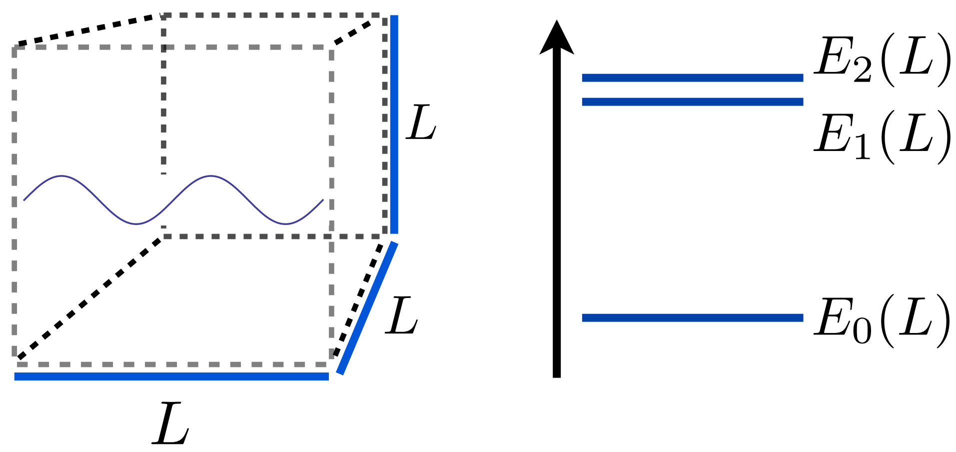
cubic, spatial volume (extent L)

periodic boundary conditions

$$\vec{p} \in (2\pi/L)\mathbb{Z}^3$$

time direction **infinite**

Finite volume



cubic, spatial volume (extent L)

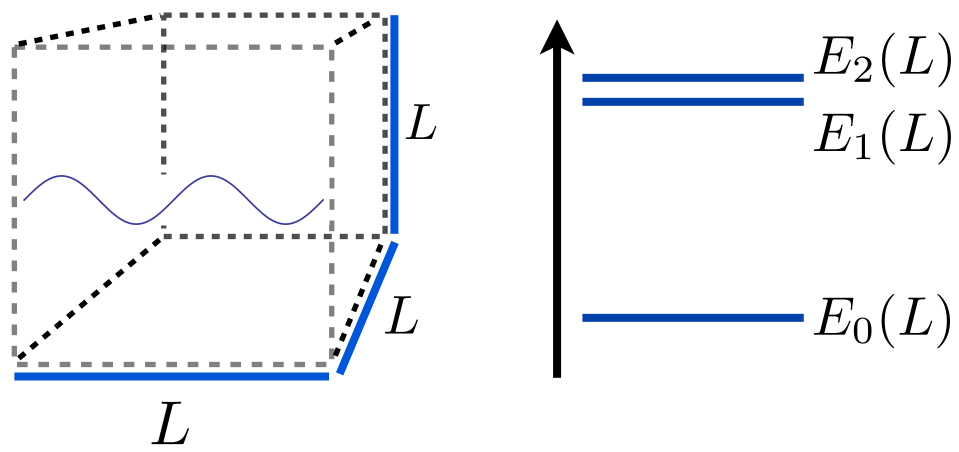
periodic boundary conditions

$$\vec{p} \in (2\pi/L)\mathbb{Z}^3$$

time direction **infinite**

L large enough to ignore e^{-mL}

Finite volume



cubic, spatial volume (extent L)

periodic boundary conditions

$$\vec{p} \in (2\pi/L)\mathbb{Z}^3$$

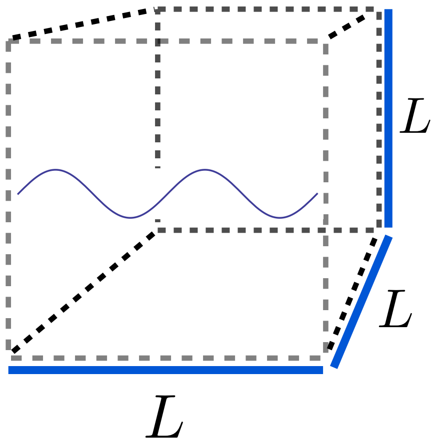
time direction **infinite**

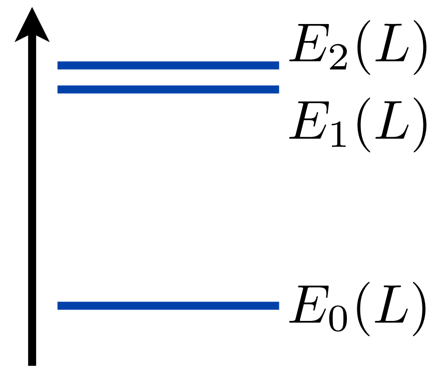
L large enough to ignore e^{-mL}

Assume lattice effects are small and accommodated elsewhere

Work in continuum field theory throughout

Finite volume





cubic, spatial volume (extent L)

periodic boundary conditions

$$\vec{p} \in (2\pi/L)\mathbb{Z}^3$$

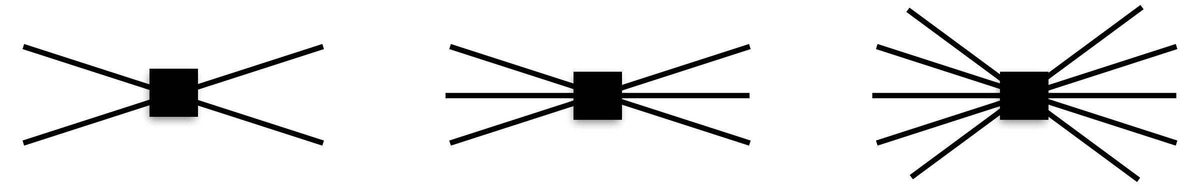
time direction **infinite**

L large enough to ignore e^{-mL}

Quantum field theory

generic relativistic QFT

1. Include all interactions

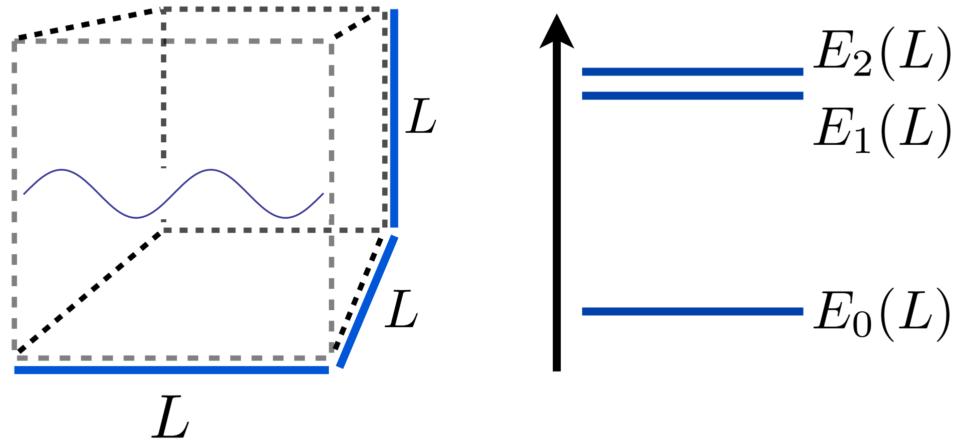


2. no power-counting scheme

Assume lattice effects are small and accommodated elsewhere

Work in continuum field theory throughout

Finite volume



cubic, spatial volume (extent L)

periodic boundary conditions

$$\vec{p} \in (2\pi/L)\mathbb{Z}^3$$

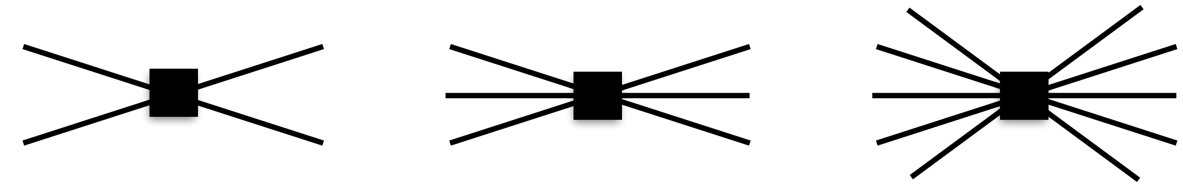
time direction **infinite**

L large enough to ignore e^{-mL}

Quantum field theory

generic relativistic QFT

1. Include all interactions



2. no power-counting scheme

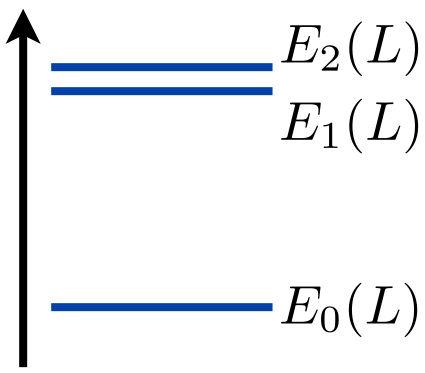
Not possible to directly calculate scattering observables to all orders

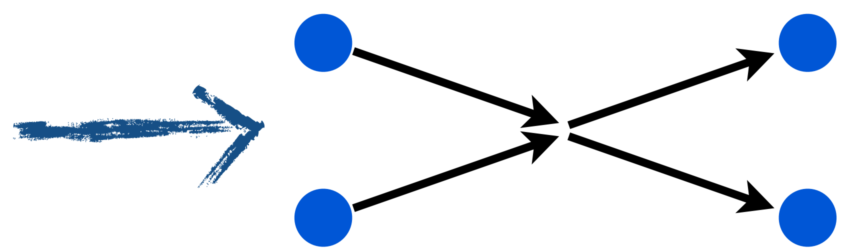
But it is possible to derive *general, all-orders relations* to finite-volume quantities

Assume lattice effects are small and accommodated elsewhere

Work in continuum field theory throughout

Two-to-two scattering

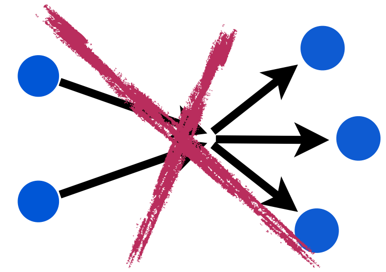




For now assume…

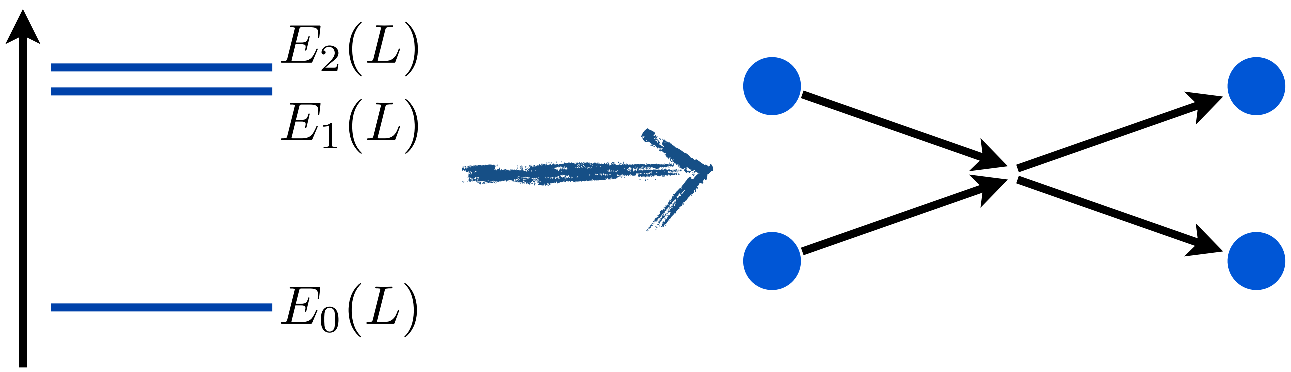
identical scalars, mass m

$\mathbb{Z}_2$ **symmetry**



Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)
Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys*. B727, 218-243 (2005)

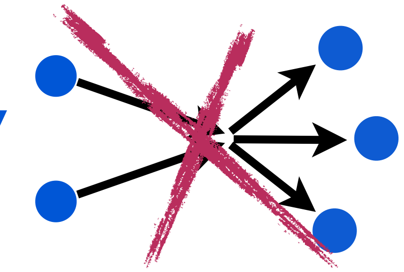
Two-to-two scattering



For now assume…

identical scalars, mass m

$\mathbb{Z}_2$ **symmetry**

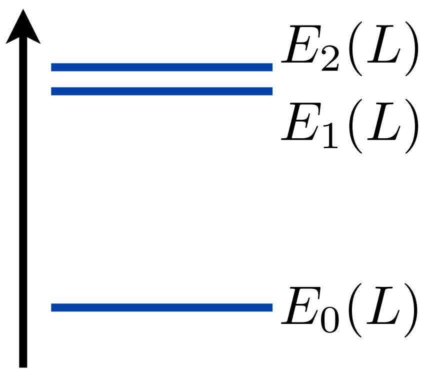


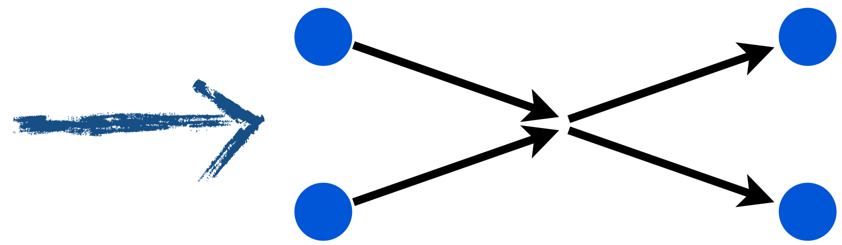
$$C_L(P) \equiv \int_L d^4x \; e^{-iPx} \langle 0|T\mathcal{O}(x)\mathcal{O}^\dagger(0)|0\rangle$$

two-particle interpolator

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)
Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

Two-to-two scattering

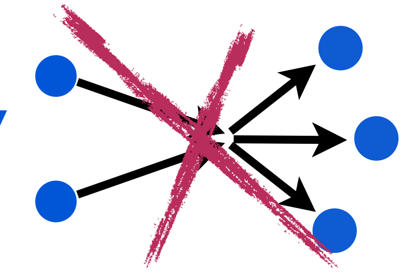




For now assume…

identical scalars, mass m

$\mathbb{Z}_2$ **symmetry**



$$C_L(P) \equiv \int_L d^4x \; e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

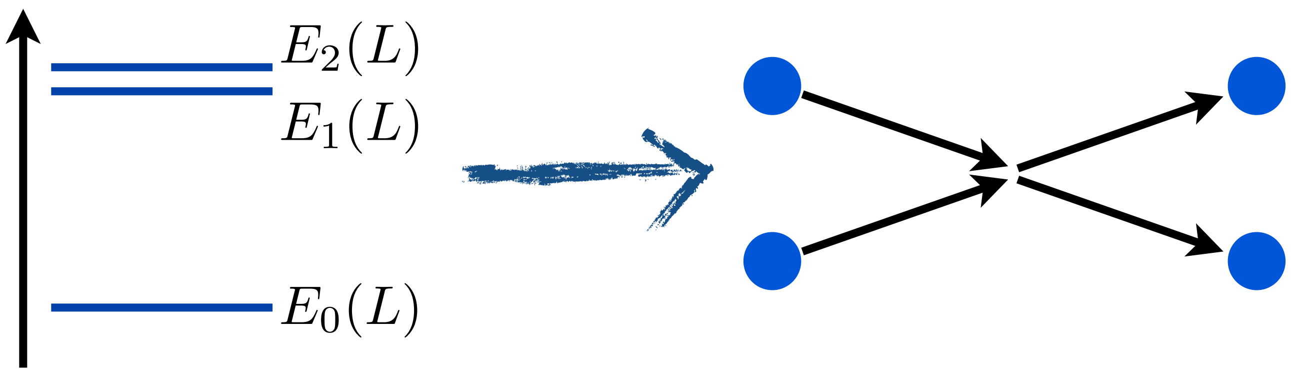
Euclidean convention **two-particle interpolator**

$$P = (P_4, \vec{P}) = (P_4, 2\pi\vec{n}/L)$$

but allow P_4 to be real or imaginary

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)
Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

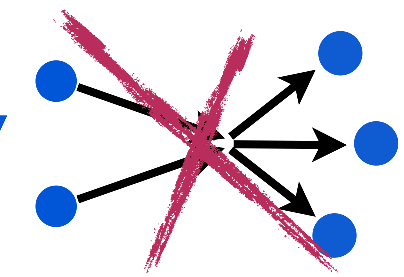
Two-to-two scattering



For now assume…

identical scalars, mass m

$\mathbb{Z}_2$ **symmetry**



$$C_L(P) \equiv \int_L d^4x \; e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

Euclidean convention

two-particle interpolator

$$P = (P_4, \vec{P}) = (P_4, 2\pi\vec{n}/L)$$

but allow P_4 to be real or imaginary

CM frame energy is then $E^{*2} = -P_4^2 - \vec{P}^2$

Require $E^* < 4m$ to isolate two-to-two scattering

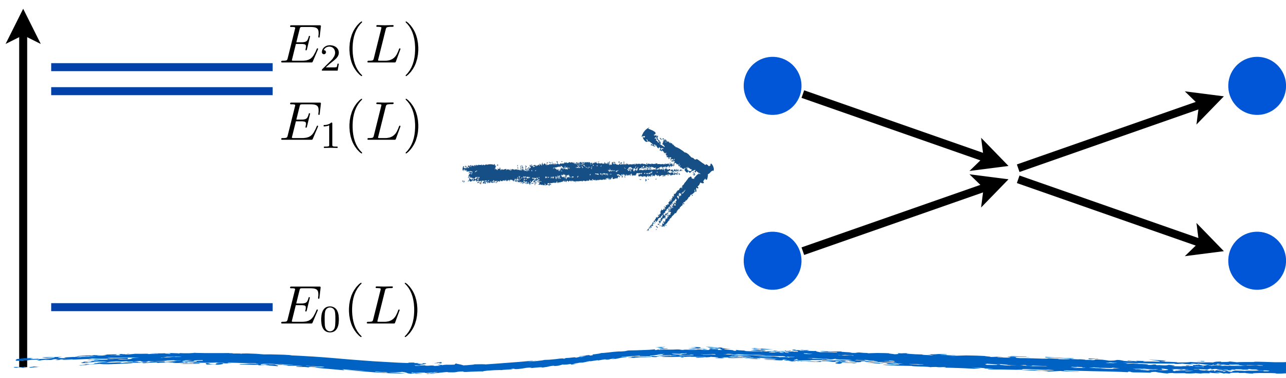
Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

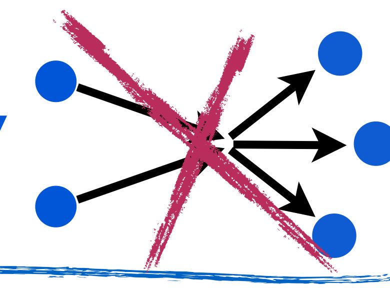
Two-to-two scattering

For now assume…

identical scalars, mass m

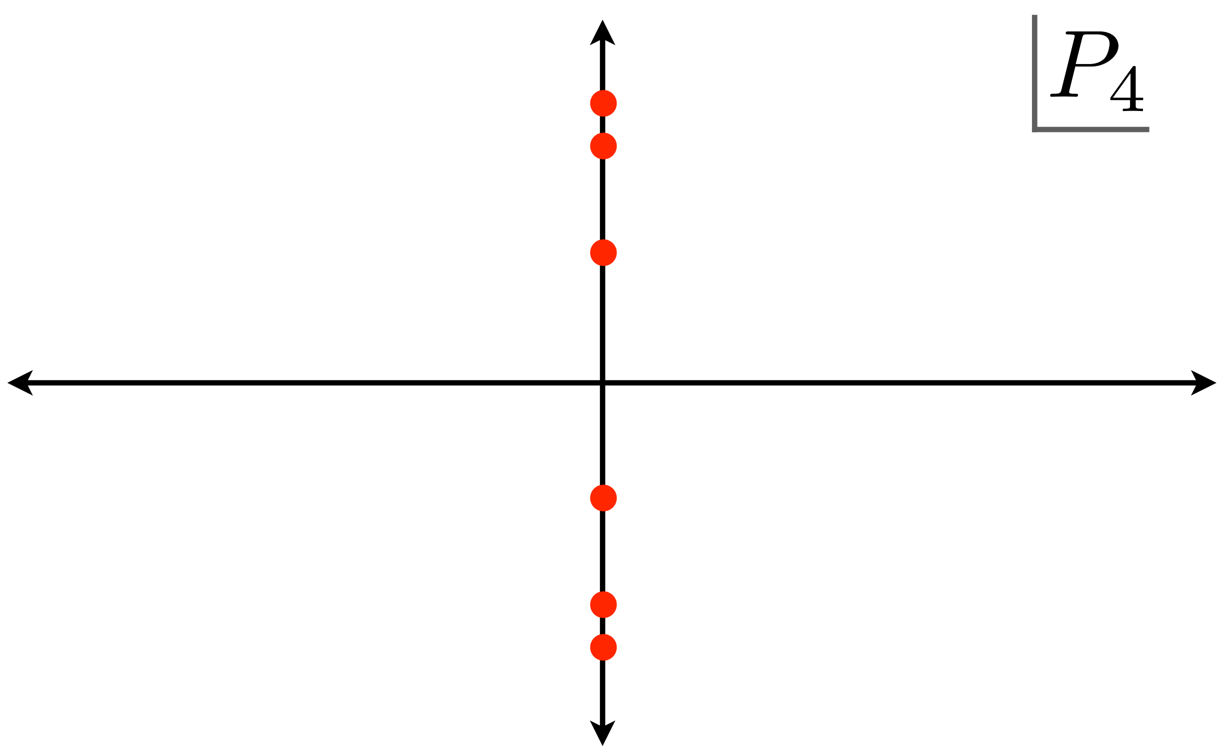


$\mathbb{Z}_2$ **symmetry**

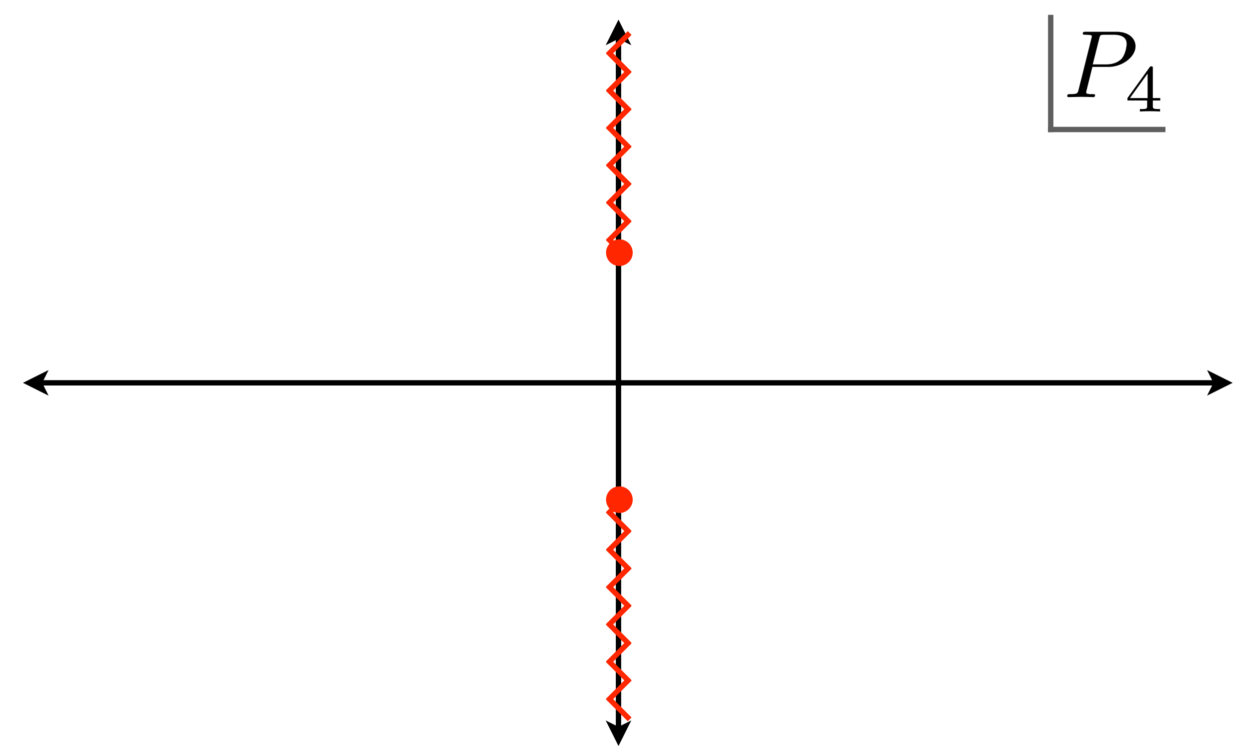


$$C_L(P) \equiv \int_L d^4x\, e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

At fixed $L, \vec{P}$, **poles in** C_L **give finite-volume spectrum**

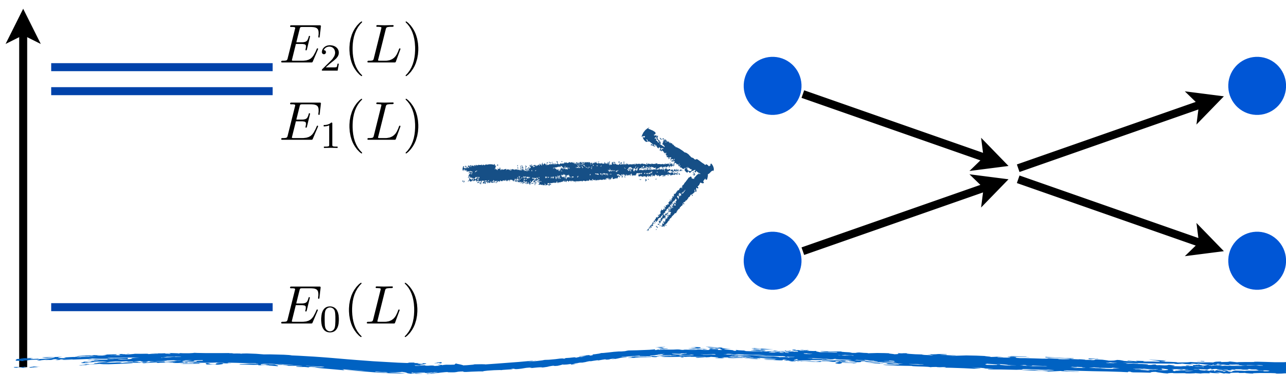


C_L **analytic structure**



C_∞ **analytic structure**

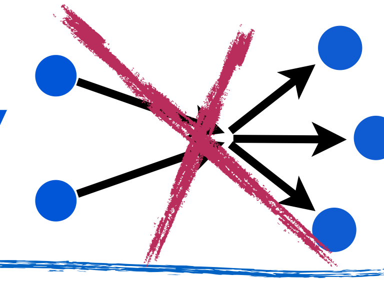
Two-to-two scattering



For now assume…

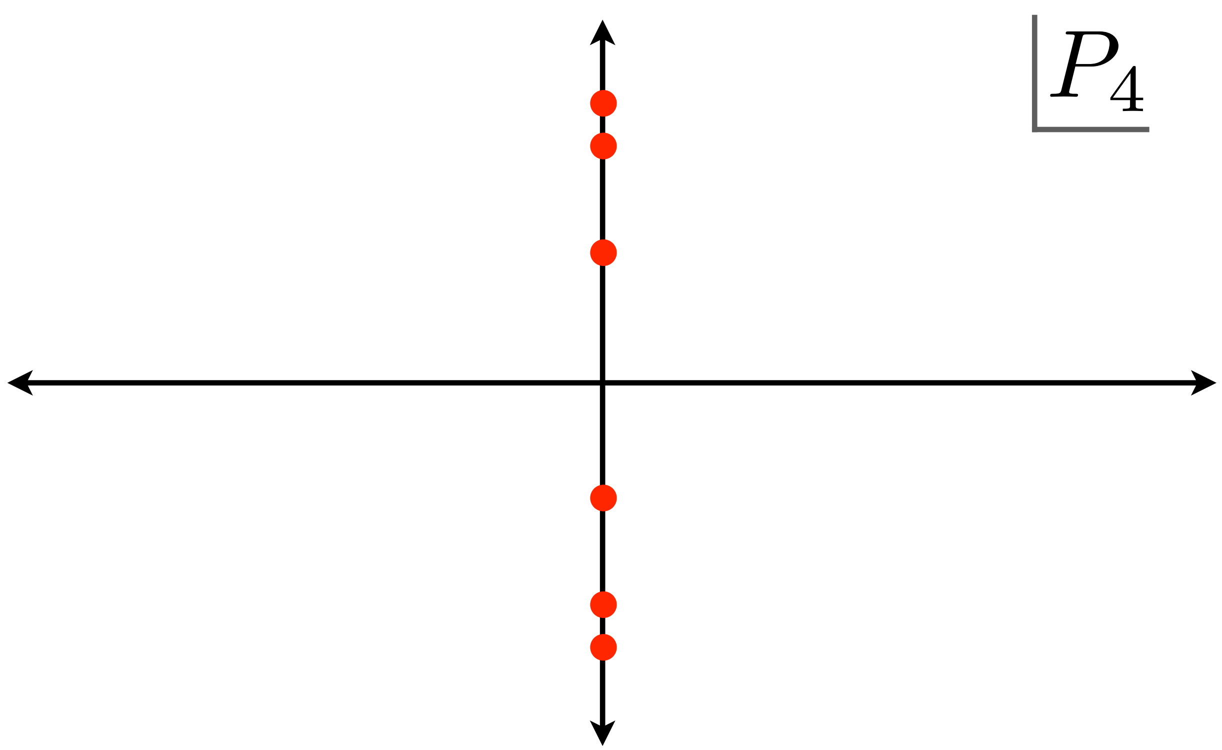
identical scalars, mass m

$\mathbb{Z}_2$ **symmetry**



$$C_L(P) \equiv \int_L d^4x \; e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

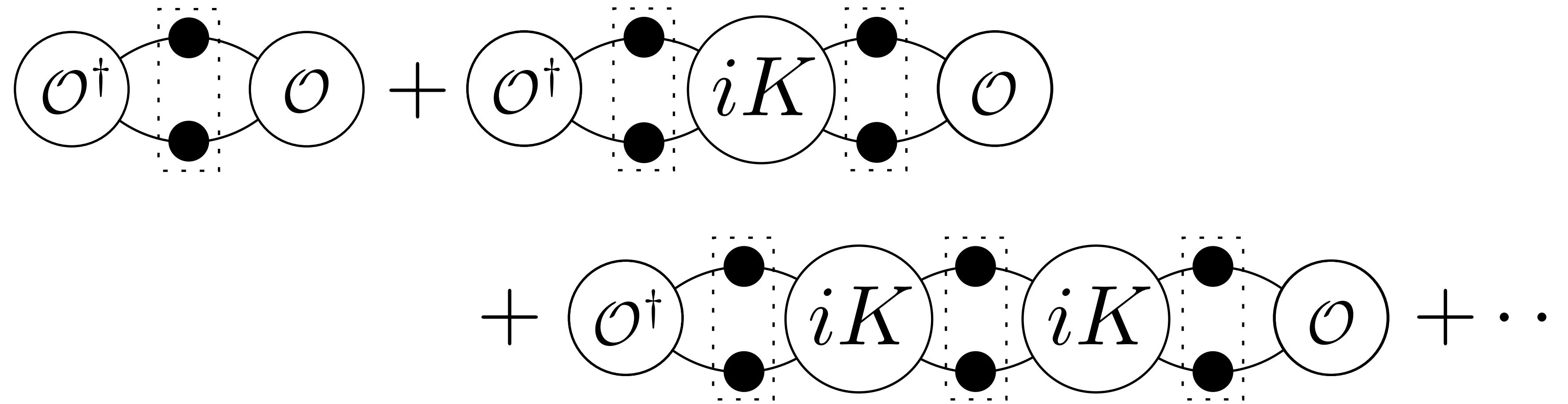
At fixed $L, \vec{P}$, **poles in** C_L **give finite-volume spectrum**



C_L **analytic structure**

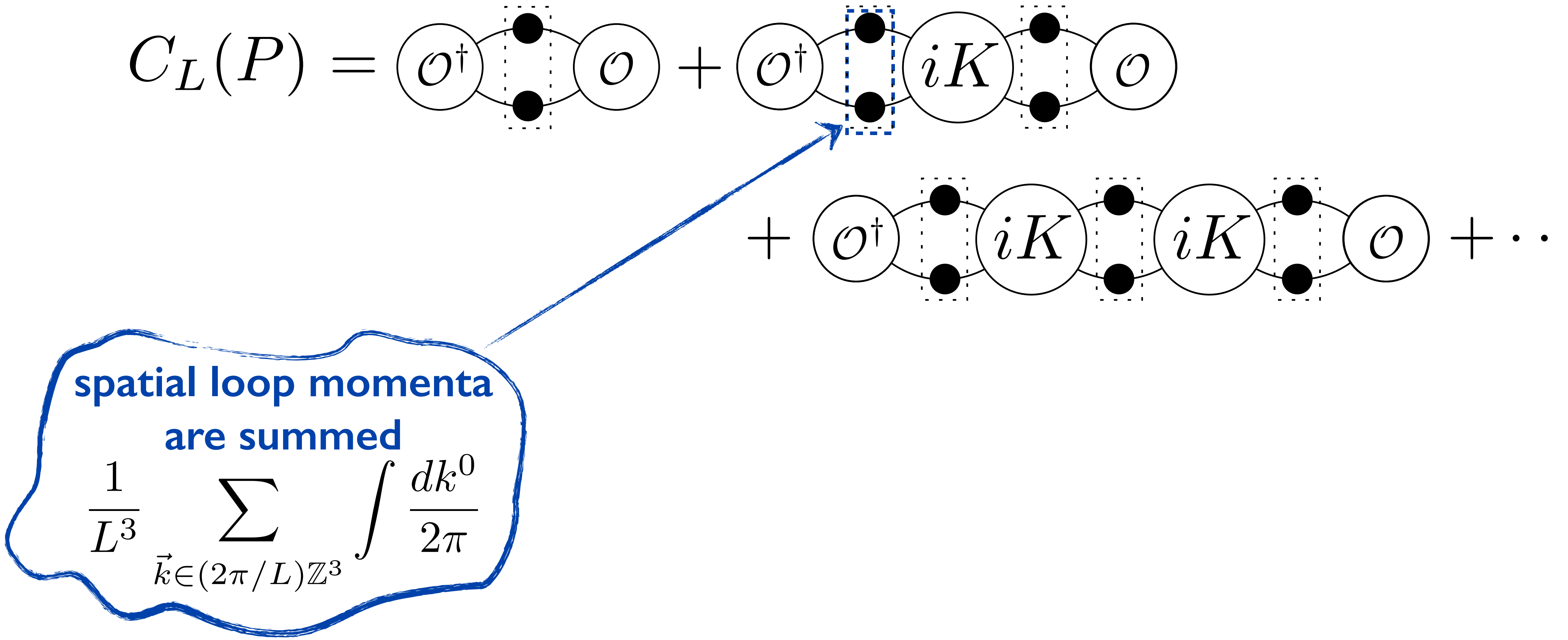
Calculate $C_L(P)$ to all orders in perturbation theory and determine locations of poles.

$C_L(P) =$



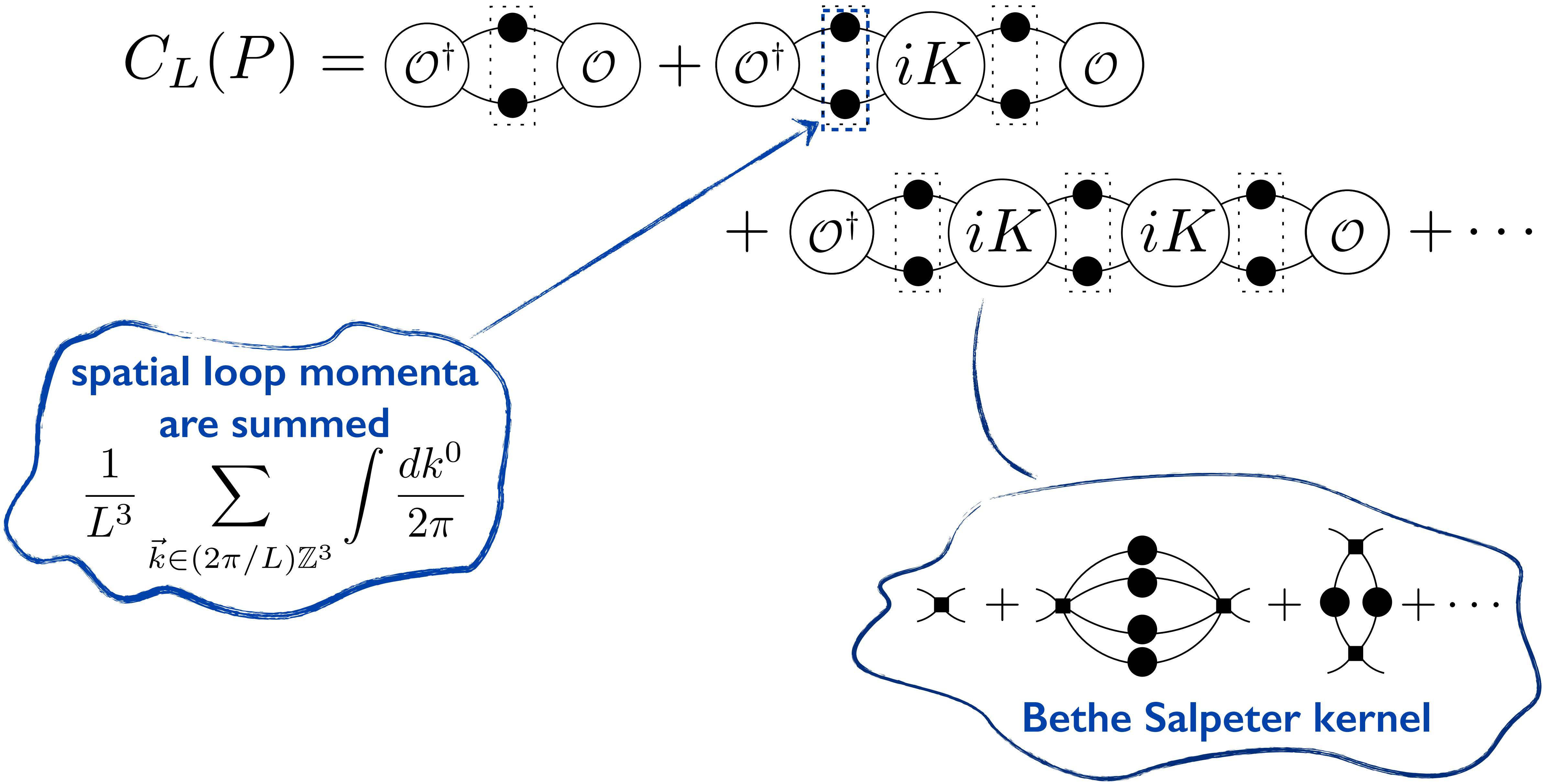
Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)



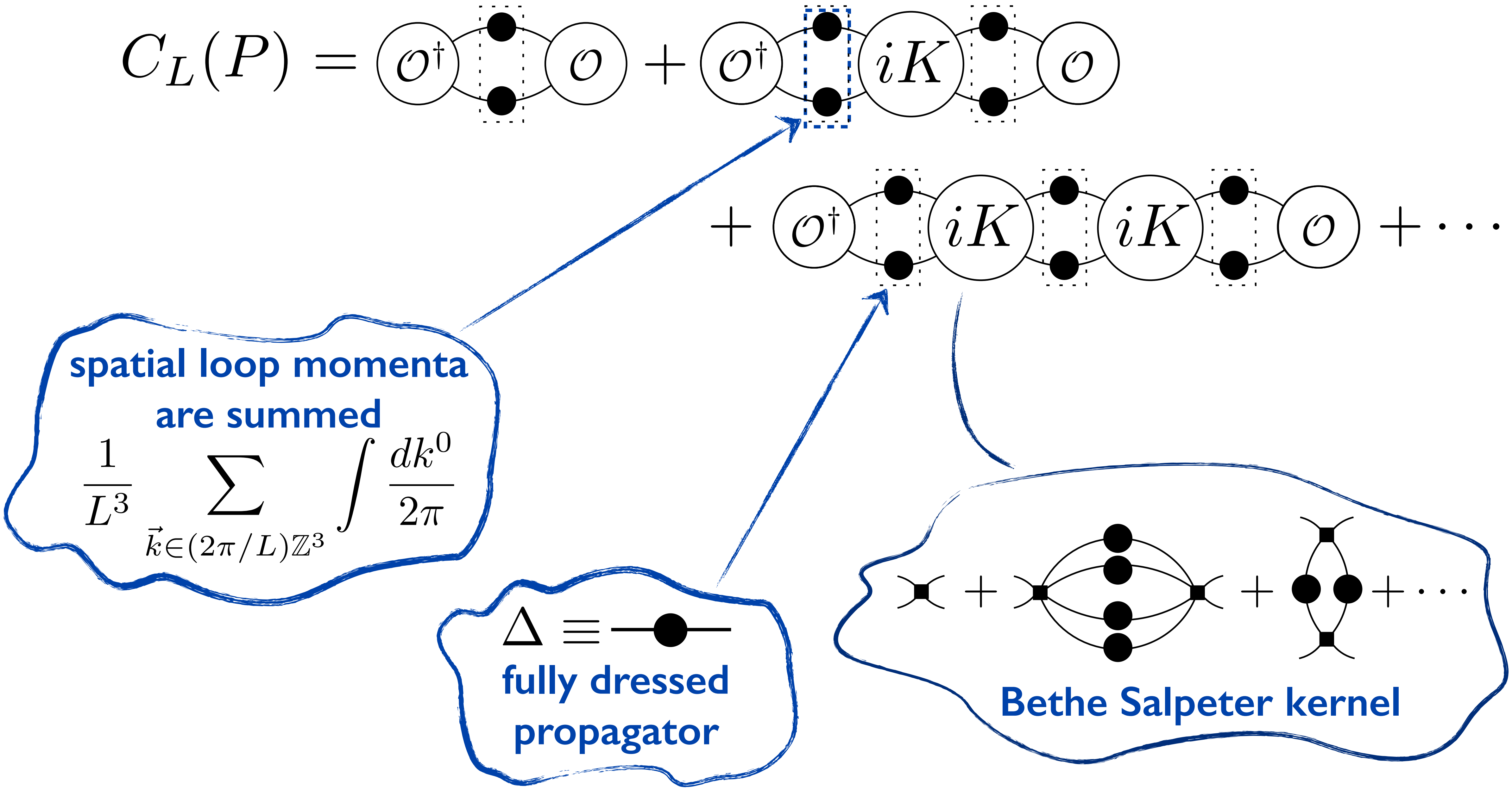
Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)



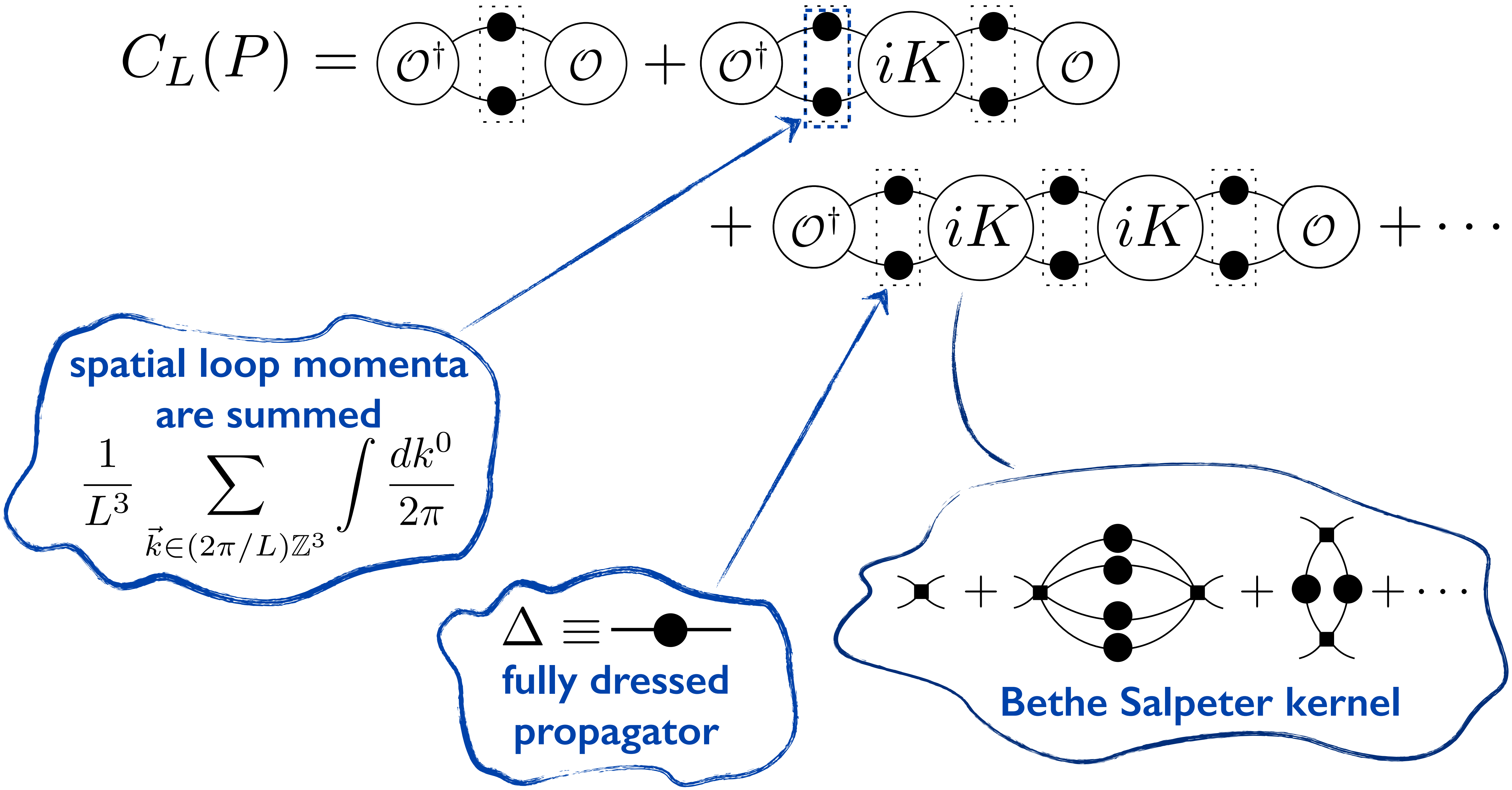
Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)



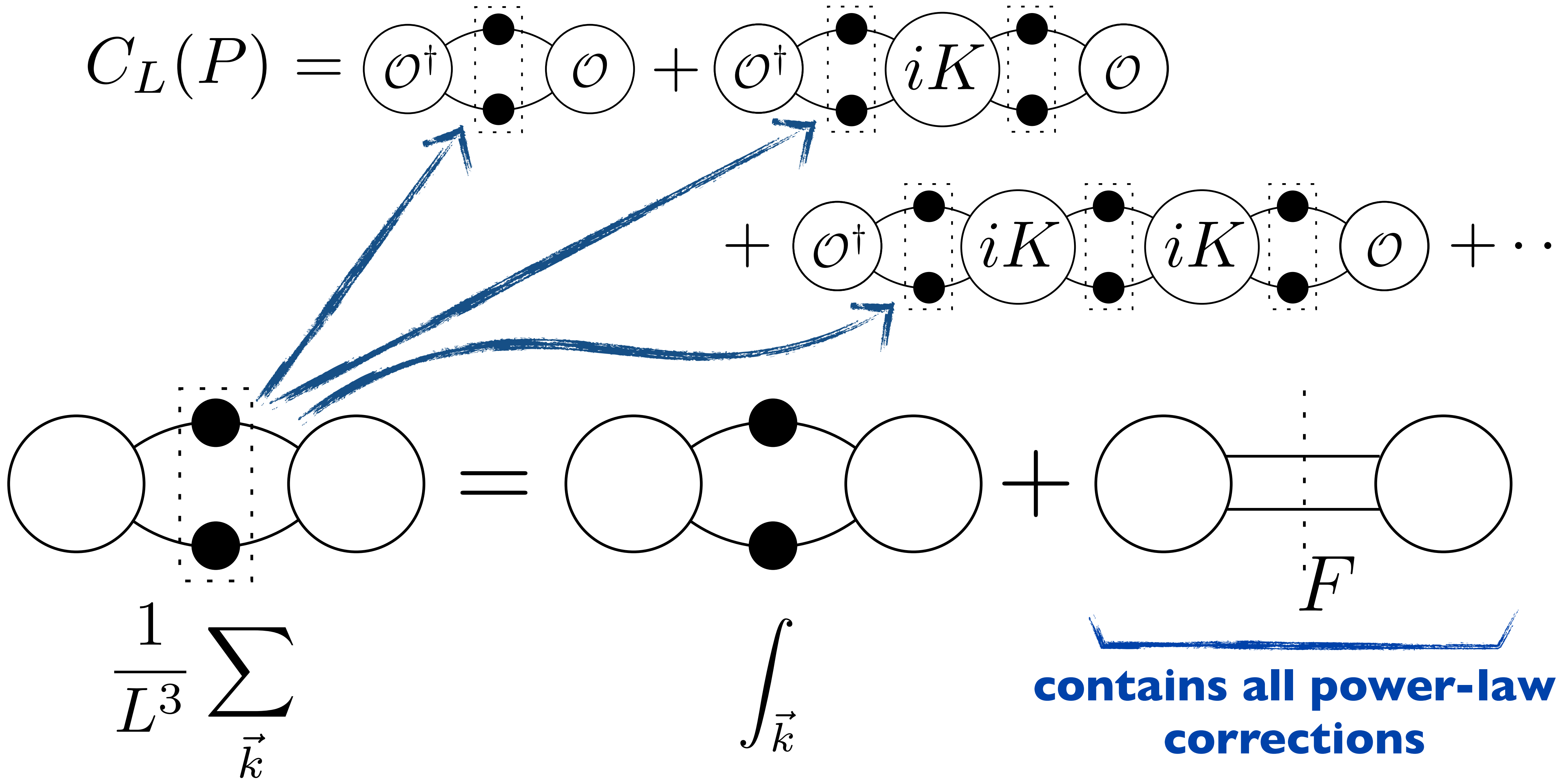
Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)



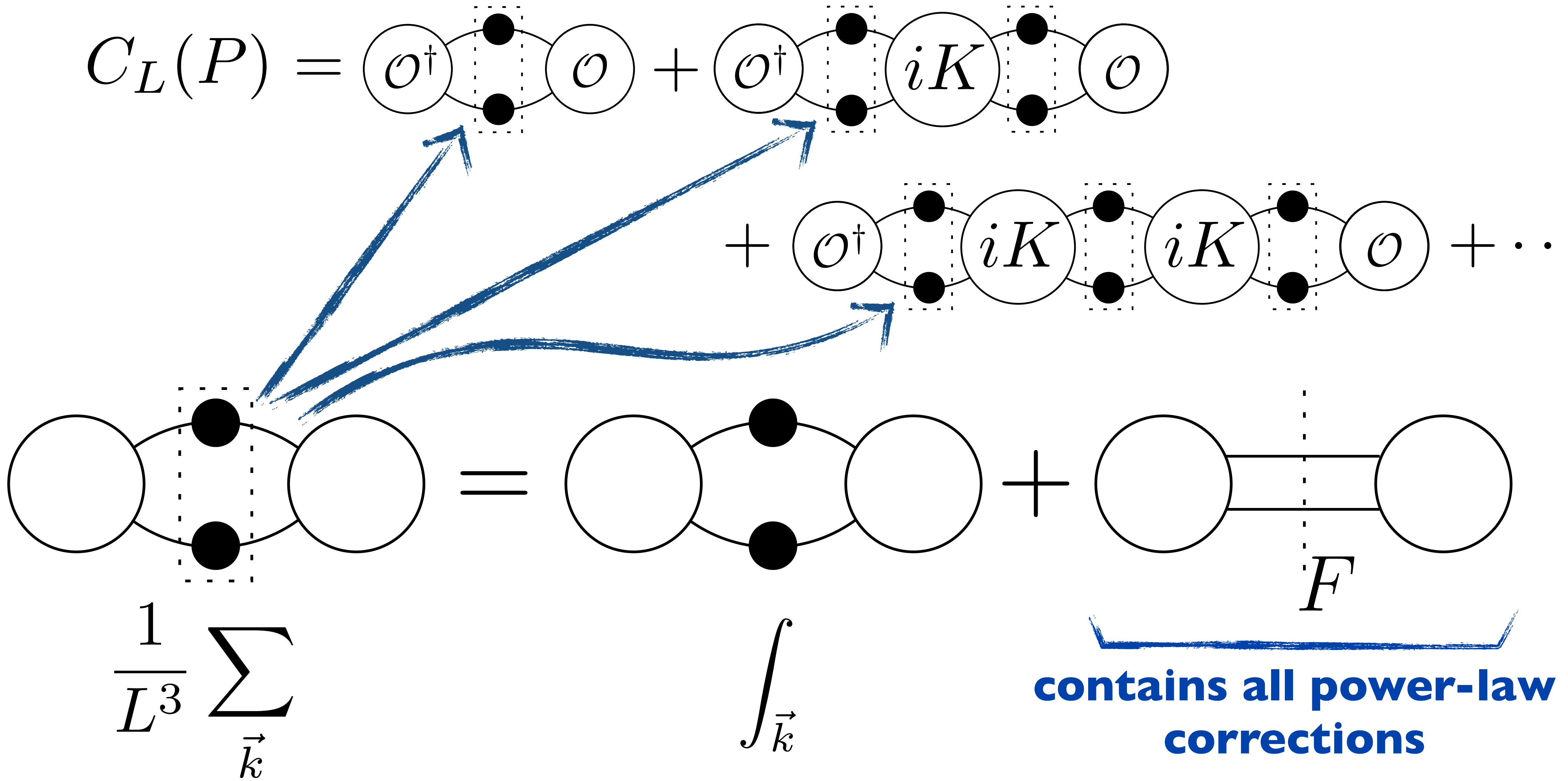
If $E^* < 4m$ **then**
$$K_L = K_\infty + \mathcal{O}(e^{-mL})$$
$$\Delta_L = \Delta_\infty + \mathcal{O}(e^{-mL})$$

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)
Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

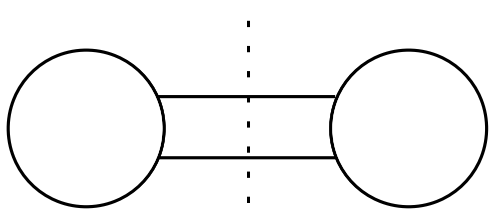


Now we introduce an important identity.

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)
Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys*. B727, 218-243 (2005)

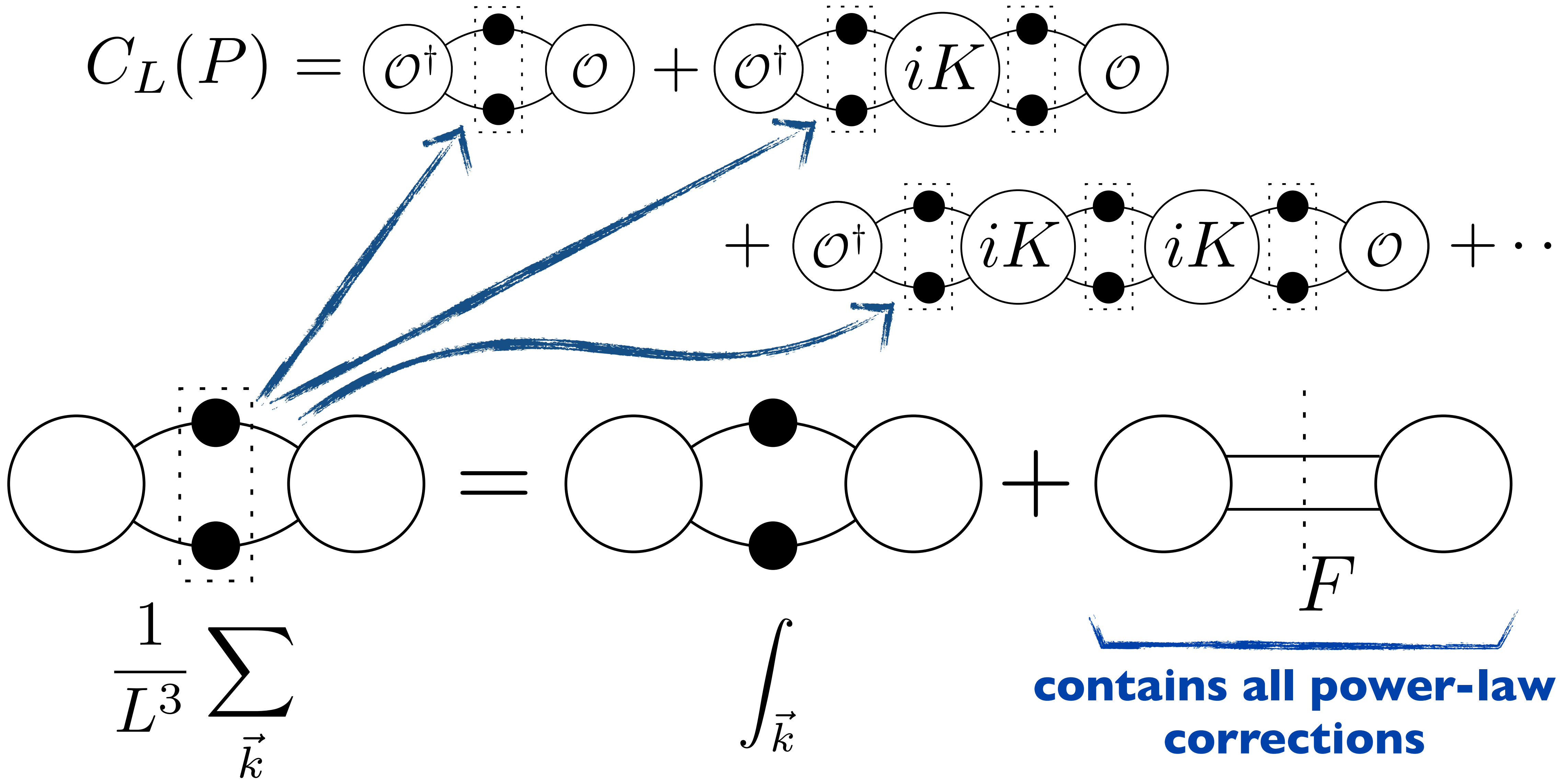


Now we introduce an important identity.

In  all four-momenta are projected on shell.

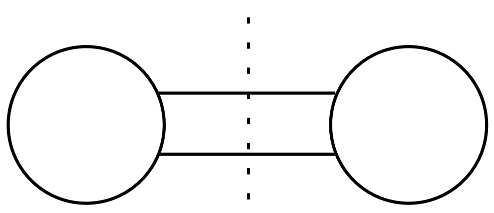
Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)



Now we introduce an important identity.

In

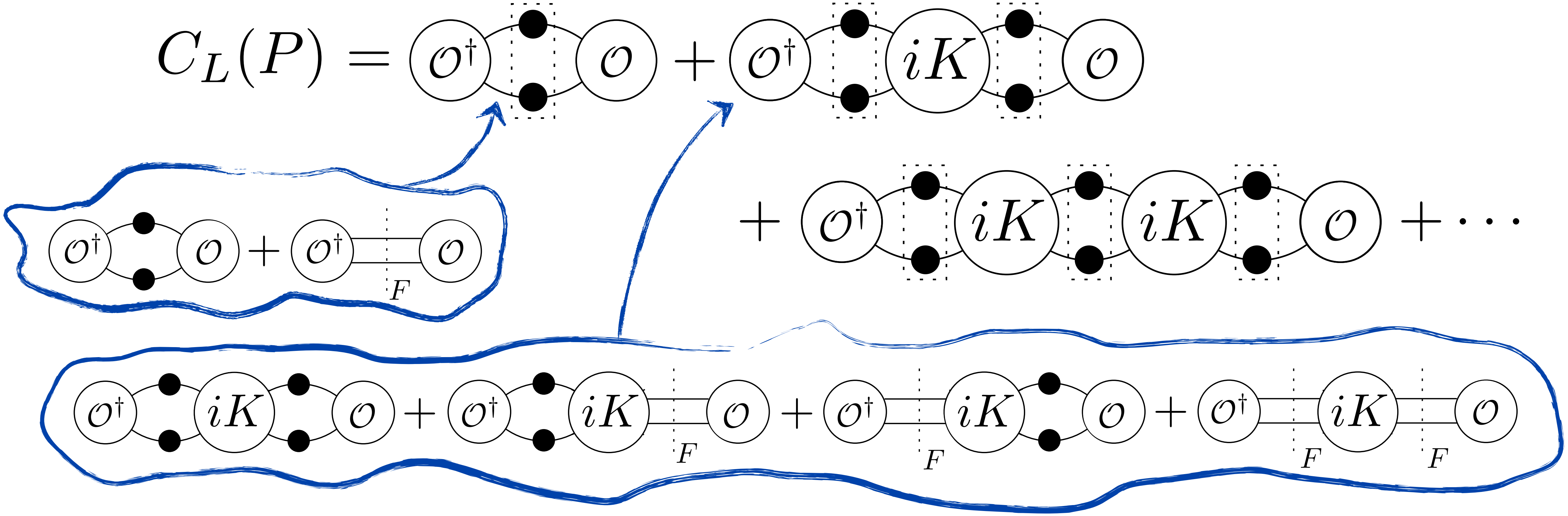


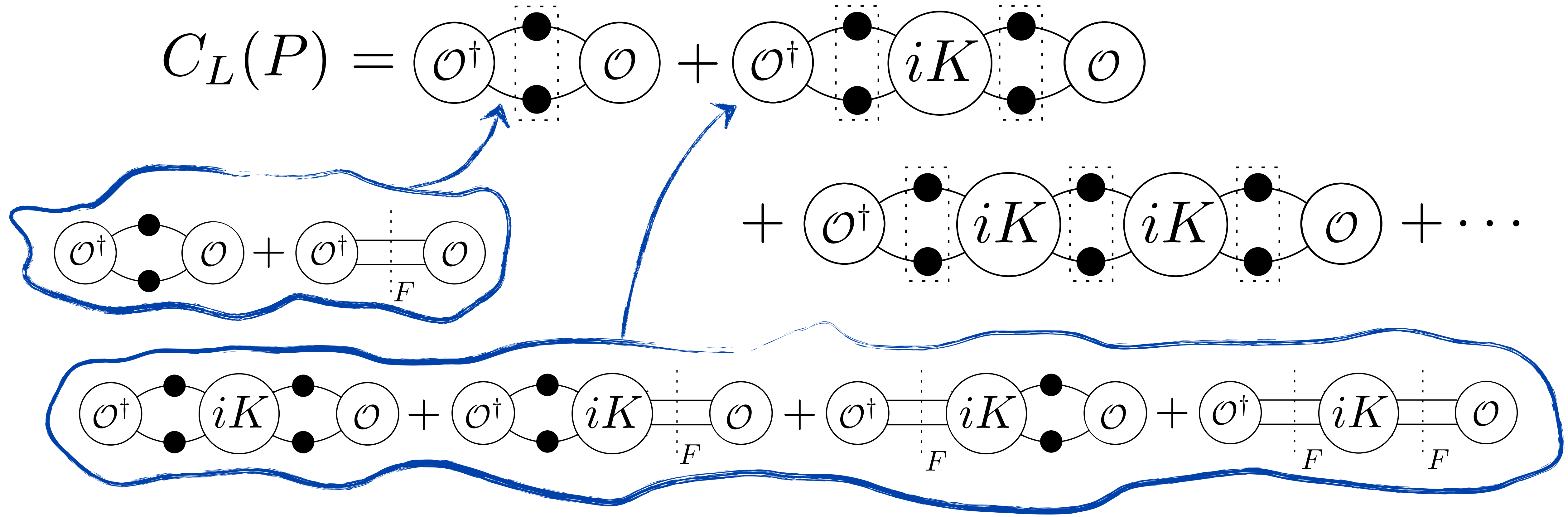
all four-momenta are projected **on shell.**

Physical, propagating states give dominate finite-volume effects.

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

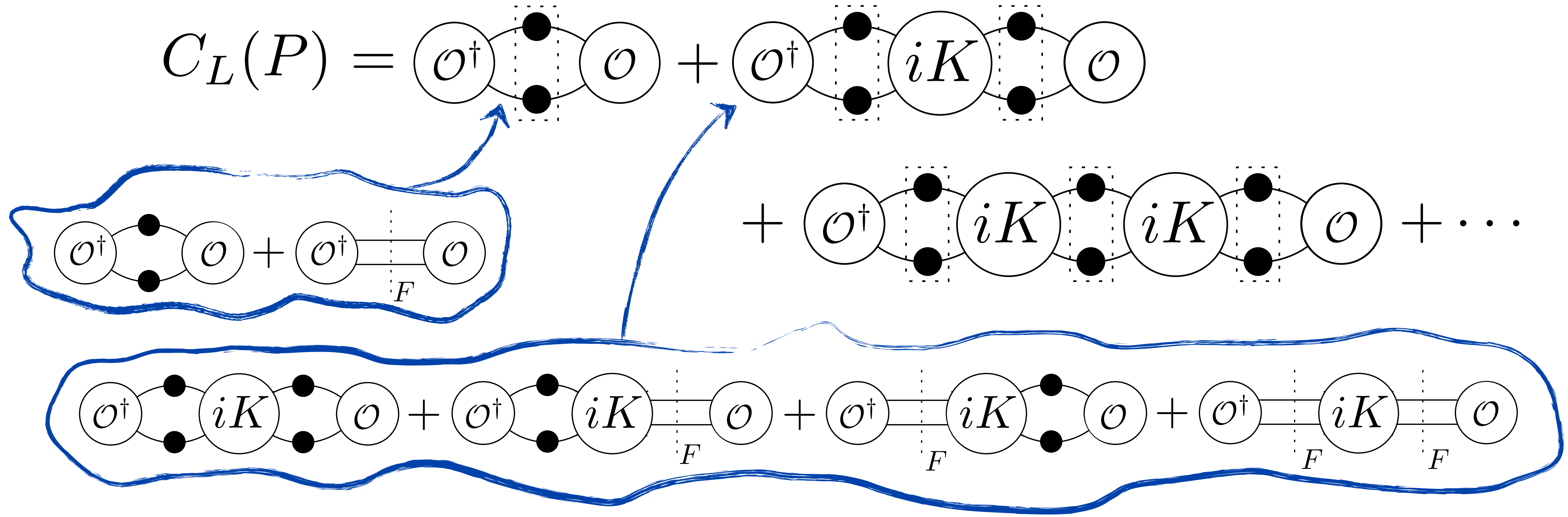

$C_L(P) =$
$\mathcal{O}^\dagger$
$\mathcal{O}$
iK
F
$+ \cdots$



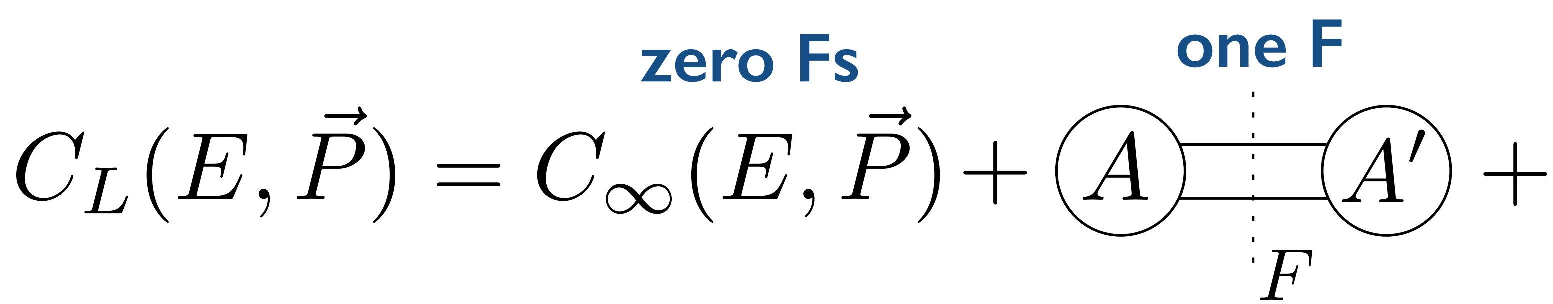
Now regroup by number of Fs

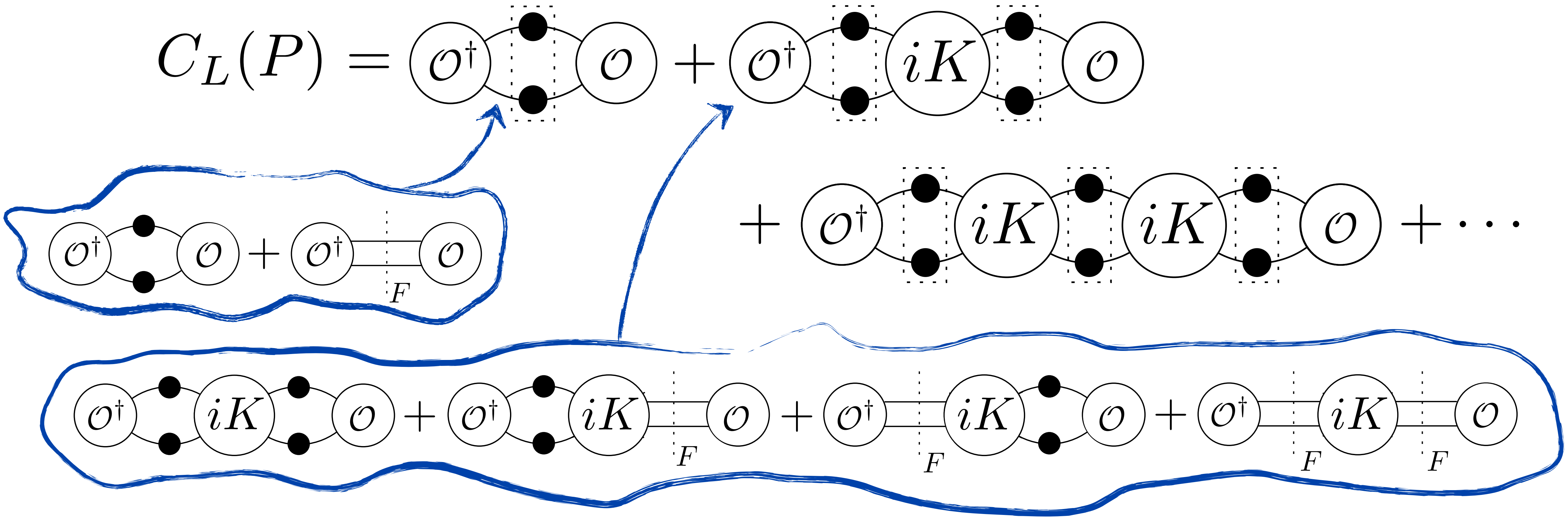
zero Fs

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) +$$

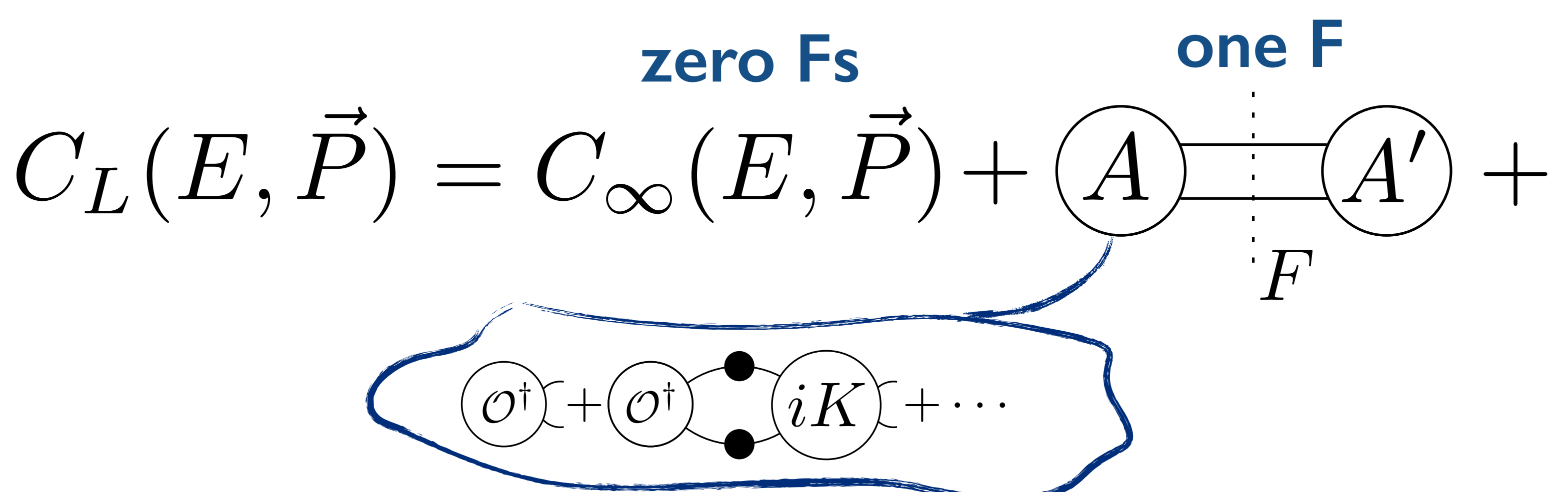


Now regroup by number of Fs

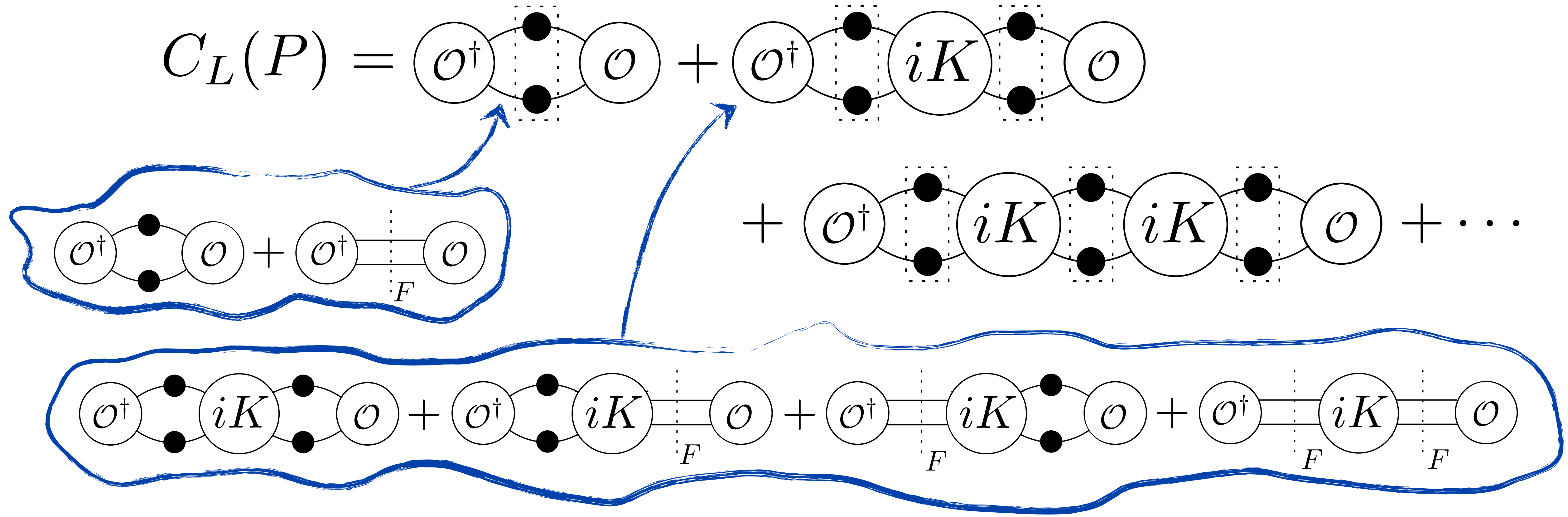




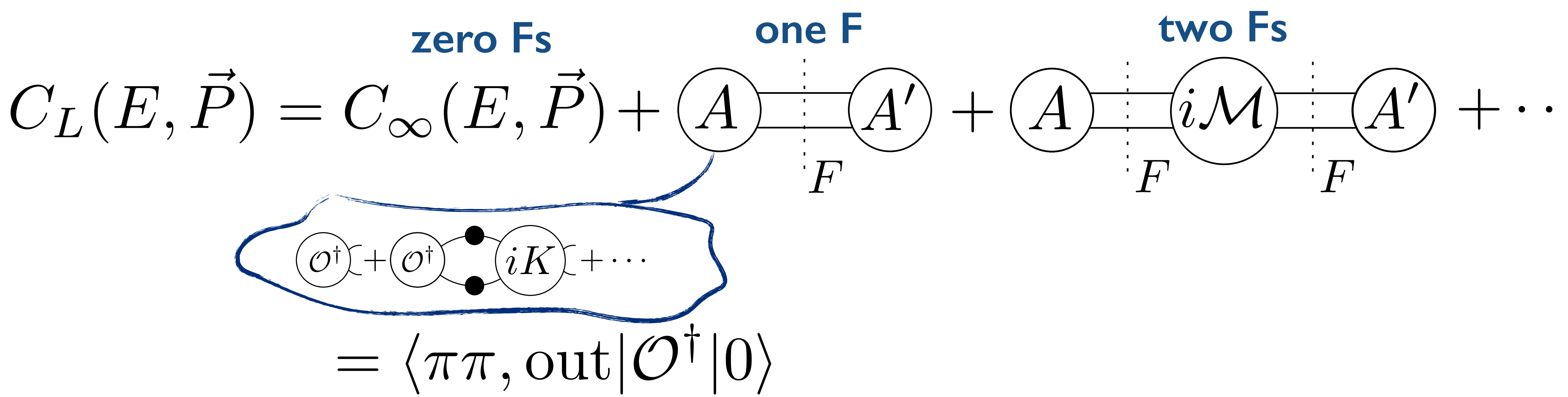
Now regroup by number of Fs

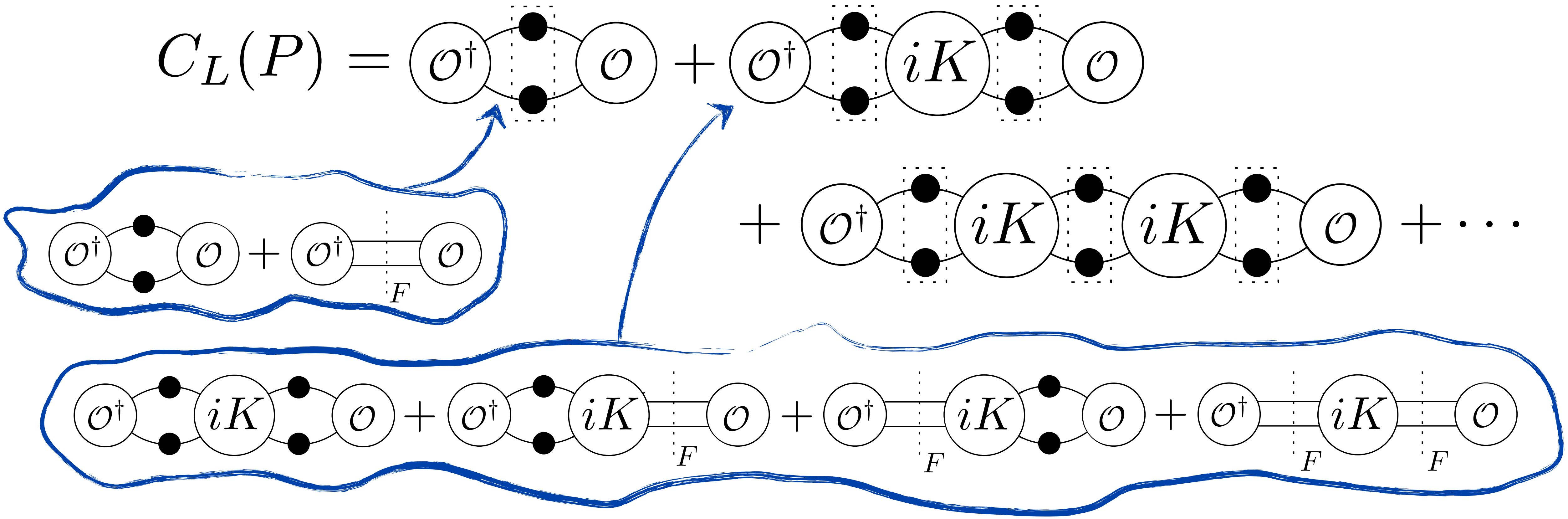


$$= \langle \pi\pi, \mathrm{out} | \mathcal{O}^\dagger | 0 \rangle$$

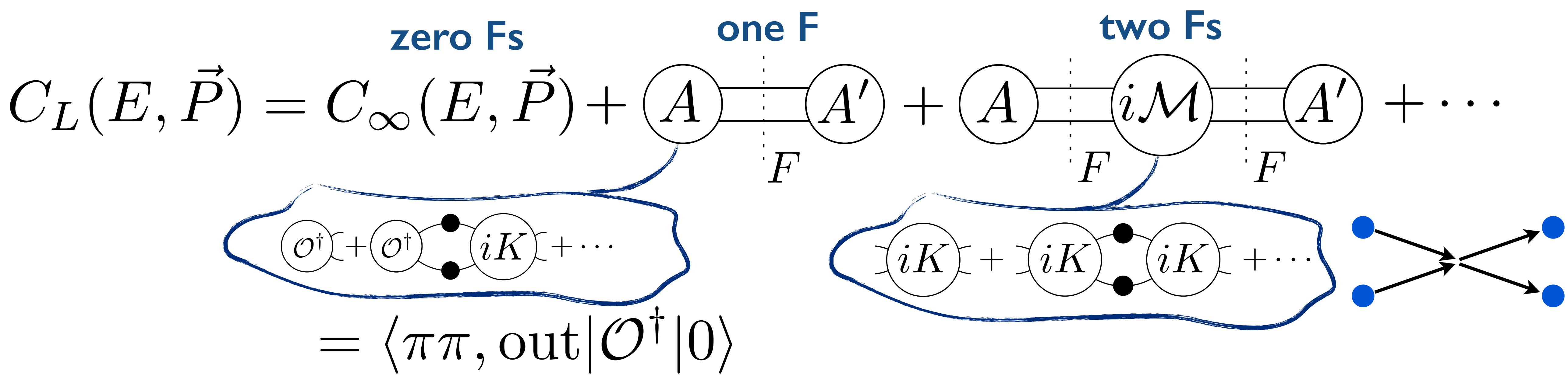


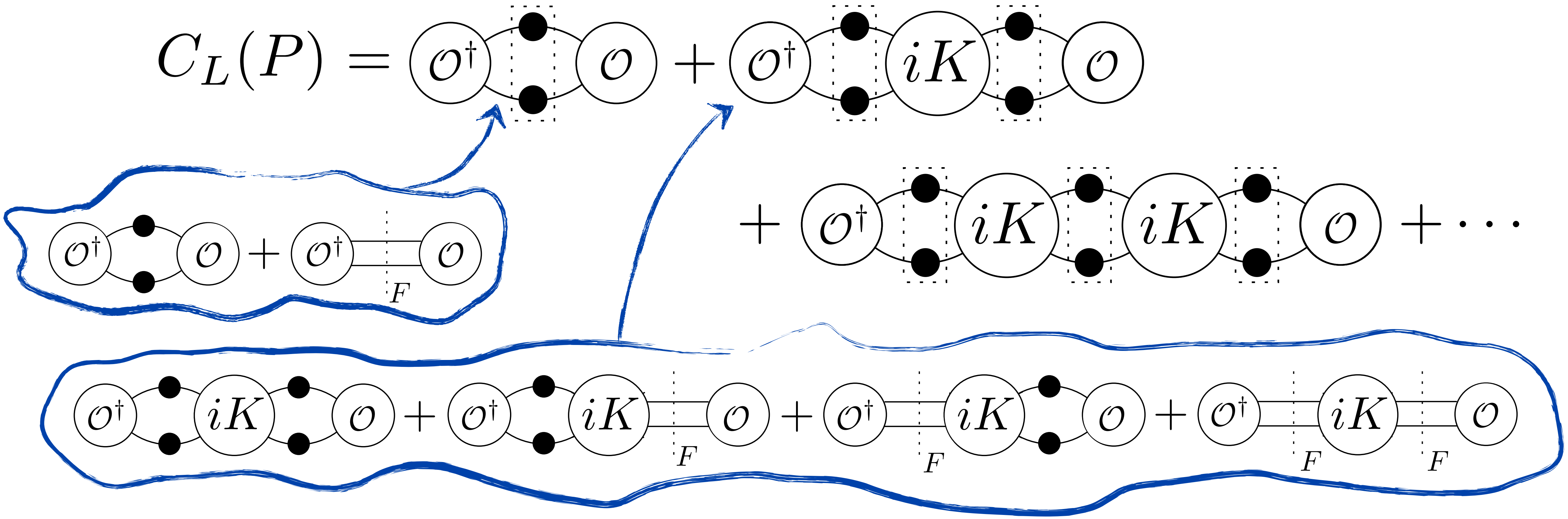
Now regroup by number of Fs



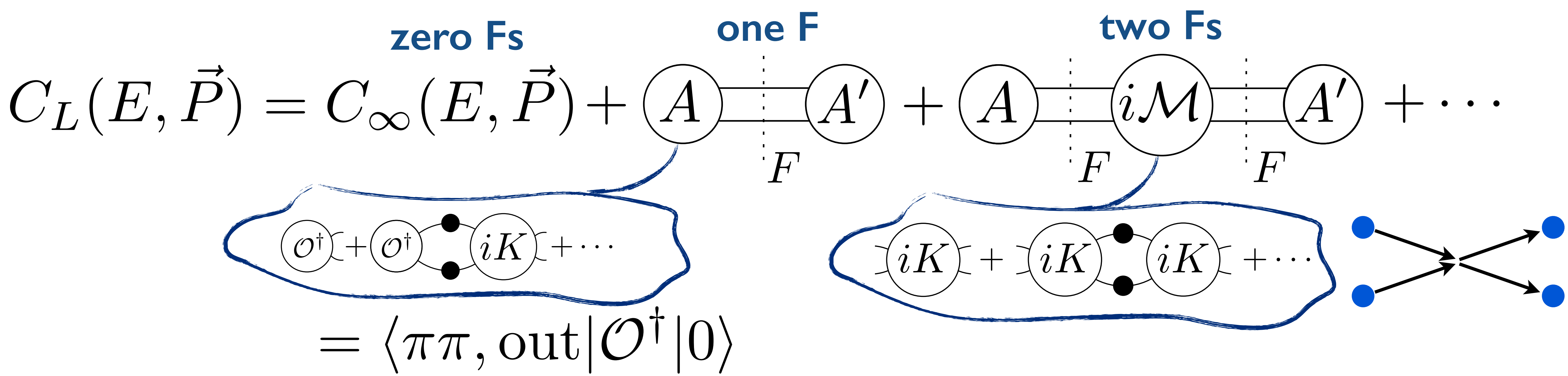


Now regroup by number of Fs





Now regroup by number of Fs



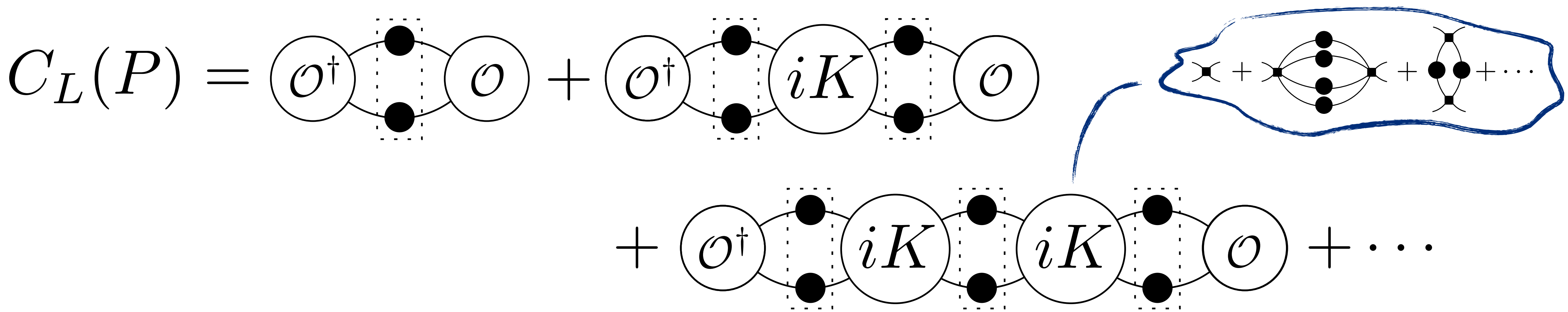
When we factorize diagrams and group infinite-volume parts…

physical observables emerge!

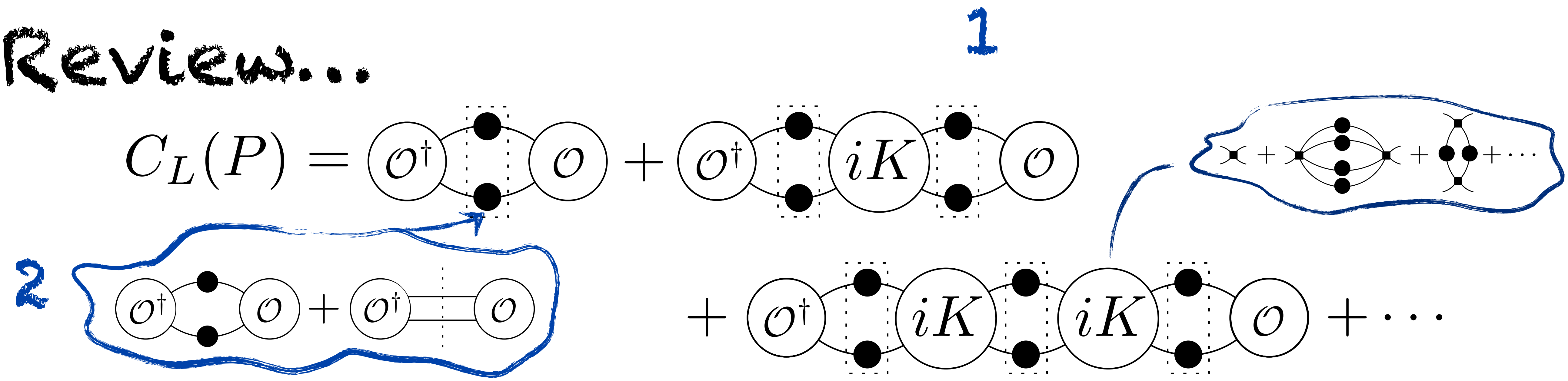
Review...

Review...

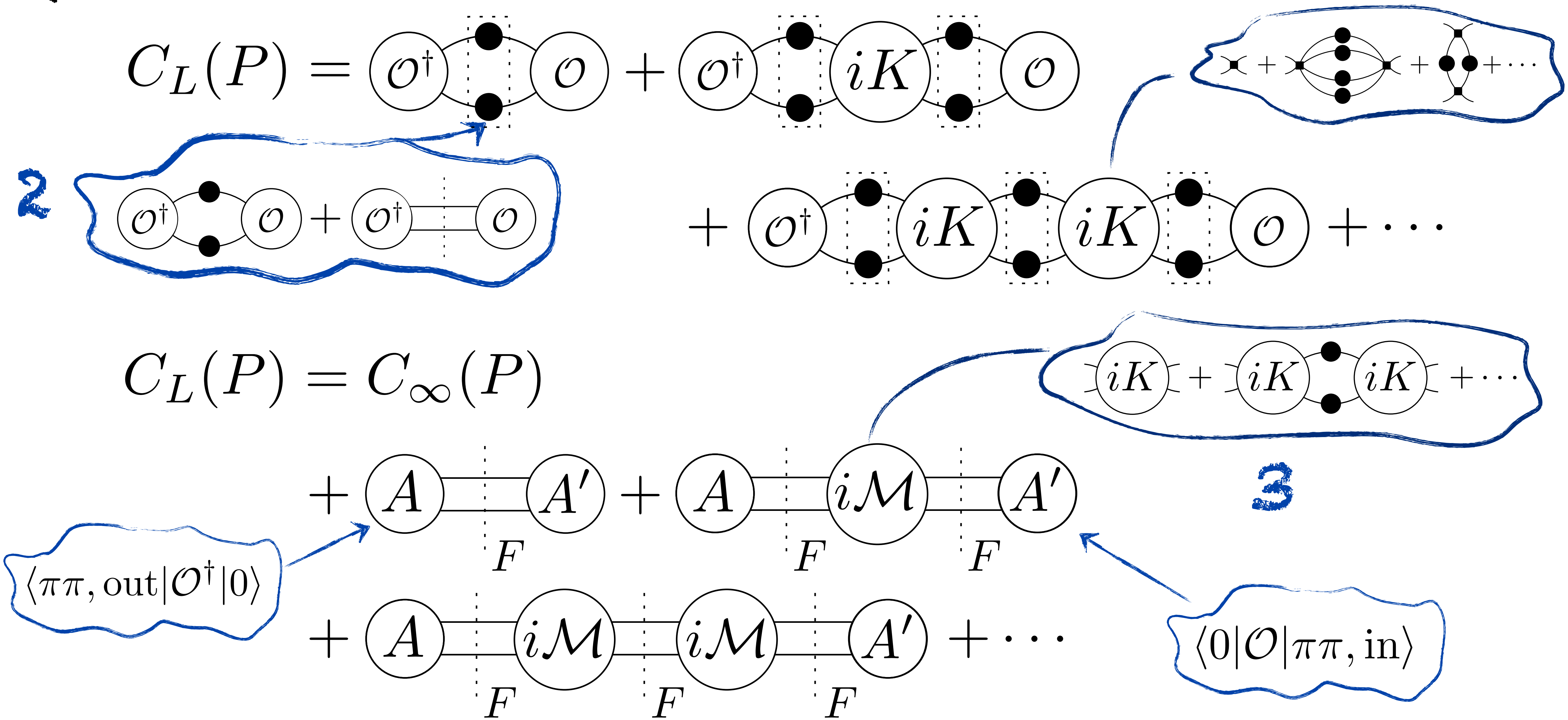
1



Review...



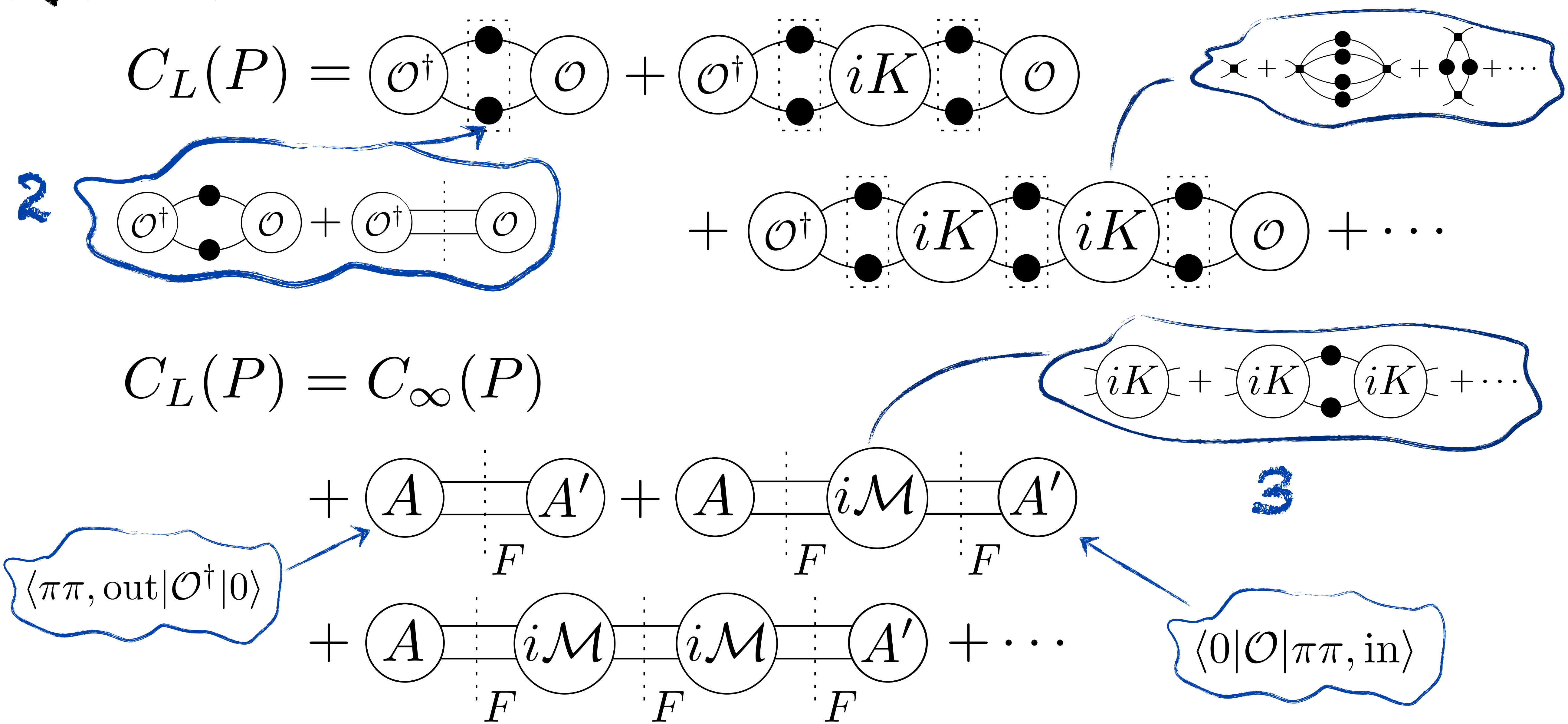
Review...



1
$C_L(P) =$
$\mathcal{O}^\dagger$
$\mathcal{O}$
iK
2
3
$C_L(P) = C_\infty(P)$
A
A'
$i\mathcal{M}$
F
$\langle \pi\pi, \mathrm{out}|\mathcal{O}^\dagger|0\rangle$
$\langle 0|\mathcal{O}|\pi\pi, \mathrm{in}\rangle$

Review...

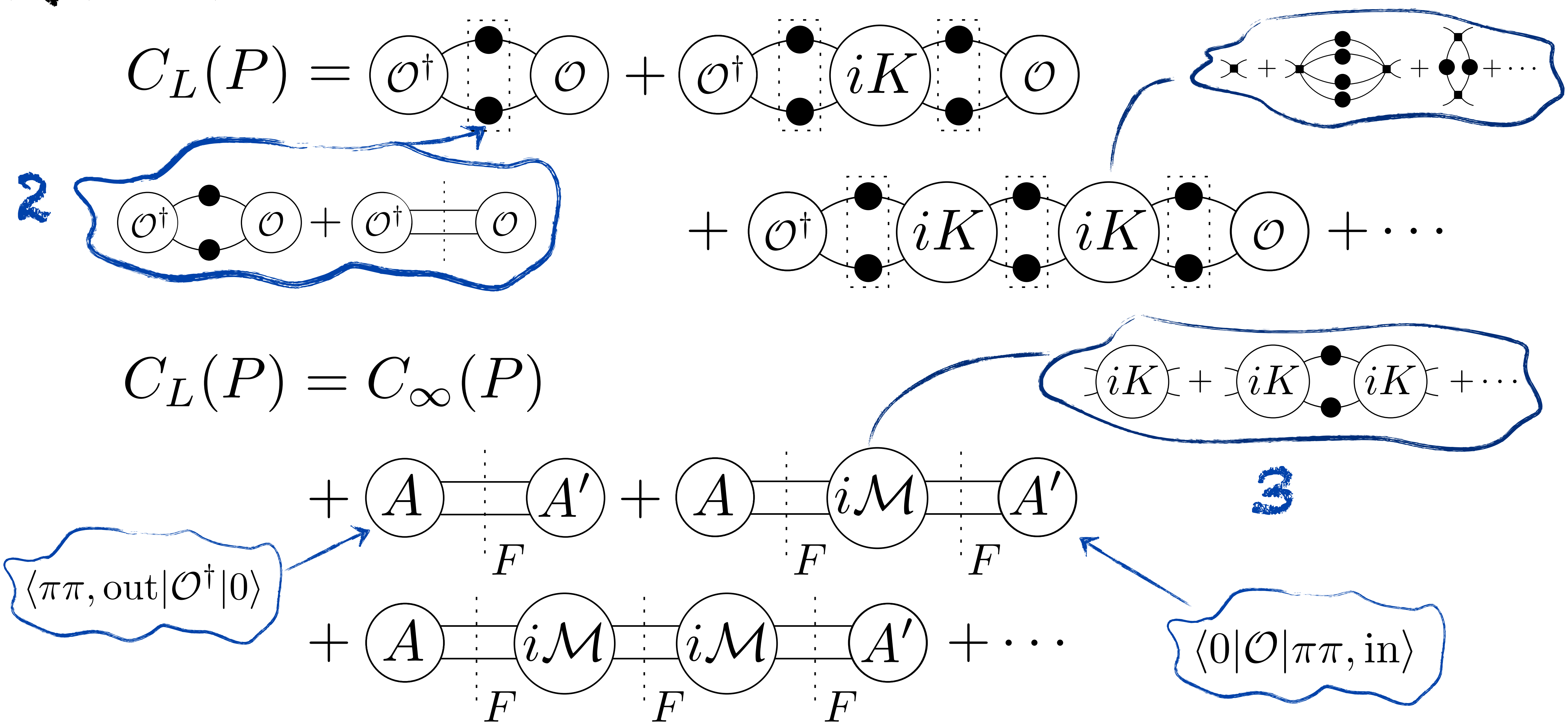
1



We deduce...

$$C_L(P) = C_\infty(P) - A' F \frac{1}{1 + \mathcal{M}_{2\to 2} F} A$$

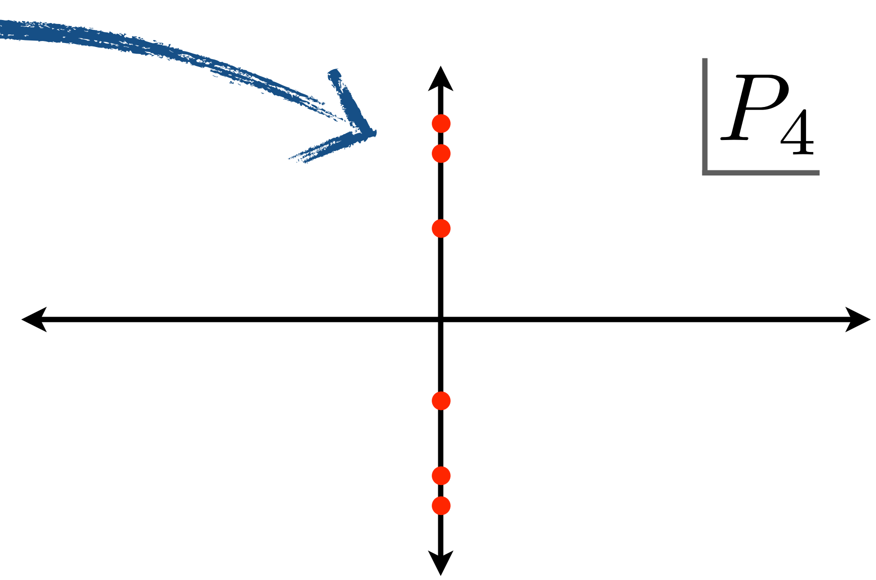
Review...



We deduce...

poles are in here

$$C_L(P) = C_\infty(P) - A' F \frac{1}{1 + \mathcal{M}_{2\to 2} F} A$$



Two-particle result

At fixed $(L, \vec{P})$, finite-volume energies are solutions to $\det[\mathcal{M}_{2\to 2}^{-1} + F] = 0$

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)
Rummukainen and Gottlieb, *Nucl. Phys.* B450, 397 (1995)
Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

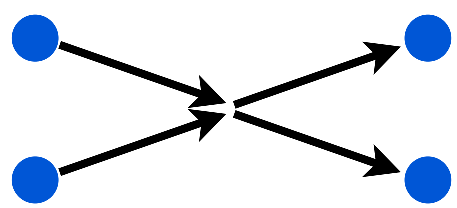
Matrices defined using angular-momentum states

Two-particle result

At fixed $(L, \vec{P})$, finite-volume energies are solutions to $\det[\mathcal{M}_{2\to 2}^{-1} + F] = 0$

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)
Rummukainen and Gottlieb, *Nucl. Phys*. B450, 397 (1995)
Kim, Sachrajda and Sharpe. *Nucl. Phys*. B727, 218-243 (2005)

Matrices defined using angular-momentum states

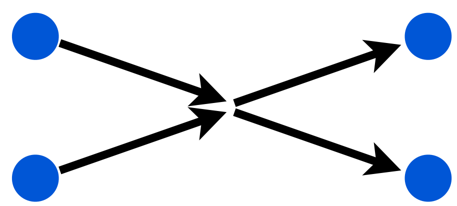
$\mathcal{M}_{2\to 2} \equiv$  **diagonal matrix, parametrized by** $\delta_\ell(E^*)$

Two-particle result

At fixed $(L, \vec{P})$, finite-volume energies are solutions to $$\det[\mathcal{M}_{2\to 2}^{-1} + F] = 0$$

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)
Rummukainen and Gottlieb, *Nucl. Phys*. B450, 397 (1995)
Kim, Sachrajda and Sharpe. *Nucl. Phys*. B727, 218-243 (2005)

Matrices defined using angular-momentum states

$\mathcal{M}_{2\to 2} \equiv$  **diagonal matrix, parametrized by** $\delta_\ell(E^*)$

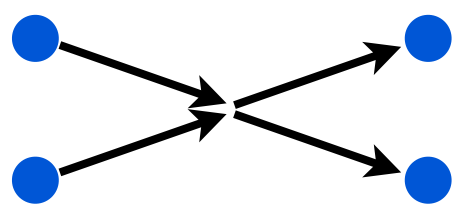
$F \equiv$ **non-diagonal matrix of known geometric functions**

Two-particle result

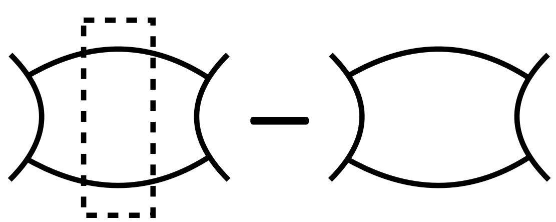
At fixed $(L, \vec{P})$, finite-volume energies are solutions to $\det[\mathcal{M}_{2\to 2}^{-1} + F] = 0$

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)
Rummukainen and Gottlieb, *Nucl. Phys.* B450, 397 (1995)
Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

Matrices defined using angular-momentum states

$\mathcal{M}_{2\to 2} \equiv$  **diagonal matrix, parametrized by** $\delta_\ell(E^*)$

$F \equiv$ **non-diagonal matrix of known geometric functions**

$\equiv$  difference of two-particle loops in finite and infinite volume

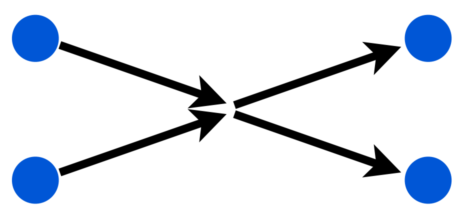
depends on $L, E, \vec{P}$

Two-particle result

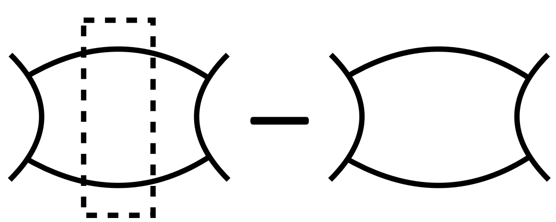
At fixed $(L, \vec{P})$, finite-volume energies are solutions to $$\det[\mathcal{M}_{2\to 2}^{-1} + F] = 0$$

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)
Rummukainen and Gottlieb, *Nucl. Phys*. B450, 397 (1995)
Kim, Sachrajda and Sharpe. *Nucl. Phys*. B727, 218-243 (2005)

Matrices defined using angular-momentum states

$\mathcal{M}_{2\to 2} \equiv$  diagonal matrix, parametrized by $\delta_\ell(E^*)$

$F \equiv$ non-diagonal matrix of known geometric functions

$\equiv$  difference of two-particle loops in finite and infinite volume

depends on $L, E, \vec{P}$

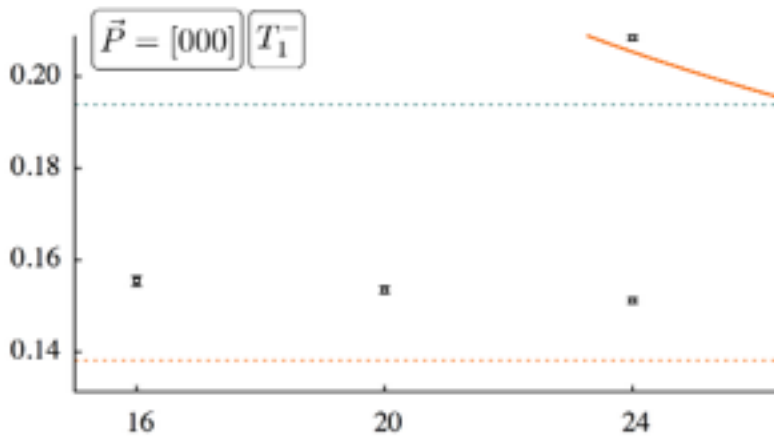
At low energies, lowest partial waves dominate $\mathcal{M}_{2\to 2}$

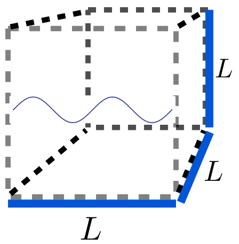
e.g. s-wave only with some rearranging $\longrightarrow$ $$\cot\delta(E_n^*) + \cot\phi(E_n, \vec{P}, L) = 0$$

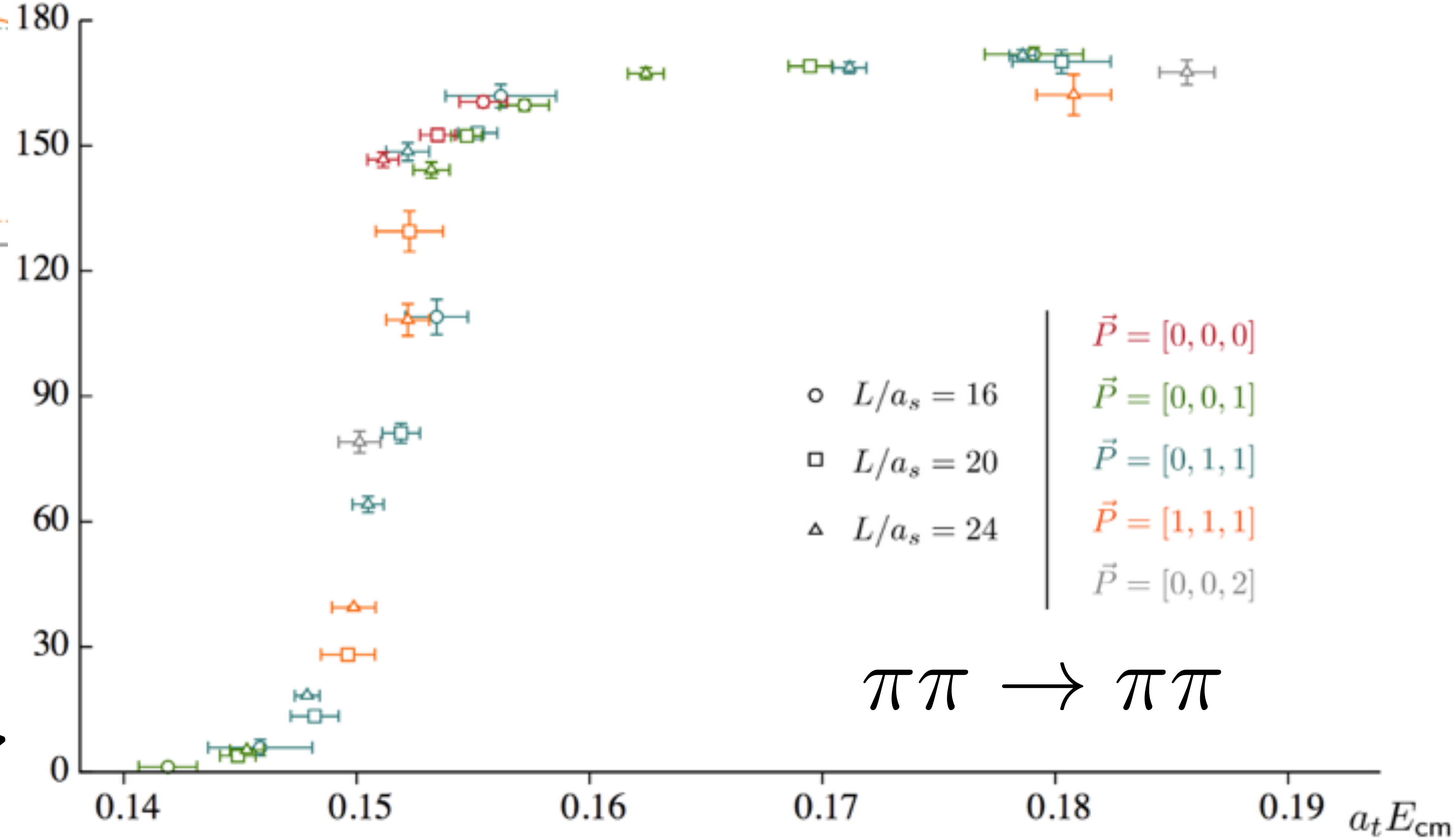
scattering phase — known function

Using the result (p-wave)

$$\cot \delta_{\ell=1}(E_n^*) + \cot \phi(E_n, \vec{P}, L) = 0$$



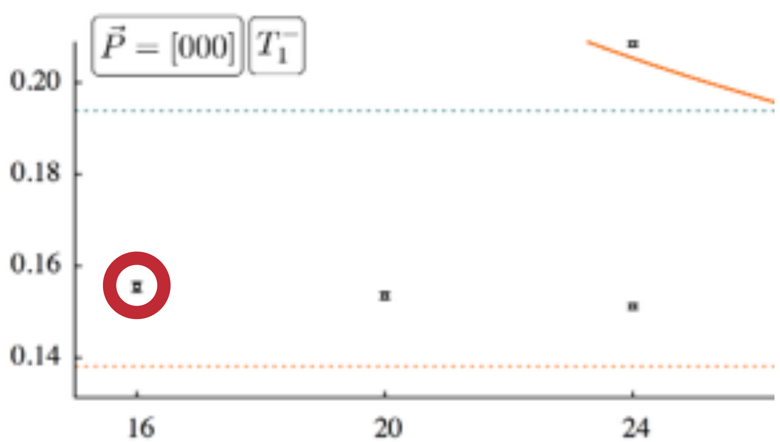


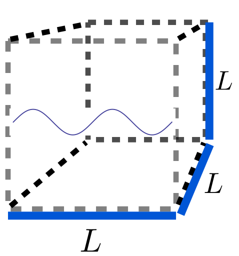


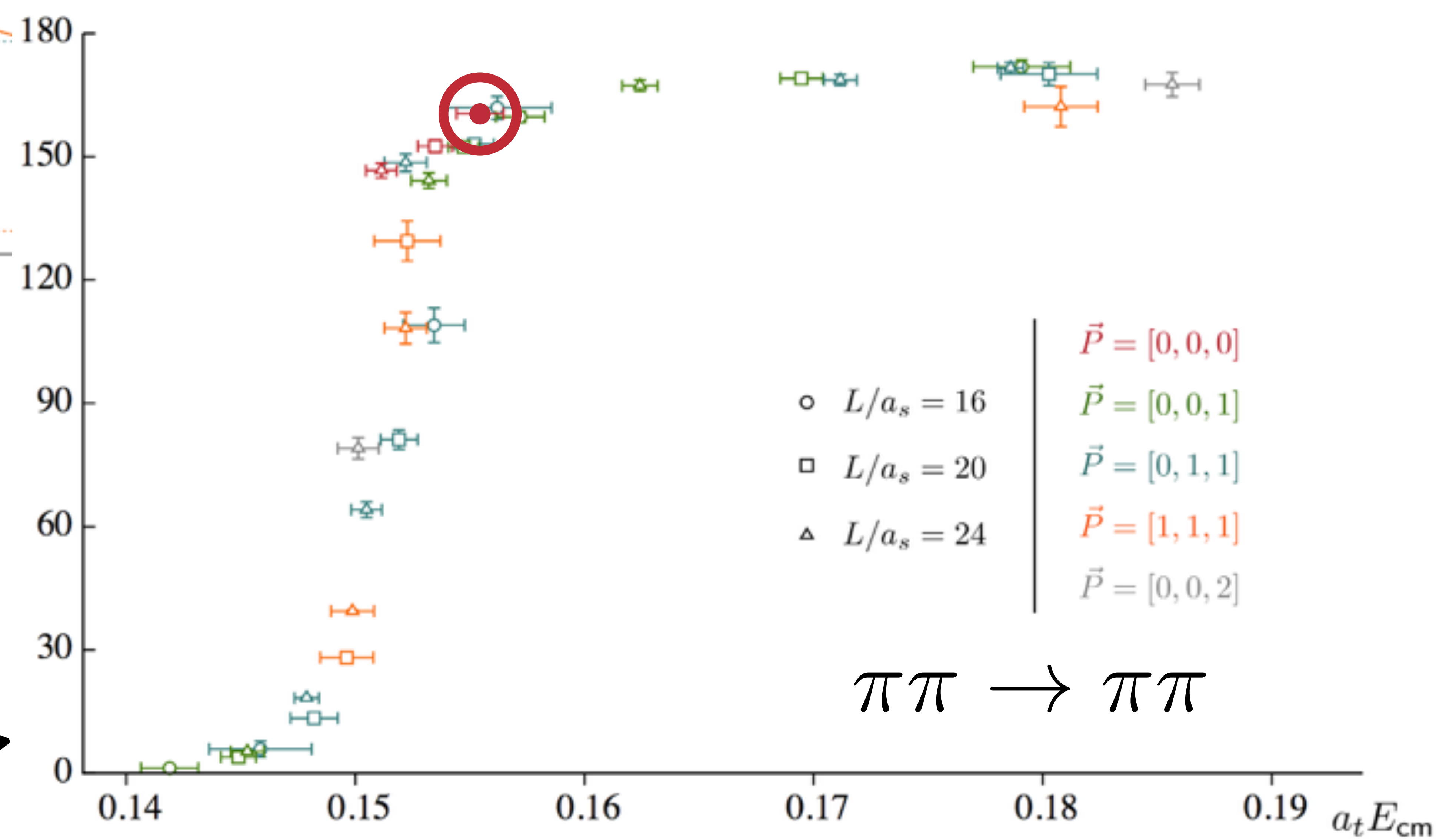
from Dudek, Edwards, Thomas in *Phys.Rev*. D87 (2013) 034505

Using the result (p-wave)

$$\cot\delta_{\ell=1}(E_n^*) + \cot\phi(E_n, \vec{P}, L) = 0$$



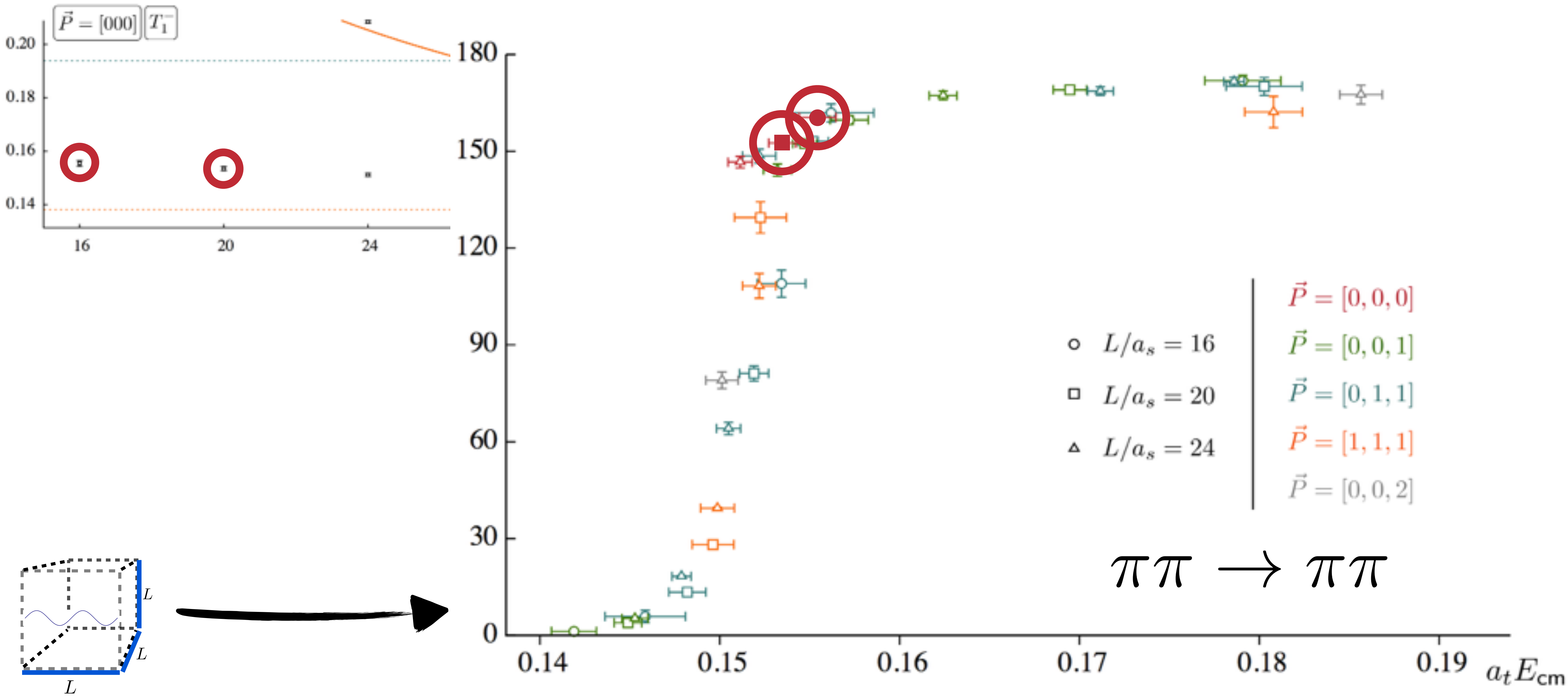




from Dudek, Edwards, Thomas in *Phys.Rev*. D87 (2013) 034505

Using the result (p-wave)

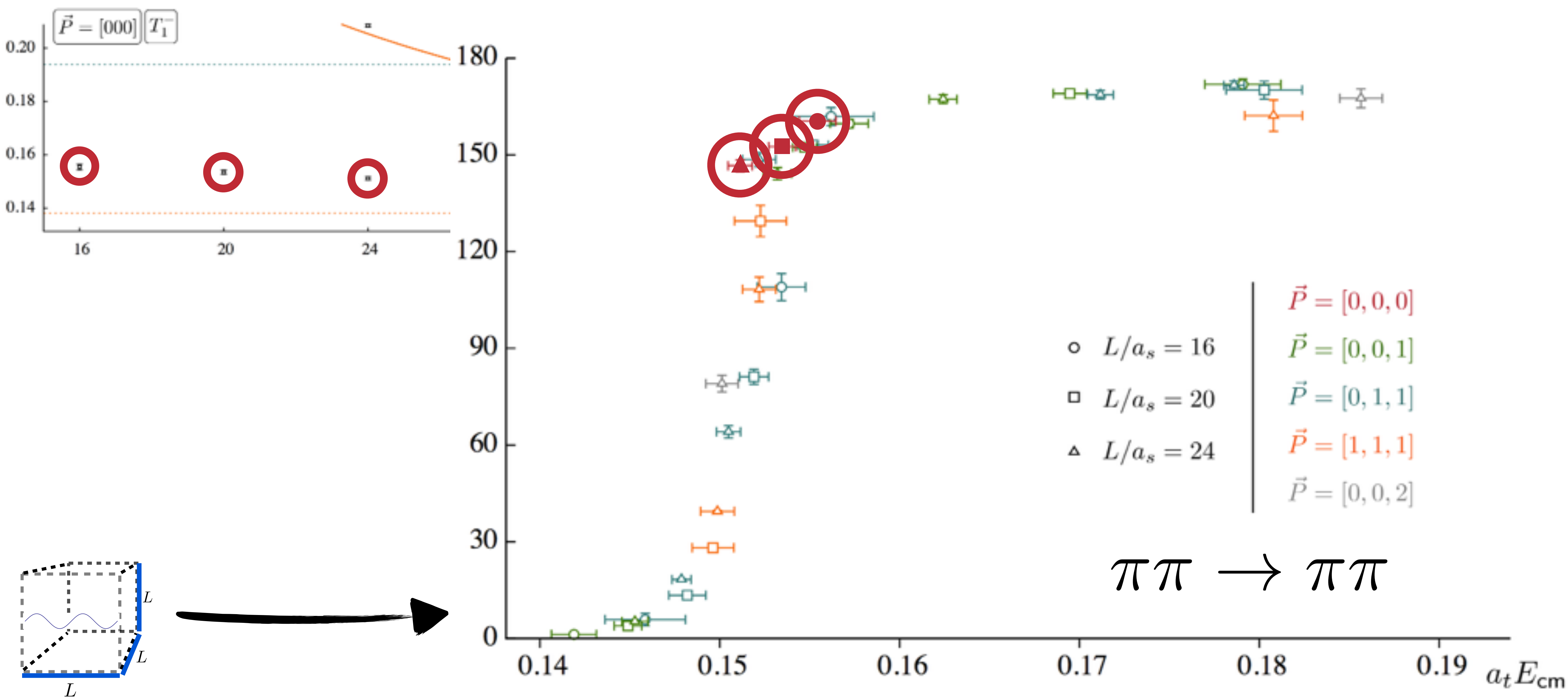
$$\cot \delta_{\ell=1}(E_n^*) + \cot \phi(E_n, \vec{P}, L) = 0$$



from Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505

Using the result (p-wave)

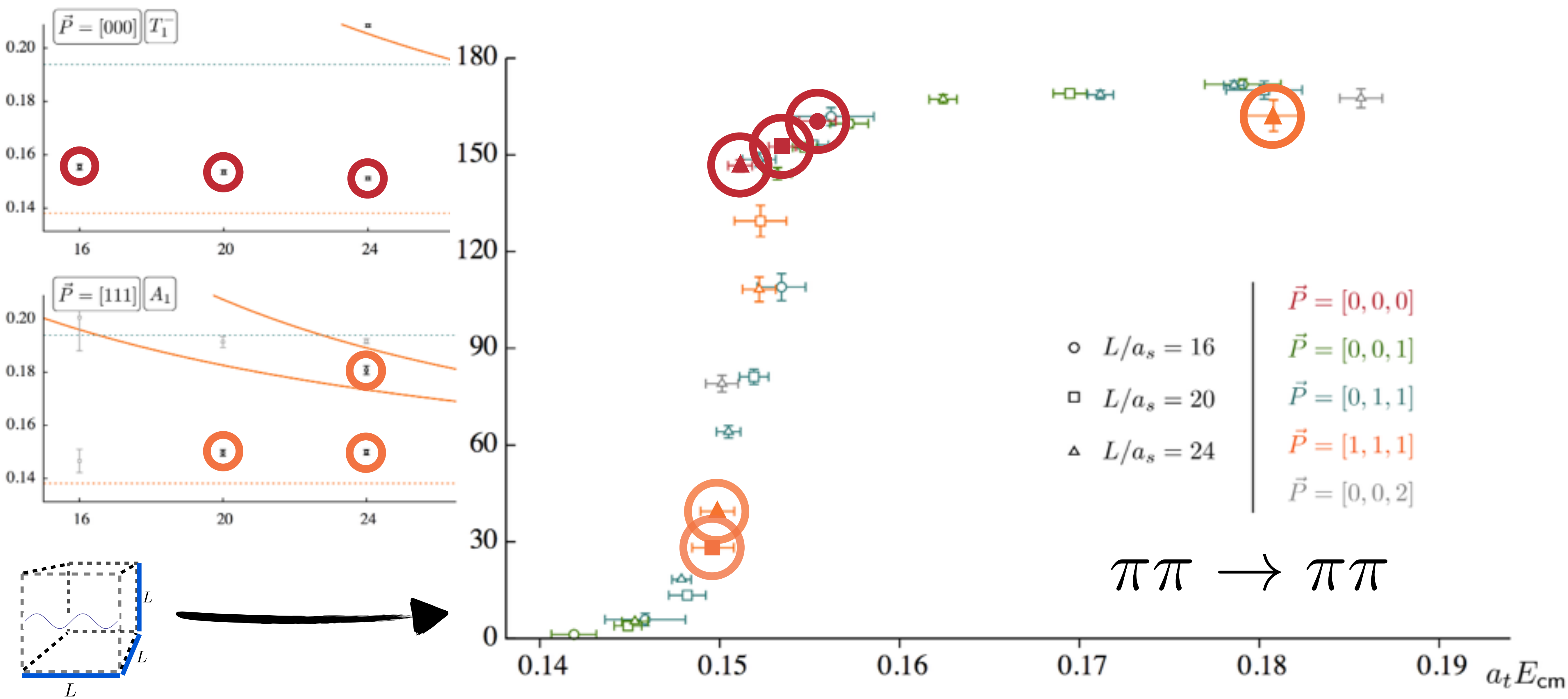
$$\cot \delta_{\ell=1}(E_n^*) + \cot \phi(E_n, \vec{P}, L) = 0$$



from Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505

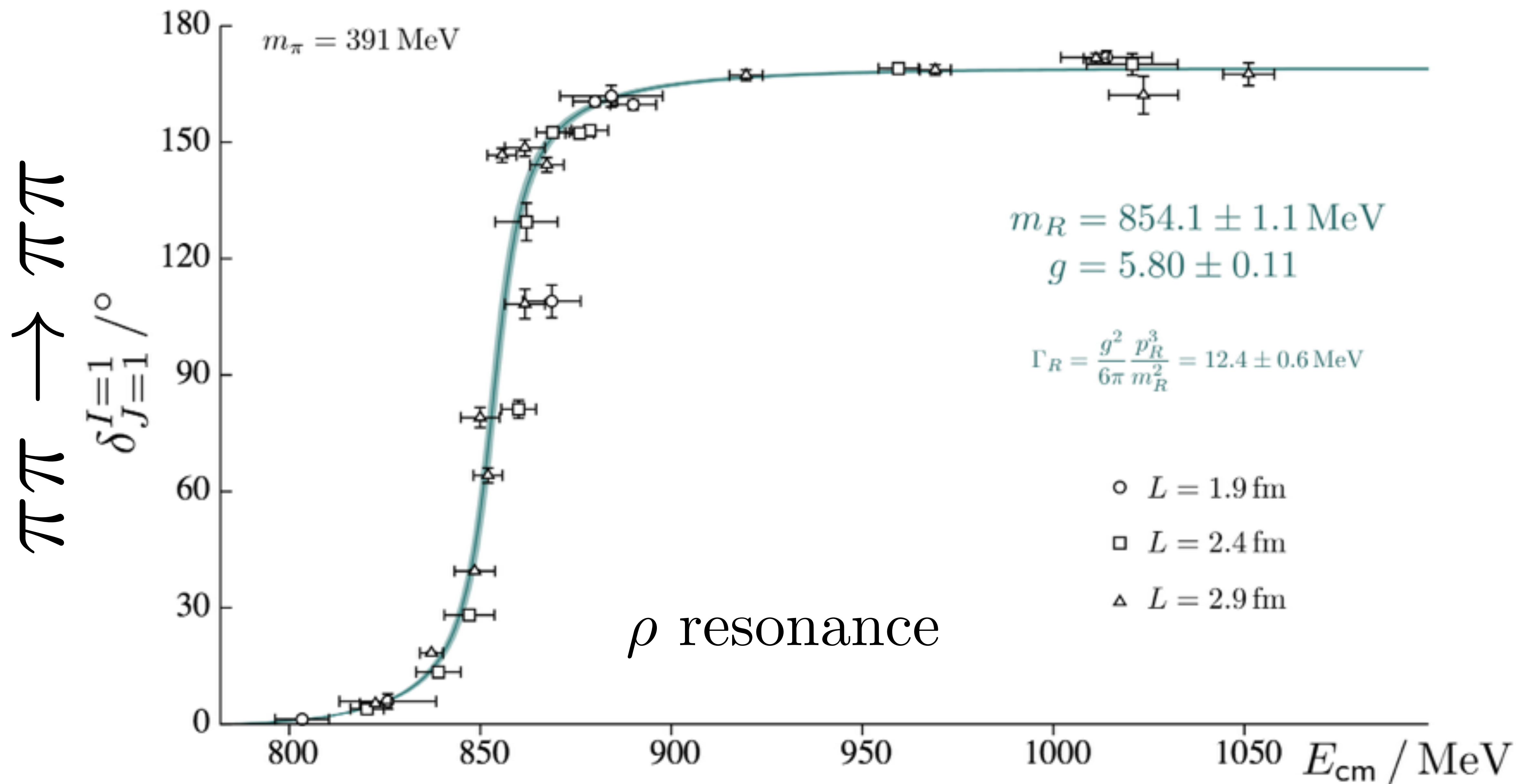
Using the result (p-wave)

$$\cot \delta_{\ell=1}(E_n^*) + \cot \phi(E_n, \vec{P}, L) = 0$$



from Dudek, Edwards, Thomas in *Phys.Rev*. D87 (2013) 034505

$$\cot \delta_{\ell=1}(E_n^*) + \cot \phi(E_n, \vec{P}, L) = 0$$



$m_\pi = 391$ MeV

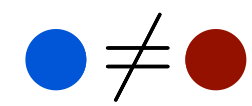
from Dudek, Edwards, Thomas in *Phys.Rev*. D87 (2013) 034505

Two-particle result

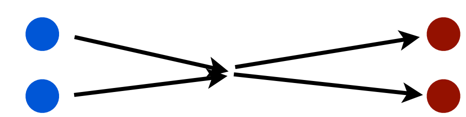
At fixed $(L, \vec{P})$, finite-volume energies are solutions to $$\det[\mathcal{M}_{2\to 2}^{-1} + F] = 0$$

Has since been generalized to include…

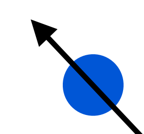
non-indentical particles



multiple two-particle channels



particles with spin



Bernard, Lage, Meißner, and Rusetsky, JHEP, 1101, 019 (2011)
MTH and Sharpe, *Phys.Rev. D86* (2012) 016007
Briceño and Davoudi, *Phys.Rev. D88* (2013) 094507
Briceño, *Phys. Rev.* D 89, 074507 (2014)

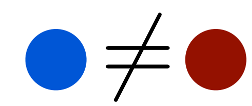
Two-particle result

At fixed $(L, \vec{P})$, finite-volume energies are solutions to

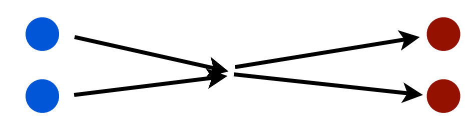
$$\det[\mathcal{M}_{2\to2}^{-1} + F] = 0$$

Has since been generalized to include…

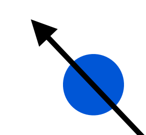
non-indentical particles



multiple two-particle channels



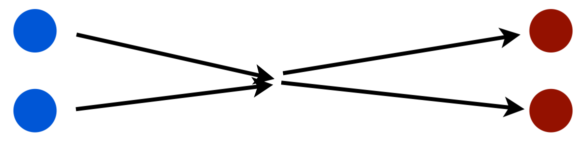
particles with spin



Bernard, Lage, Meißner, and Rusetsky, JHEP, 1101, 019 (2011)
MTH and Sharpe, *Phys.Rev. D86* (2012) 016007
Briceño and Davoudi, *Phys.Rev. D88* (2013) 094507
Briceño, *Phys. Rev.* D 89, 074507 (2014)

The basic form of the equation stays the same, but the matrix space and definition of F change

Multiple two-particle channels

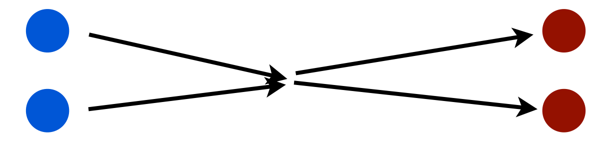


Must now include a channel index

MTH and Sharpe/Briceño and Davoudi

$$\det\left[\begin{pmatrix}\mathcal{M}_{a\to a} & \mathcal{M}_{a\to b}\\ \mathcal{M}_{b\to a} & \mathcal{M}_{b\to b}\end{pmatrix}^{-1}+\begin{pmatrix}F_a & 0\\ 0 & F_b\end{pmatrix}\right]=0$$

Multiple two-particle channels



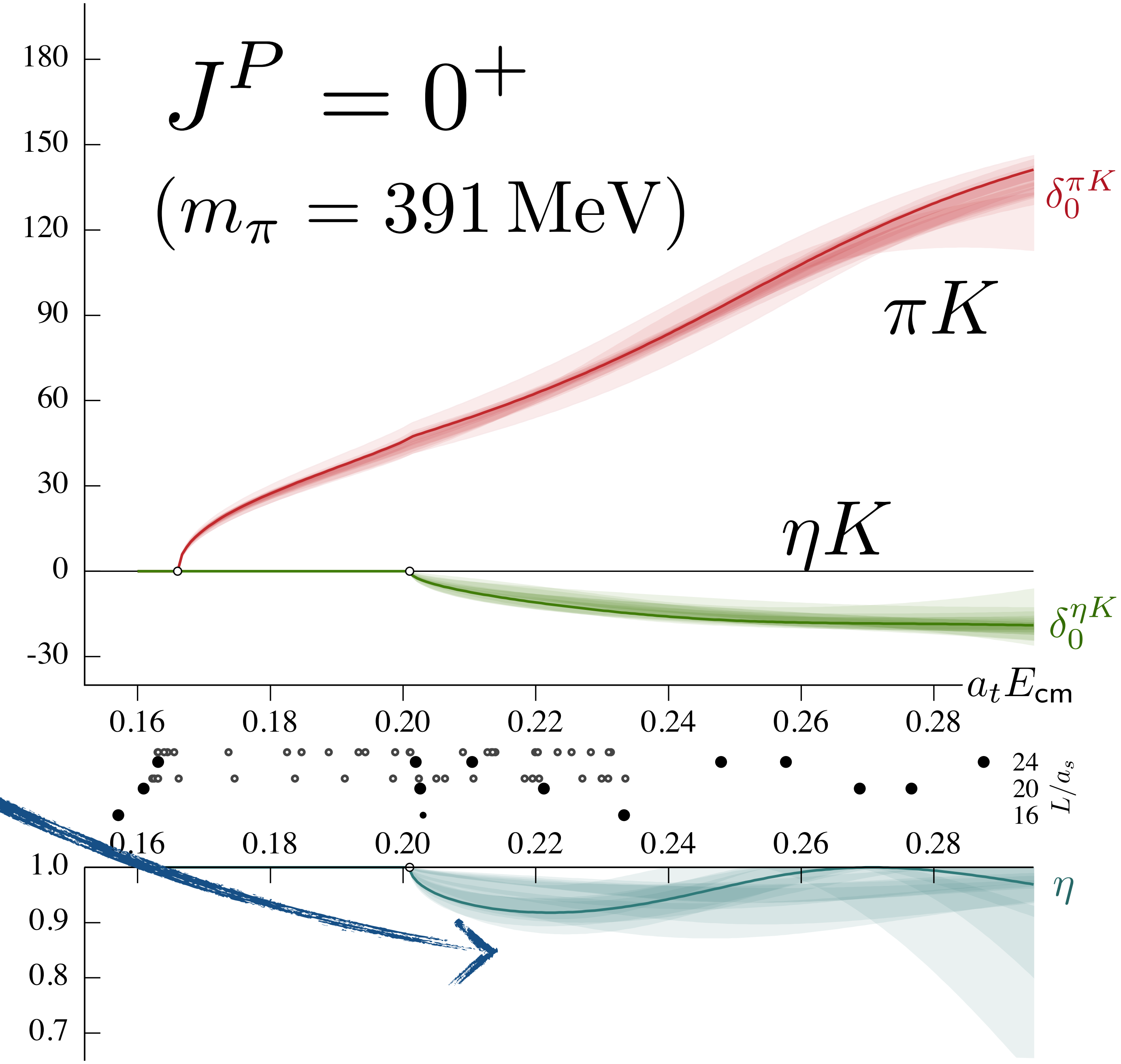
Must now include a channel index

MTH and Sharpe/Briceño and Davoudi

$$\det\left[\begin{pmatrix}\mathcal{M}_{a\to a} & \mathcal{M}_{a\to b}\\ \mathcal{M}_{b\to a} & \mathcal{M}_{b\to b}\end{pmatrix}^{-1}+\begin{pmatrix}F_a & 0\\ 0 & F_b\end{pmatrix}\right]=0$$

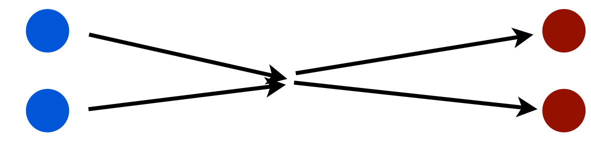
Already used in JLab study of
$\pi K,\ \eta K$

$$\mathcal{M}(\pi K\to\eta K)\sim\sqrt{1-\eta^2}$$



Wilson, Dudek, Edwards, Thomas,
Phys. Rev. D 91, 054008 (2015)
arXiv: 1411.2004

Multiple two-particle channels



Must now include a channel index

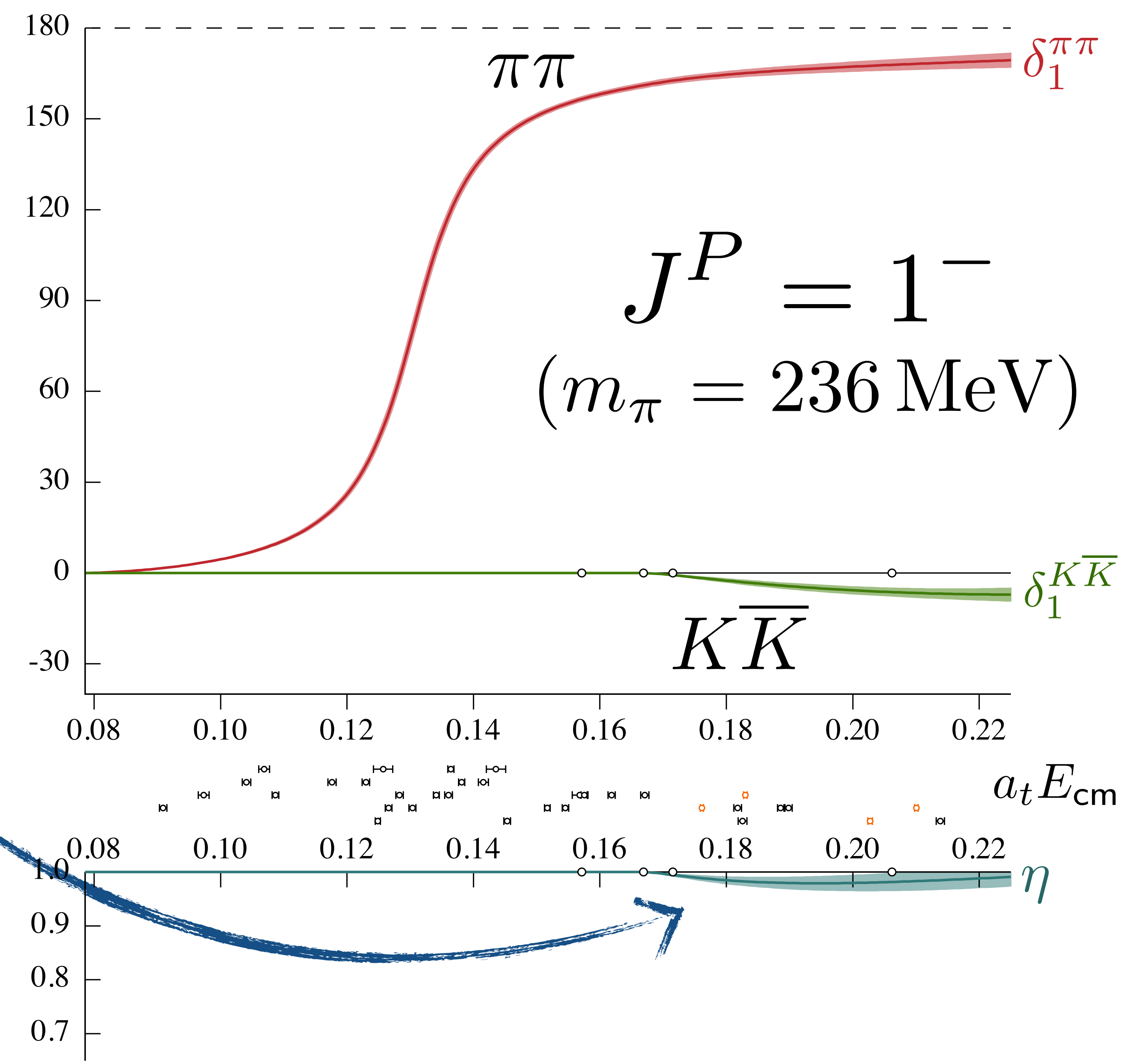
MTH and Sharpe/Briceño and Davoudi

$$\det\left[\begin{pmatrix}\mathcal{M}_{a\to a} & \mathcal{M}_{a\to b}\\ \mathcal{M}_{b\to a} & \mathcal{M}_{b\to b}\end{pmatrix}^{-1}+\begin{pmatrix}F_a & 0\\ 0 & F_b\end{pmatrix}\right]=0$$

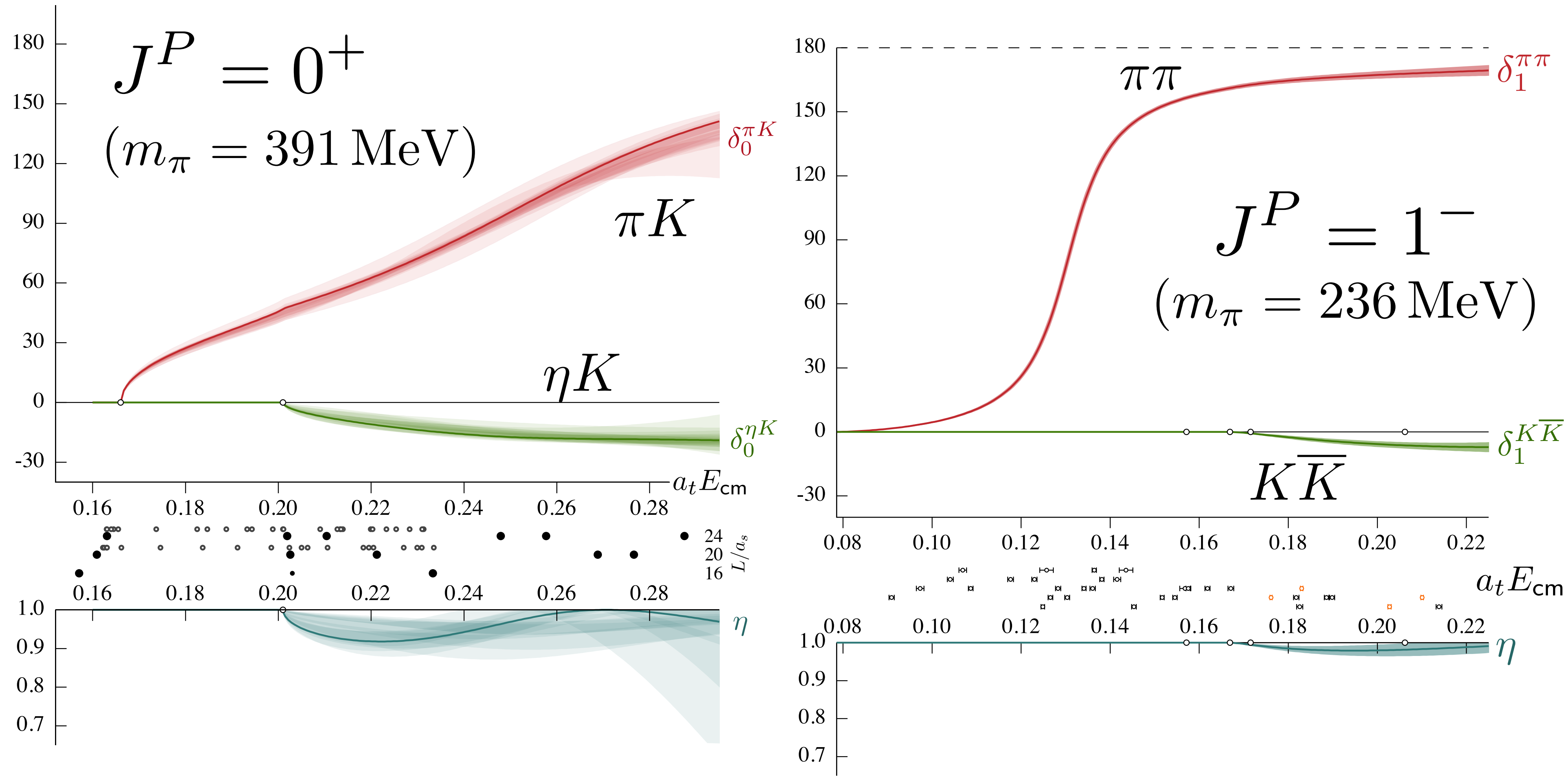
As well as JLab rho study with

$$\pi\pi,\ K\overline{K}$$

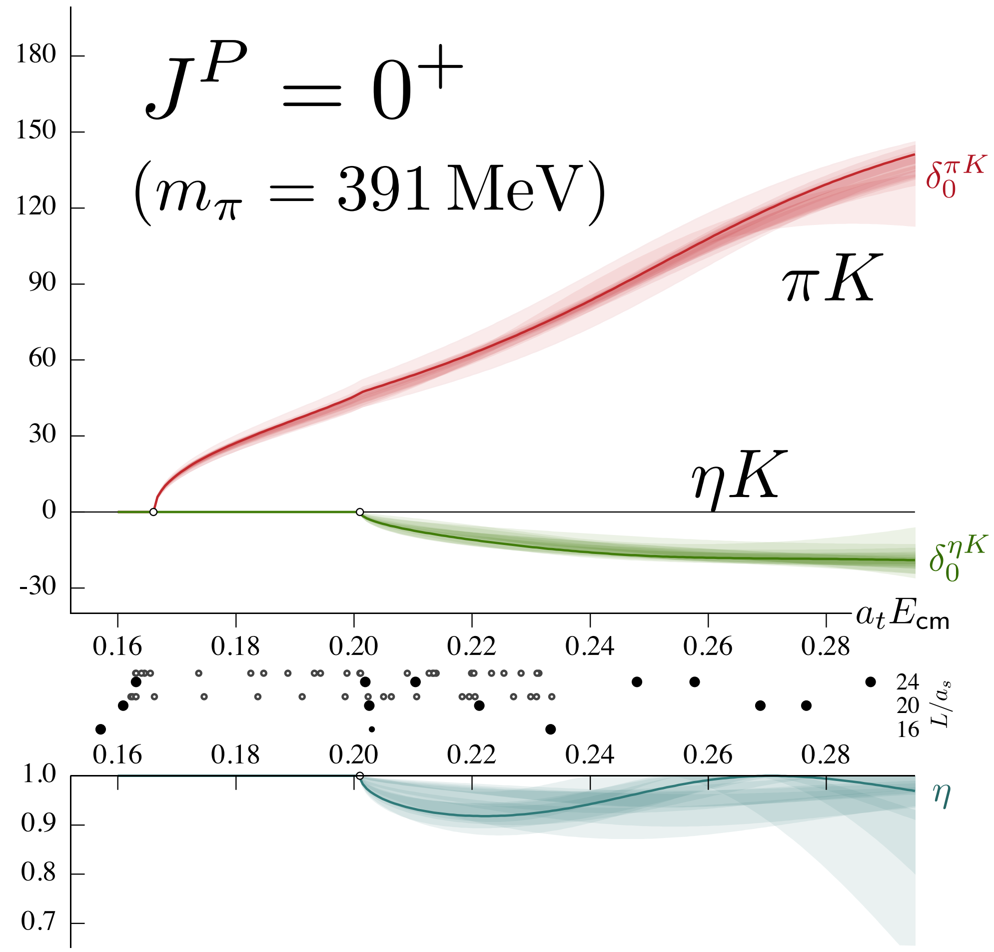
$$\mathcal{M}(\pi\pi\to K\overline{K})\sim\sqrt{1-\eta^2}$$

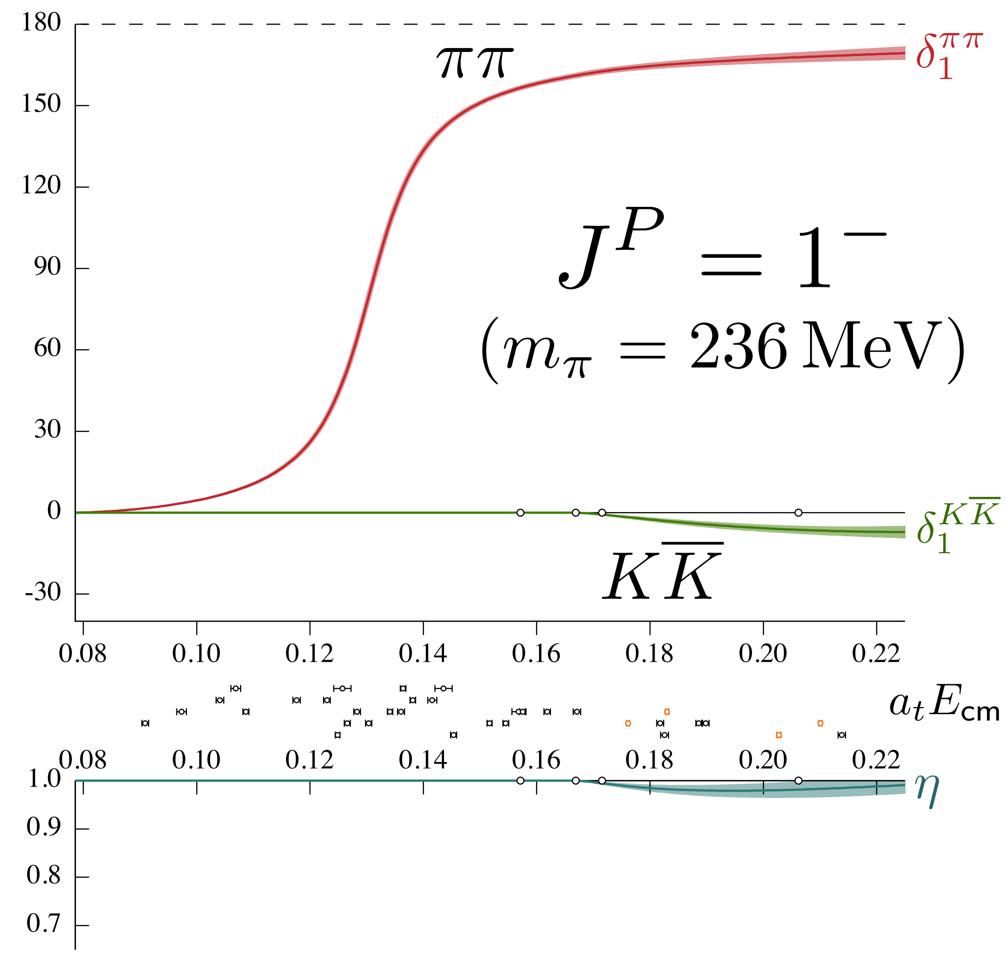


Wilson, Briceño, Dudek, Edwards, Thomas, arXiv:1507:02599



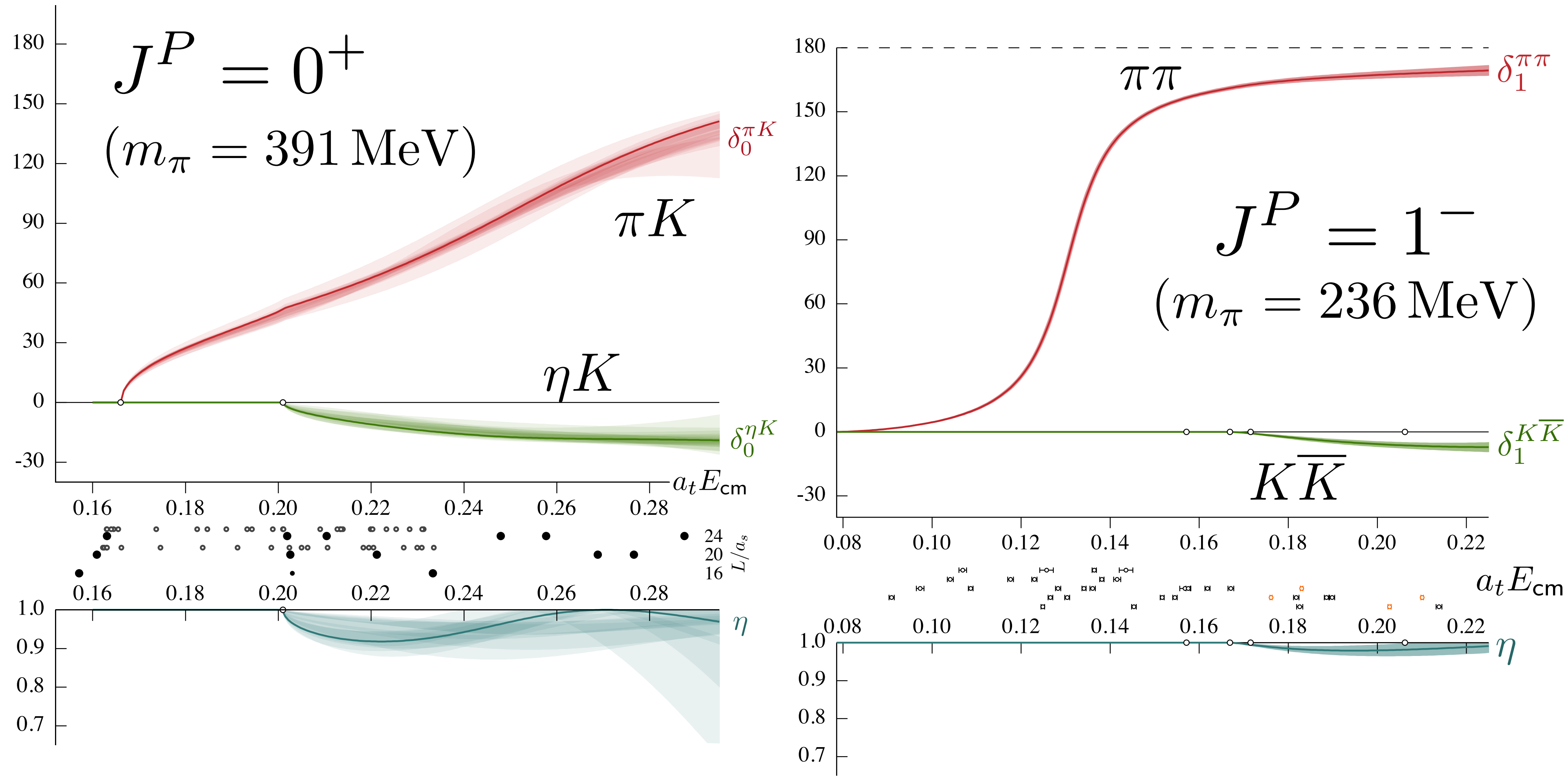
$J^P = 0^+$
$(m_\pi = 391\,\mathrm{MeV})$
πK
ηK
$\delta_0^{\pi K}$
$\delta_0^{\eta K}$
$a_t E_{\mathsf{cm}}$
L/a_s
η
$J^P = 1^-$
$(m_\pi = 236\,\mathrm{MeV})$
$\pi\pi$
$K\overline{K}$
$\delta_1^{\pi\pi}$
$\delta_1^{K\overline{K}}$
$a_t E_{\mathsf{cm}}$
η





Three volumes are used to calculate many points on phase shift curve

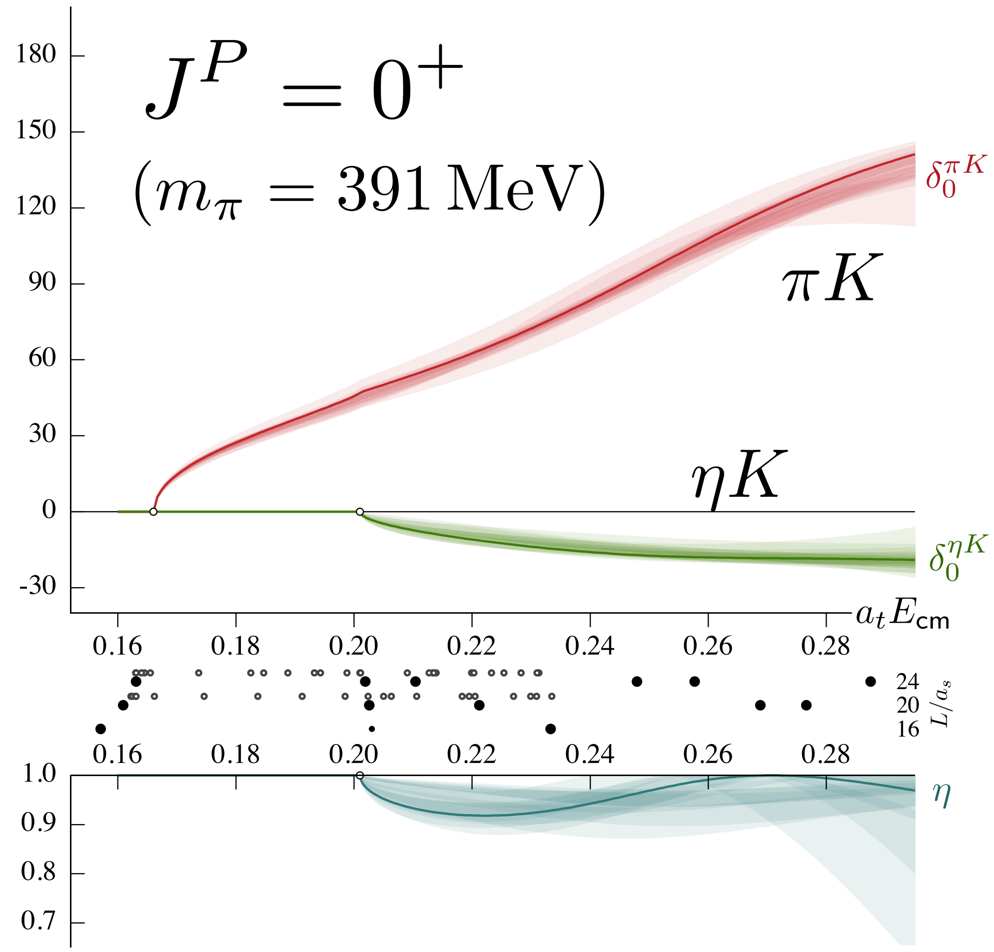
but could still have $e^{-M_\pi L}$ effects

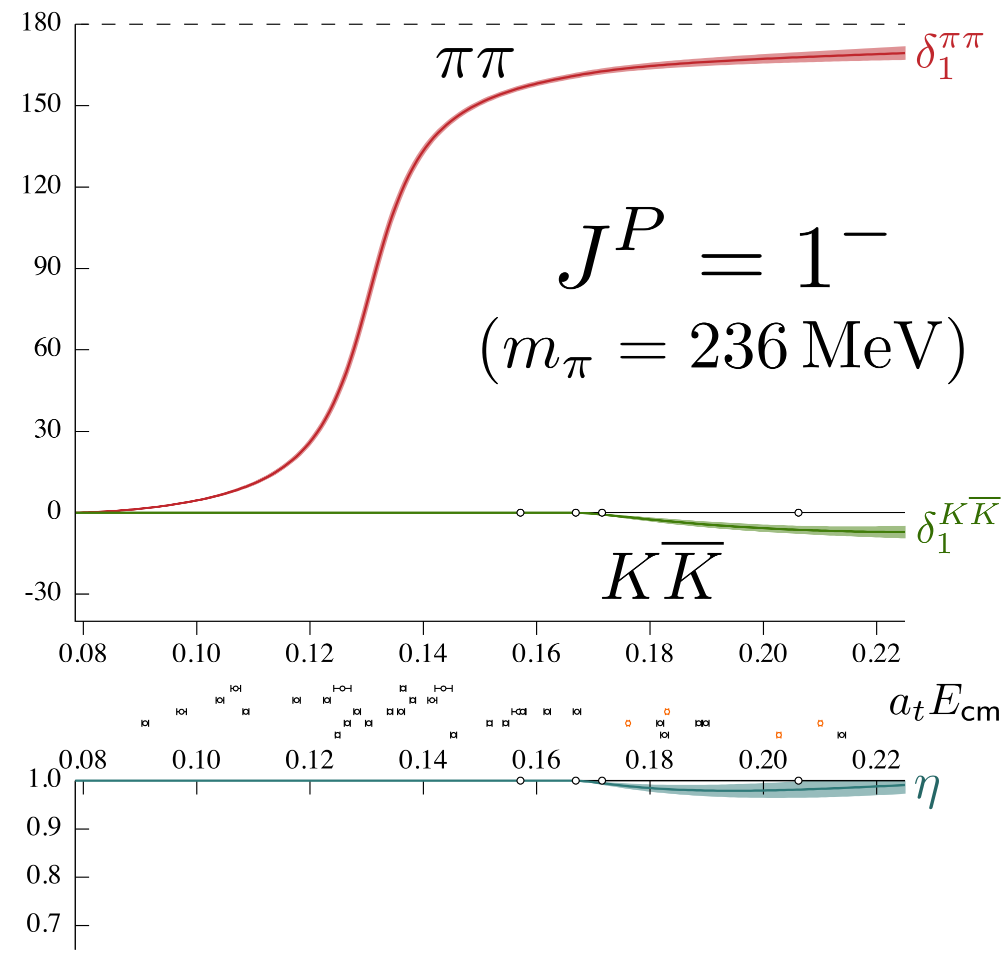


Three volumes are used to calculate many points on phase shift curve

but could still have $e^{-M_\pi L}$ effects

Three and four-particle thresholds





Three volumes are used to calculate many points on phase shift curve

but could still have $e^{-M_\pi L}$ effects

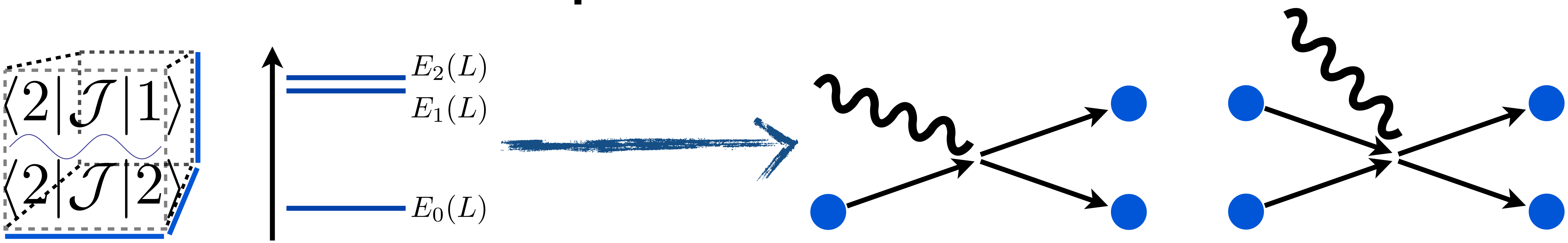
Three and four-particle thresholds

One lattice spacing

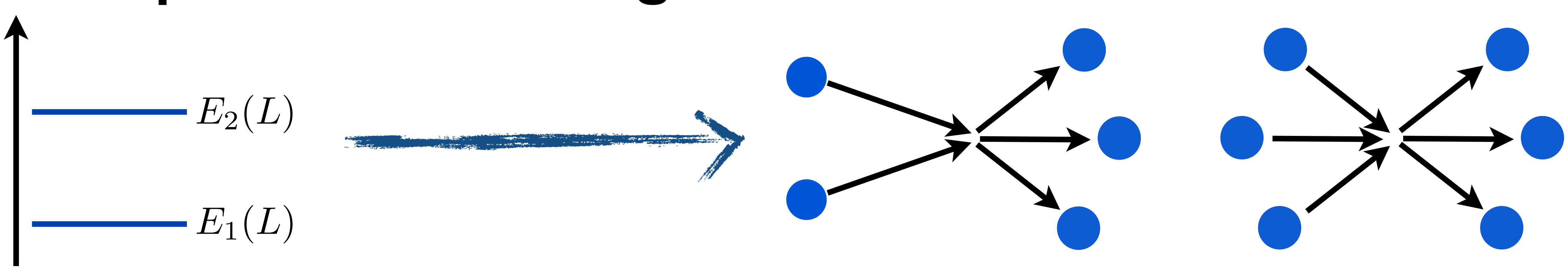
Chiral extrapolation performed in the 1- channel
but not in 0+

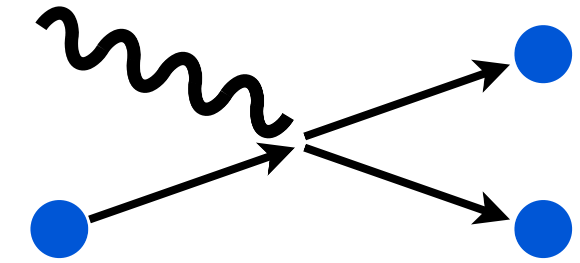
Two-particle scattering

Photo- and electroproduction



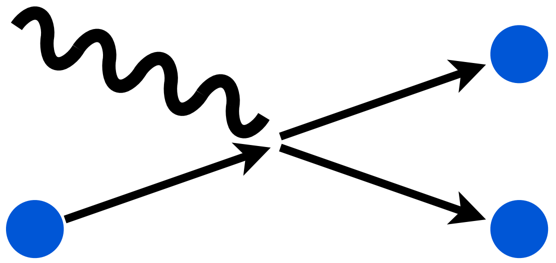
Three-particle scattering



Photoproduction $\langle \pi\pi, \mathrm{out} | \mathcal{J}_\mu | \pi \rangle \equiv$ 

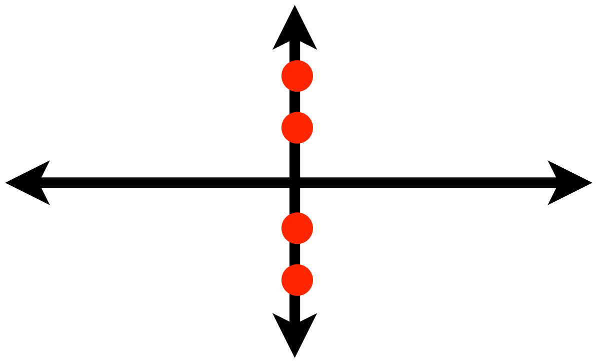
How can we get this from finite-volume observables?

Photoproduction $\langle \pi\pi, \mathrm{out} | \mathcal{J}_\mu | \pi \rangle \equiv$

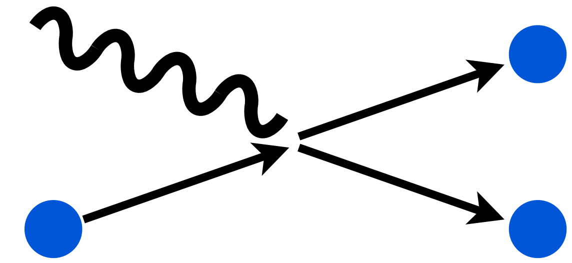


How can we get this from finite-volume observables?

Why did we expect $C_L(P)$ *to have poles?*

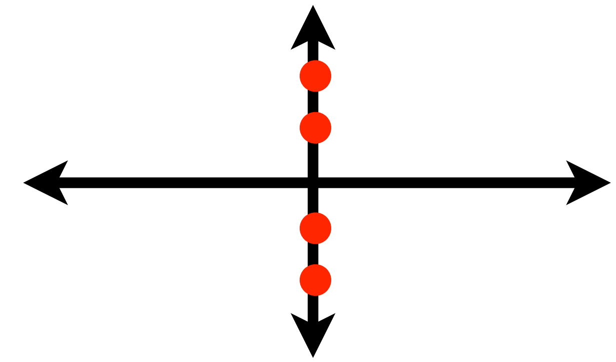


$$C_L(P) \equiv \int_L d^4x \; e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

Photoproduction $\langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu|\pi\rangle \equiv$ 

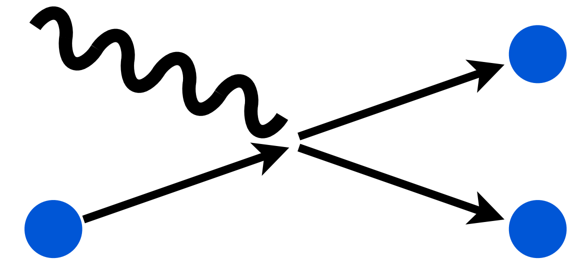
How can we get this from finite-volume observables?

Why did we expect $C_L(P)$ *to have poles?*



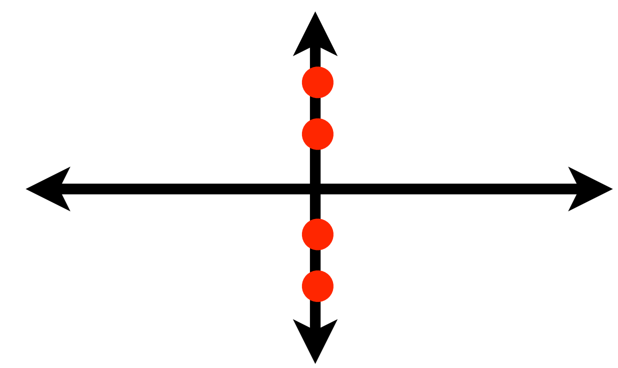
$$C_L(P) \equiv \int_L d^4x \; e^{-iPx} \langle 0|T\mathcal{O}(x)\mathcal{O}^\dagger(0)|0\rangle$$

Insert a complete set finite-volume of states

Photoproduction $\langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu|\pi\rangle \equiv$ 

How can we get this from finite-volume observables?

Why did we expect $C_L(P)$ ***to have poles?***

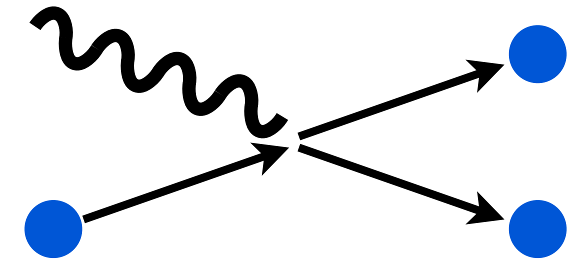


$$C_L(P) \equiv \int_L d^4x \; e^{-iPx} \langle 0|T\mathcal{O}(x)\mathcal{O}^\dagger(0)|0\rangle$$

Insert a complete set finite-volume of states

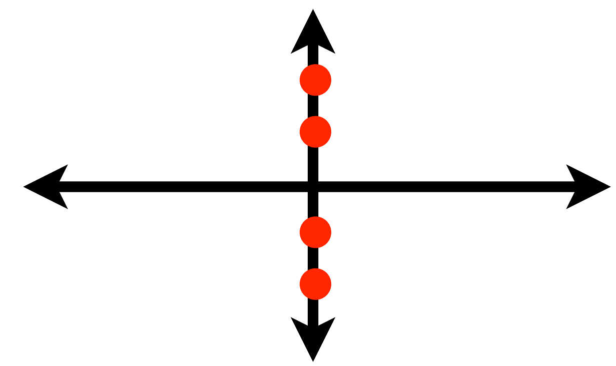
$$C_L(P) \xrightarrow[P_4 \to iE_n]{} \frac{L^3 \langle 0|\mathcal{O}(0)|n, \vec{P}, L\rangle\langle n, \vec{P}, L|\mathcal{O}^\dagger(0)|0\rangle}{(E_n + iP_4)}$$

Photoproduction $\langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu|\pi\rangle \equiv$



How can we get this from finite-volume observables?

Why did we expect $C_L(P)$ *to have poles?*



$$C_L(P) \equiv \int_L d^4x \; e^{-iPx} \langle 0|T\mathcal{O}(x)\mathcal{O}^\dagger(0)|0\rangle$$

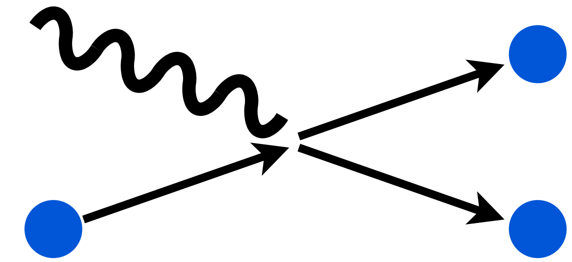
Insert a complete set finite-volume of states

$$C_L(P) \xrightarrow[P_4 \to iE_n]{} \frac{L^3 \langle 0|\mathcal{O}(0)|n, \vec{P}, L\rangle\langle n, \vec{P}, L|\mathcal{O}^\dagger(0)|0\rangle}{(E_n + iP_4)}$$

Now compare this to our factorized result

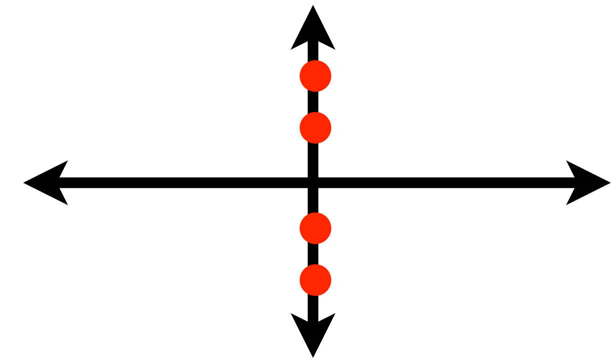
$$C_L(P) = C_\infty(P) - A' F \frac{1}{1 + \mathcal{M}_{2\to 2} F} A$$

Photoproduction $\langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu|\pi\rangle \equiv$



How can we get this from finite-volume observables?

Why did we expect $C_L(P)$ to have poles?



$$C_L(P) \equiv \int_L d^4x \; e^{-iPx} \langle 0|T\mathcal{O}(x)\mathcal{O}^\dagger(0)|0\rangle$$

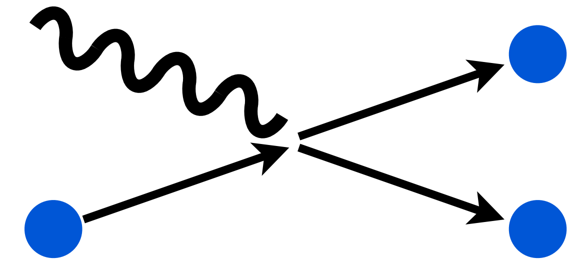
Insert a complete set finite-volume of states

$$C_L(P) \xrightarrow[P_4 \to iE_n]{} \frac{L^3 \langle 0|\mathcal{O}(0)|n, \vec{P}, L\rangle\langle n, \vec{P}, L|\mathcal{O}^\dagger(0)|0\rangle}{(E_n + iP_4)}$$

Now compare this to our factorized result

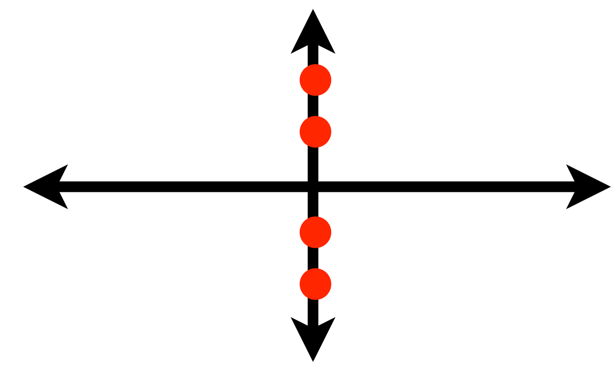
$$C_L(P) = C_\infty(P) - A' F \frac{1}{1 + \mathcal{M}_{2\to 2} F} A$$

$$\xrightarrow[P_4 \to iE_n]{} \frac{\langle 0|\mathcal{O}(0)|\pi\pi, \mathrm{in}\rangle \mathcal{R}(E_n, \vec{P}, L) \langle \pi\pi, \mathrm{out}|\mathcal{O}^\dagger(0)|0\rangle}{(E_n + iP_4)}$$

Photoproduction $\langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu|\pi\rangle \equiv$ 

How can we get this from finite-volume observables?

Why did we expect $C_L(P)$ to have poles?



$$C_L(P) \equiv \int_L d^4x \; e^{-iPx} \langle 0|T\mathcal{O}(x)\mathcal{O}^\dagger(0)|0\rangle$$

Insert a complete set finite-volume of states

$$C_L(P) \xrightarrow[P_4 \to iE_n]{} \frac{L^3 \langle 0|\mathcal{O}(0)|n, \vec{P}, L\rangle\langle n, \vec{P}, L|\mathcal{O}^\dagger(0)|0\rangle}{(E_n + iP_4)}$$

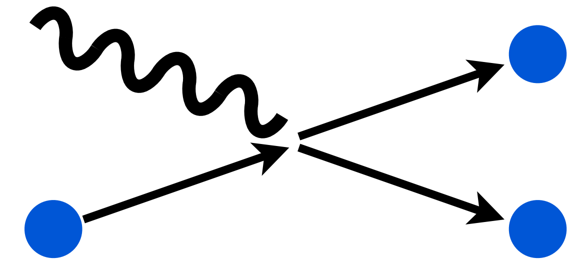
Now compare this to our factorized result

$$C_L(P) = C_\infty(P) - A' \boxed{F \frac{1}{1 + \mathcal{M}_{2\to2}F}} A$$

$\mathcal{R}$ **is the residue of this matrix**

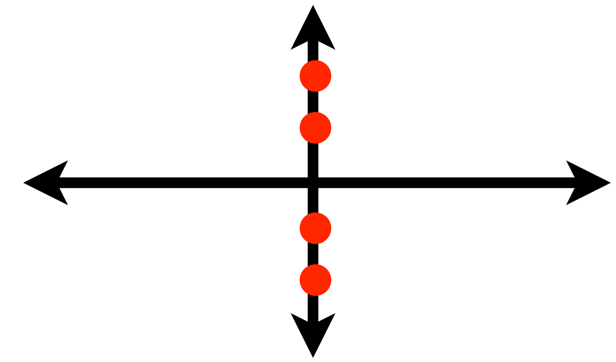
$$\xrightarrow[P_4 \to iE_n]{} \frac{\langle 0|\mathcal{O}(0)|\pi\pi, \mathrm{in}\rangle \mathcal{R}(E_n, \vec{P}, L)\langle \pi\pi, \mathrm{out}|\mathcal{O}^\dagger(0)|0\rangle}{(E_n + iP_4)}$$

Photoproduction $\langle \pi\pi, \mathrm{out} | \mathcal{J}_\mu | \pi \rangle \equiv$



How can we get this from finite-volume observables?

Why did we expect $C_L(P)$ to have poles?



$$C_L(P) \equiv \int_L d^4x \; e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

Insert a complete set finite-volume of states

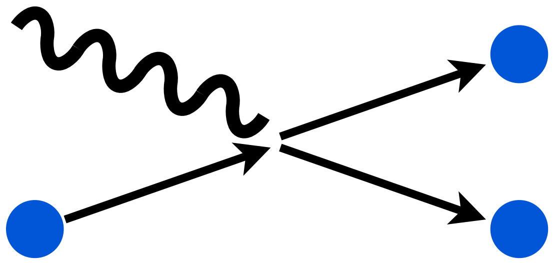
$$C_L(P) \xrightarrow[P_4 \to iE_n]{} \frac{L^3 \langle 0 | \mathcal{O}(0) | n, \vec{P}, L \rangle \langle n, \vec{P}, L | \mathcal{O}^\dagger(0) | 0 \rangle}{(E_n + iP_4)}$$

Now compare this to our factorized result

$$C_L(P) = C_\infty(P) - A' F \frac{1}{1 + \mathcal{M}_{2\to2} F} A$$

$\mathcal{R}$ is the residue of this matrix

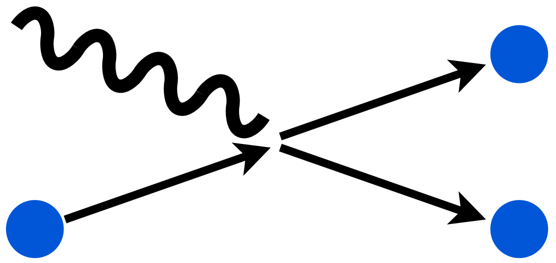
$$\xrightarrow[P_4 \to iE_n]{} \frac{\langle 0 | \mathcal{O}(0) | \pi\pi, \mathrm{in} \rangle \mathcal{R}(E_n, \vec{P}, L) \langle \pi\pi, \mathrm{out} | \mathcal{O}^\dagger(0) | 0 \rangle}{(E_n + iP_4)}$$

Photoproduction $\langle \pi\pi, \mathrm{out} | \mathcal{J}_\mu | \pi \rangle \equiv$ 

How can we get this from finite-volume observables?

$$L^3 \langle 0 | \mathcal{O}(0) | n, \vec{P}, L \rangle \langle n, \vec{P}, L | \mathcal{O}^\dagger(0) | 0 \rangle =$$

$$\langle 0 | \mathcal{O}(0) | \pi\pi, \mathrm{in} \rangle \mathcal{R}(E_n, \vec{P}, L) \langle \pi\pi, \mathrm{out} | \mathcal{O}^\dagger(0) | 0 \rangle$$

Photoproduction $\langle \pi\pi, \mathrm{out} | \mathcal{J}_\mu | \pi \rangle \equiv$ 

How can we get this from finite-volume observables?

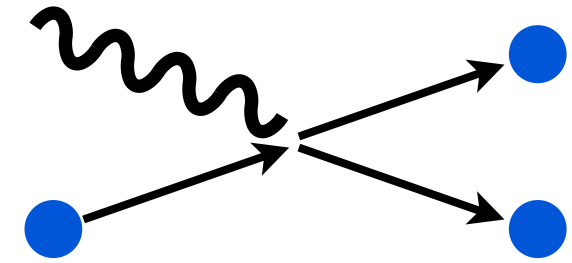
$$L^3 \langle 0 | \mathcal{O}(0) | n, \vec{P}, L \rangle \langle n, \vec{P}, L | \mathcal{O}^\dagger(0) | 0 \rangle =$$

$$\langle 0 | \mathcal{O}(0) | \pi\pi, \mathrm{in} \rangle \mathcal{R}(E_n, \vec{P}, L) \langle \pi\pi, \mathrm{out} | \mathcal{O}^\dagger(0) | 0 \rangle$$

One has the freedom to choose $\mathcal{O}^\dagger$ such that $\mathcal{O}^\dagger | 0 \rangle = \mathcal{J}_\mu | \pi \rangle$.

(Finite-volume effects are exponentially suppressed for single particles.)

Photoproduction $\langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu|\pi\rangle \equiv$



How can we get this from finite-volume observables?

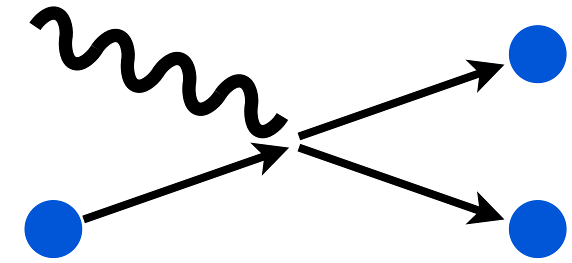
$$L^3\langle 0|\mathcal{O}(0)|n,\vec{P},L\rangle\langle n,\vec{P},L|\mathcal{O}^\dagger(0)|0\rangle = \langle 0|\mathcal{O}(0)|\pi\pi,\mathrm{in}\rangle\mathcal{R}(E_n,\vec{P},L)\langle \pi\pi,\mathrm{out}|\mathcal{O}^\dagger(0)|0\rangle$$

One has the freedom to choose $\mathcal{O}^\dagger$ such that $\mathcal{O}^\dagger|0\rangle = \mathcal{J}_\mu|\pi\rangle$.

(Finite-volume effects are exponentially suppressed for single particles.)

$$2\omega_\pi L^6|\langle n,\vec{P},L|\mathcal{J}_\mu(0)|\pi,L\rangle|^2 = \langle \pi|\mathcal{J}_\mu(0)|\pi\pi,\mathrm{in}\rangle\mathcal{R}(E_n,\vec{P},L)\langle \pi\pi,\mathrm{out}|\mathcal{J}_\mu(0)|\pi\rangle$$

R. A. Briceño, MTH, A. Walker-Loud, *Phys. Rev.* D91, 034501 (2015)
R. A. Briceño, MTH, *Phys. Rev.* D92, 074509 (2015)

Photoproduction $\langle \pi\pi, \mathrm{out} | \mathcal{J}_\mu | \pi \rangle \equiv$ 

How can we get this from finite-volume observables?

$$L^3 \langle 0 | \mathcal{O}(0) | n, \vec{P}, L \rangle \langle n, \vec{P}, L | \mathcal{O}^\dagger(0) | 0 \rangle =$$

$$\langle 0 | \mathcal{O}(0) | \pi\pi, \mathrm{in} \rangle \mathcal{R}(E_n, \vec{P}, L) \langle \pi\pi, \mathrm{out} | \mathcal{O}^\dagger(0) | 0 \rangle$$

One has the freedom to choose $\mathcal{O}^\dagger$ such that $\mathcal{O}^\dagger |0\rangle = \mathcal{J}_\mu |\pi\rangle$.

(Finite-volume effects are exponentially suppressed for single particles.)

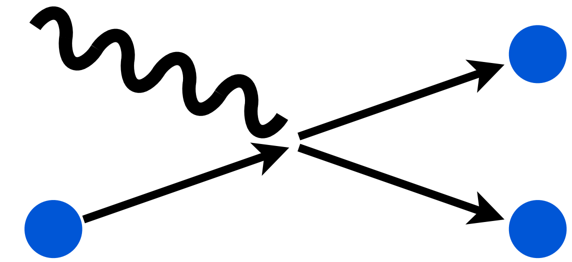
get this from the lattice

$$2\omega_\pi L^6 |\langle n, \vec{P}, L | \mathcal{J}_\mu(0) | \pi, L \rangle|^2 =$$

experimental observable

$$\langle \pi | \mathcal{J}_\mu(0) | \pi\pi, \mathrm{in} \rangle \mathcal{R}(E_n, \vec{P}, L) \langle \pi\pi, \mathrm{out} | \mathcal{J}_\mu(0) | \pi \rangle$$

R. A. Briceño, MTH, A. Walker-Loud, *Phys. Rev.* D91, 034501 (2015)
R. A. Briceño, MTH, *Phys. Rev.* D92, 074509 (2015)

Photoproduction $\langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu|\pi\rangle \equiv$ 

How can we get this from finite-volume observables?

$$L^3 \langle 0|\mathcal{O}(0)|n, \vec{P}, L\rangle\langle n, \vec{P}, L|\mathcal{O}^\dagger(0)|0\rangle =$$
$$\langle 0|\mathcal{O}(0)|\pi\pi, \mathrm{in}\rangle \mathcal{R}(E_n, \vec{P}, L)\langle \pi\pi, \mathrm{out}|\mathcal{O}^\dagger(0)|0\rangle$$

One has the freedom to choose $\mathcal{O}^\dagger$ such that $\mathcal{O}^\dagger|0\rangle = \mathcal{J}_\mu|\pi\rangle$.

(Finite-volume effects are exponentially suppressed for single particles.)

get this from the lattice

$$2\omega_\pi L^6 |\langle n, \vec{P}, L|\mathcal{J}_\mu(0)|\pi, L\rangle|^2 =$$

experimental observable

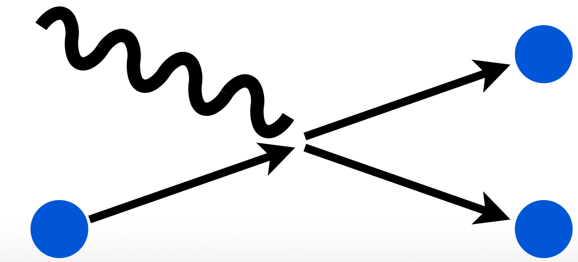
$$\langle \pi|\mathcal{J}_\mu(0)|\pi\pi, \mathrm{in}\rangle \mathcal{R}(E_n, \vec{P}, L)\langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu(0)|\pi\rangle$$

$$\mathcal{R}(E_n, \vec{P}, L) = -\underset{E_n}{\mathrm{Residue}}\left[\frac{1}{F^{-1} + \mathcal{M}_{2\to 2}}\right]$$

R. A. Briceño, MTH, A. Walker-Loud, *Phys. Rev.* D91, 034501 (2015)

R. A. Briceño, MTH, *Phys. Rev.* D92, 074509 (2015)

Photoproduction $\langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu|\pi\rangle \equiv$



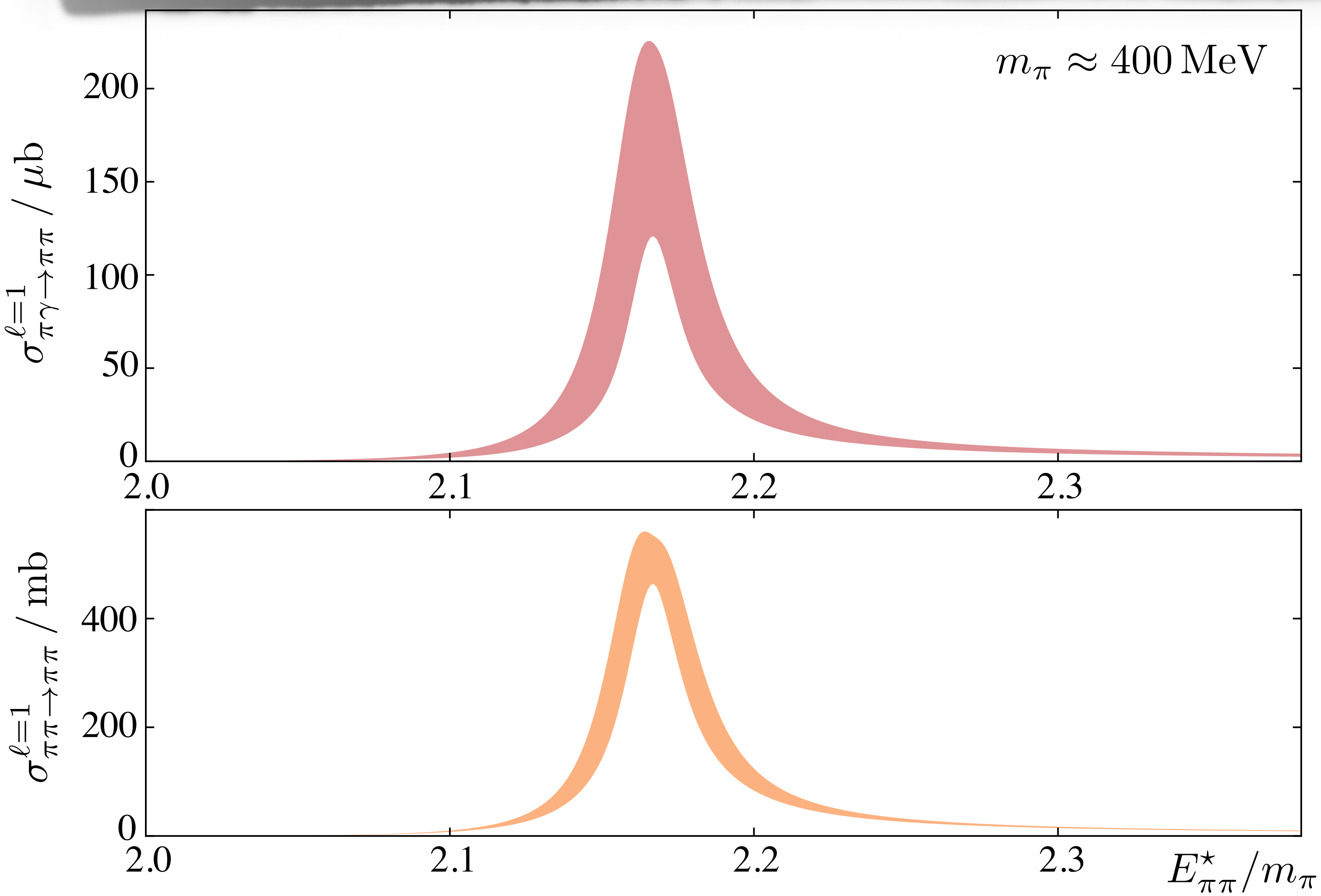
get this from the lattice

$$2\omega_\pi L^6 |\langle n, \vec{P}, L|\mathcal{J}_\mu(0)|\pi, L\rangle|^2 =$$

experimental observable

$$\langle \pi|\mathcal{J}_\mu(0)|\pi\pi, \mathrm{in}\rangle \mathcal{R}(E_n, \vec{P}, L)\langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu(0)|\pi\rangle$$

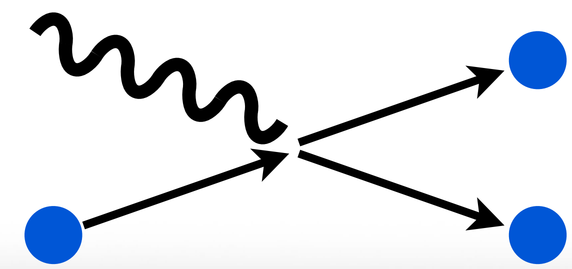
Briceño, MTH, Walker-Loud/Briceño, MTH



Photoproduction in the rho channel

Briceño, Dudek, Edwards, Schultz, Thomas, Wilson arXiv: 1507.6622

Photoproduction $\langle \pi\pi, \text{out} | \mathcal{J}_\mu | \pi \rangle \equiv$

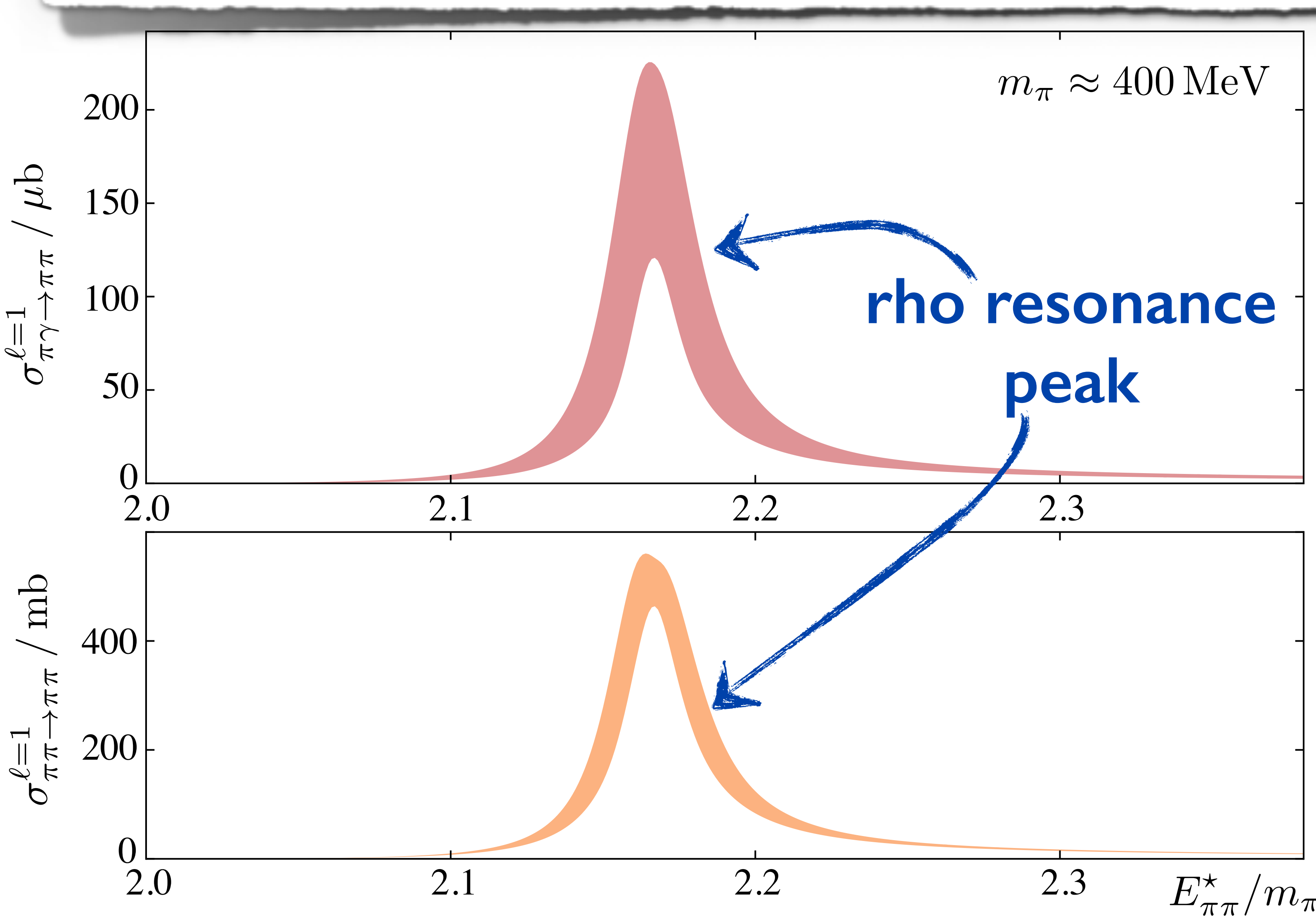


get this from the lattice

$$2\omega_\pi L^6 |\langle n, \vec{P}, L | \mathcal{J}_\mu(0) | \pi, L\rangle|^2 = \langle \pi | \mathcal{J}_\mu(0) | \pi\pi, \text{in}\rangle \mathcal{R}(E_n, \vec{P}, L) \langle \pi\pi, \text{out} | \mathcal{J}_\mu(0) | \pi \rangle$$

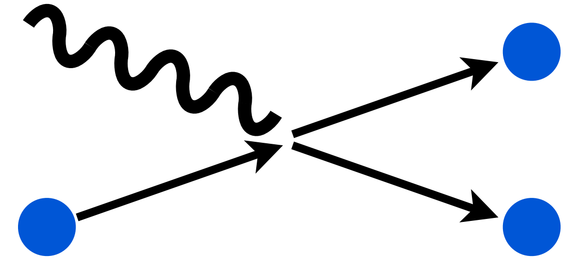
experimental observable

Briceño, MTH, Walker-Loud/Briceño, MTH



Photoproduction in the rho channel

Briceño, Dudek, Edwards, Schultz, Thomas, Wilson
arXiv: 1507.6622

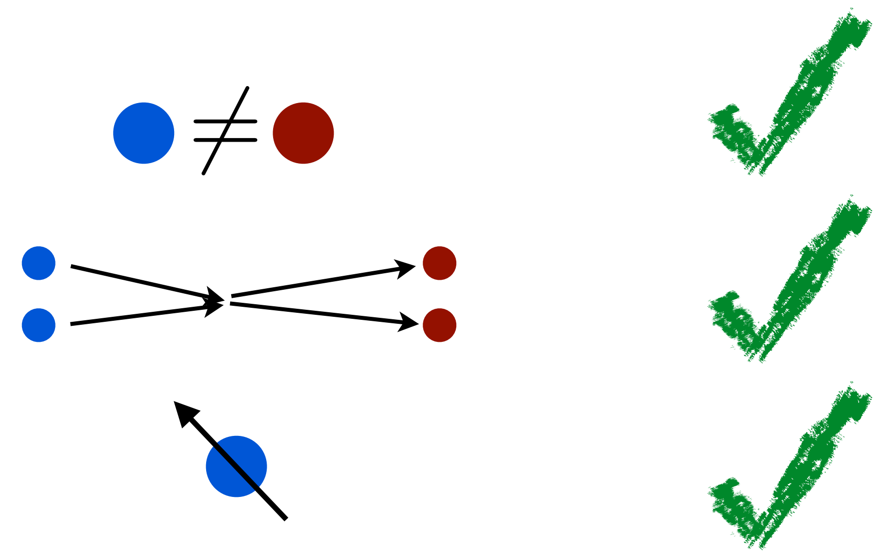
Photoproduction $\langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu|\pi\rangle \equiv$ 

Result is very general

non-indentical particles

multiple two-particle channels

particles with spin



H. B. Meyer, Eur.Phys.J. A49, 84 (2013)
Agadjanov, Bernard, Meißner and Rusetsky, (2014), Nucl.Phys. B886, 1199 (2014).
R. A. Briceño, MTH, A. Walker-Loud, *Phys. Rev.* D91, 034501 (2015)
R. A. Briceño, MTH, *Phys. Rev.* D92, 074509 (2015)

all generalizations of
L. Lellouch and M. Lüscher, *Commun. Math. Phys.* 219, 31 (2001)

Two-particle scattering

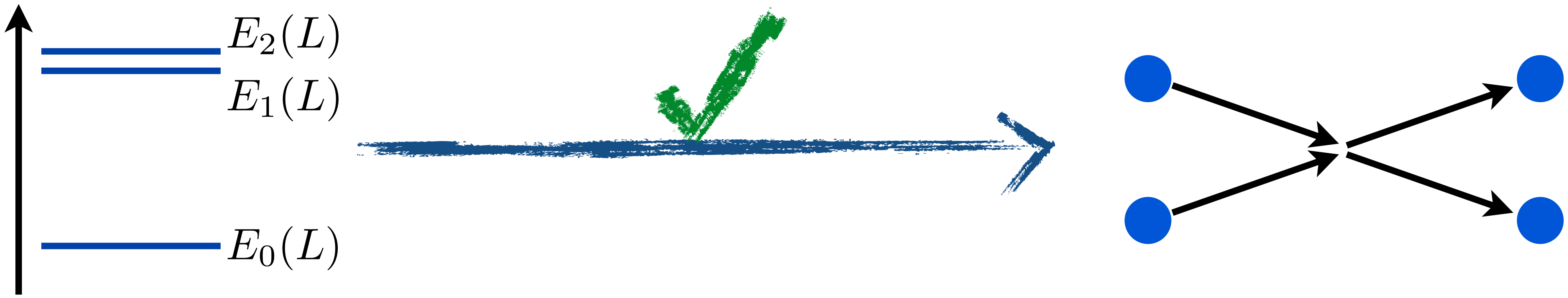
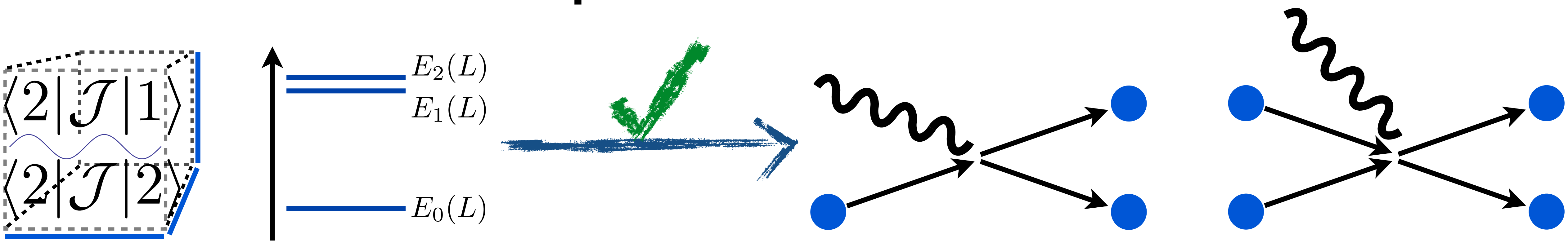
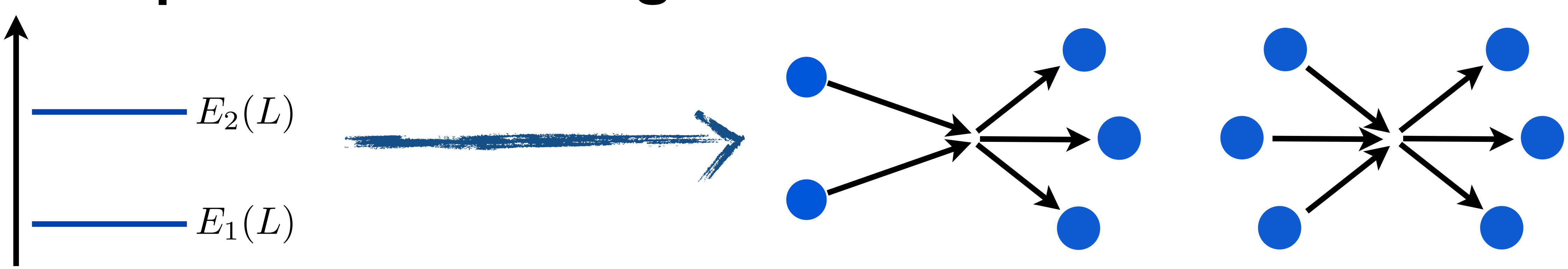


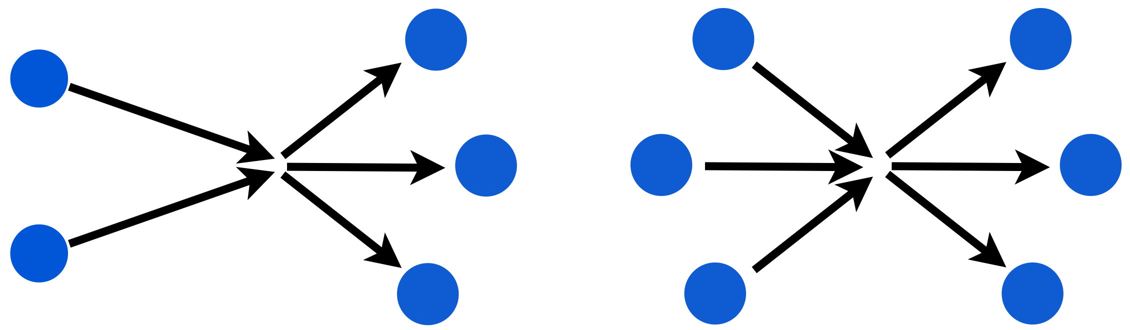
Photo- and electroproduction



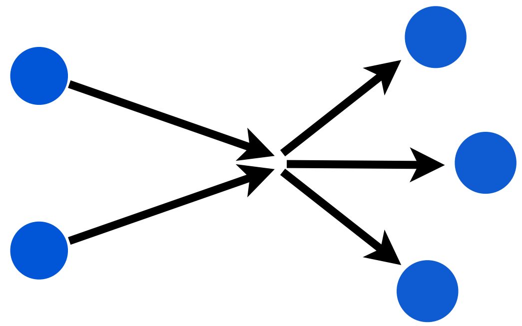
Three-particle scattering

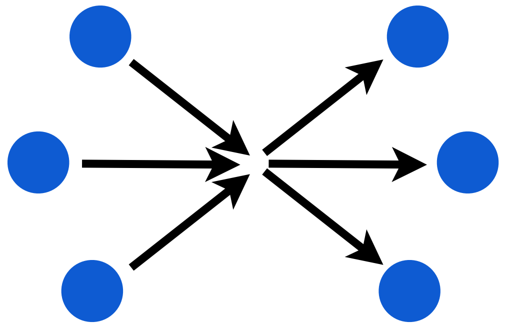


Begin by considering the infinite-volume observables



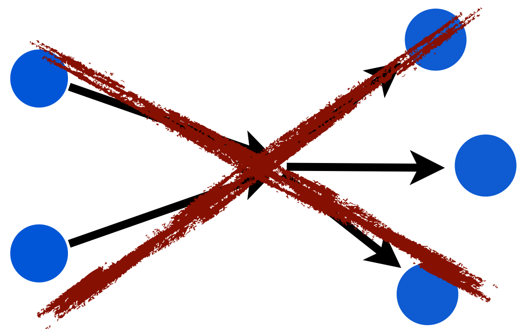
Begin by considering the infinite-volume observables

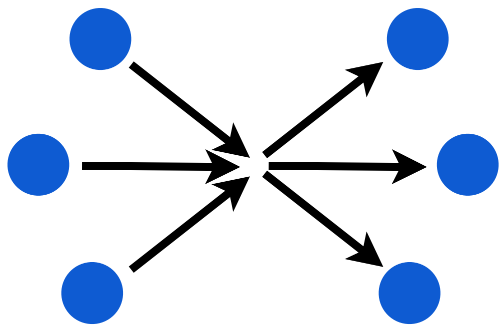




Because of "finite-volume rescattering" it is not possible to access two-to-three without also accessing three-to-three

Begin by considering the infinite-volume observables

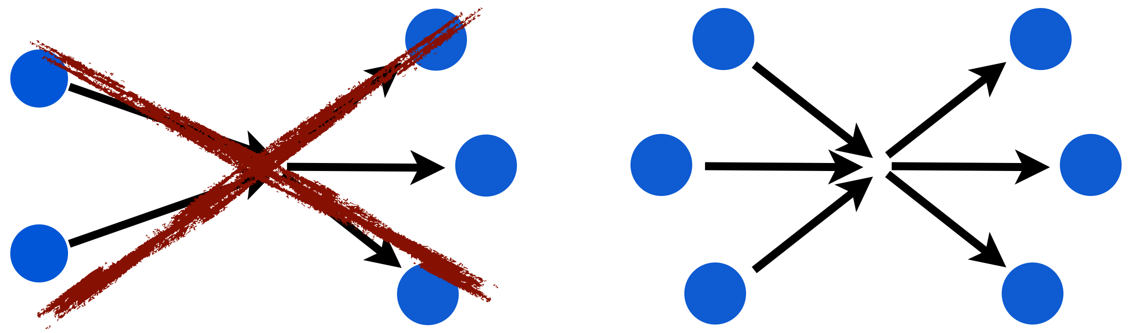




Because of "finite-volume rescattering" it is not possible to access two-to-three without also accessing three-to-three

For now we turn off two-to-three scattering using a symmetry

Begin by considering the infinite-volume observables



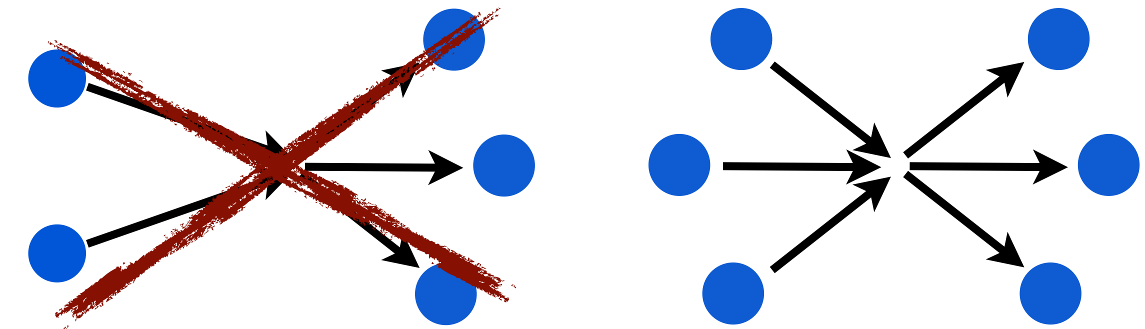
Because of "finite-volume rescattering" it is not possible to access two-to-three without also accessing three-to-three

For now we turn off two-to-three scattering using a symmetry

Three-to-three amplitude has kinematic singularities

$i\mathcal{M}_{3\to3} \equiv$ fully connected correlator with six external legs amputated and projected on shell

Begin by considering the infinite-volume observables

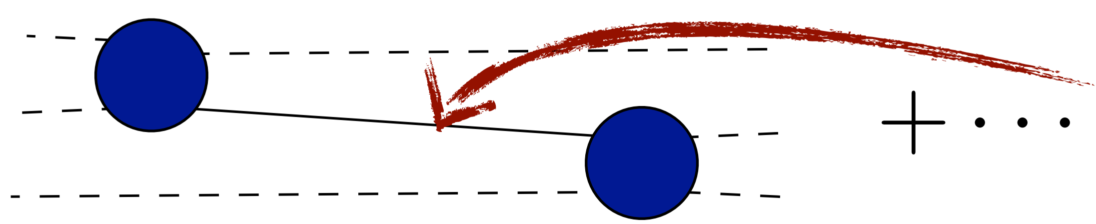


Because of "finite-volume rescattering" it is not possible to access two-to-three without also accessing three-to-three

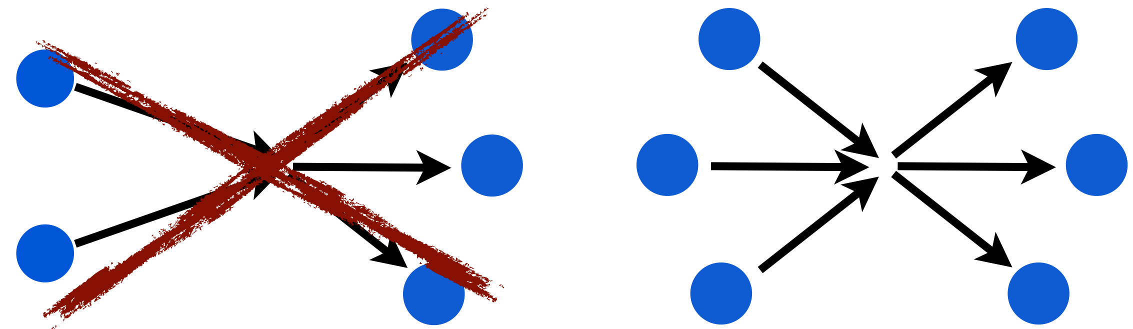
For now we turn off two-to-three scattering using a symmetry

Three-to-three amplitude has kinematic singularities

$i\mathcal{M}_{3\to3} \equiv$ fully connected correlator with six external legs amputated and projected on shell

$=$  $+ \cdots$

Begin by considering the infinite-volume observables

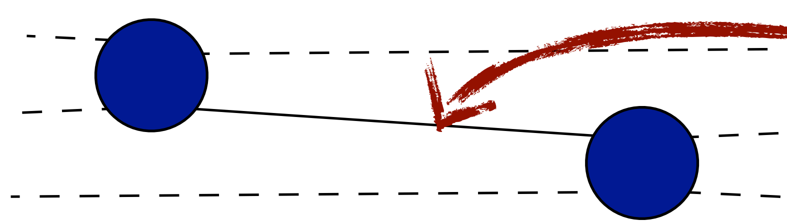


Because of "finite-volume rescattering" it is not possible to access two-to-three without also accessing three-to-three

For now we turn off two-to-three scattering using a symmetry

Three-to-three amplitude has kinematic singularities

$i\mathcal{M}_{3\to3} \equiv$ fully connected correlator with six external legs amputated and projected on shell

$=$  $+\cdots$

Certain external momenta put this on-shell!

Three-to-three amplitude has more degrees of freedom

8 degrees of freedom including total energy

Compared with 2 for the two-to-two amplitude

How can we possibly hope to extract a singular, eight-coordinate function using finite-volume energies?

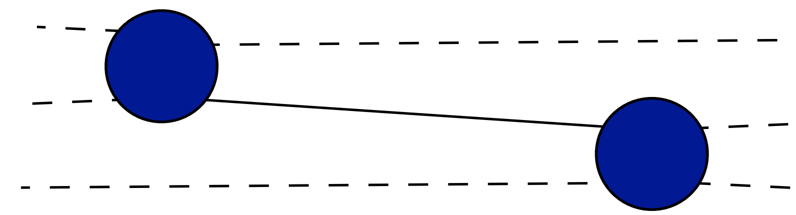
How can we possibly hope to extract a singular, eight-coordinate function using finite-volume energies?

Short answer…

How can we possibly hope to extract a singular, eight-coordinate function using finite-volume energies?

Short answer…

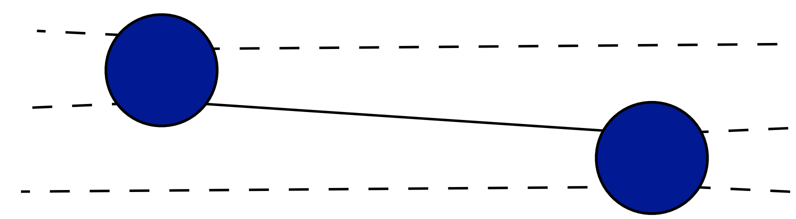
(1). We found that the spectrum depends on a modified quantity with singularities removed

$$\mathcal{K}_{\mathrm{df},3} \not\supset$$ 

How can we possibly hope to extract a singular, eight-coordinate function using finite-volume energies?

Short answer…

(1). We found that the spectrum depends on a modified quantity with singularities removed

$$\mathcal{K}_{\mathrm{df},3} \not\supset$$ 

(a) Same degrees of freedom as $\mathcal{M}_{3\to3}$.

(b) Relation to $\mathcal{M}_{3\to3}$ is known (depends only on on-shell $\mathcal{M}_{2\to2}$)

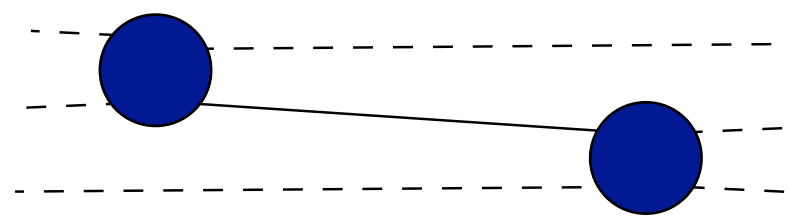
(c) Smooth function (allows harmonic decomposition)

How can we possibly hope to extract a singular, eight-coordinate function using finite-volume energies?

Short answer…

(1). We found that the spectrum depends on a modified quantity with singularities removed

$$\mathcal{K}_{\mathrm{df},3} \not\supset$$



(a) Same degrees of freedom as $\mathcal{M}_{3\to3}$.

(b) Relation to $\mathcal{M}_{3\to3}$ is known (depends only on on-shell $\mathcal{M}_{2\to2}$)

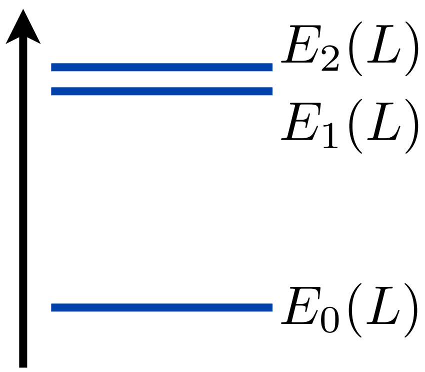
(c) Smooth function (allows harmonic decomposition)

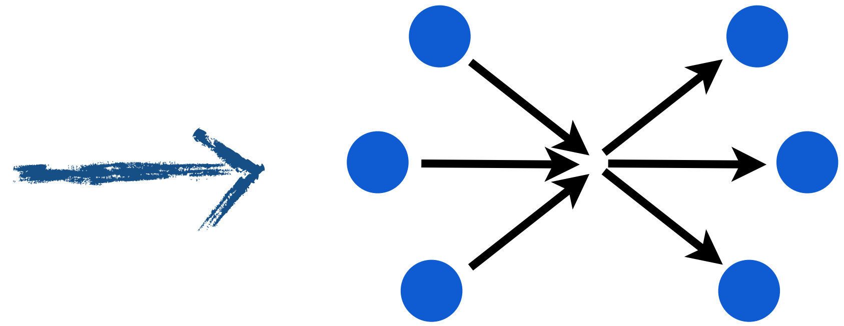
(2). Degrees of freedom encoded in an extended matrix space

($\vec{k}$ is restricted to finite-volume momenta)

$$\vec{k}, \ell, m$$

Three-to-three scattering

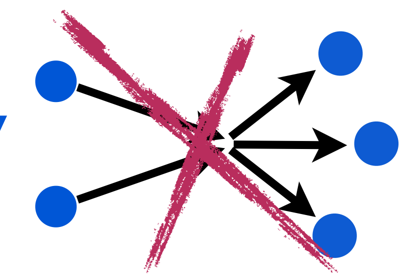




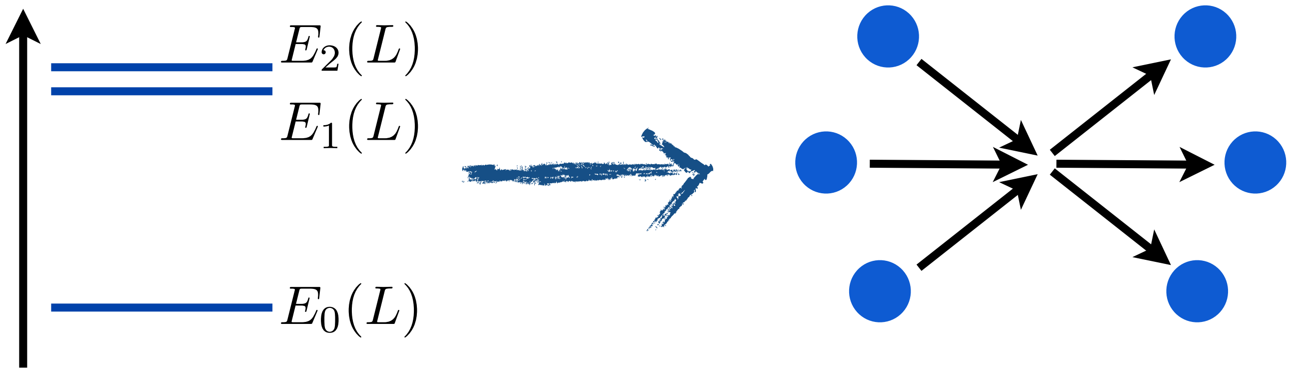
For now assume…

identical scalars, mass m

$\mathbb{Z}_2$ **symmetry**



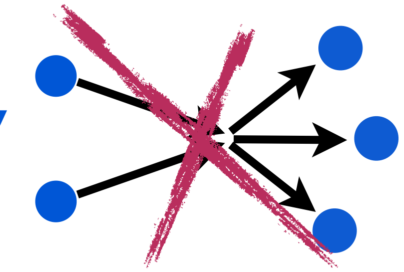
Three-to-three scattering



For now assume…

identical scalars, mass m

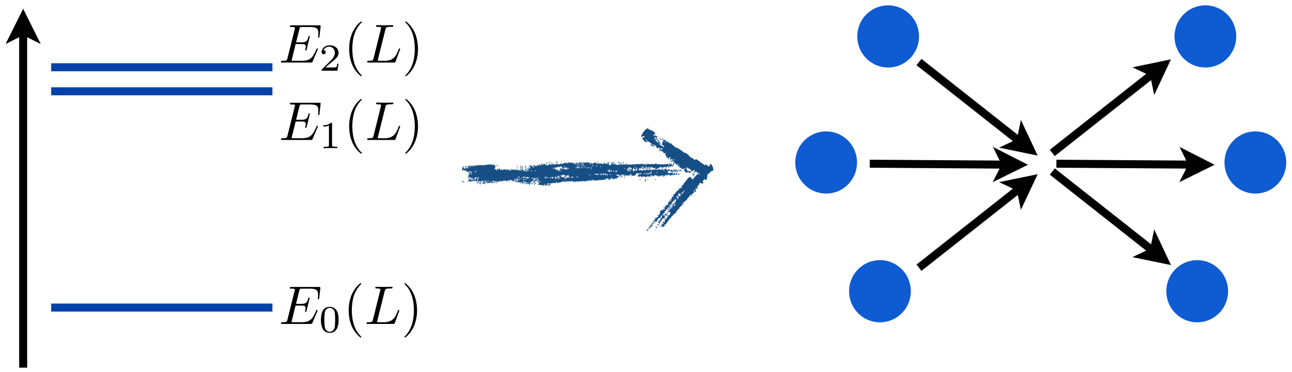
$\mathbb{Z}_2$ **symmetry**



$$C_L(P) \equiv \int_L d^4x \; e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

three-pion interpolator

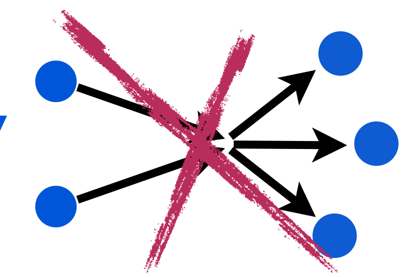
Three-to-three scattering



For now assume...

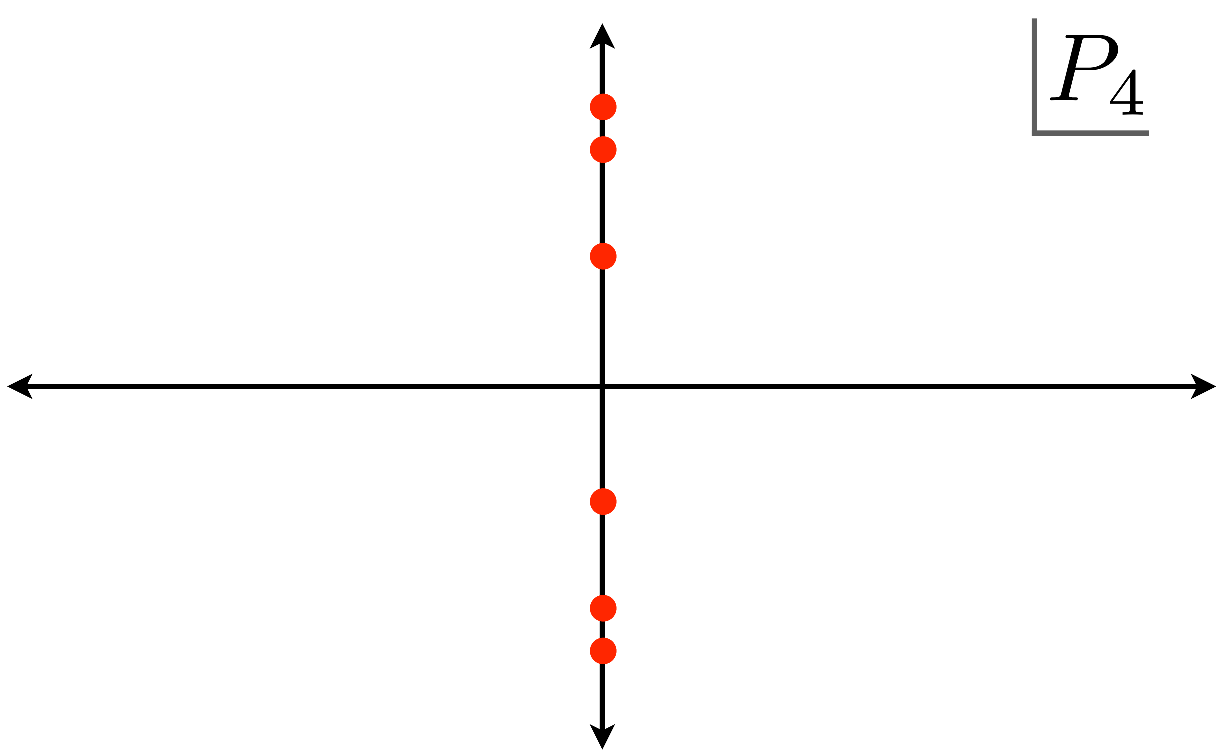
identical scalars, mass m

$\mathbb{Z}_2$ **symmetry**



$$C_L(P) \equiv \int_L d^4x \; e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

three-pion interpolator



Calculate $C_L(P)$ to all orders in perturbation theory and determine locations of poles.

Require $m < E^* < 5m$ to isolate three-particle states

Three-particle result

At fixed $(L, \vec{P})$, finite-volume energies are solutions to

$$\det_{k,\ell,m}\left[\mathcal{K}_{\text{df},3}^{-1} + F_3\right] = 0$$

MTH and Sharpe, *Phys. Rev.* D90, 116003 (2014)

$F_3 \equiv$ **matrix that depends on known geometric functions as well as $\mathcal{M}_{2\to2}$.**

Three-particle result

At fixed $(L, \vec{P})$, finite-volume energies are solutions to $\det_{k,\ell,m}\left[\mathcal{K}_{\mathrm{df},3}^{-1} + F_3\right] = 0$

MTH and Sharpe, *Phys. Rev.* D90, 116003 (2014)

$F_3 \equiv$ matrix that depends on known geometric functions as well as $\mathcal{M}_{2\to 2}$.

---

(1). Use two-particle quantization condition to constrain $\mathcal{M}_{2\to 2}$ and thus determine $F_3(E, \vec{P}, L)$

Three-particle result

At fixed $(L, \vec{P})$, finite-volume energies are solutions to $\det_{k,\ell,m}\left[\mathcal{K}_{\mathrm{df},3}^{-1} + F_3\right] = 0$

MTH and Sharpe, *Phys. Rev.* D90, 116003 (2014)

$F_3 \equiv$ **matrix that depends on known geometric functions as well as $\mathcal{M}_{2\to2}$.**

---

(1). Use two-particle quantization condition to constrain $\mathcal{M}_{2\to2}$ and thus determine $F_3(E, \vec{P}, L)$

(2). Use harmonic decomposition + various parametrizations to express $\mathcal{K}_{\mathrm{df},3}(E^*)$ in terms of N unknown parameters

Three-particle result

At fixed $(L, \vec{P})$, finite-volume energies are solutions to $\det_{k,\ell,m}\left[\mathcal{K}_{\mathrm{df},3}^{-1} + F_3\right] = 0$

MTH and Sharpe, *Phys. Rev.* D90, 116003 (2014)

$F_3 \equiv$ **matrix that depends on known geometric functions as well as** $\mathcal{M}_{2\to 2}$ **.**

---

(1). Use two-particle quantization condition to constrain $\mathcal{M}_{2\to 2}$ **and thus determine** $F_3(E, \vec{P}, L)$

(2). Use harmonic decomposition + various parametrizations to express $\mathcal{K}_{\mathrm{df},3}(E^*)$ **in terms of** N **unknown parameters**

(3). Use quantization condition with lattice (or otherwise) determined energies to determine all parameters

Three-particle result

At fixed $(L, \vec{P})$, finite-volume energies are solutions to $\det_{k,\ell,m}\left[\mathcal{K}_{\mathrm{df},3}^{-1} + F_3\right] = 0$

MTH and Sharpe, *Phys. Rev.* D90, 116003 (2014)

$F_3 \equiv$ matrix that depends on known geometric functions as well as $\mathcal{M}_{2\to2}$.

---

(1). Use two-particle quantization condition to constrain $\mathcal{M}_{2\to2}$ and thus determine $F_3(E, \vec{P}, L)$

(2). Use harmonic decomposition + various parametrizations to express $\mathcal{K}_{\mathrm{df},3}(E^*)$ in terms of N unknown parameters

(3). Use quantization condition with lattice (or otherwise) determined energies to determine all parameters

(4). Use known relation to recover $\mathcal{M}_{3\to3}$

MTH and Sharpe, *Phys. Rev.* D92, 114509 (2015)

Three-particle result

At fixed $(L, \vec{P})$, finite-volume energies are solutions to

$$\det_{k,\ell,m}\left[\mathcal{K}_{\text{df},3}^{-1} + F_3\right] = 0$$

MTH and Sharpe, *Phys. Rev.* D90, 116003 (2014)

Some nice features…

Matrices automatically truncated in the $\vec{k}$ index

truncate angular momentum space → **solvable system**

Three-particle result

At fixed $(L, \vec{P})$, finite-volume energies are solutions to

$$\det_{k,\ell,m}\left[\mathcal{K}_{\text{df},3}^{-1} + F_3\right] = 0$$

MTH and Sharpe, *Phys. Rev.* D90, 116003 (2014)

Some nice features...

Matrices automatically truncated in the $\vec{k}$ index

truncate angular momentum space → **solvable system**

Expanding about weak interactions gives an important check

$$E = 3m + \frac{a_3}{L^3} + \frac{a_4}{L^4} + \frac{a_5}{L^5} + \frac{a_6}{L^6} + \mathcal{O}(1/L^7)$$

Our result agrees with existing results for $a_{3\to5}$ and gives a prediction for a_6

K. Huang and C. Yang, *Phys. Rev.* 105 (1957) 767-775

Beane, Detmold, Savage, *Phys. Rev.* D76 (2007) 074507

MTH and Sharpe, *Phys. Rev.* D 93, 096006 (2016)

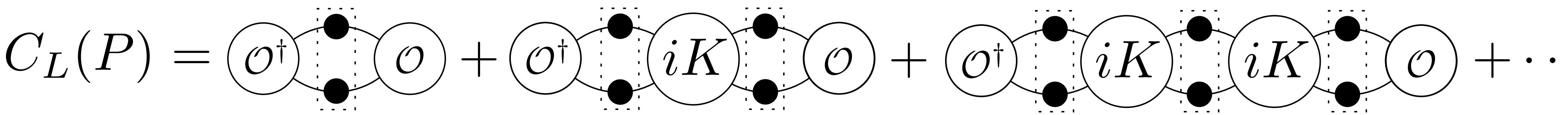
Three-particle result $\det_{k,\ell,m}\left[\mathcal{K}_{\mathrm{df},3}^{-1} + F_3\right] = 0$

Sketch of the derivation…

Three-particle result $\det_{k,\ell,m}\left[\mathcal{K}_{\mathrm{df},3}^{-1} + F_3\right] = 0$

Sketch of the derivation…

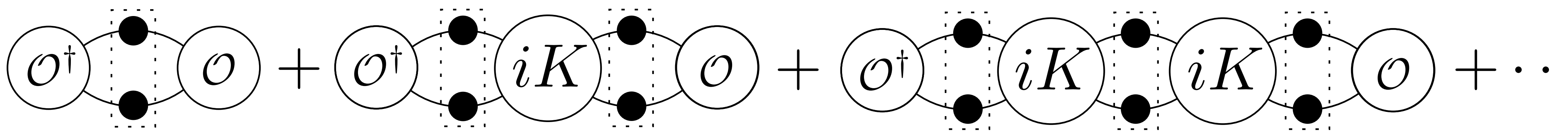
Recall for two particles we started with a "skeleton expansion"



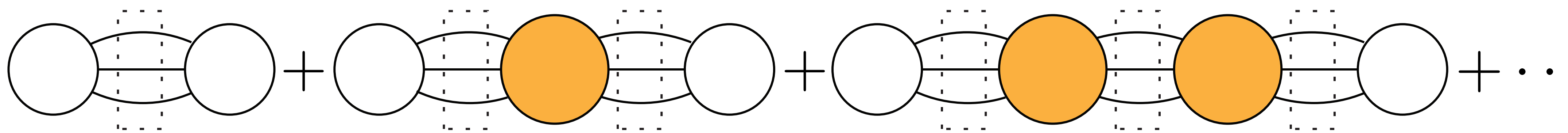
Three-particle result $\det_{k,\ell,m}\left[\mathcal{K}_{\mathrm{df},3}^{-1} + F_3\right] = 0$

Sketch of the derivation...

Recall for two particles we started with a "skeleton expansion"

$C_L(P) =$ 

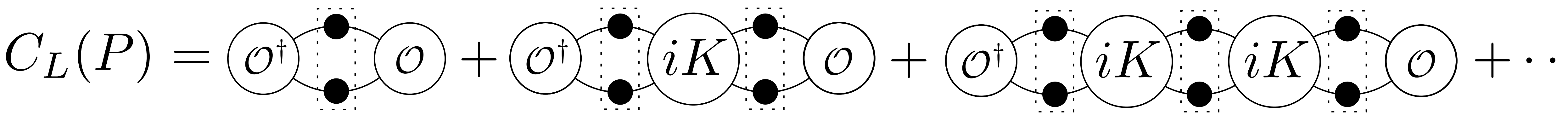
So now we need the same for three...

$C_L(E, \vec{P}) \stackrel{?}{=}$ 

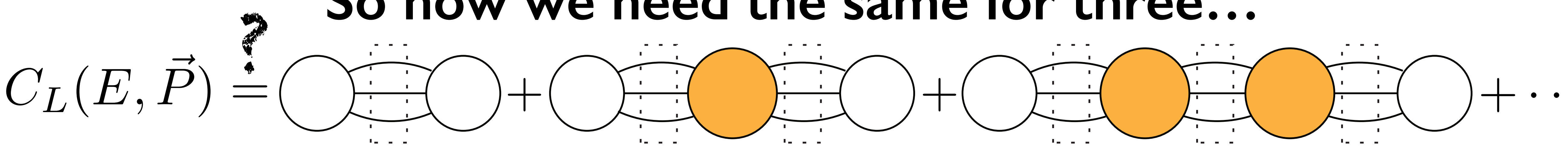
Three-particle result $\det_{k,\ell,m}\left[\mathcal{K}_{\mathrm{df},3}^{-1} + F_3\right] = 0$

Sketch of the derivation…

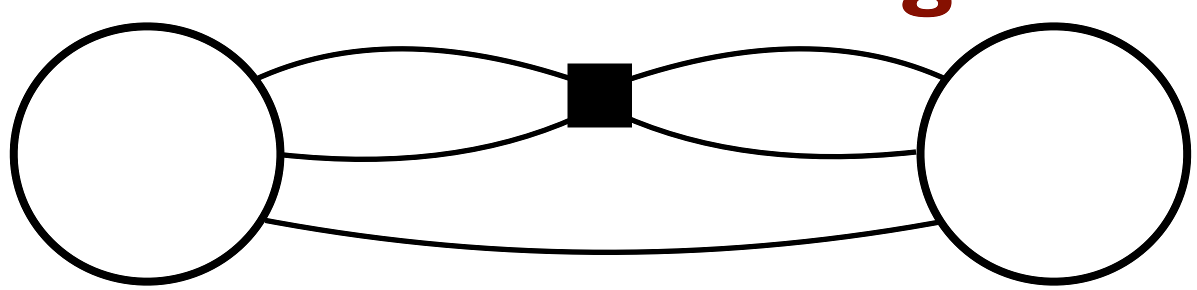
Recall for two particles we started with a "skeleton expansion"

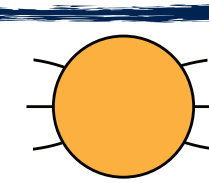


So now we need the same for three…



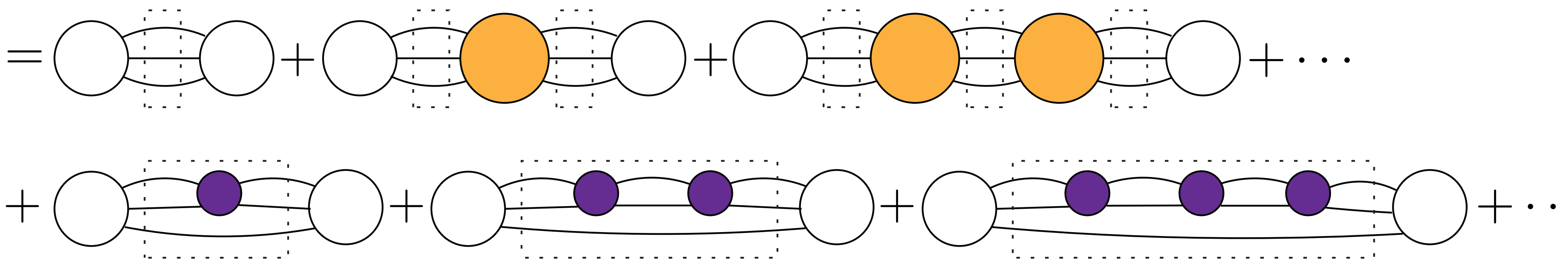
No! We also need diagrams like



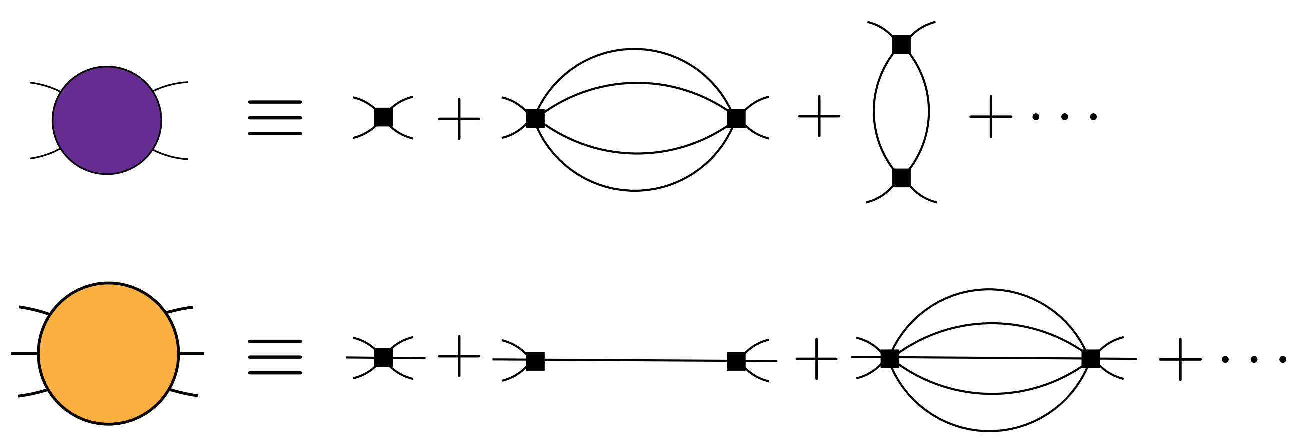
Disconnected diagrams in  **lead to singularities that invalidate the derivation**

New skeleton expansion

$C_L(E, \vec{P}) =$

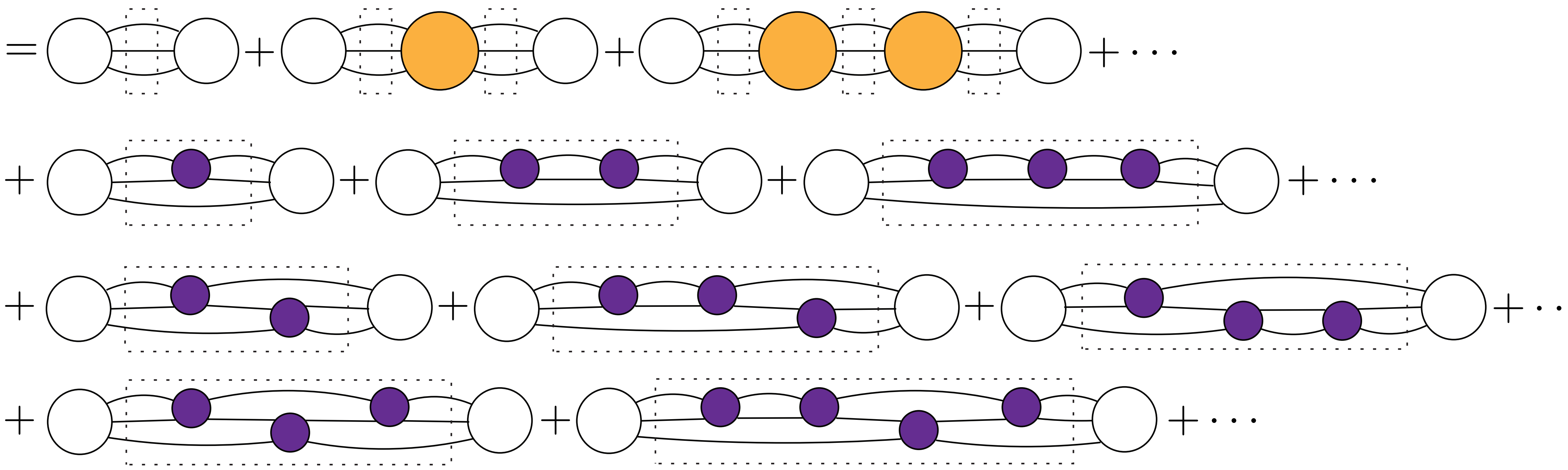


Kernel definitions:

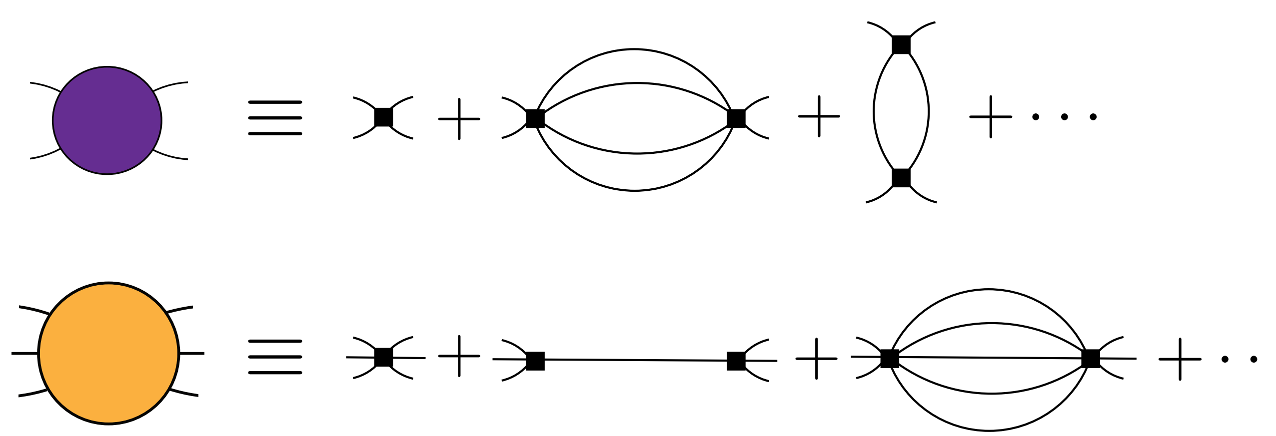


New skeleton expansion

$C_L(E, \vec{P}) =$

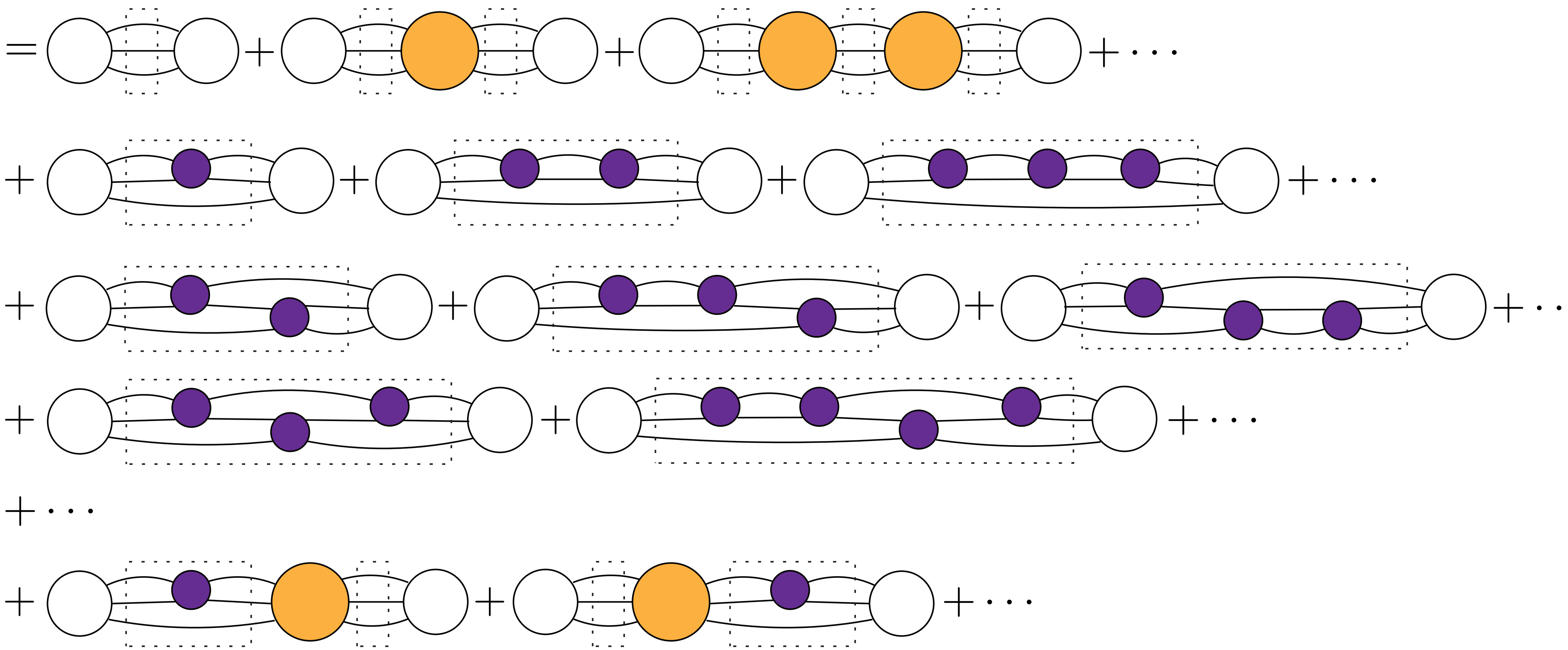


Kernel definitions:

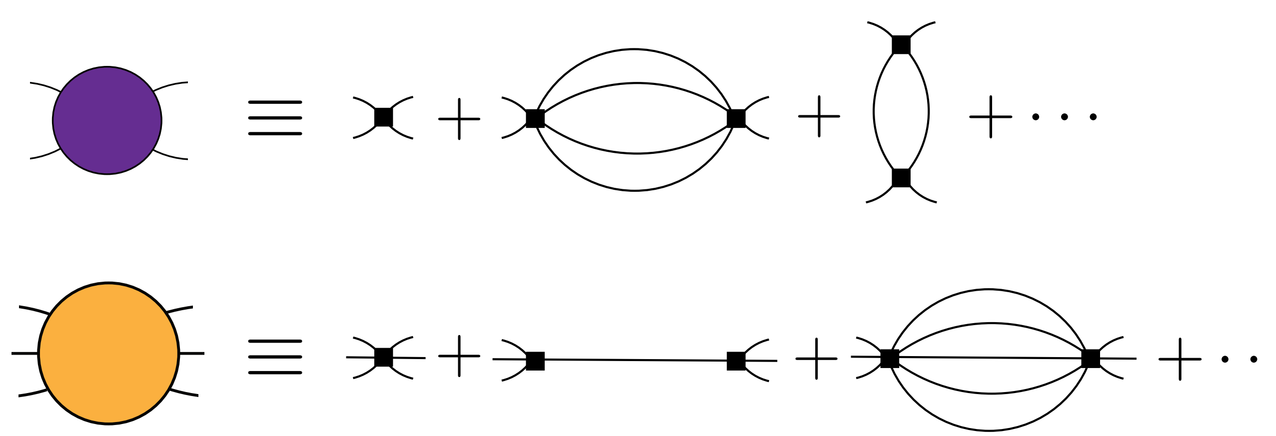


New skeleton expansion

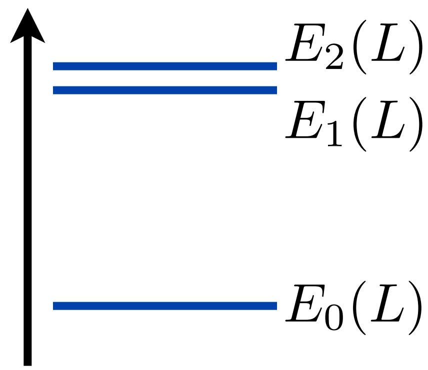
$C_L(E, \vec{P}) =$

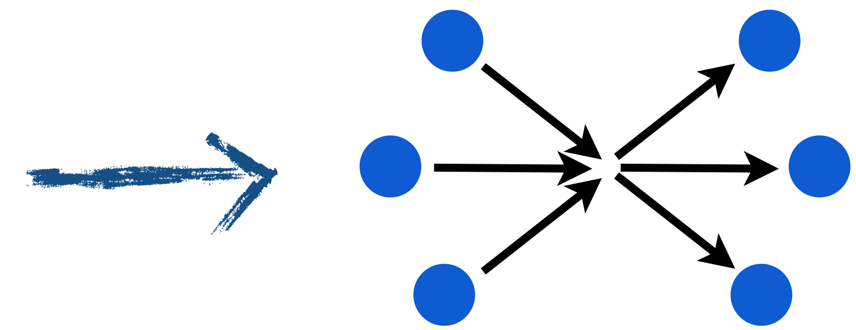


Kernel definitions:

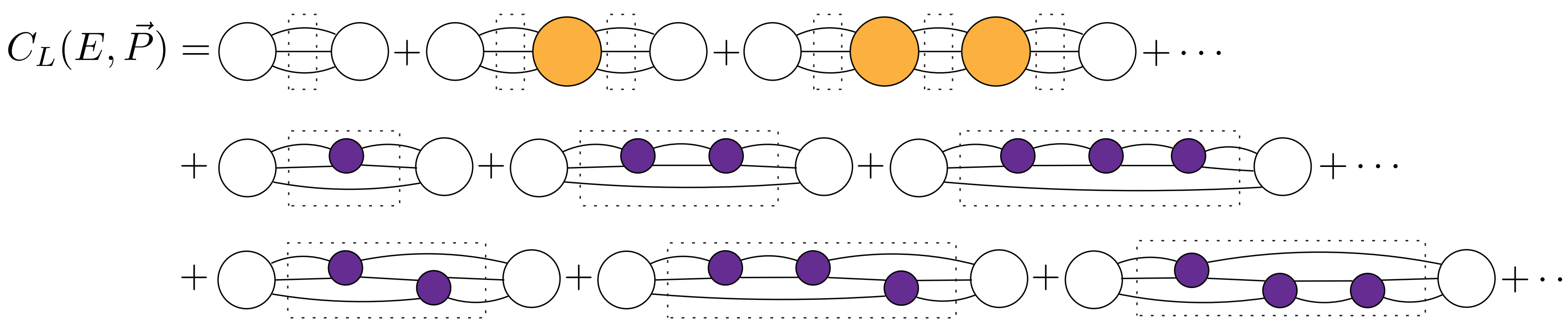


Three-to-three scattering





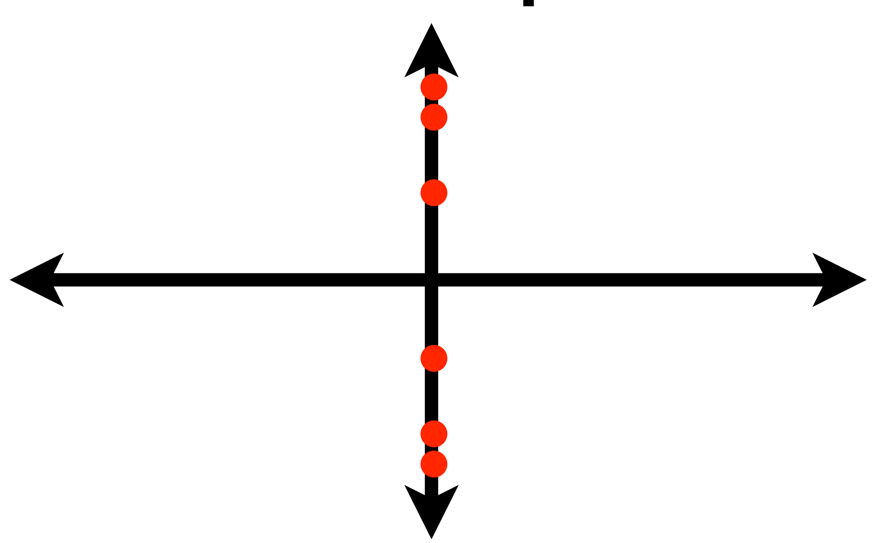
1. **Work out the three particle skeleton expansion**



2. **Break diagrams into finite- and infinite-volume parts**

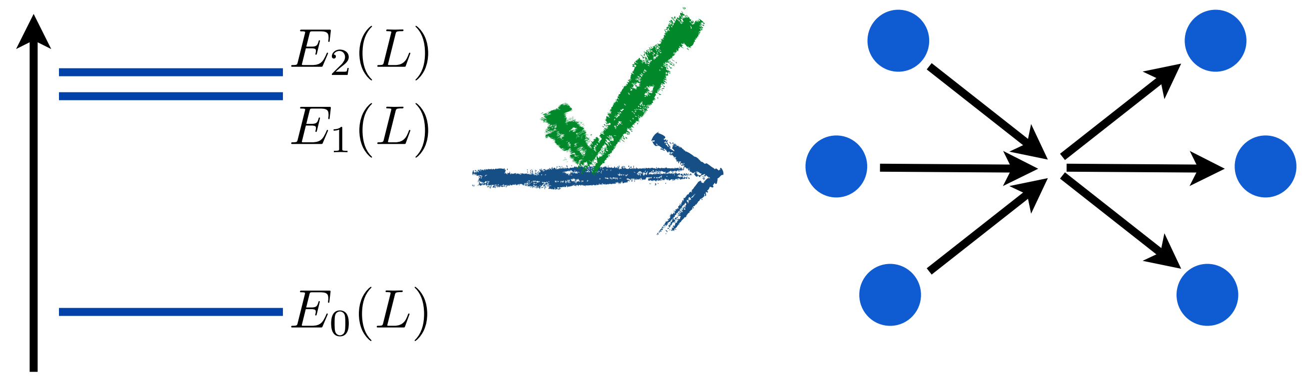
3. **Sum subsets of terms to identify** infinite-volume quantities

4. **Relate these to poles in the finite-volume correlator**



$$\det_{k,\ell,m}\left[\mathcal{K}_{\mathrm{df},3}^{-1} + F_3\right] = 0$$

Three-to-three scattering



Current status:

Formalism is complete for the simplest three-scalar system

General, model-independent relation between
finite-volume energies and three-to-three scattering amplitude

Derived using a generic relativistic field theory

MTH and Sharpe, *Phys. Rev.* D90, 116003 (2014)
MTH and Sharpe, *Phys. Rev.* D92, 114509 (2015)

Important caveats:

Identical particles with no two-to-three transitions

$$\pi\pi\pi \longrightarrow \pi\pi\pi$$

Requires that two-particle scattering phase is bounded

$$|\delta_\ell(E)| < \pi/2$$

Currently underway:

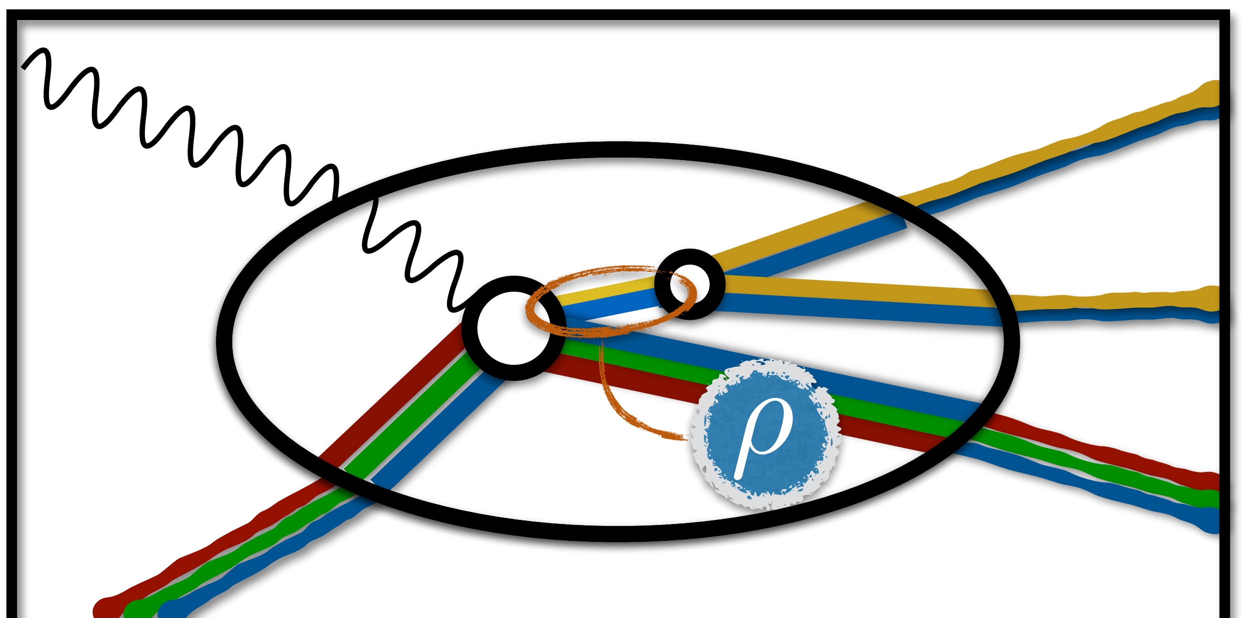
Relax all simplifying assumptions:

Allow all particle types, allow two-to-three couplings, remove bound on phase shift

$K\pi \to K\pi\pi$ $\quad N\pi \to N\pi\pi$ $\quad NNN \to NNN$

Briceño, MTH, Sharpe, *in development*

Derive formalism for three-particle transition amplitudes



$$p\gamma \to N\rho \to N\pi\pi$$

Also want to make connections to other work...

Polejaeva and Rusetsky, *Eur. Phys. J.* A48, 67 (2012)

Briceño and Davoudi, *Phys. Rev.* D87, 094507 (2013)

Meißner, Rios and Rusektsky. *Phys. Rev. Lett.* 114, 091602 (2015)

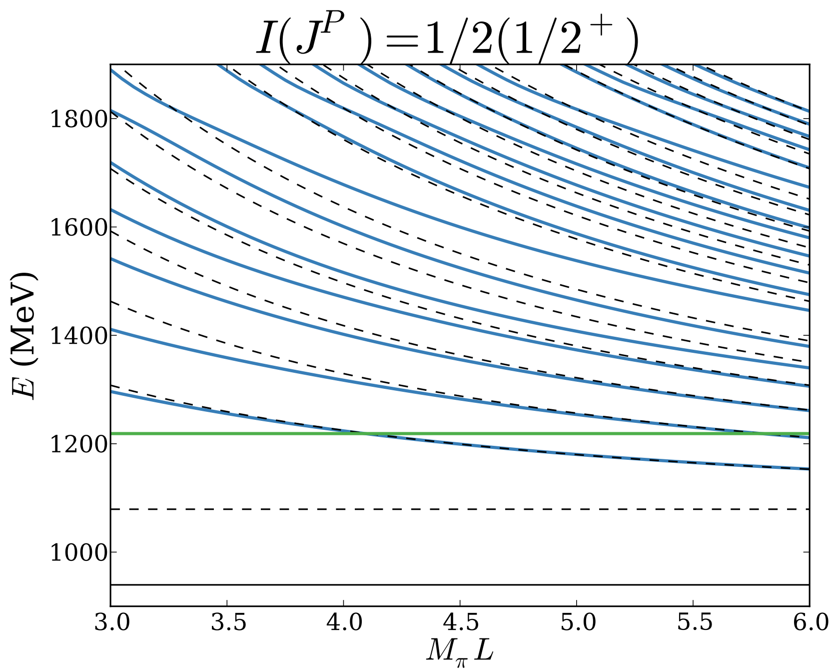
What lattice needs for resonances…

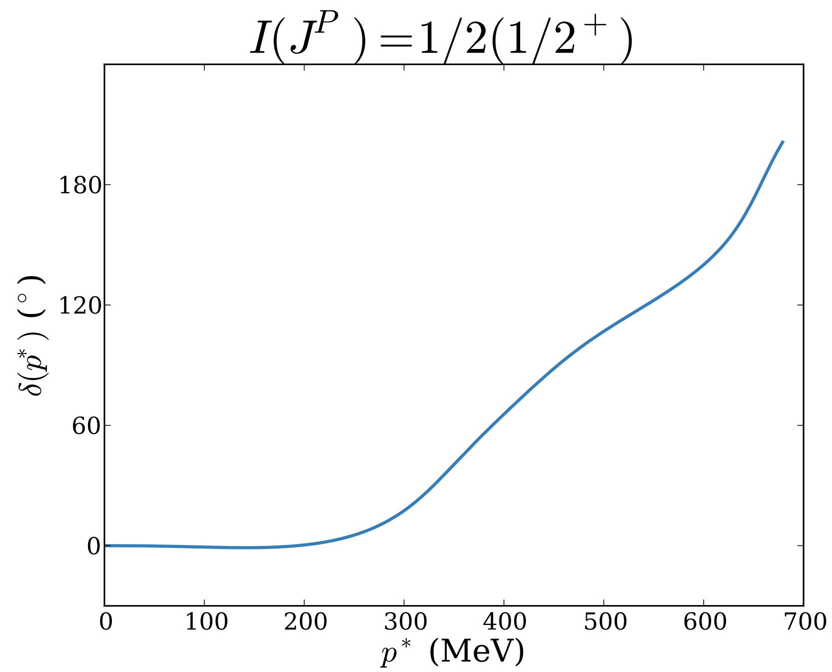
What lattice needs for resonances…

As much information as possible about the finite-volume spectrum

Can functional methods be used to calculate energies in various volumes?

Given energies in one volume can one “bootstrap” to energies in a different volume?



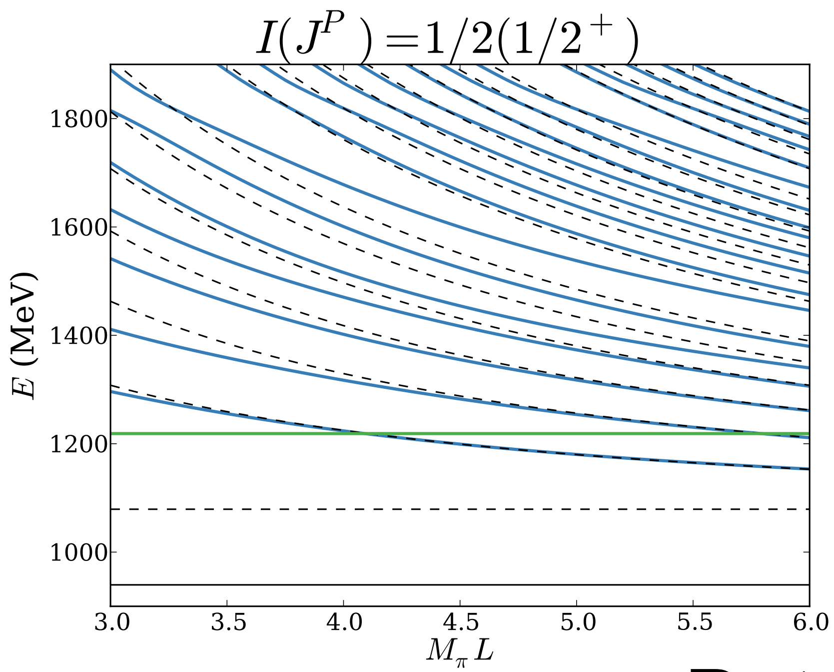


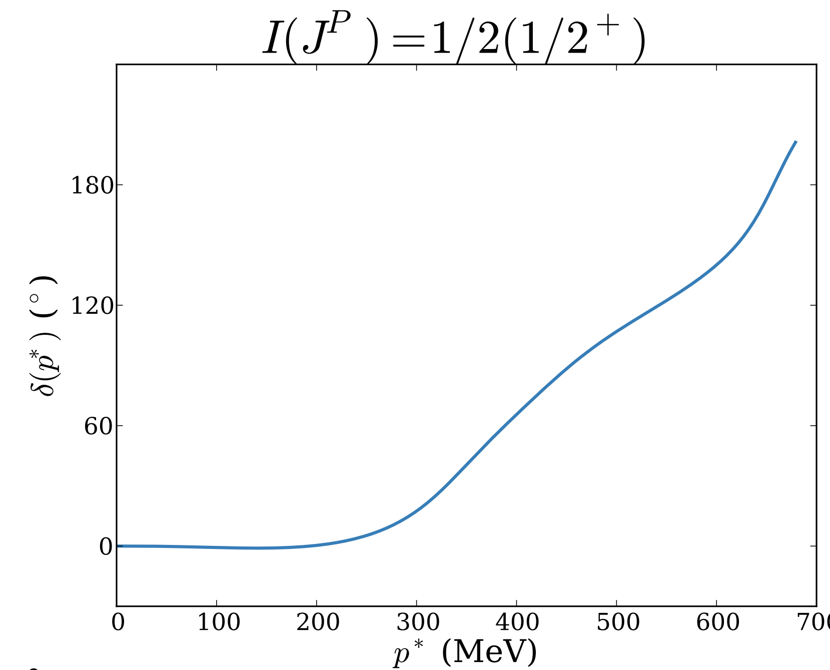
What lattice needs for resonances…

As much information as possible about the finite-volume spectrum

Can functional methods be used to calculate energies in various volumes?

Given energies in one volume can one "bootstrap" to energies in a different volume?





Better chiral extrapolations

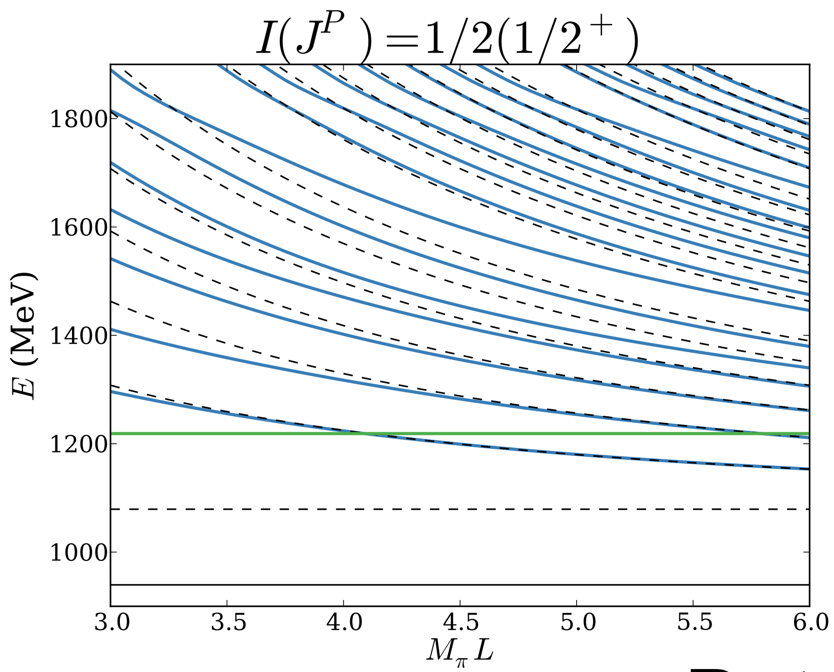
Can functional methods be used to supplement ChPT in interpolating to the physical point?

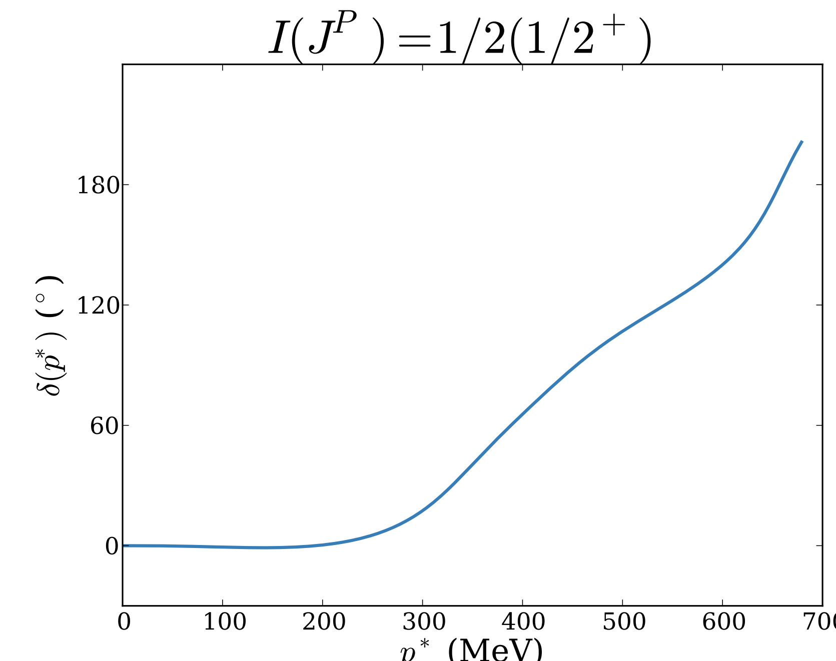
What lattice needs for resonances…

As much information as possible about the finite-volume spectrum

Can functional methods be used to calculate energies in various volumes?

Given energies in one volume can one "bootstrap" to energies in a different volume?





Better chiral extrapolations

Can functional methods be used to supplement ChPT in interpolating to the physical point?

Help applying the three-particle formalism

We have a systematic technique for extracting $\mathcal{K}_{\text{df},3}(E^*)$ from the finite-volume spectrum.

Can functional methods help solve the set of Fadeev-like equations that relate it to the scattering amplitude?