

Extracting scattering and resonance properties from the lattice

Maxwell T. Hansen

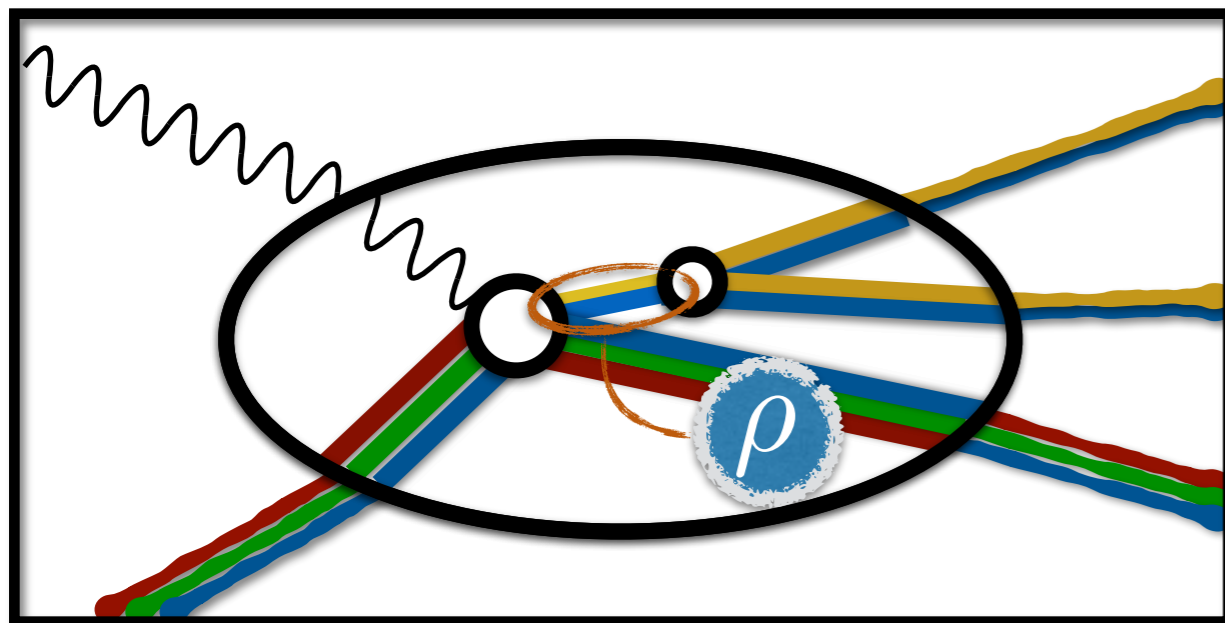
Institut für Kernphysik and Helmholtz-Institut Mainz

Johannes Gutenberg Universität

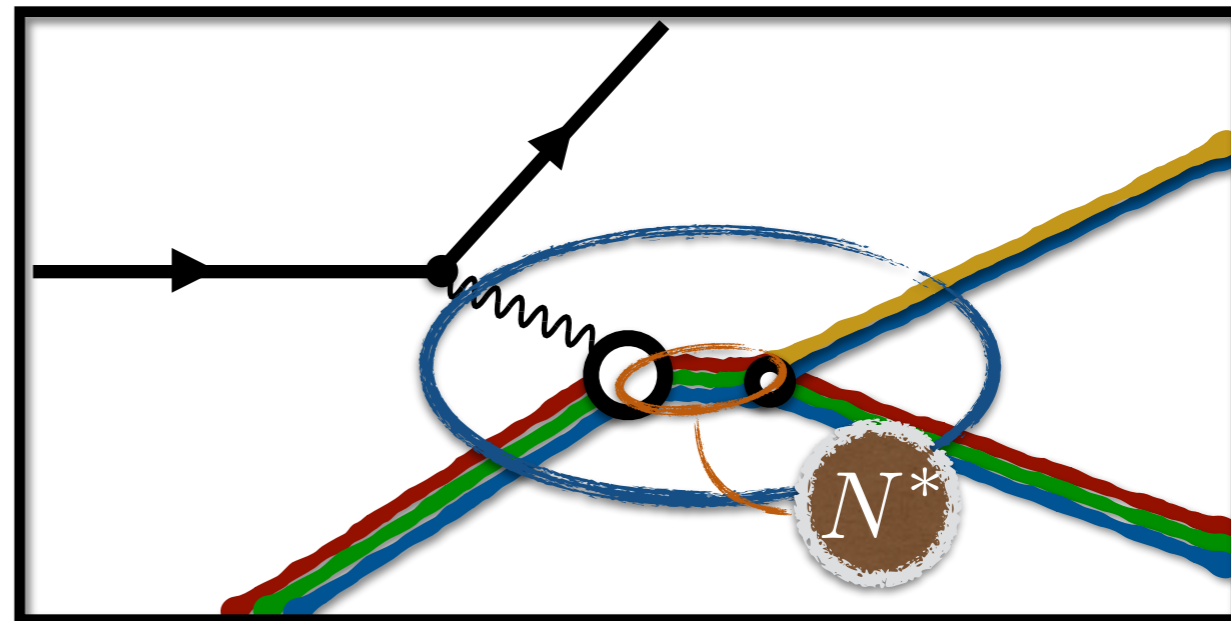
Mainz, Germany

July 6th, 2016





$$p\gamma \rightarrow N\rho \rightarrow N\pi\pi$$



$$p\gamma^* \rightarrow N^* \rightarrow N\pi, N\eta$$

Resonances are not directly detected.

Outgoing hadrons are used to reconstruct resonance properties.

It is thus highly valuable to predict these transition amplitudes from the underlying theory of QCD

Combining accurate, model-independent predictions with experiment will lead to a deeper understanding of QCD's rich resonance structure

What can we extract from the lattice?

We are trying to evaluate a difficult integral numerically

$$\text{observable} = \int \mathcal{D}\phi e^{iS} \left[\begin{array}{l} \text{interpolator} \\ \text{for observable} \end{array} \right]$$

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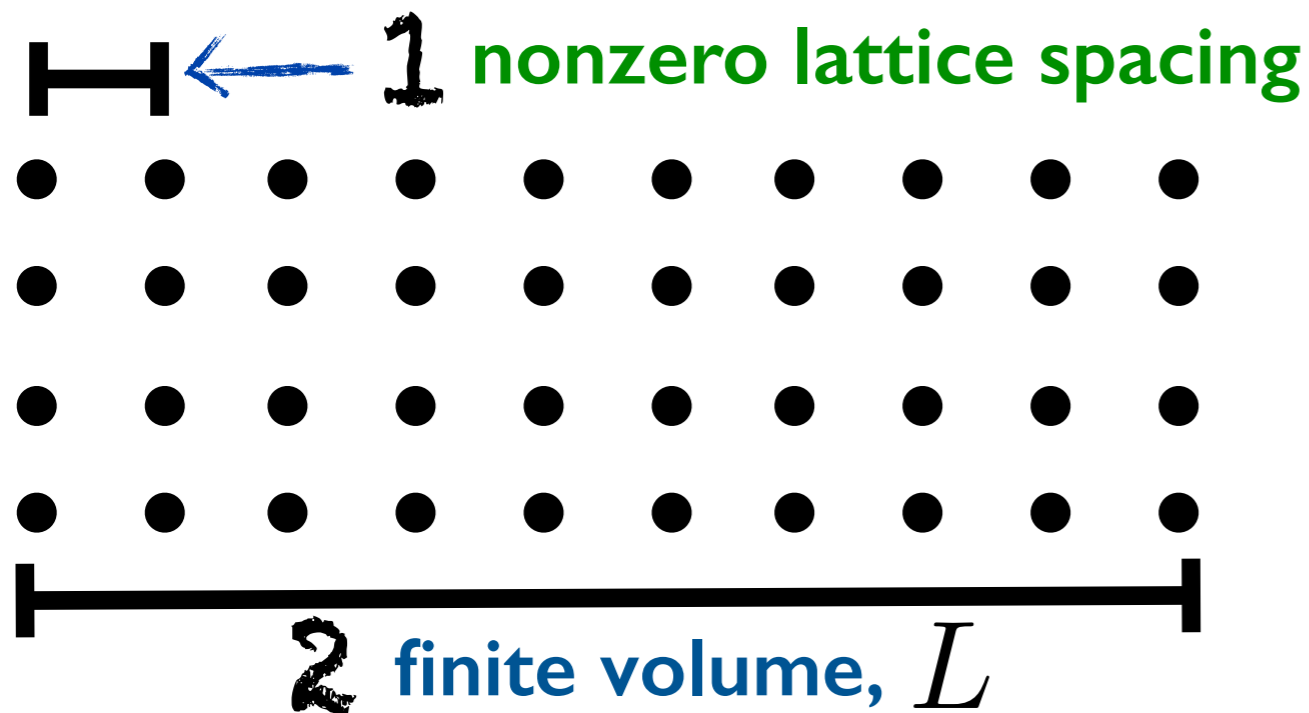
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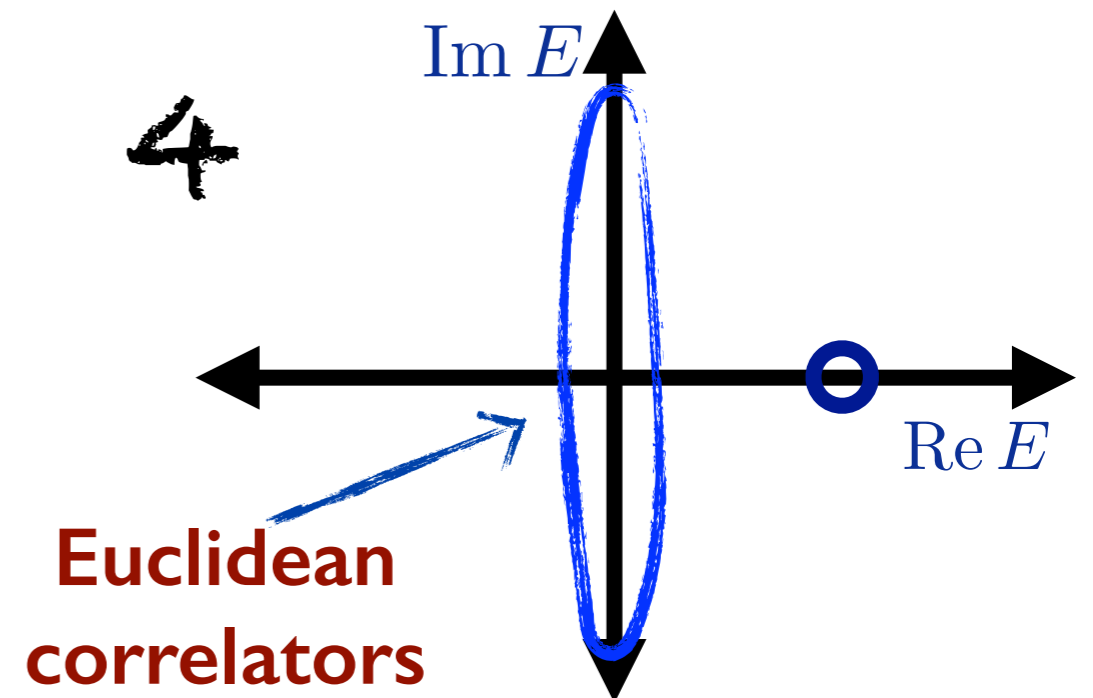
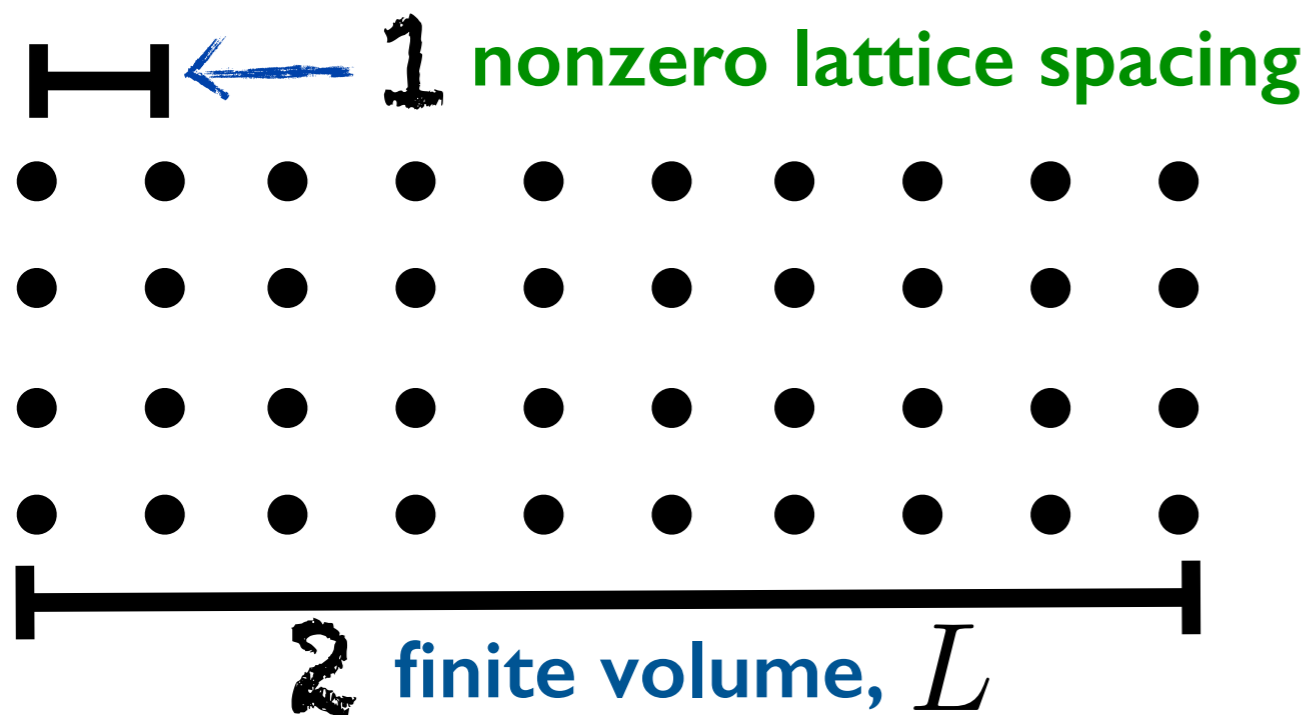


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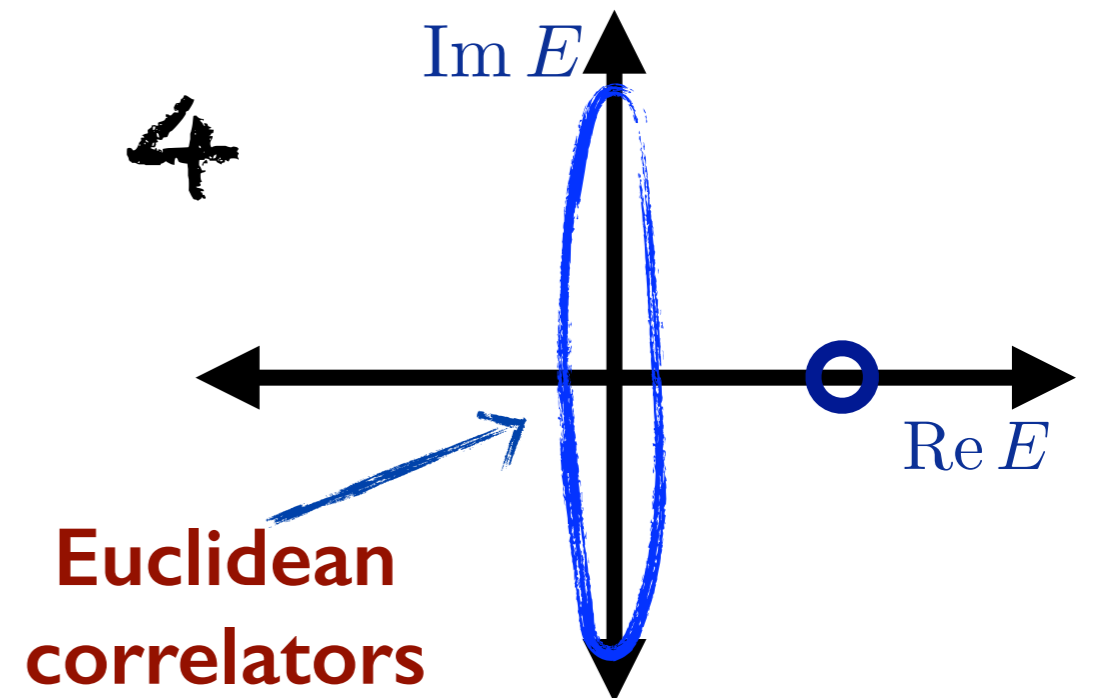
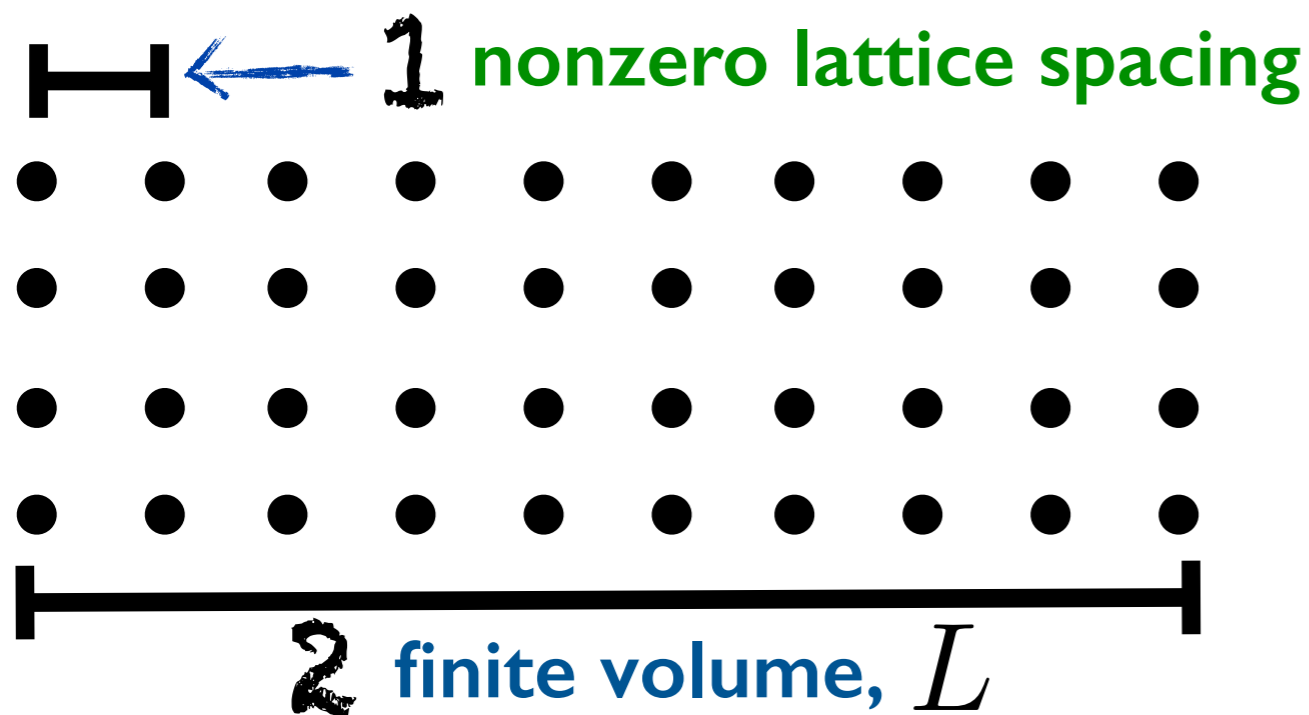
But calculations at the physical pion mass do now exist

What can we extract from the lattice?

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$$\left(\begin{array}{l} \text{observable?} \\ \text{discretized, finite volume,} \\ \text{Euclidean, heavy pions} \end{array} \right) = \int \prod_i^N d\phi_i e^{-S} \left[\begin{array}{l} \text{interpolator} \\ \text{for observable} \end{array} \right]$$

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Stable particle masses

$$C(\tau) \equiv \langle 0 | \mathcal{O}(\tau) \mathcal{O}^\dagger(0) | 0 \rangle = \int \prod d\phi e^{-S} \mathcal{O}(\tau) \mathcal{O}^\dagger(0)$$

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The correlator is equal to a sum of decaying exponentials

$$\begin{aligned} C(\tau) &= \sum_n \langle 0 | e^{H\tau} \mathcal{O}(0) e^{-H\tau} | E_n \rangle \langle E_n | \mathcal{O}^\dagger(0) | 0 \rangle \\ &= \sum_n |\langle 0 | \mathcal{O}(0) | E_n \rangle|^2 e^{-E_n \tau} \xrightarrow{\tau \rightarrow \infty} |\langle 0 | \mathcal{O}(0) | E_1 \rangle|^2 e^{-E_1 \tau} \end{aligned}$$

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One can fit to the long time behavior to extract the ground state

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$$\mathcal{O} = \bar{u} \gamma_5 d$$

**Then in the infinite-volume,
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$$E_1(a, L, m_q) \longrightarrow M_\pi(m_q)$$

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Stable particle masses

Full error budget calculation of isospin splittings

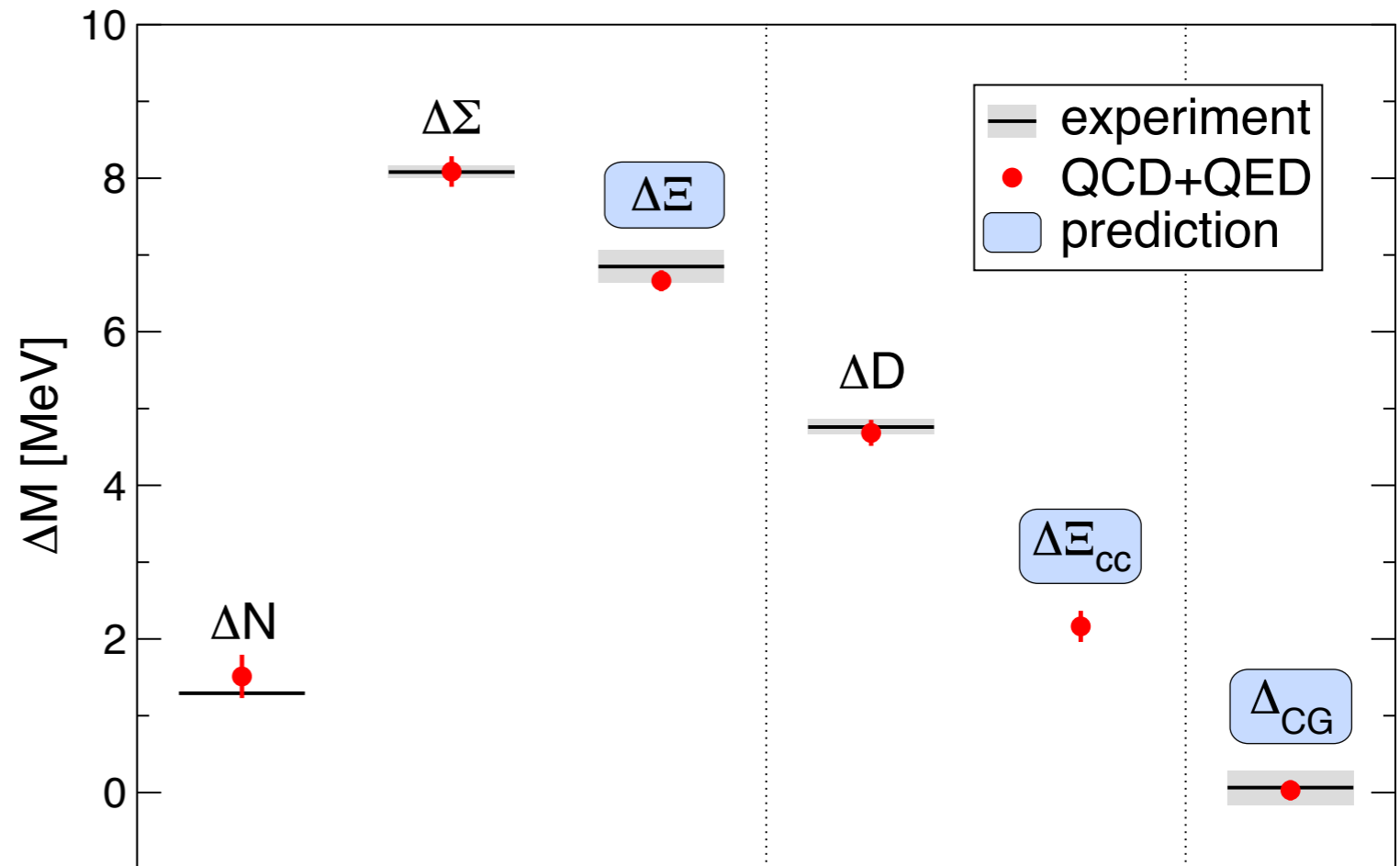
Four lattice spacings

0.06fm to 0.10fm

Four volumes, ranging up to
8fm

Many pion masses,
ranging down to

197MeV



Dynamical up, down, strange and charm quarks + QED

Decay constants

$$\langle 0|A_\mu(0)\mathcal{O}^\dagger(-\tau)|0\rangle = \int \prod d\phi e^{-S} A_\mu(0)\mathcal{O}^\dagger(-\tau)$$

$$\langle 0|A_\mu(0)\mathcal{O}^\dagger(-\tau)|0\rangle \xrightarrow{\tau \rightarrow \infty} ip_\mu f$$

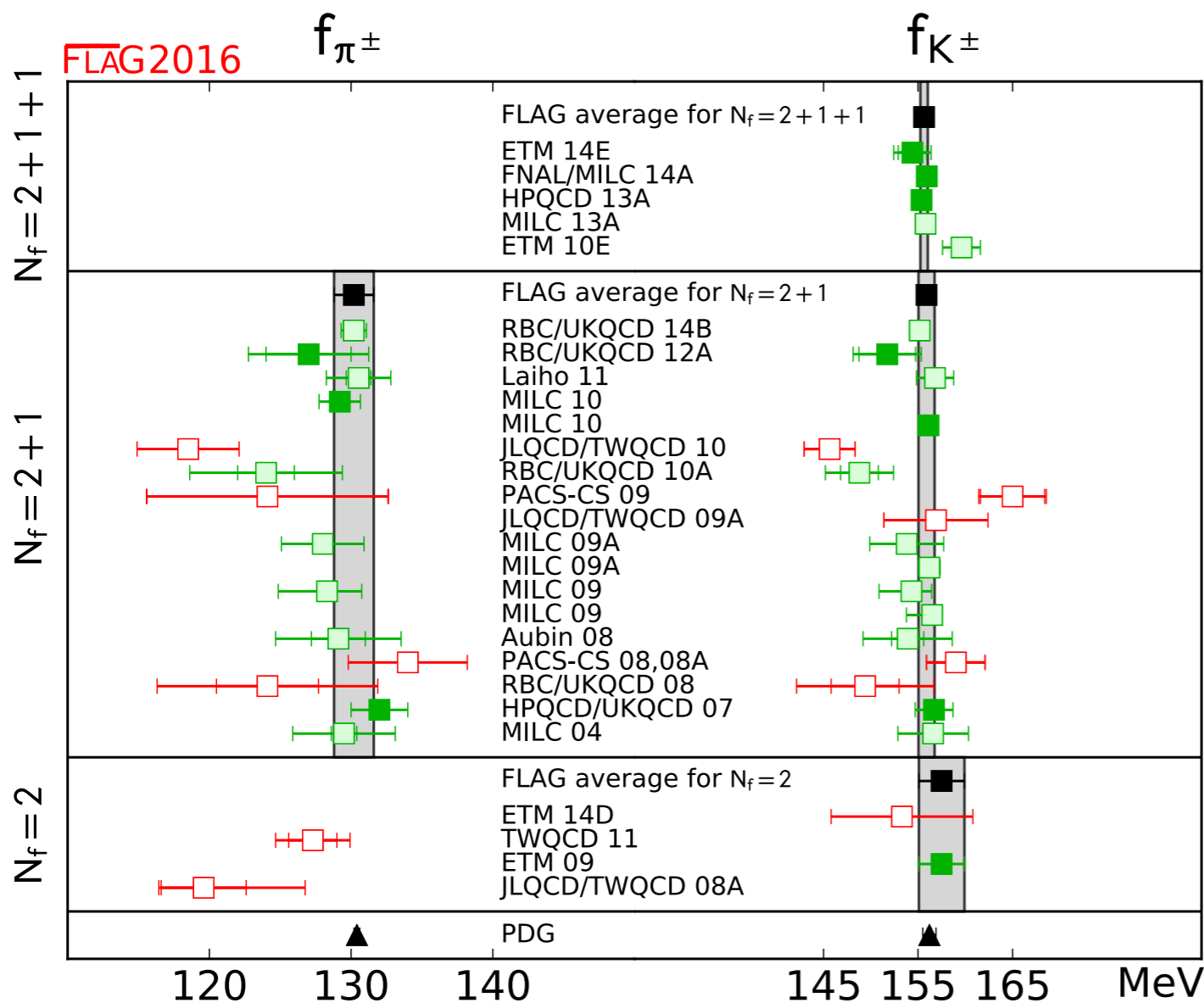
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FLAG

(Flavor Lattice Averaging Group)

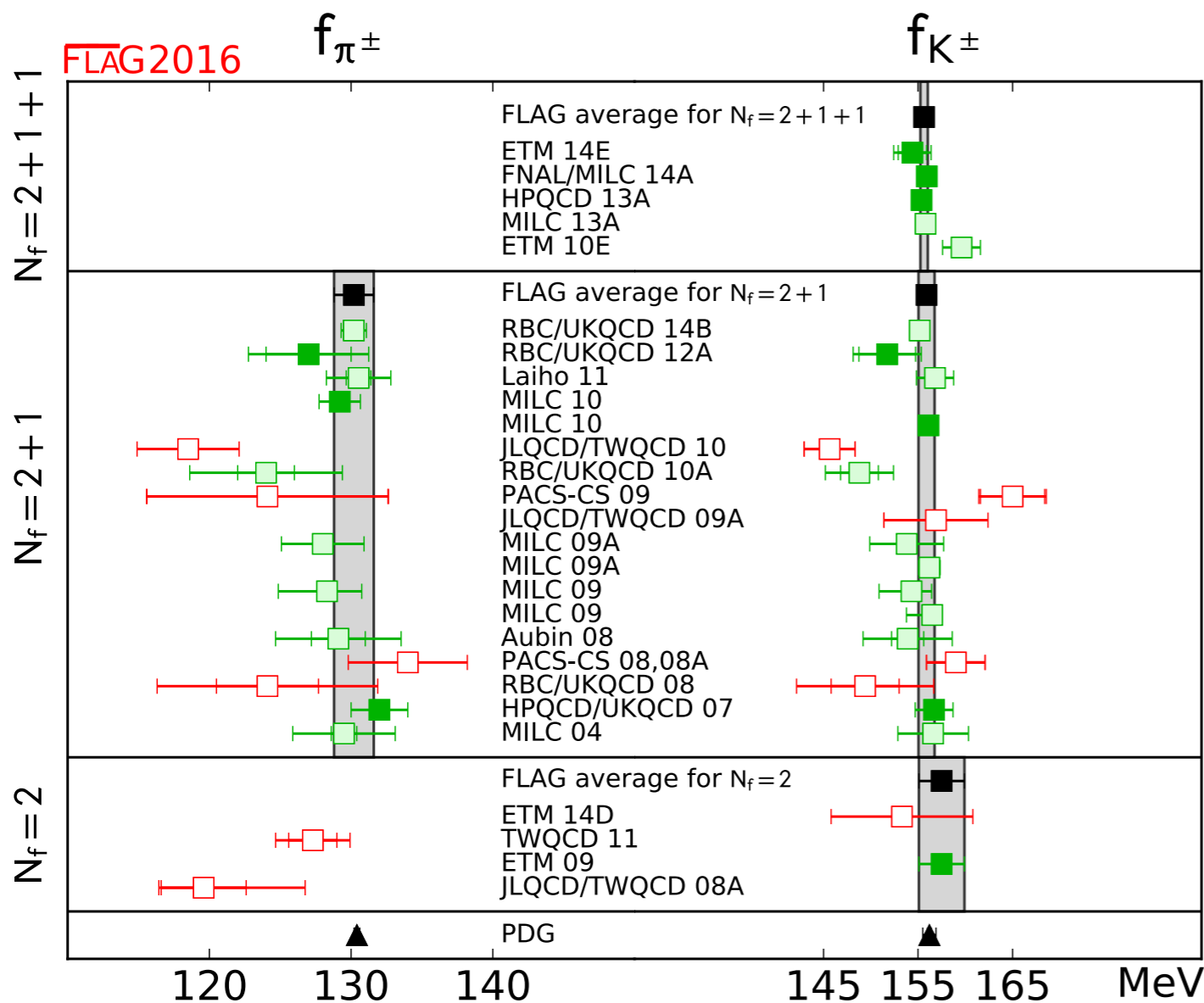
Collaboration for systematically averaging results from different collaborations

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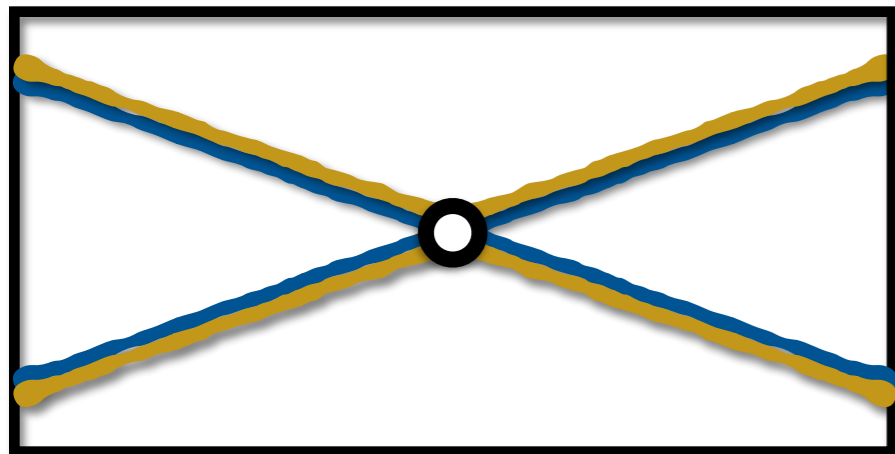
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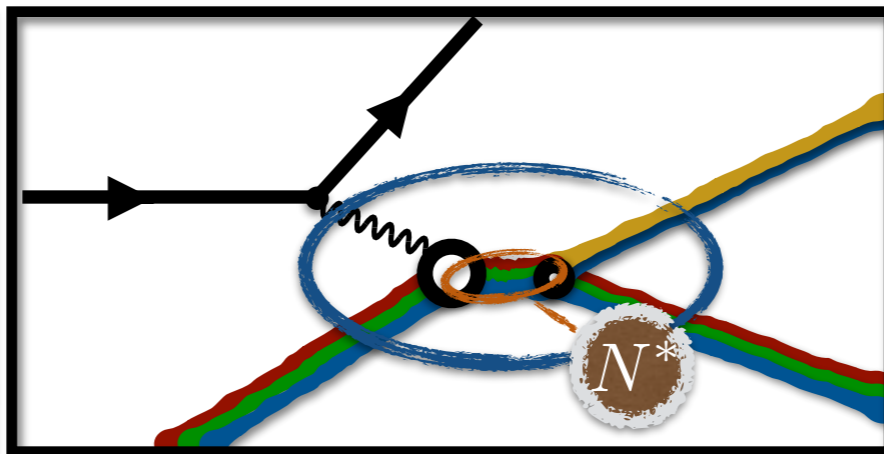
Results are only included if they meet certain standards:

1. Chiral extrapolation, pion below 400MeV
2. Continuum extrapolation, minimum two lattices (below 0.1 fm, sufficiently different)
3. Two volumes or demonstrably small effect
4. Non-perturbative renormalization

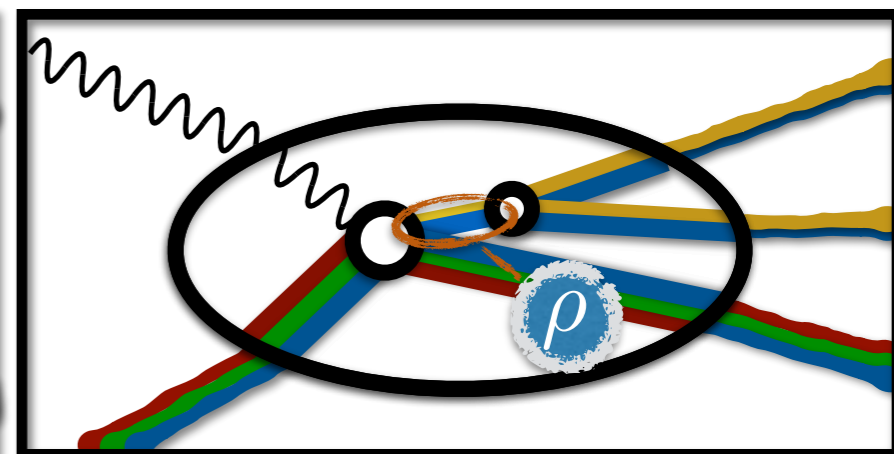
What can we extract from the lattice?



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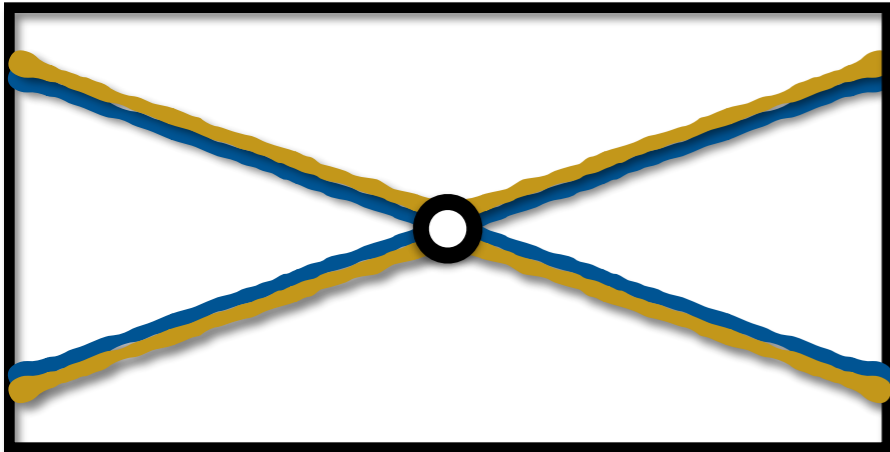
Resonances are fundamentally different from stable particles

$$C(\tau) = \langle 0 | \mathcal{O}_\rho(\tau) \mathcal{O}_\rho^\dagger(0) | 0 \rangle = \sum_n |\langle 0 | \mathcal{O}_\rho(0) | E_n \rangle|^2 e^{-E_n \tau}$$

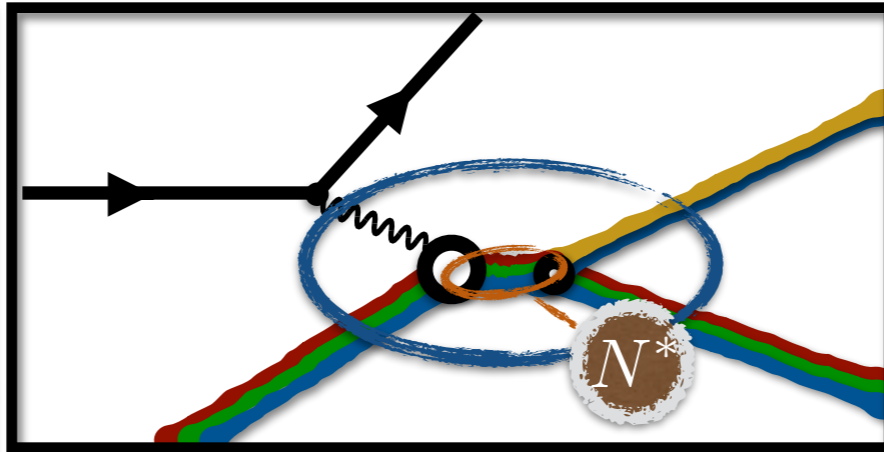
$$\lim_{L \rightarrow \infty} E_n(L) \neq M_\rho$$

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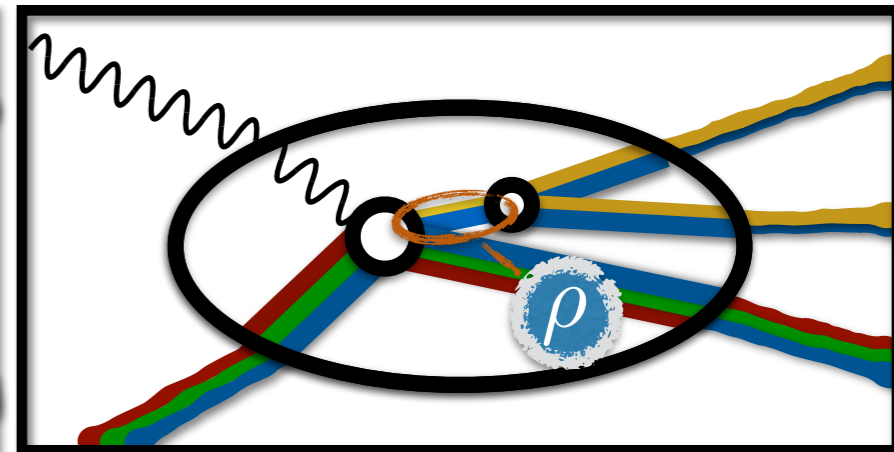
Not possible to directly calculate



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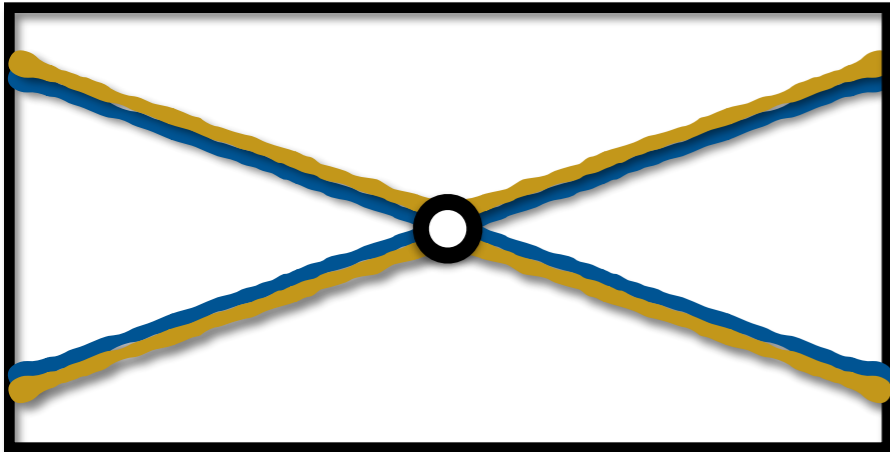
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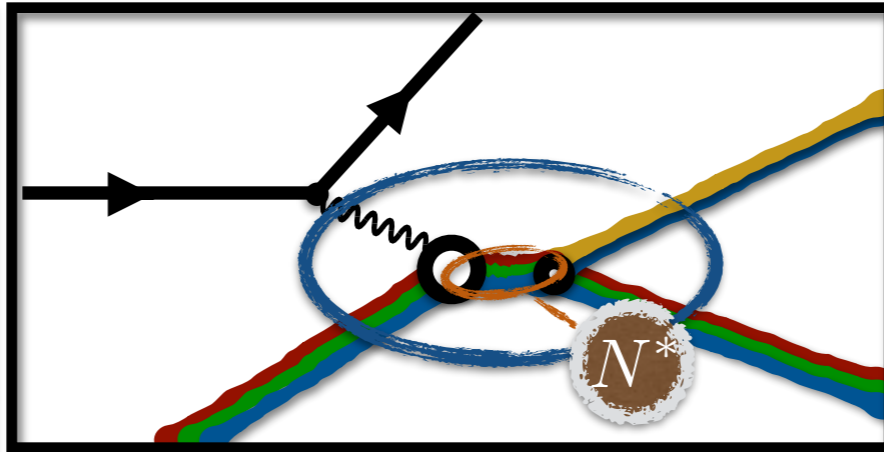
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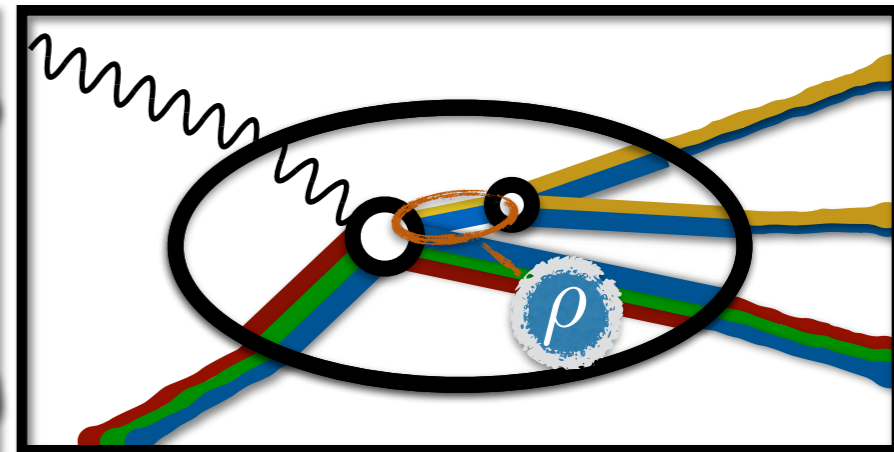
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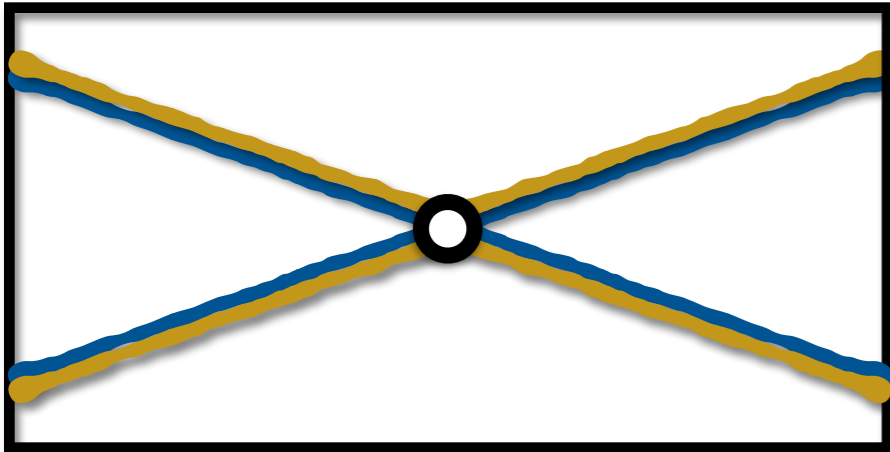


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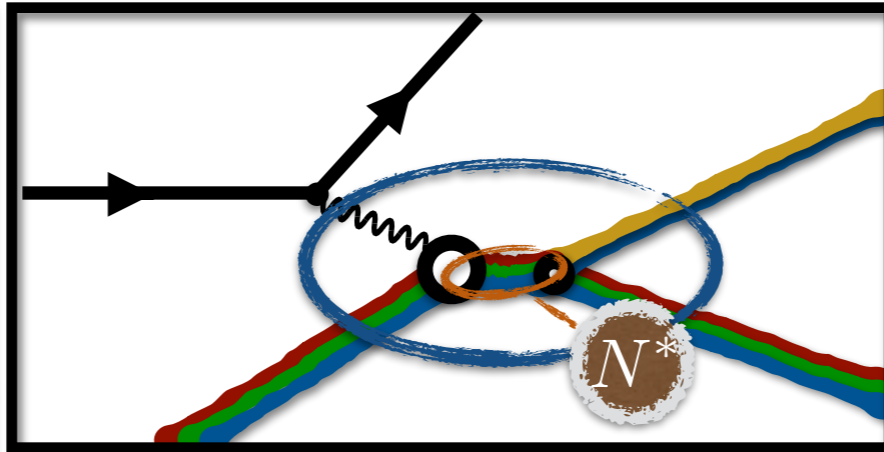
multi-particle in- and outstates

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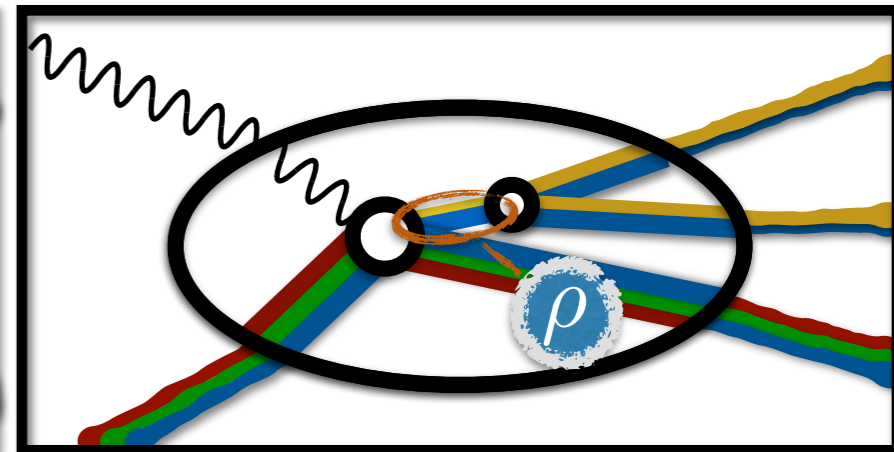
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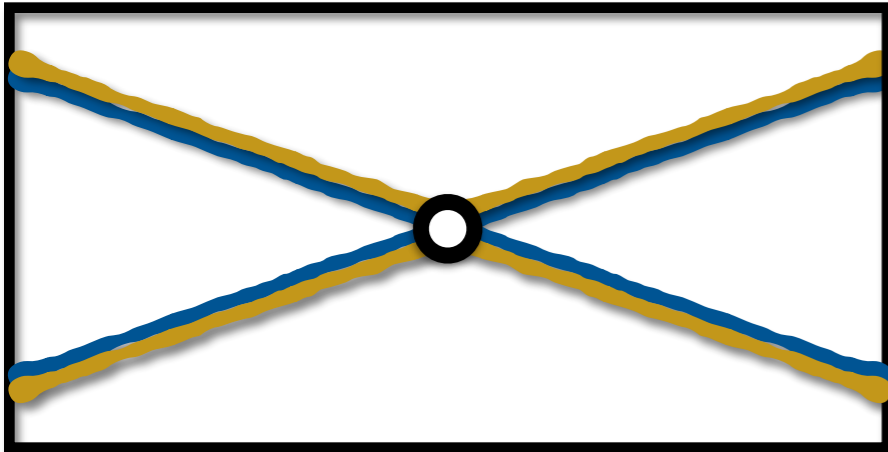
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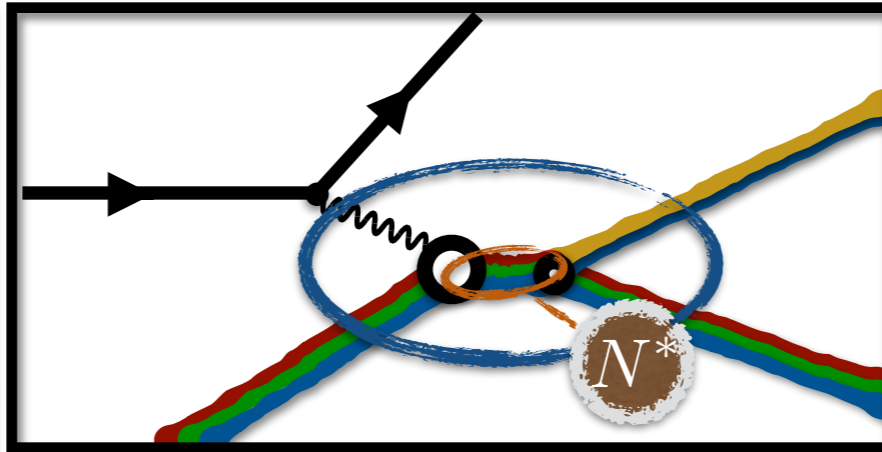
$$\langle \pi\pi, \text{out} | \pi\pi, \text{in} \rangle = \frac{\text{amputate and put on-shell}}{\langle 0 | \tilde{\pi}(p') \tilde{\pi}(k') \tilde{\pi}(p) \tilde{\pi}(k) | 0 \rangle}$$

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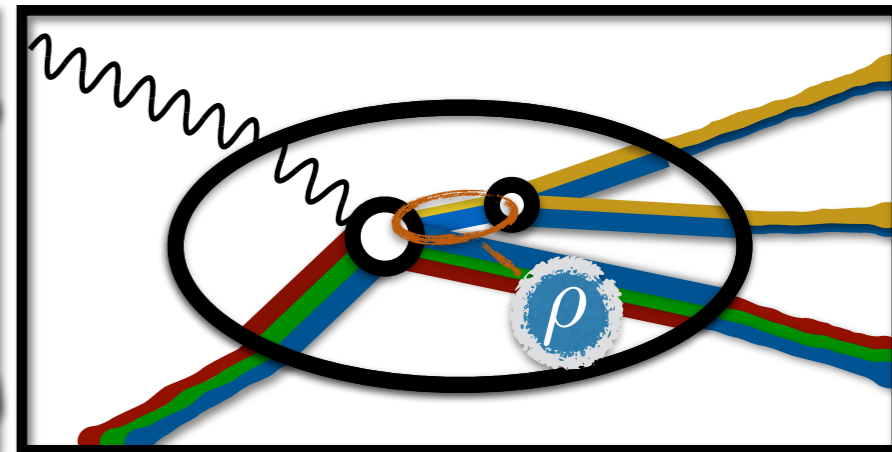
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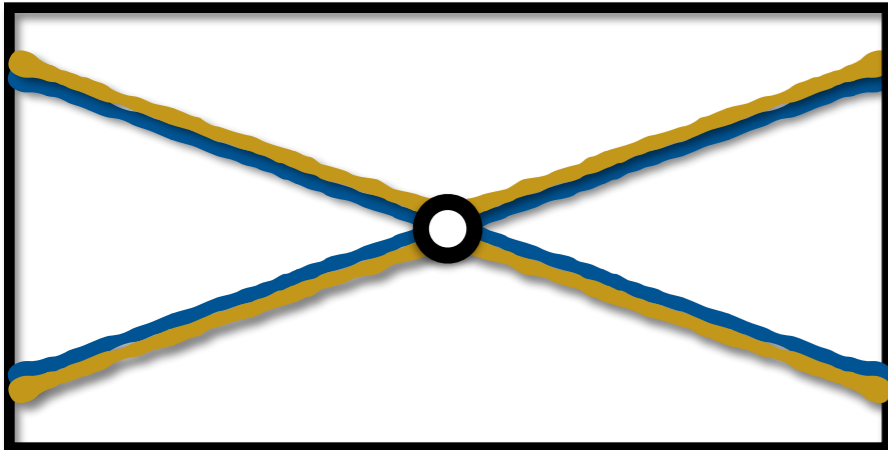
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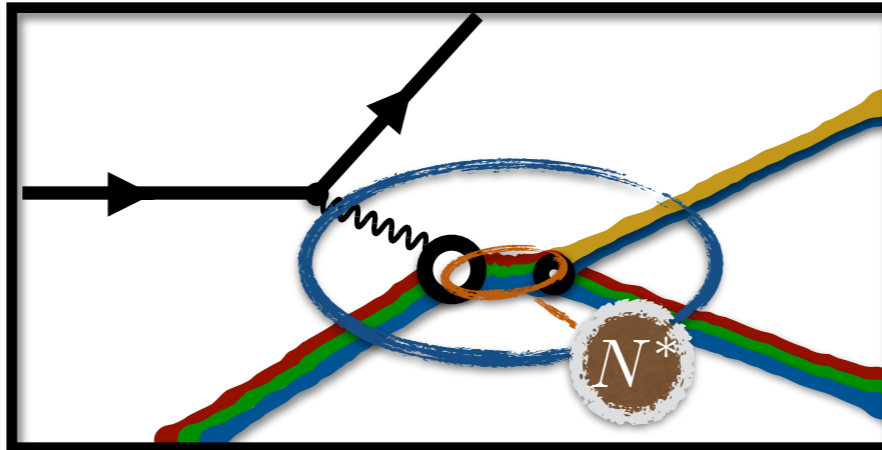
$$\langle N\pi\pi, \text{out} | \mathcal{J}_\mu(x) | N \rangle = \text{amputate and put on-shell} \langle 0 | \tilde{N}(p'_1) \tilde{\pi}(p'_2) \tilde{\pi}(p'_3) \mathcal{J}_\mu(x) \tilde{N}(P) | 0 \rangle$$

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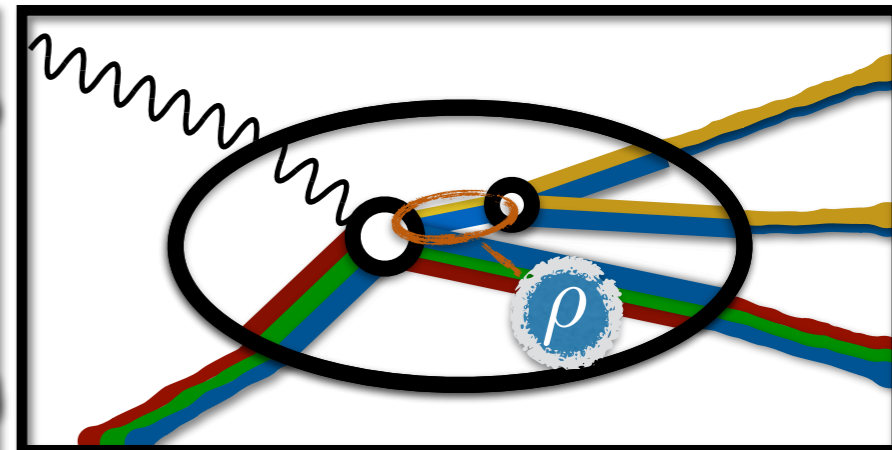
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Requires Minkowski momenta and infinite volume

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Instead we can only access

$$H_{\text{QCD}}|n, L\rangle = |n, L\rangle \underline{E_n(L)} \quad \underline{\langle n, L, \text{“}N\pi\pi\text{”} | \mathcal{J}_\mu(x) | \text{“}N\text{”}, L \rangle}$$

finite-volume energies and matrix elements

labels in quotes indicate quantum numbers

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How can we determine

$$\underline{\langle \pi\pi, \text{out} | \pi\pi, \text{in} \rangle \quad \text{and} \quad \langle N\pi\pi, \text{out} | \mathcal{J}_\mu(x) | N \rangle}$$

from

$$\underline{E_n(L) \quad \text{and} \quad \langle n, L, \text{"}N\pi\pi\text{"} | \mathcal{J}_\mu(x) | \text{"}N\text{"}, L \rangle ?}$$

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Two-particle scattering

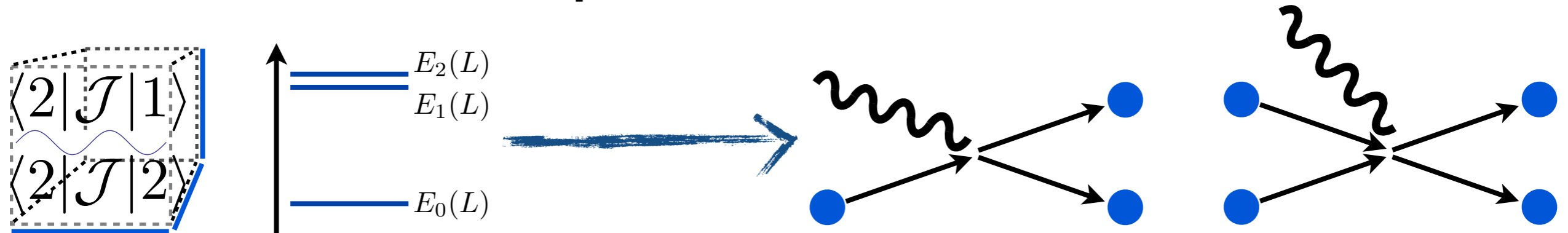


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Photo- and electroproduction

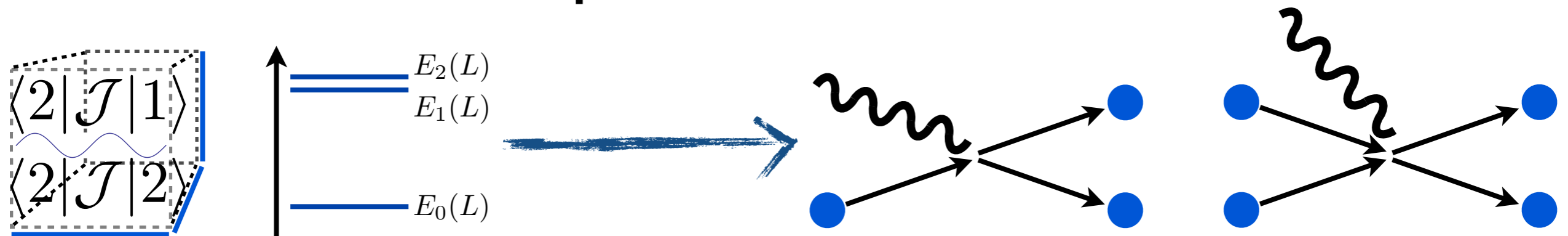


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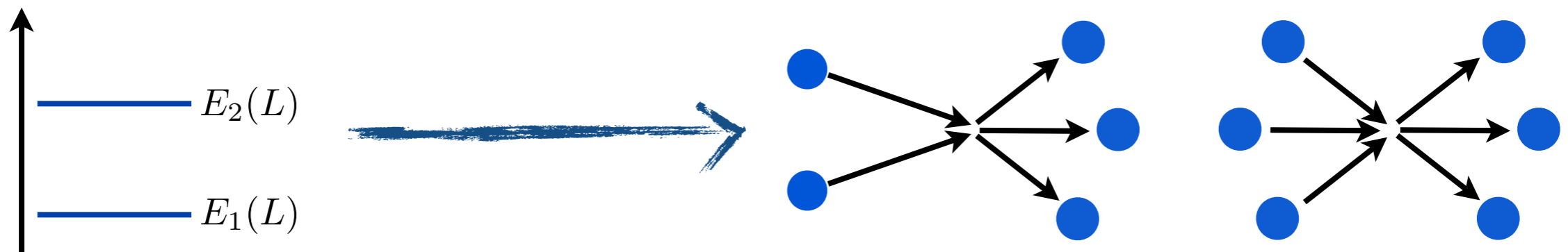
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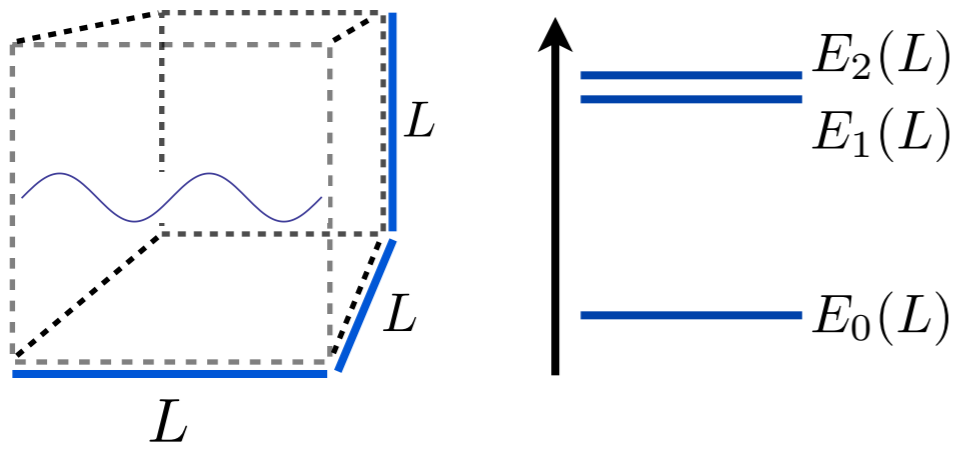
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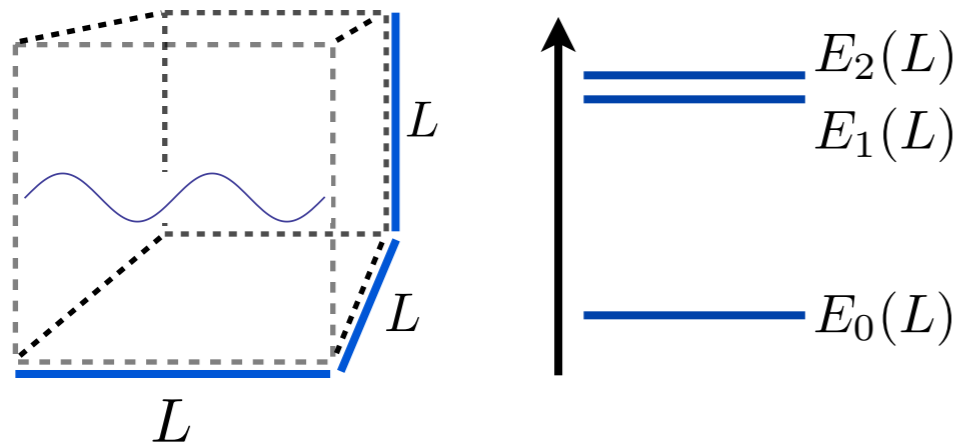
Three-particle scattering



Finite volume



Finite volume



cubic, spatial volume (extent L)

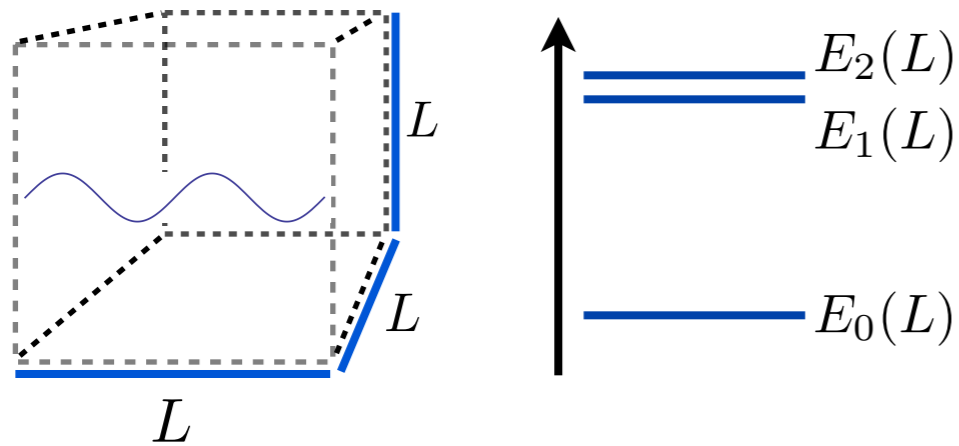
periodic boundary conditions

$$\vec{p} \in (2\pi/L)\mathbb{Z}^3$$

time direction **infinite**



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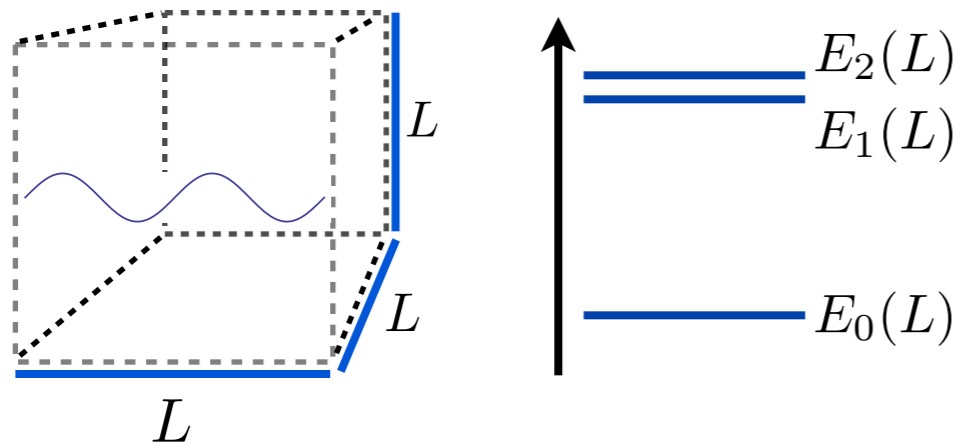
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L large enough to ignore e^{-mL}



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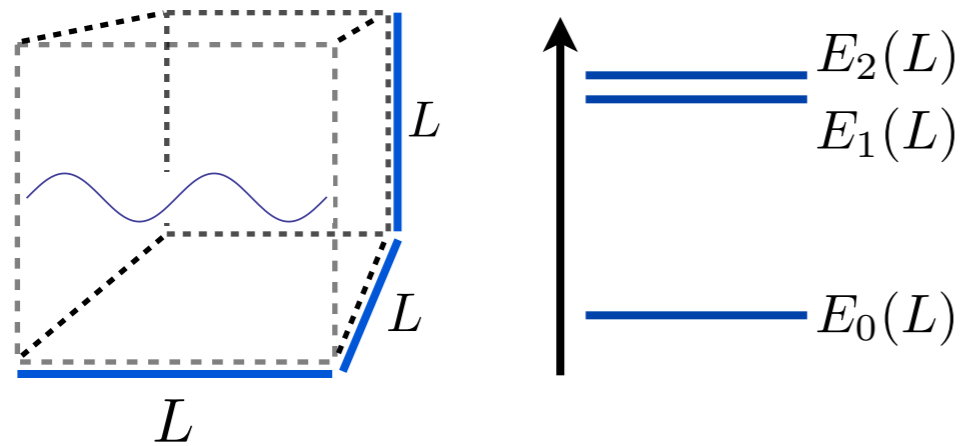


L large enough to ignore e^{-mL}

Assume lattice effects are small and accommodated elsewhere

Work in continuum field theory throughout

Finite volume



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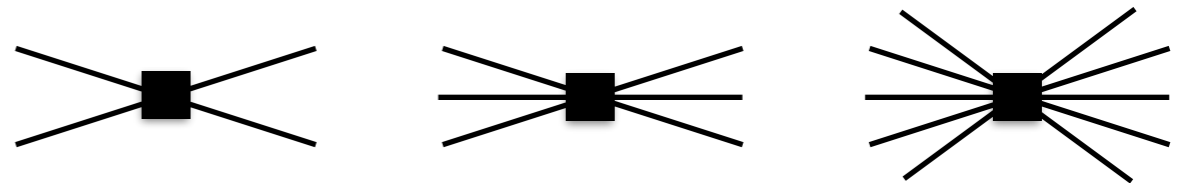
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Quantum field theory

generic relativistic QFT

1. Include all interactions

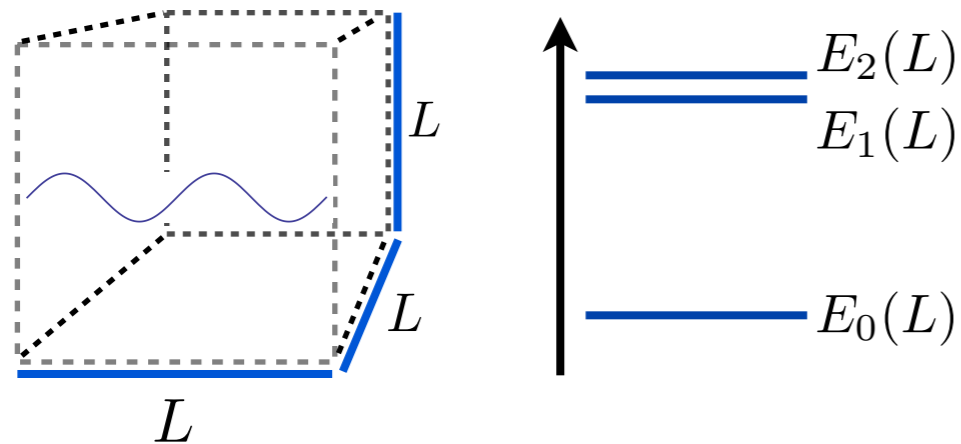


2. no power-counting scheme

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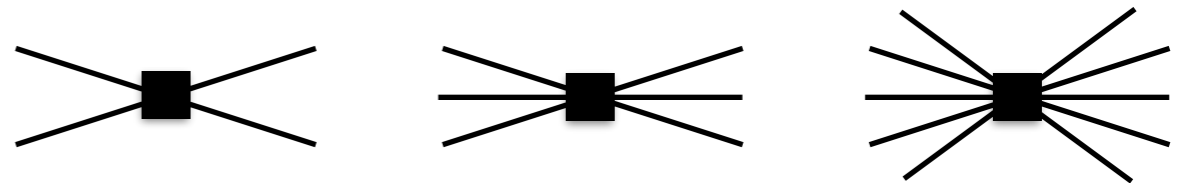


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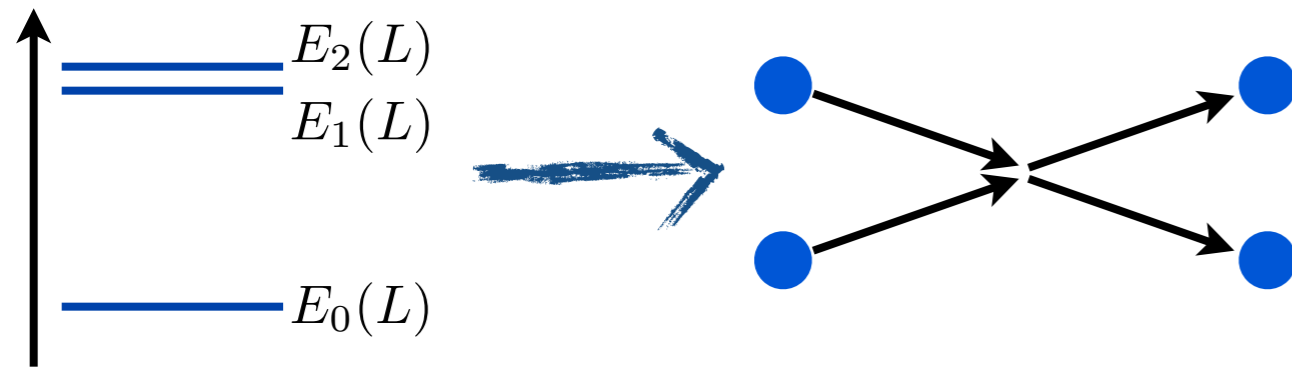
Not possible to directly calculate scattering observables to all orders

But it is possible to derive general, all-orders relations to finite-volume quantities

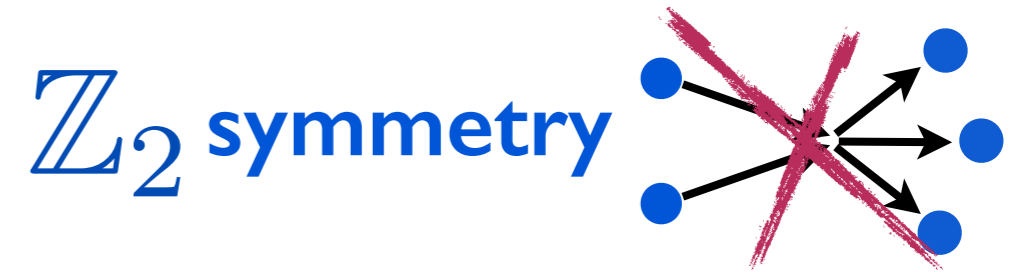
Assume lattice effects are small and accommodated elsewhere

Work in continuum field theory throughout

Two-to-two scattering



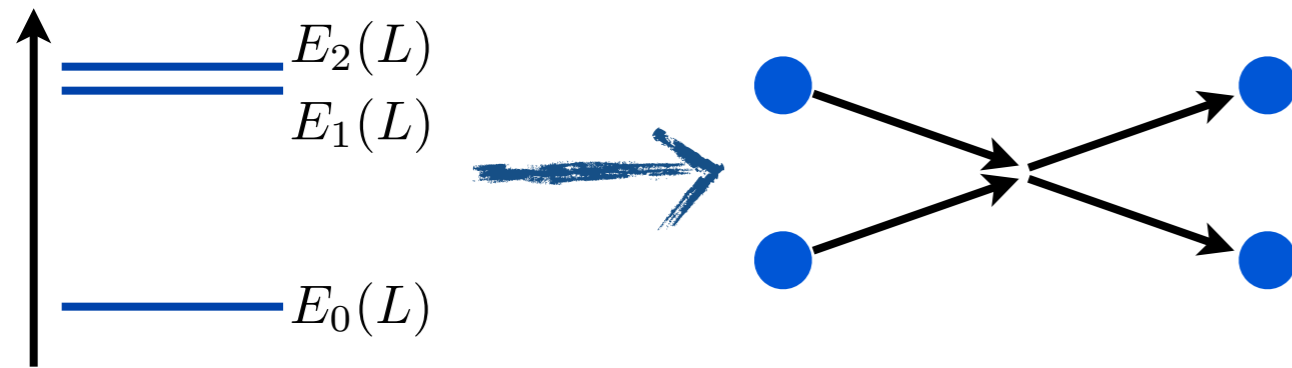
For now assume...
identical scalars, mass m



Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

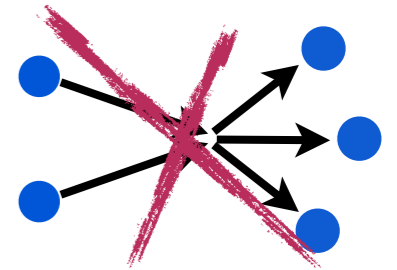
Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

Two-to-two scattering



For now assume...
identical scalars, mass m

\mathbb{Z}_2 symmetry



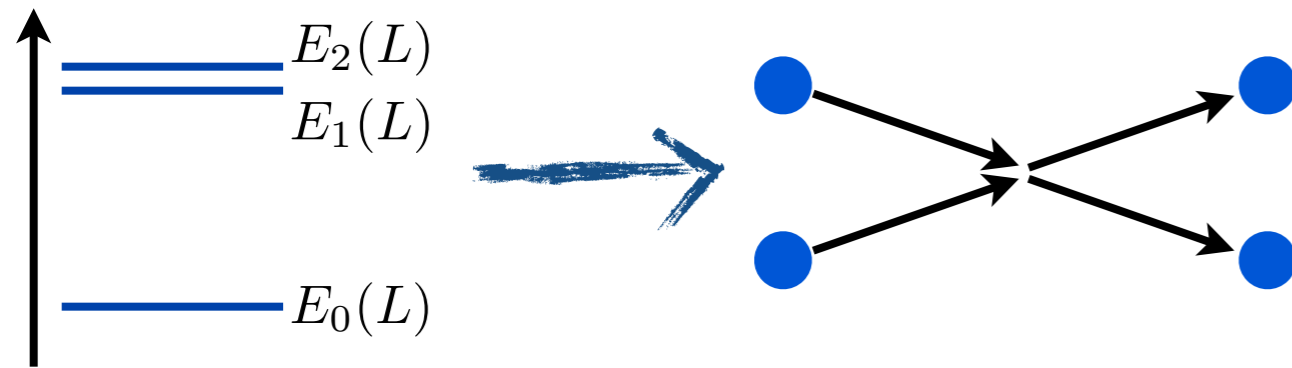
$$C_L(P) \equiv \int_L d^4x e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

two-particle interpolator

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

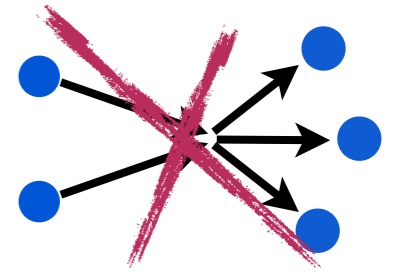
Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

Two-to-two scattering



For now assume...
identical scalars, mass m

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$$C_L(P) \equiv \int_L d^4x e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

Euclidean convention

two-particle interpolator

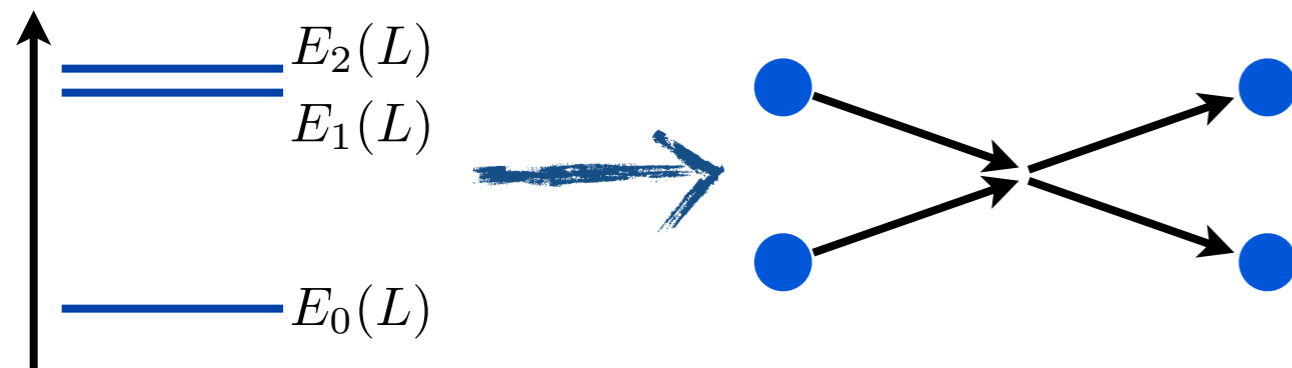
$$P = (P_4, \vec{P}) = (P_4, 2\pi\vec{n}/L)$$

but allow P_4 to be real or imaginary

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

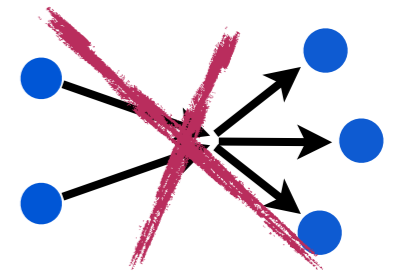
Two-to-two scattering



For now assume...

identical scalars, mass m

\mathbb{Z}_2 symmetry



$$C_L(P) \equiv \int_L d^4x e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

Euclidean convention

two-particle interpolator

$$P = (P_4, \vec{P}) = (P_4, 2\pi\vec{n}/L)$$

but allow P_4 to be real or imaginary

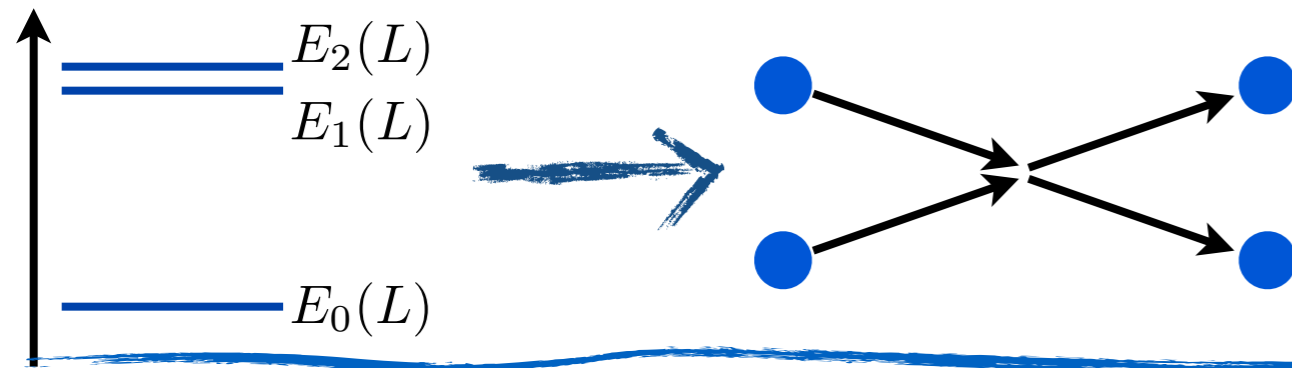
CM frame energy is then $E^{*2} = -P_4^2 - \vec{P}^2$

Require $E^* < 4m$ to isolate two-to-two scattering

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

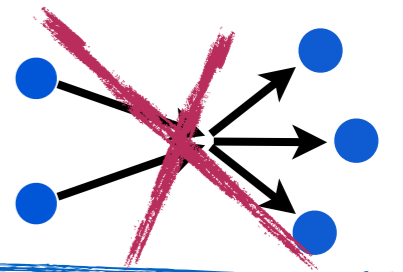
Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

Two-to-two scattering



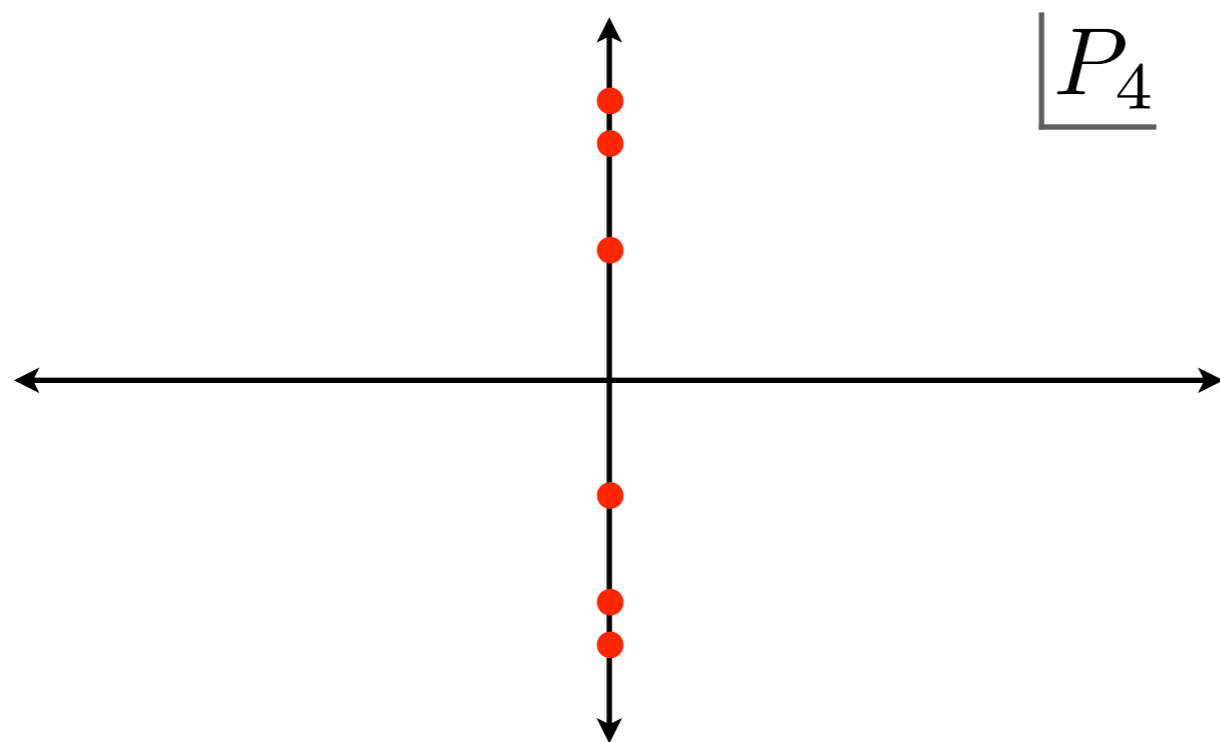
For now assume...
identical scalars, mass m

\mathbb{Z}_2 symmetry

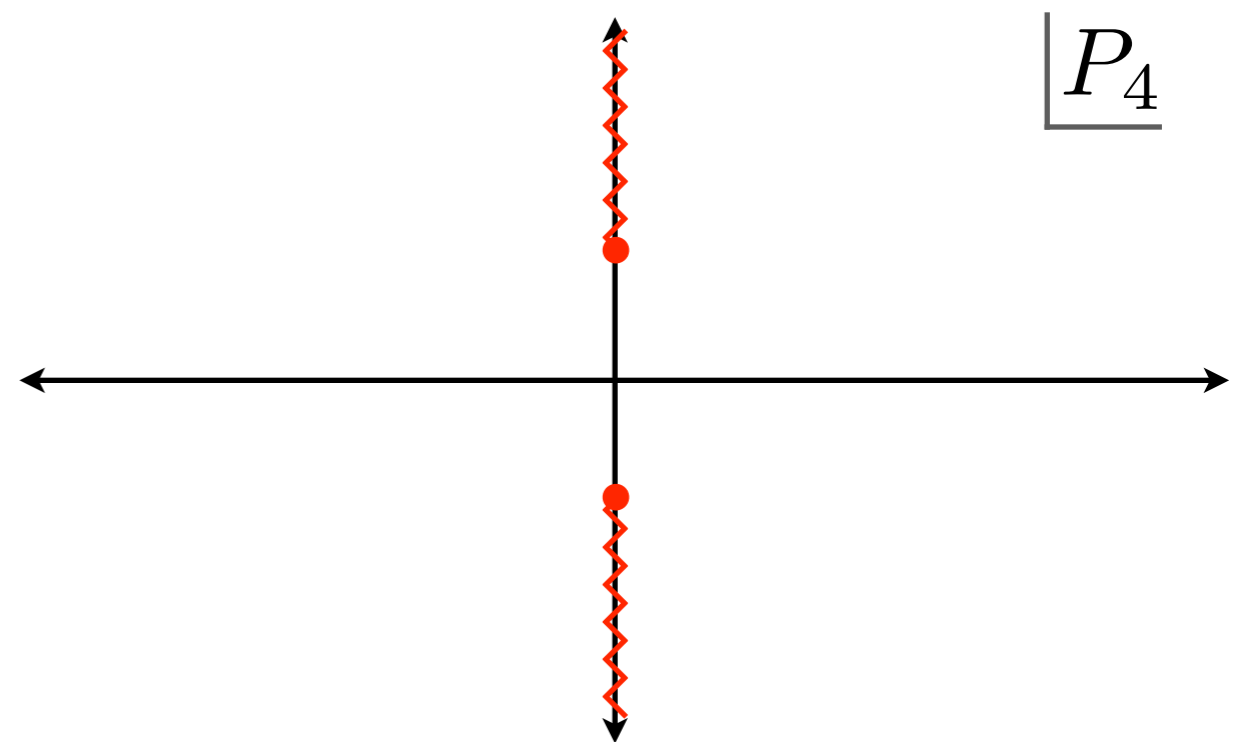


$$C_L(P) \equiv \int_L d^4x e^{-iPx} \langle 0|T\mathcal{O}(x)\mathcal{O}^\dagger(0)|0\rangle$$

At fixed L, \vec{P} , poles in C_L give finite-volume spectrum

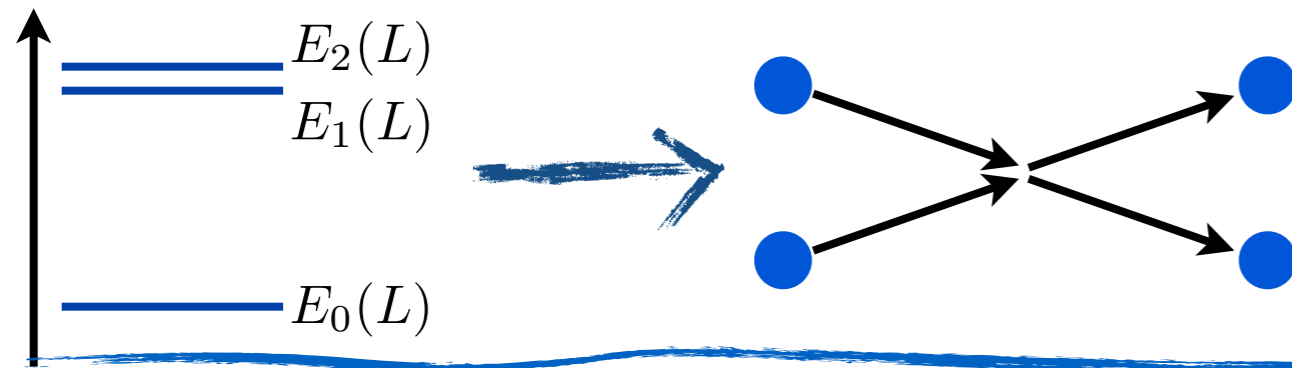


C_L analytic structure



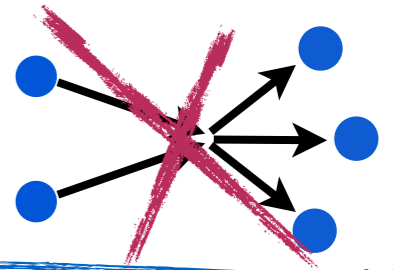
C_∞ analytic structure

Two-to-two scattering



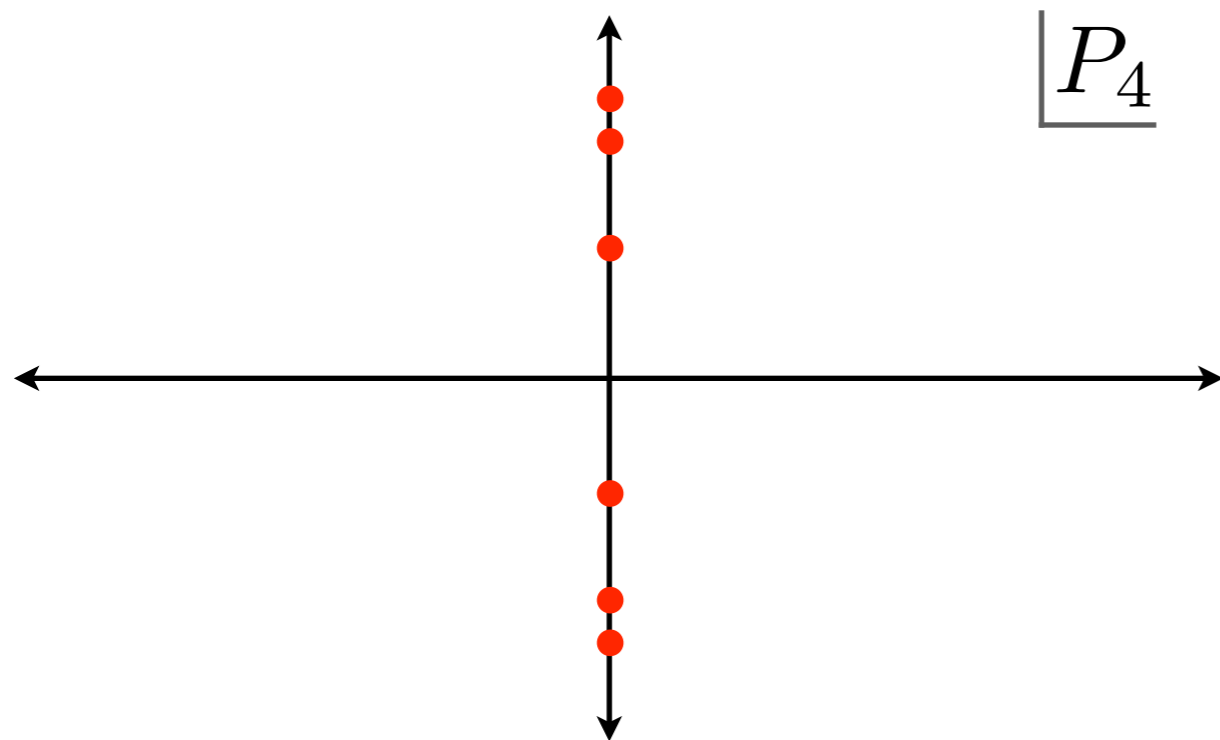
For now assume...
identical scalars, mass m

\mathbb{Z}_2 symmetry



$$C_L(P) \equiv \int_L d^4x e^{-iPx} \langle 0|T\mathcal{O}(x)\mathcal{O}^\dagger(0)|0\rangle$$

At fixed L, \vec{P} , poles in C_L give finite-volume spectrum



C_L analytic structure

Calculate $C_L(P)$ to all orders in perturbation theory and determine locations of poles.

$$\begin{aligned}
C_L(P) = & \text{Diagram 1} + \text{Diagram 2} \\
& + \text{Diagram 3} + \dots
\end{aligned}$$

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

The diagram shows a series of terms in a sum. Each term consists of a sequence of circles connected by arcs. The first circle in each term is labeled \mathcal{O}^\dagger and the last is labeled \mathcal{O} . The first term has two black dots between \mathcal{O}^\dagger and \mathcal{O} . The second term has two black dots between \mathcal{O}^\dagger and a circle labeled iK , and two black dots between iK and \mathcal{O} . The third term has two black dots between \mathcal{O}^\dagger and iK , two between the first iK and a second iK , and two between the second iK and \mathcal{O} . A blue dashed box highlights the two dots between \mathcal{O}^\dagger and iK in the second term, with a blue arrow pointing to a text box below.

spatial loop momenta
are summed

$$\frac{1}{L^3} \sum_{\vec{k} \in (2\pi/L)\mathbb{Z}^3} \int \frac{dk^0}{2\pi}$$

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

The equation shows a series of Feynman diagrams representing the correlation function $C_L(P)$. The first diagram consists of two circles, \mathcal{O}^\dagger and \mathcal{O} , connected by two lines, with two black dots on each line. The second diagram is similar but includes a central circle labeled iK . The third diagram includes two iK circles. Blue dashed boxes and an arrow highlight the iK circles in the first two diagrams, indicating they are summed over spatial loop momenta.

spatial loop momenta
are summed

$$\frac{1}{L^3} \sum_{\vec{k} \in (2\pi/L)\mathbb{Z}^3} \int \frac{dk^0}{2\pi}$$



Bethe Salpeter kernel

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

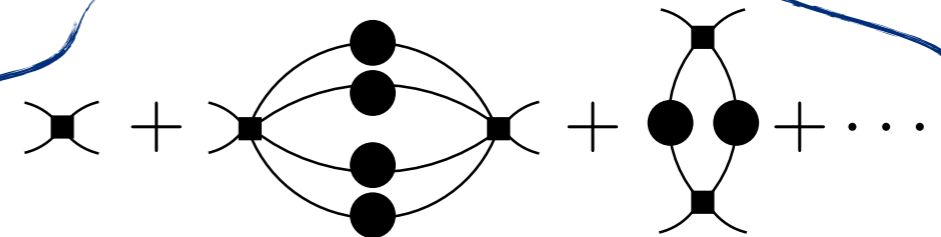
$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

The equation shows a series of diagrams representing the correlation function $C_L(P)$. The first diagram consists of two circles, \mathcal{O}^\dagger and \mathcal{O} , connected by two lines, each with a black dot. The second diagram is similar but includes a circle labeled iK between the two lines. The third diagram includes two iK circles. Blue dashed boxes highlight the internal lines in each diagram, and blue arrows point from these boxes to the definitions below.

spatial loop momenta
are summed

$$\frac{1}{L^3} \sum_{\vec{k} \in (2\pi/L)\mathbb{Z}^3} \int \frac{dk^0}{2\pi}$$

$\Delta \equiv$ 
fully dressed
propagator



Bethe Salpeter kernel

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

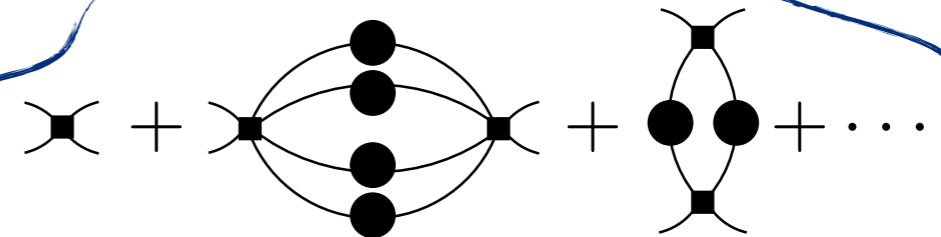
$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

The first diagram shows a circle labeled \mathcal{O}^\dagger connected to a circle labeled \mathcal{O} via two vertices, each with two external legs. The second diagram is similar but includes a circle labeled iK between the two vertices. The third diagram includes two iK circles in series. Blue dashed boxes highlight the vertices in the first two diagrams, and a blue arrow points from the first diagram to the second.

spatial loop momenta
are summed

$$\frac{1}{L^3} \sum_{\vec{k} \in (2\pi/L)\mathbb{Z}^3} \int \frac{dk^0}{2\pi}$$

$\Delta \equiv$ 
fully dressed
propagator



Bethe Salpeter kernel

if $E^* < 4m$ **then**

$$K_L = K_\infty + \mathcal{O}(e^{-mL})$$

$$\Delta_L = \Delta_\infty + \mathcal{O}(e^{-mL})$$

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

$\frac{1}{L^3} \sum_{\vec{k}} \int_{\vec{k}}$

contains all power-law corrections

Now we introduce an important identity.

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

The equation shows a series of diagrams representing the expansion of $C_L(P)$. Each diagram consists of two large circles (representing nucleons) connected by two lines (representing pions). The first diagram has two black dots on the lines between the circles. The second diagram has two black dots on the lines and a central circle labeled iK . The third diagram has two black dots on the lines, two central circles labeled iK , and a final circle labeled \mathcal{O} . Blue arrows point from the first two diagrams of the expansion to the first two terms of the equation above.

$$\frac{1}{L^3} \sum_{\vec{k}} \int_{\vec{k}} \text{diagram}_1 = \text{diagram}_2 + \underbrace{\text{diagram}_3}_F$$

The diagram on the left shows two large circles connected by two lines with two black dots. This is equal to the sum of two diagrams: one with two black dots and one with two horizontal lines connecting the circles, labeled F . A blue bracket under F is labeled "contains all power-law corrections".

Now we introduce an important identity.

In  all four-momenta are projected on shell.

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

The equation shows a series of diagrams representing terms in a sum. Each diagram consists of two large circles (representing operators \mathcal{O}^\dagger and \mathcal{O}) connected by two lines. The first diagram has two black dots on the lines between the circles. The second diagram has two black dots on the lines and a central circle labeled iK . The third diagram has two black dots on the lines, two central circles labeled iK , and a final circle labeled \mathcal{O} . Blue arrows point from the first two diagrams of the sum to the first two diagrams of the equation below.

$$\frac{1}{L^3} \sum_{\vec{k}} \text{diagram}_1 = \text{diagram}_2 + \underbrace{\text{diagram}_3}_F$$

The diagram on the left is the same as the first diagram in the sum above. The diagram on the right is the same as the second diagram in the sum above. The diagram on the far right is a horizontal line connecting two large circles, with a vertical dashed line through the center labeled F . Below the F diagram is the text "contains all power-law corrections" in blue.

Now we introduce an important identity.

In  all four-momenta are projected on shell.

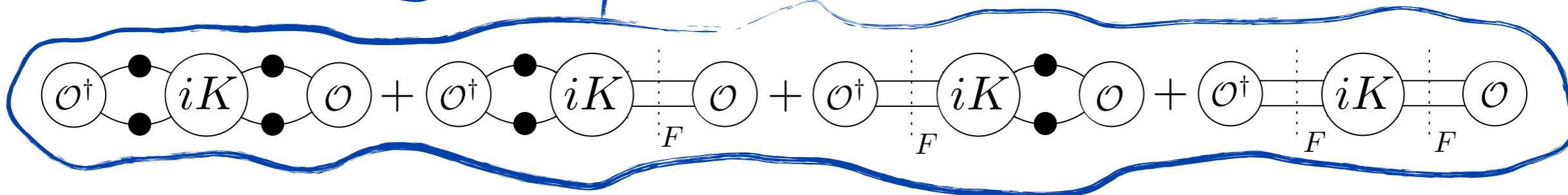
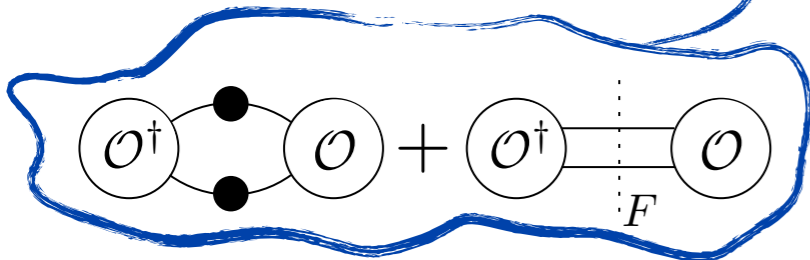
Physical, propagating states give dominate finite-volume effects.

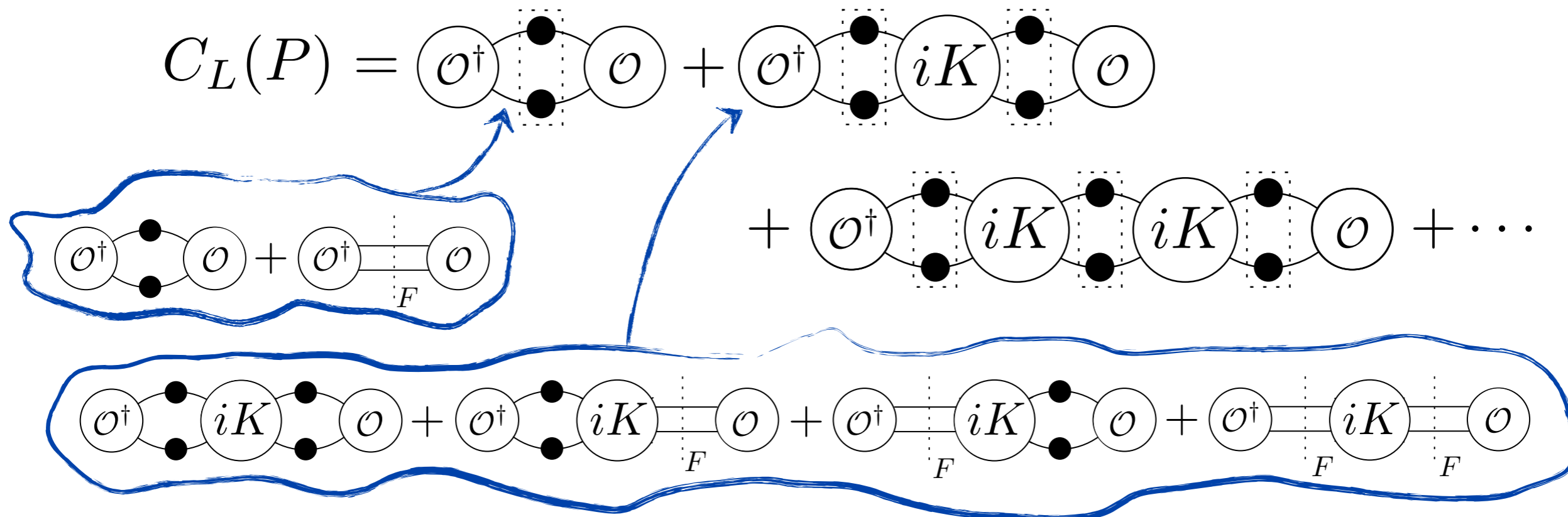
Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

$$C_L(P) = \text{diagram}_1 + \text{diagram}_2$$

$$+ \text{diagram}_3 + \dots$$

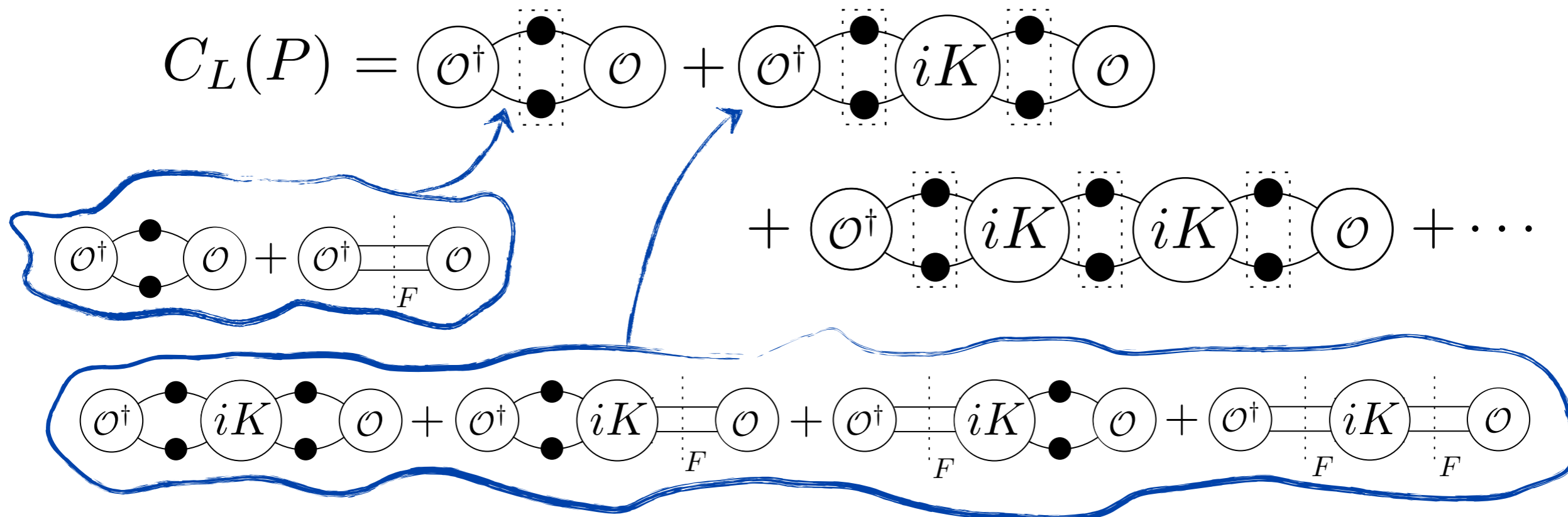




Now regroup by number of Fs

zero Fs

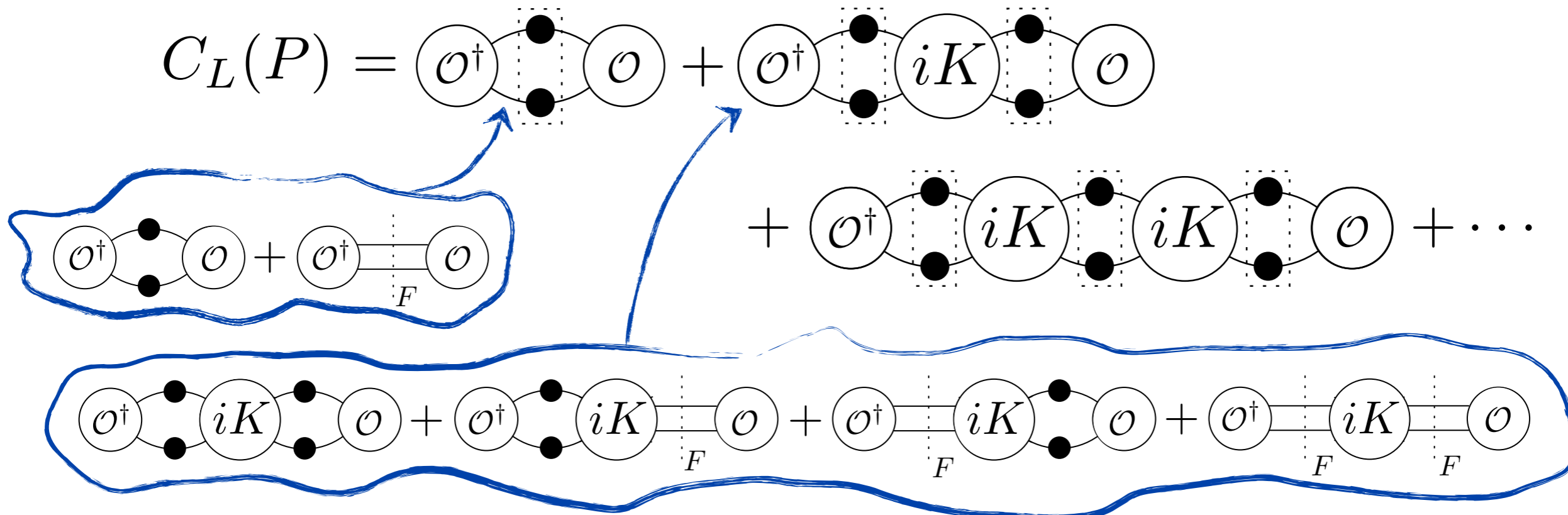
$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) +$$



Now regroup by number of Fs

$$C_L(E, \vec{P}) = \overset{\text{zero Fs}}{C_\infty(E, \vec{P})} + \overset{\text{one F}}{\text{diagram 6}} + \dots$$

Diagram 6: A and A' with a horizontal line between them and a vertical dashed line labeled F below the line.

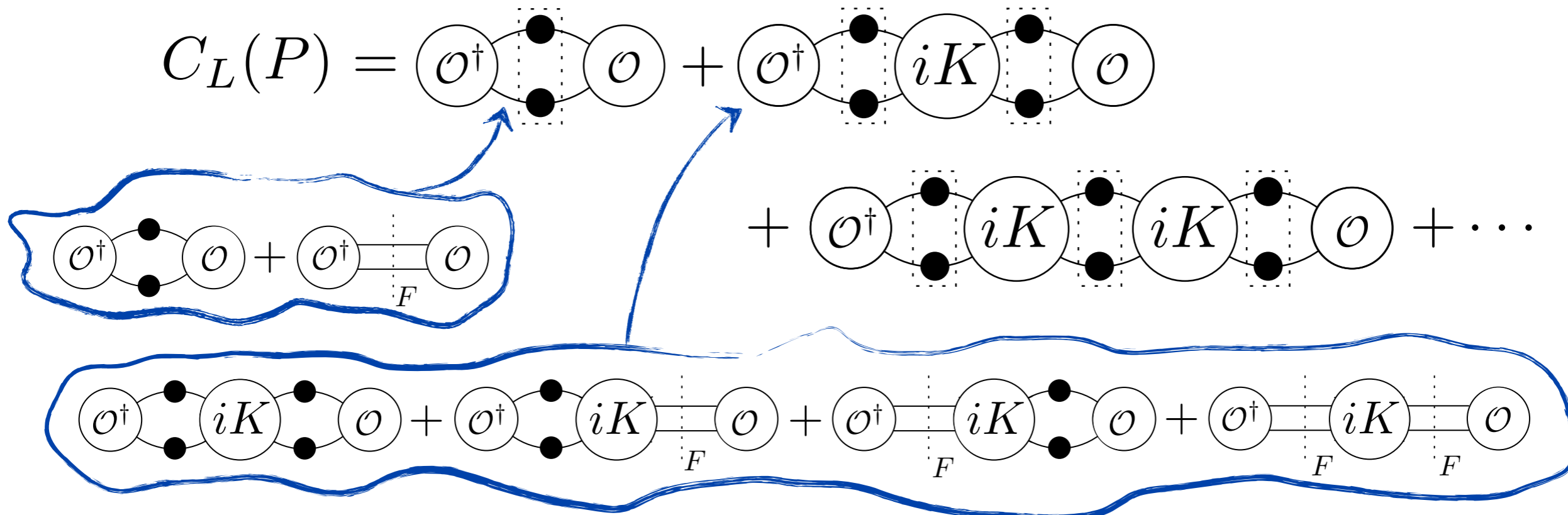


Now regroup by number of Fs

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \text{diagram with one F}$$

zero Fs
one F

$$= \langle \pi\pi, \text{out} | \mathcal{O}^\dagger | 0 \rangle$$



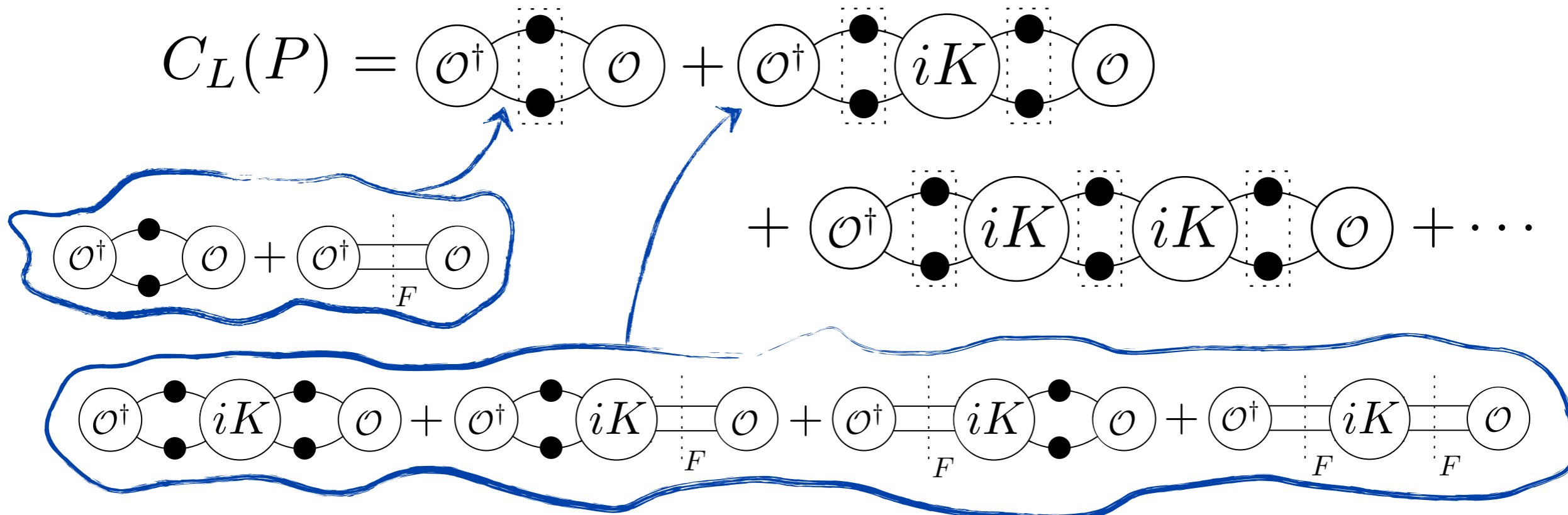
Now regroup by number of Fs

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \text{one } F + \text{two } Fs + \dots$$

The diagram shows the expansion of $C_L(E, \vec{P})$ as a sum of terms. The first term is $C_\infty(E, \vec{P})$. The second term is labeled "one F" and shows a diagram with one F line. The third term is labeled "two Fs" and shows a diagram with two F lines. Ellipses indicate further terms.

$$= \langle \pi\pi, \text{out} | \mathcal{O}^\dagger | 0 \rangle$$

The diagram shows a blue box containing the terms $\mathcal{O}^\dagger + \mathcal{O}^\dagger + iK + \dots$, which is then equated to the vacuum expectation value $\langle \pi\pi, \text{out} | \mathcal{O}^\dagger | 0 \rangle$.



Now regroup by number of Fs

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \begin{array}{c} \text{zero Fs} \\ \text{one F} \\ \text{two Fs} \end{array} \begin{array}{c} \text{diagram 1} \\ \text{diagram 2} \\ \text{diagram 3} \end{array} + \dots$$

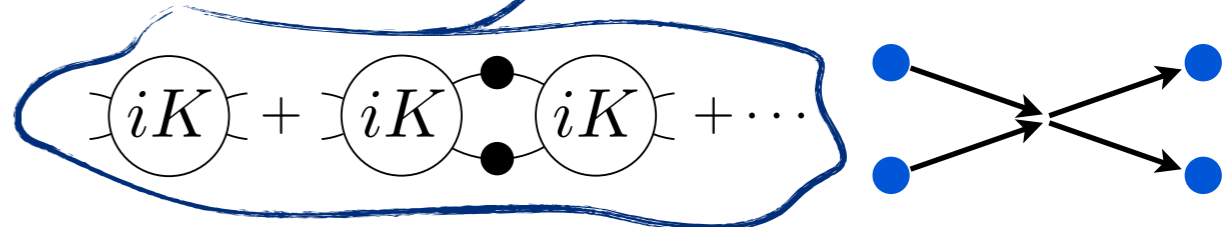
Diagram 1: A and A' with a vertical dashed line labeled F .

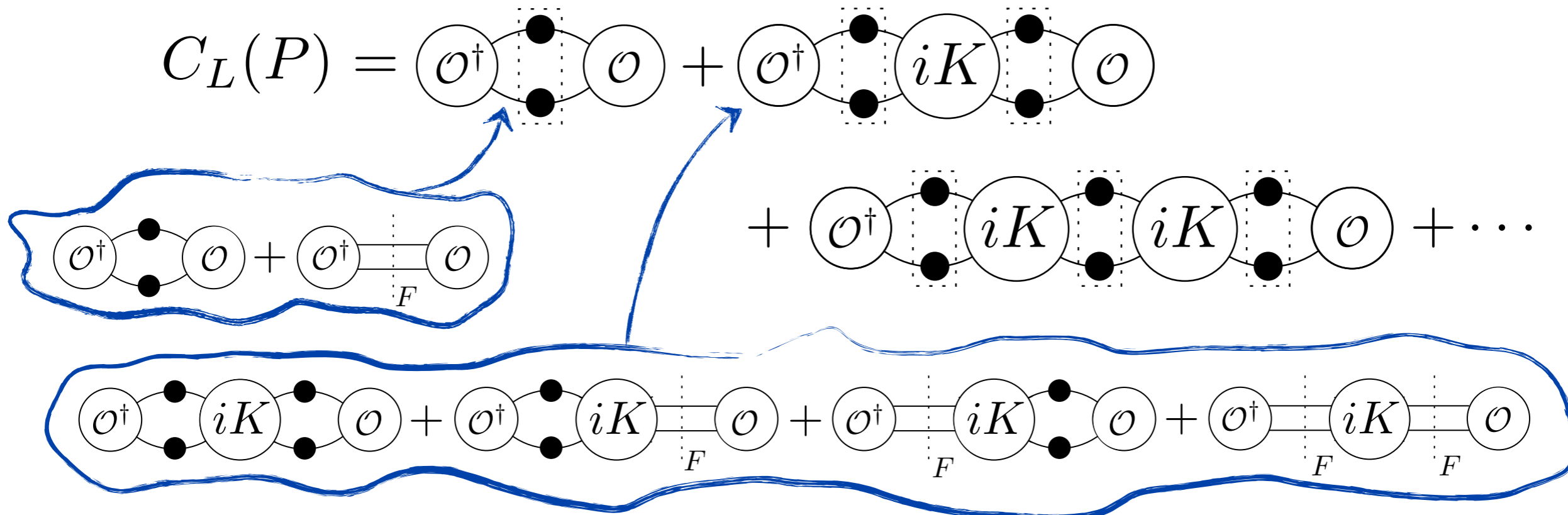
Diagram 2: A , iM , and A' with two vertical dashed lines labeled F .

$$\begin{array}{c} \text{diagram 1} + \text{diagram 2} + \dots \\ = \langle \pi\pi, \text{out} | \mathcal{O}^\dagger | 0 \rangle \end{array}$$

Diagram 1: \mathcal{O}^\dagger and \mathcal{O} .

Diagram 2: \mathcal{O}^\dagger , iK , and \mathcal{O} .





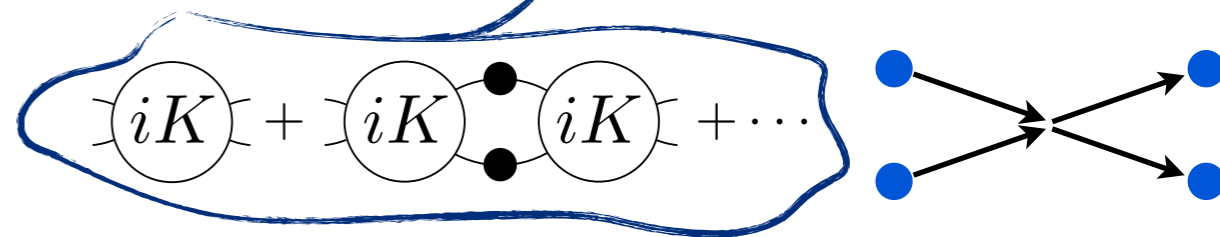
Now regroup by number of Fs

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \text{diagram}_1 + \text{diagram}_2 + \dots$$

The diagram shows the expansion of $C_L(E, \vec{P})$ regrouped by the number of F lines. The first term is $C_\infty(E, \vec{P})$. The second term has one F line between vertices A and A' . The third term has two F lines between vertices A , iM , and A' . Vertical dashed lines labeled F separate the vertices.

$$= \langle \pi\pi, \text{out} | \mathcal{O}^\dagger | 0 \rangle$$

The diagram shows the first term in the expansion, $\langle \pi\pi, \text{out} | \mathcal{O}^\dagger | 0 \rangle$, represented as a chain of vertices \mathcal{O}^\dagger , \mathcal{O} , and iK .



When we factorize diagrams and group infinite-volume parts...

physical observables emerge!

Review..

Review...

1

$$C_L(P) = \begin{array}{c} \text{Diagram 1} + \text{Diagram 2} \\ + \text{Diagram 3} + \dots \end{array}$$

The diagrammatic equation for $C_L(P)$ is shown. The first row contains two terms: \mathcal{O}^\dagger connected to \mathcal{O} via two vertices (top and bottom) enclosed in a dashed box, and \mathcal{O}^\dagger connected to iK via two vertices (top and bottom) enclosed in a dashed box, which is then connected to \mathcal{O} via two vertices (top and bottom) enclosed in a dashed box. The second row contains a term where \mathcal{O}^\dagger is connected to the first iK vertex, which is connected to a second iK vertex, which is then connected to \mathcal{O} . Ellipses follow. A blue callout box highlights a sub-diagram showing a sequence of vertices and lines, with a blue arrow pointing from the first iK vertex in the second row to the first vertex in the callout.

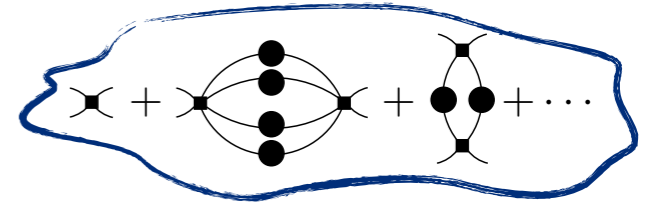
Review...

1

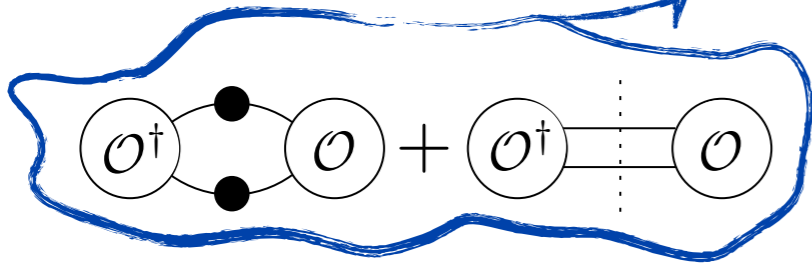
$$C_L(P) = \text{diagram 1} + \text{diagram 2}$$

The first diagram shows a circle labeled \mathcal{O}^\dagger on the left and a circle labeled \mathcal{O} on the right. Two black dots are positioned between them, enclosed in a vertical dashed rectangle. Two arcs connect the dots to the \mathcal{O}^\dagger circle, and two arcs connect the dots to the \mathcal{O} circle.

The second diagram is identical to the first, but the central circle is labeled iK .



2



$$+ \text{diagram 3} + \text{diagram 4} + \dots$$

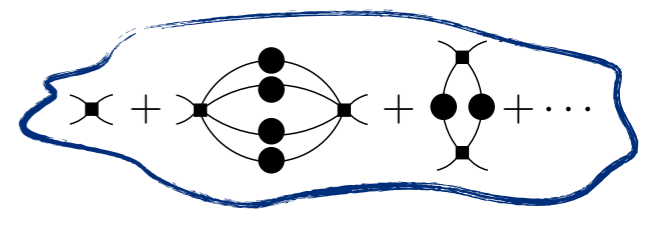
The third diagram shows a circle labeled \mathcal{O}^\dagger on the left and a circle labeled \mathcal{O} on the right, with two iK circles in between. Each iK circle has two dots, and the dots are enclosed in vertical dashed rectangles. Arcs connect the dots to the \mathcal{O}^\dagger and \mathcal{O} circles.

The fourth diagram is identical to the third, but the second iK circle is also labeled iK .

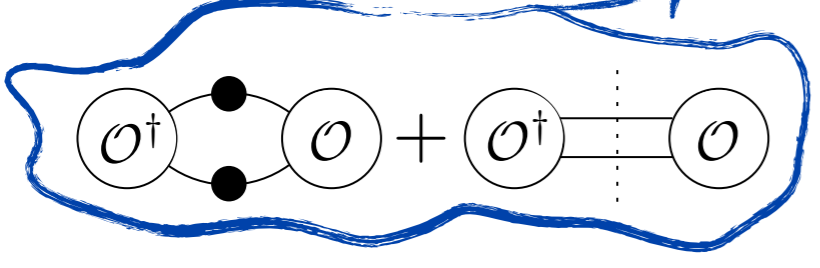
Review...

1

$$C_L(P) = \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} \mathcal{O} + \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} \mathcal{O}$$

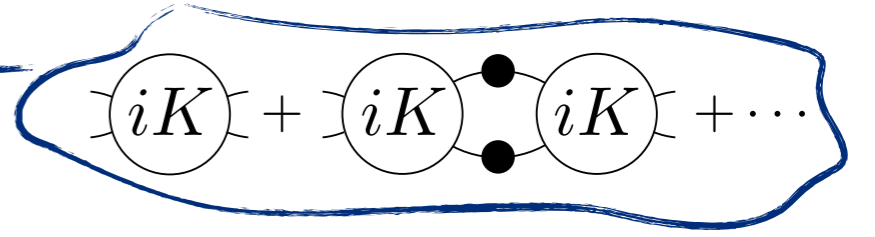


2

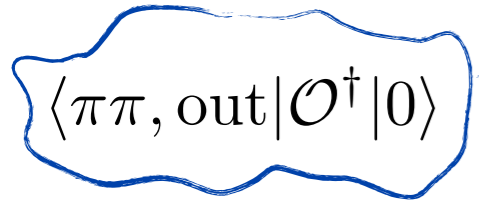


$$+ \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} \mathcal{O} + \dots$$

$$C_L(P) = C_\infty(P)$$

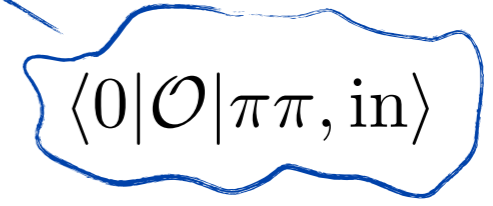


3



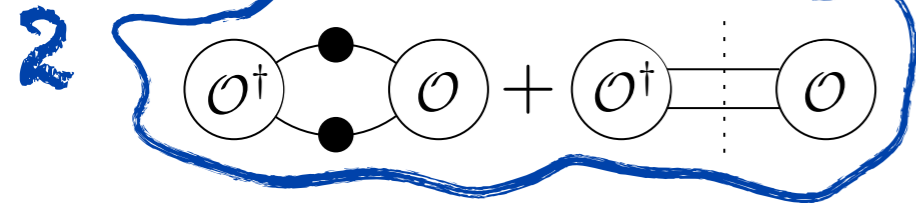
$$+ \begin{array}{c} \mathcal{A} \text{---} \mathcal{A}' \\ F \end{array} + \begin{array}{c} \mathcal{A} \text{---} i\mathcal{M} \text{---} \mathcal{A}' \\ F \quad F \quad F \end{array}$$

$$+ \begin{array}{c} \mathcal{A} \text{---} i\mathcal{M} \text{---} i\mathcal{M} \text{---} \mathcal{A}' \\ F \quad F \quad F \end{array} + \dots$$

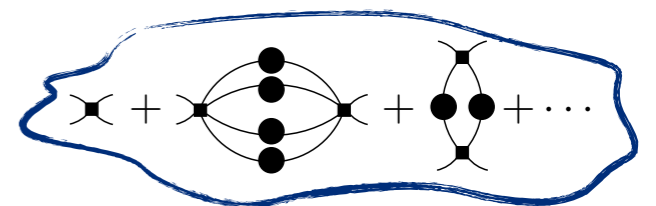


Review...

$$C_L(P) = \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \mathcal{O} + \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \mathcal{O}$$

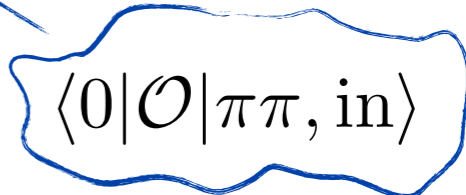
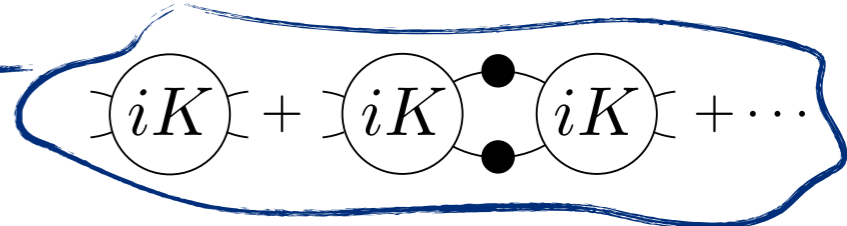
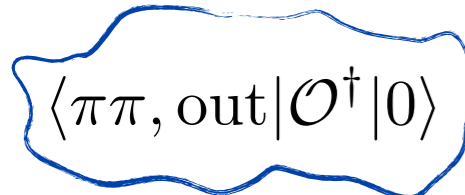


$$+ \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \mathcal{O} + \dots$$



$$C_L(P) = C_\infty(P)$$

$$+ \begin{array}{c} A \\ \text{---} \\ F \end{array} \begin{array}{c} A' \\ \text{---} \\ F \end{array} + \begin{array}{c} A \\ \text{---} \\ F \end{array} \begin{array}{c} i\mathcal{M} \\ \text{---} \\ F \end{array} \begin{array}{c} A' \\ \text{---} \\ F \end{array} \\ + \begin{array}{c} A \\ \text{---} \\ F \end{array} \begin{array}{c} i\mathcal{M} \\ \text{---} \\ F \end{array} \begin{array}{c} i\mathcal{M} \\ \text{---} \\ F \end{array} \begin{array}{c} A' \\ \text{---} \\ F \end{array} + \dots$$

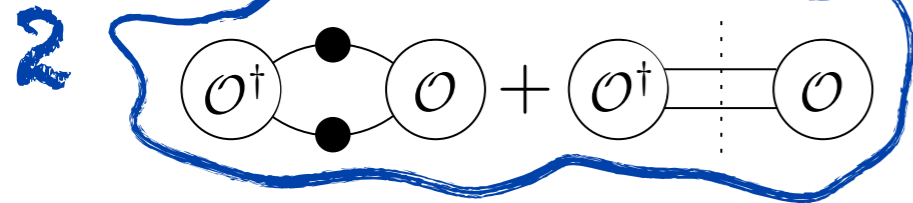


We deduce...

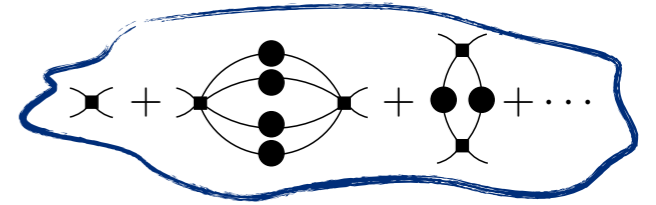
$$C_L(P) = C_\infty(P) - A' F \frac{1}{1 + \mathcal{M}_{2 \rightarrow 2} F} A$$

Review...

$$C_L(P) = \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} \mathcal{O} + \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} \mathcal{O}$$

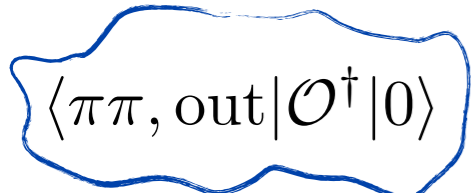


$$+ \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} \mathcal{O} + \dots$$

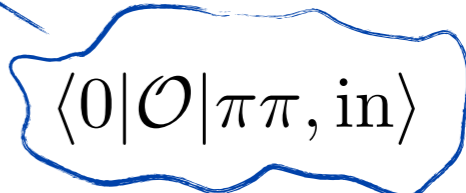
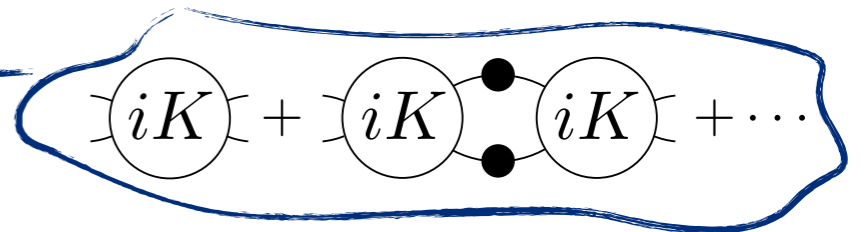


$$C_L(P) = C_\infty(P)$$

$$+ \begin{array}{c} A \\ \vdots \\ F \end{array} A' + \begin{array}{c} A \\ \vdots \\ F \end{array} i\mathcal{M} \begin{array}{c} \vdots \\ F \end{array} A'$$



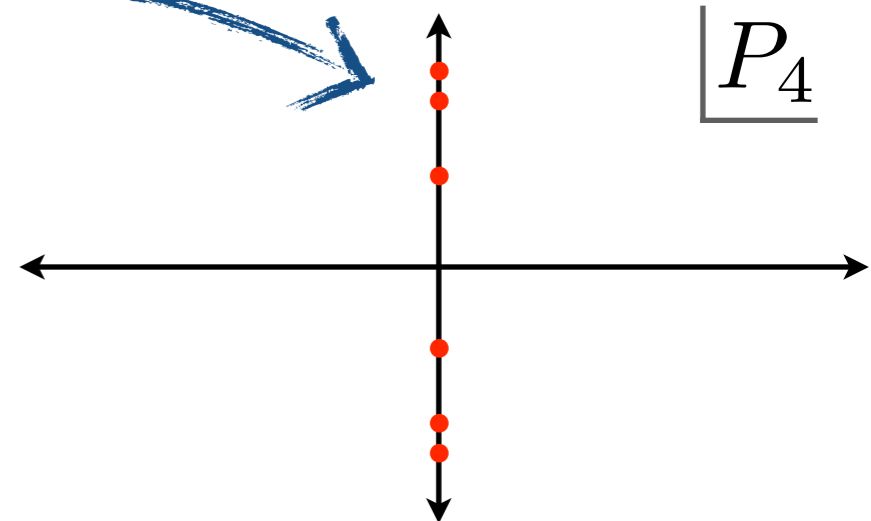
$$+ \begin{array}{c} A \\ \vdots \\ F \end{array} i\mathcal{M} \begin{array}{c} \vdots \\ F \end{array} i\mathcal{M} \begin{array}{c} \vdots \\ F \end{array} A' + \dots$$



We deduce...

$$C_L(P) = C_\infty(P) - A' F \frac{1}{1 + \mathcal{M}_{2 \rightarrow 2} F} A$$

poles are in here



Two-particle result

At fixed (L, \vec{P}) , finite-volume energies are solutions to $\det[\mathcal{M}_{2 \rightarrow 2}^{-1} + F] = 0$

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

Rummukainen and Gottlieb, *Nucl. Phys.* B450, 397 (1995)

Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

Matrices defined using angular-momentum states

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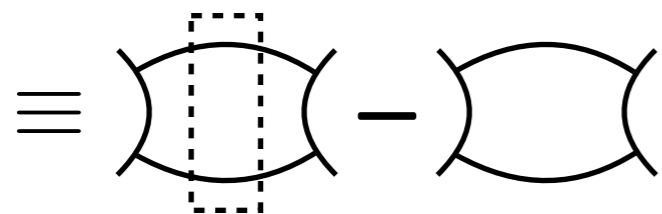
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**difference of two-particle loops
in finite and infinite volume**

**depends on
 L, E, \vec{P}**

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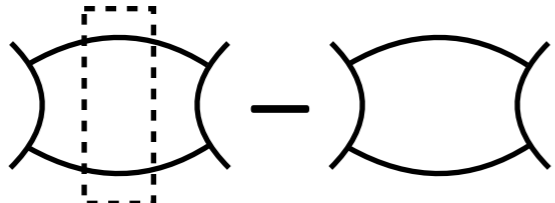
Rummukainen and Gottlieb, *Nucl. Phys.* B450, 397 (1995)

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$F \equiv$ non-diagonal matrix of known geometric functions

\equiv  difference of two-particle loops in finite and infinite volume depends on L, E, \vec{P}

At low energies, lowest partial waves dominate $\mathcal{M}_{2 \rightarrow 2}$

e.g. s-wave only
with some
rearranging

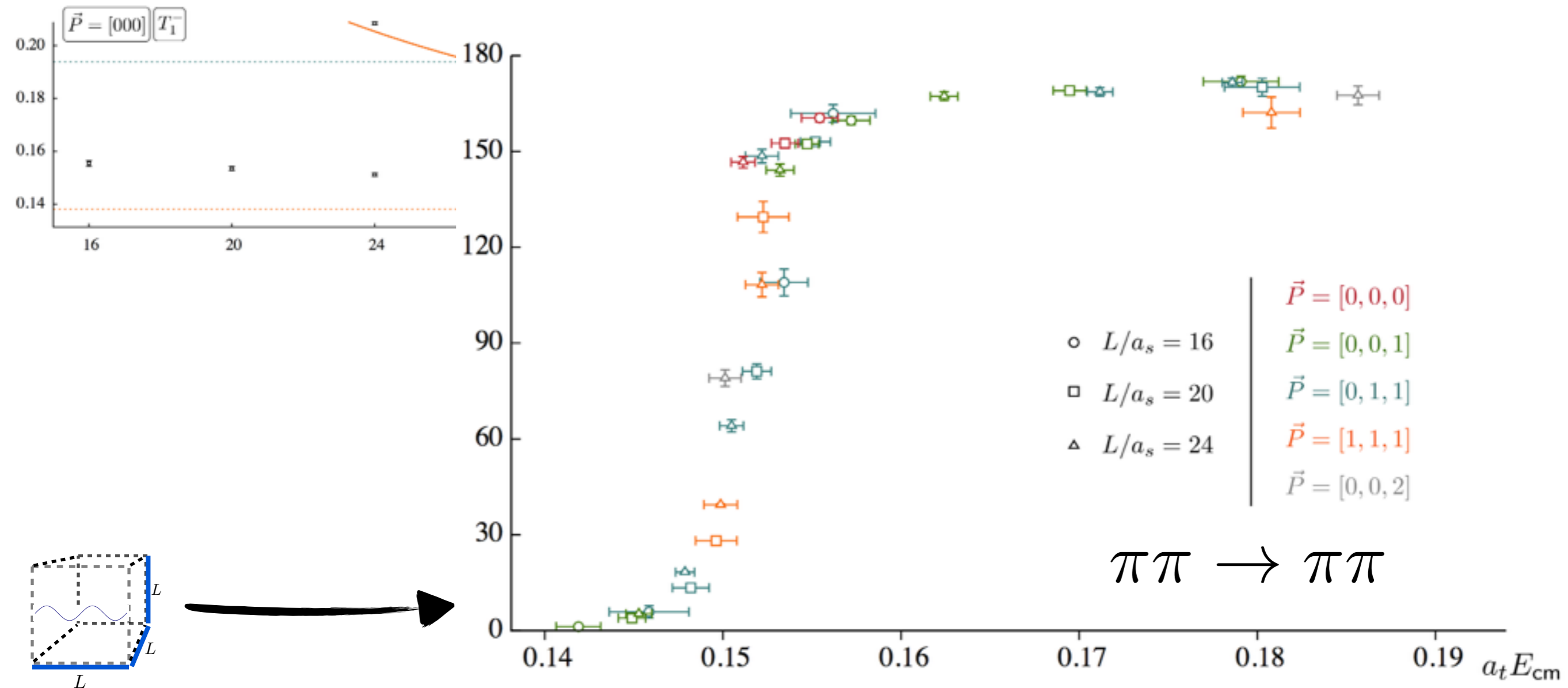
$$\rightarrow \cot \delta(E_n^*) + \cot \phi(E_n, \vec{P}, L) = 0$$

scattering phase

known function

Using the result (p-wave)

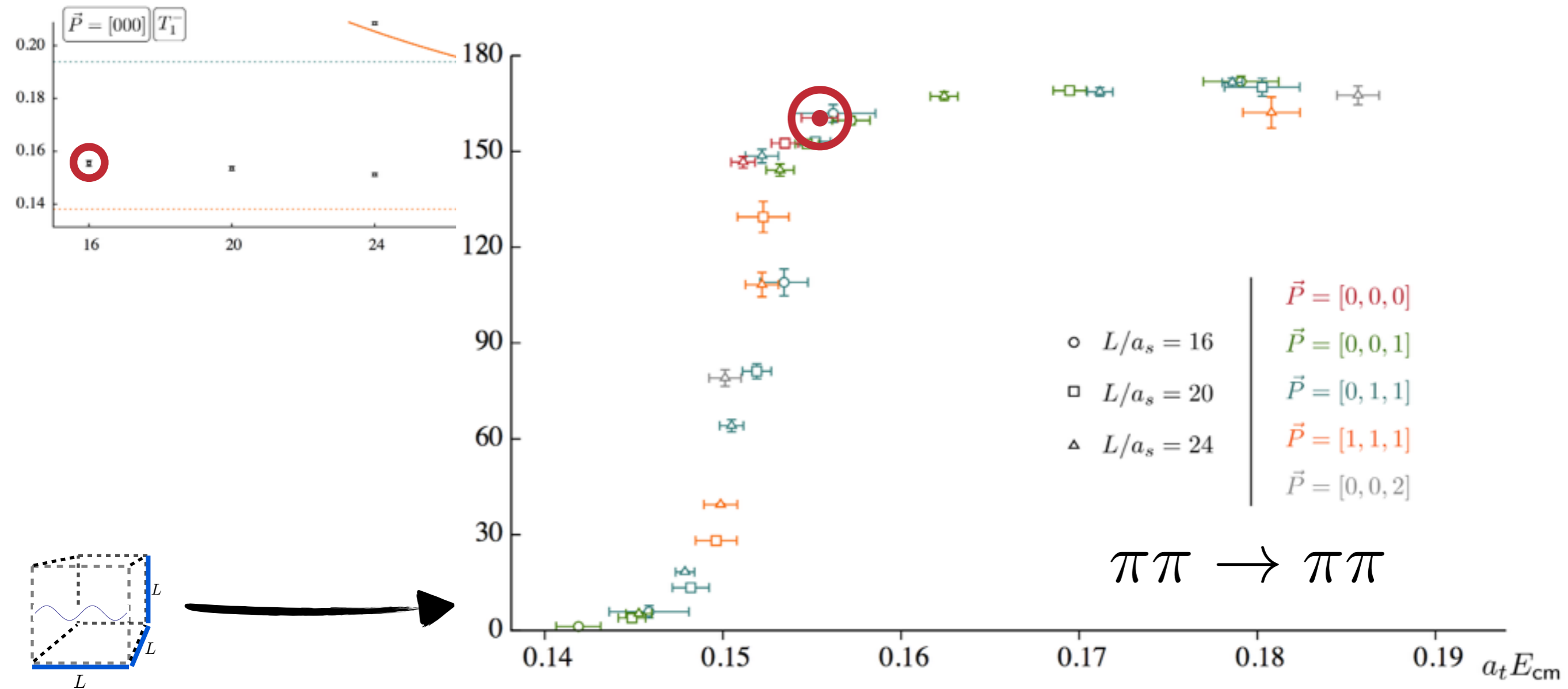
$$\cot \delta_{\ell=1}(E_n^*) + \cot \phi(E_n, \vec{P}, L) = 0$$



from Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505

Using the result (p-wave)

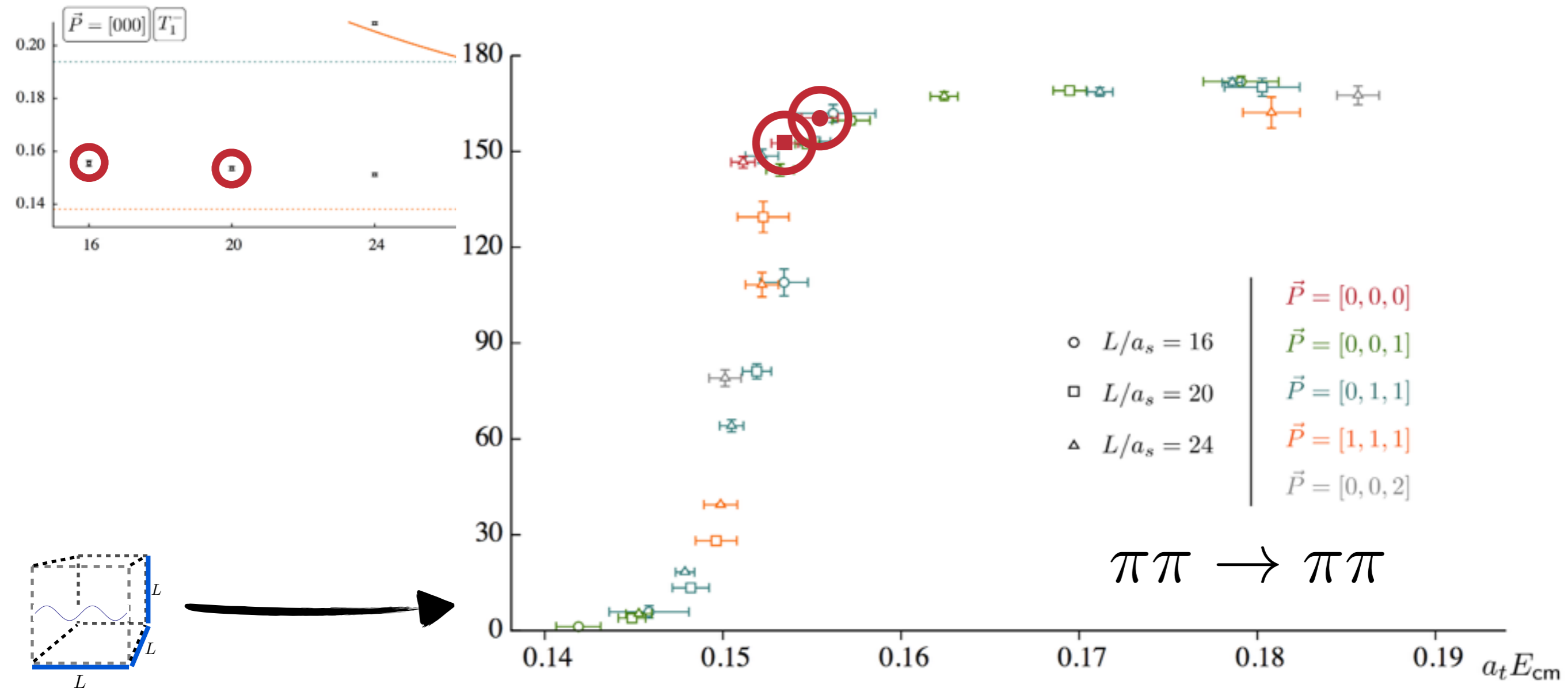
$$\cot \delta_{\ell=1}(E_n^*) + \cot \phi(E_n, \vec{P}, L) = 0$$



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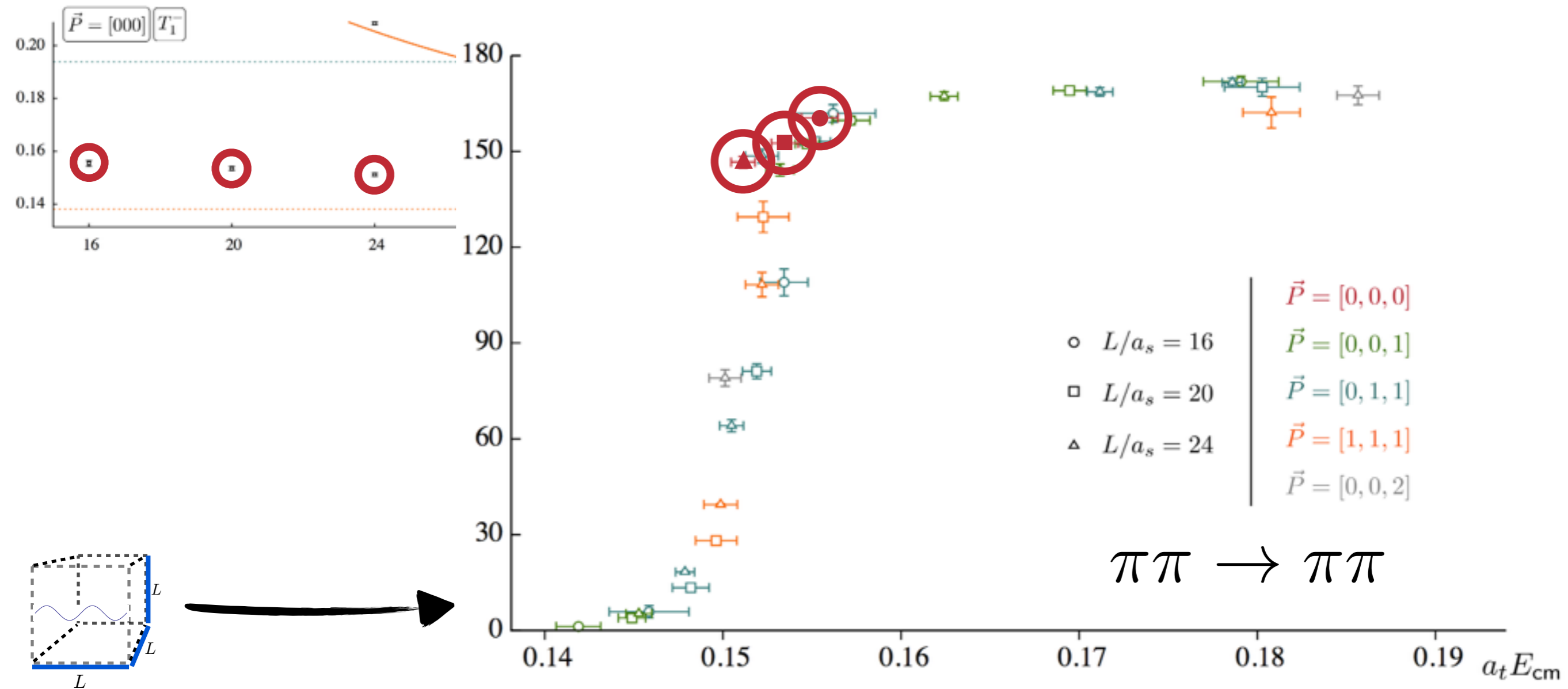
$$\cot \delta_{\ell=1}(E_n^*) + \cot \phi(E_n, \vec{P}, L) = 0$$



from Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505

Using the result (p-wave)

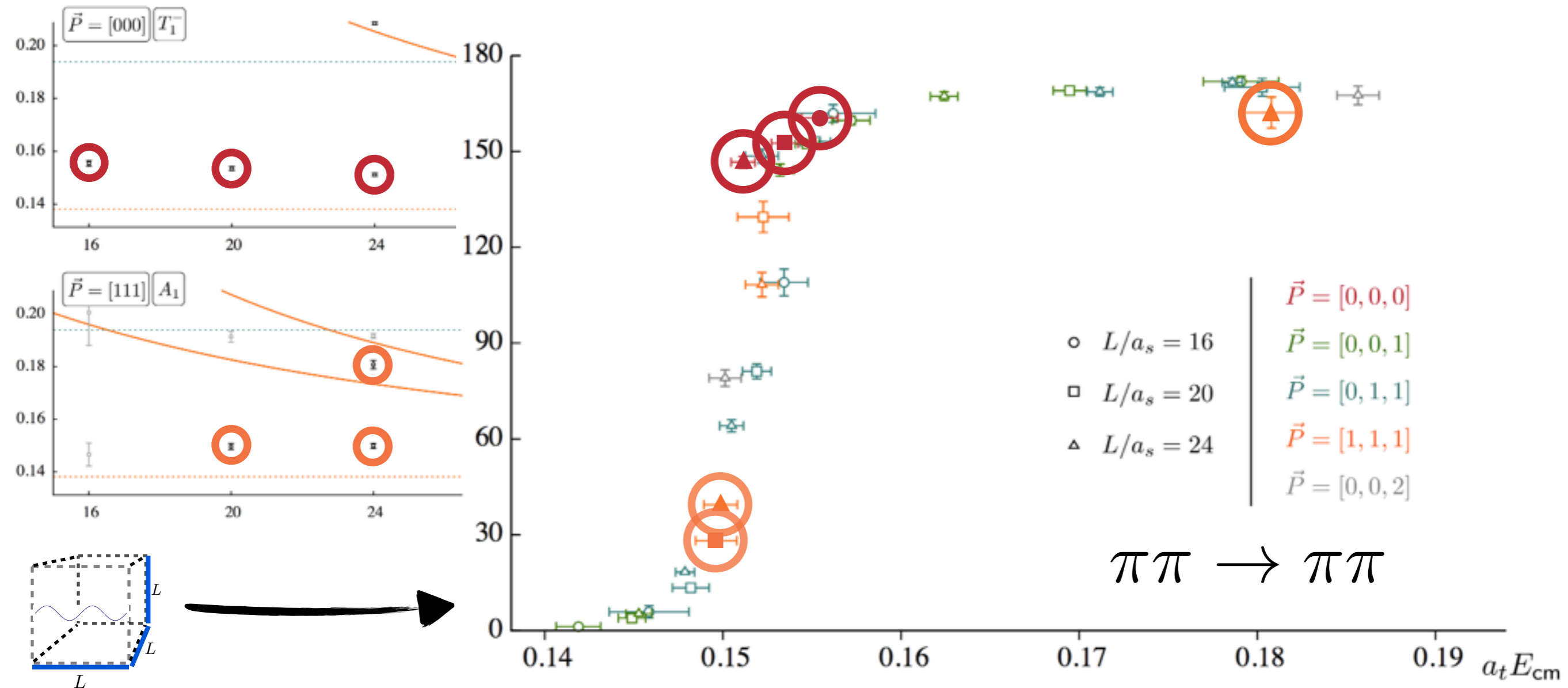
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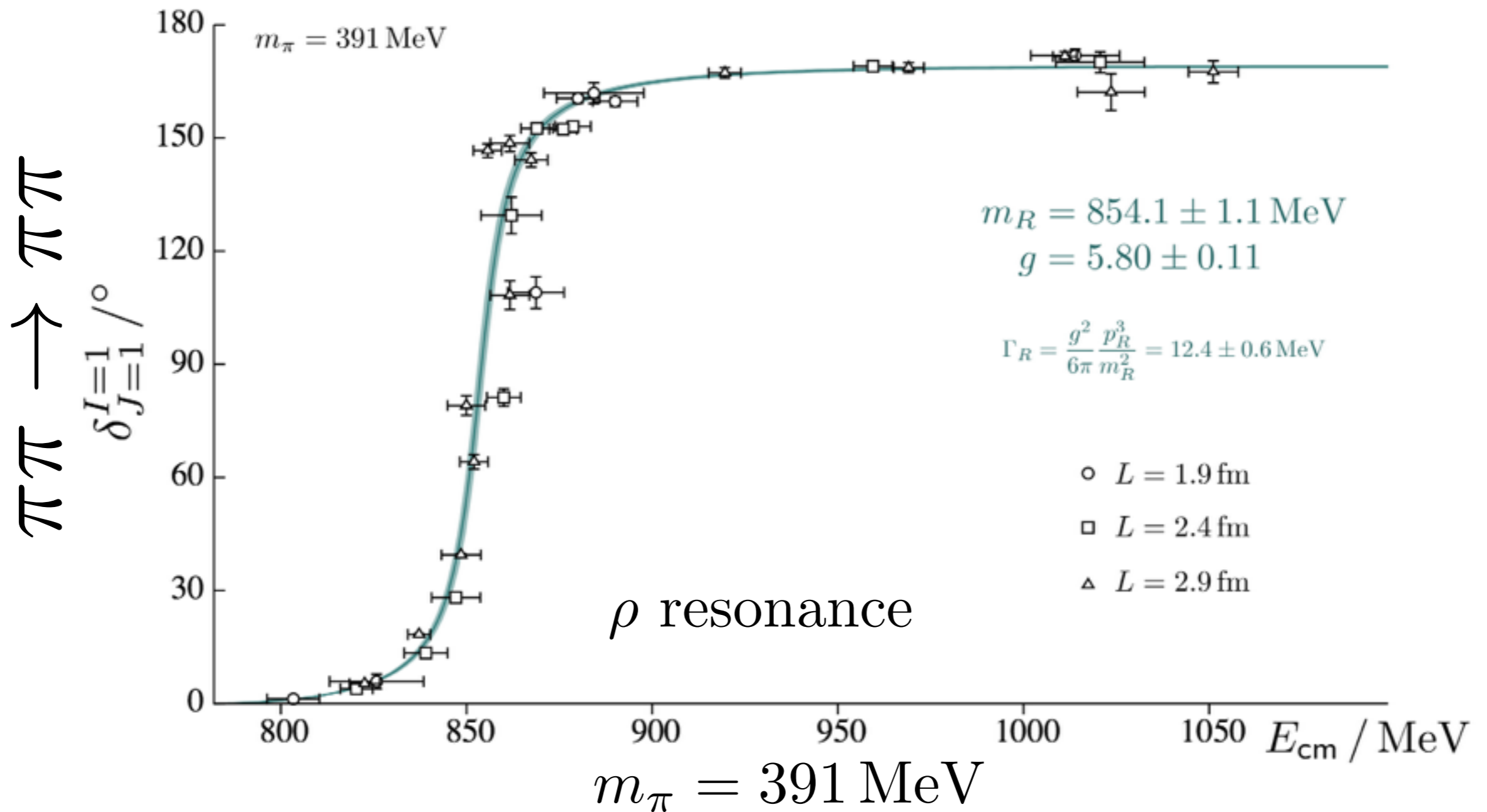
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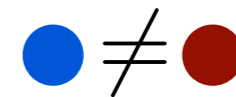
from Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505

Two-particle result

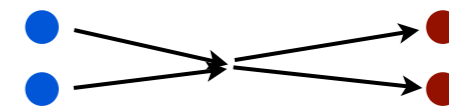
At fixed (L, \vec{P}) , finite-volume energies are solutions to $\det[\mathcal{M}_{2 \rightarrow 2}^{-1} + F] = 0$

Has since been generalized to include...

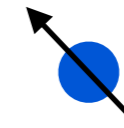
non-identical particles



multiple two-particle channels



particles with spin



Bernard, Lage, Meißner, and Rusetsky, JHEP, 1101, 019 (2011)

MTH and Sharpe, *Phys.Rev. D*86 (2012) 016007

Briceño and Davoudi, *Phys.Rev. D*88 (2013) 094507

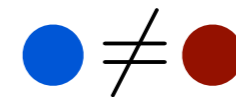
Briceño, *Phys. Rev. D* 89, 074507 (2014)

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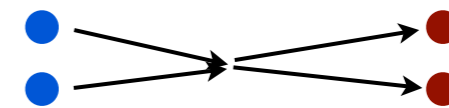
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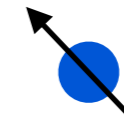
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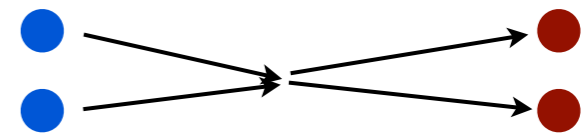
MTH and Sharpe, *Phys.Rev. D86* (2012) 016007

Briceño and Davoudi, *Phys.Rev. D88* (2013) 094507

Briceño, *Phys. Rev. D 89*, 074507 (2014)

The basic form of the equation stays the same,
but the **matrix space** and **definition of F** change

Multiple two-particle channels

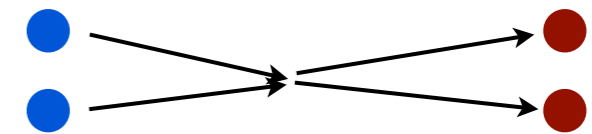


Must now include
a channel index

MTH and Sharpe/Briceño and Davoudi

$$\det \left[\begin{pmatrix} \mathcal{M}_{a \rightarrow a} & \mathcal{M}_{a \rightarrow b} \\ \mathcal{M}_{b \rightarrow a} & \mathcal{M}_{b \rightarrow b} \end{pmatrix}^{-1} + \begin{pmatrix} F_a & 0 \\ 0 & F_b \end{pmatrix} \right] = 0$$

Multiple two-particle channels



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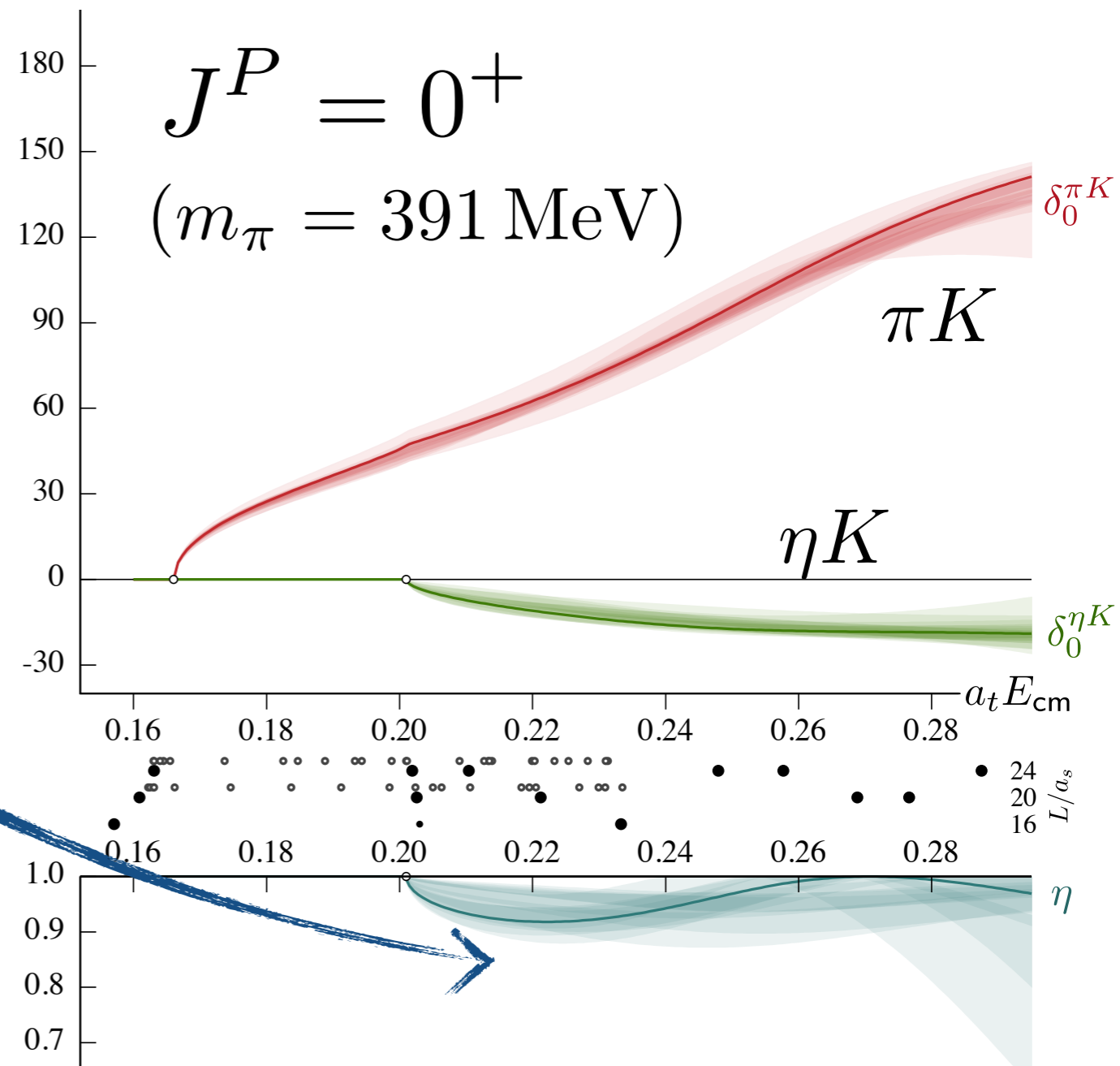
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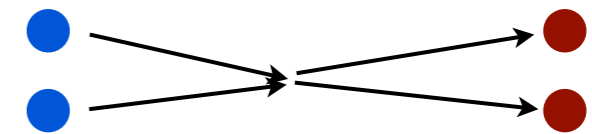
Already used in JLab study of
 $\pi K, \eta K$

$$\mathcal{M}(\pi K \rightarrow \eta K) \sim \sqrt{1 - \eta^2}$$

Wilson, Dudek, Edwards, Thomas,
Phys. Rev. D 91, 054008 (2015)
arXiv: 1411.2004



Multiple two-particle channels



Must now include
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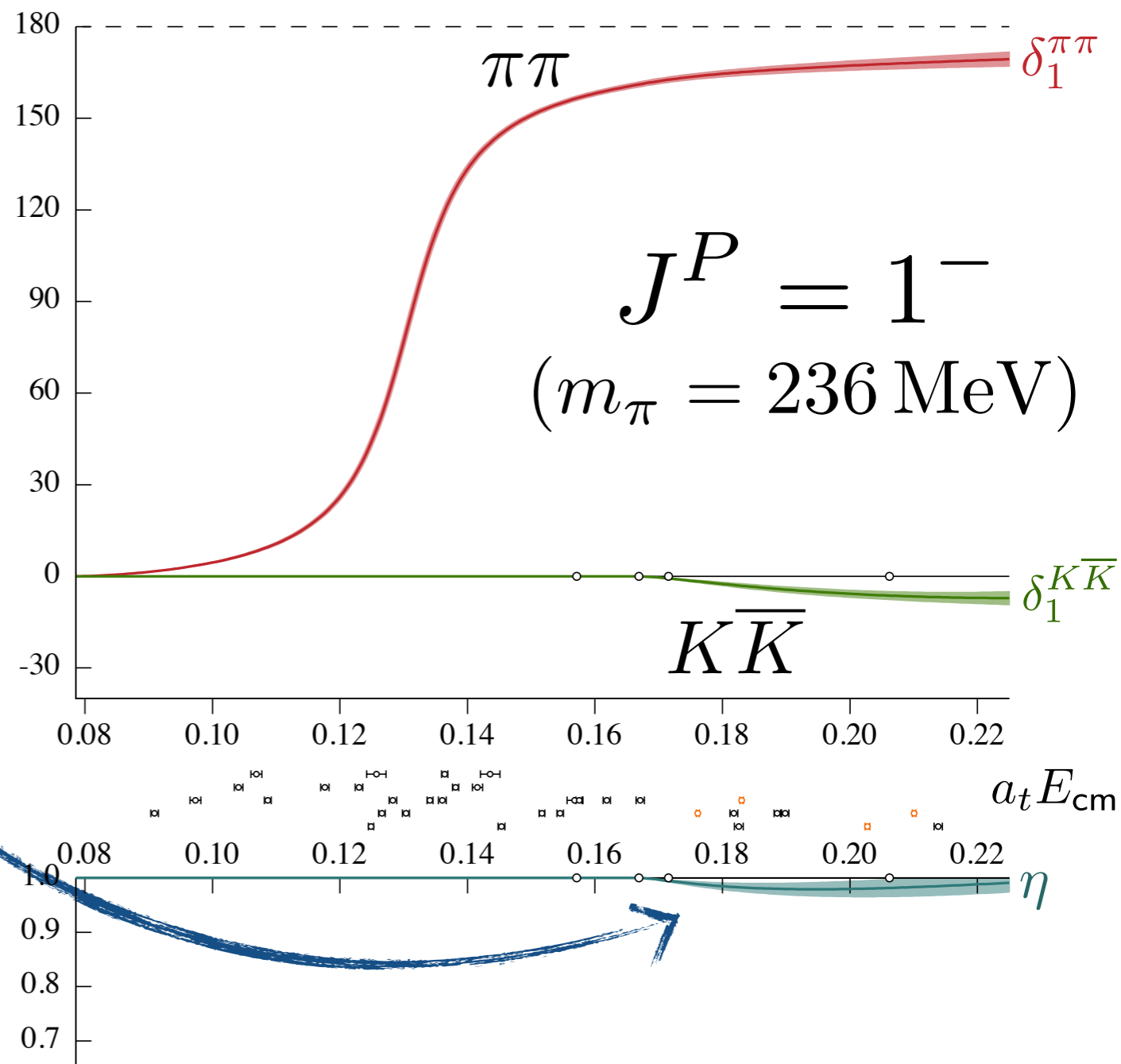
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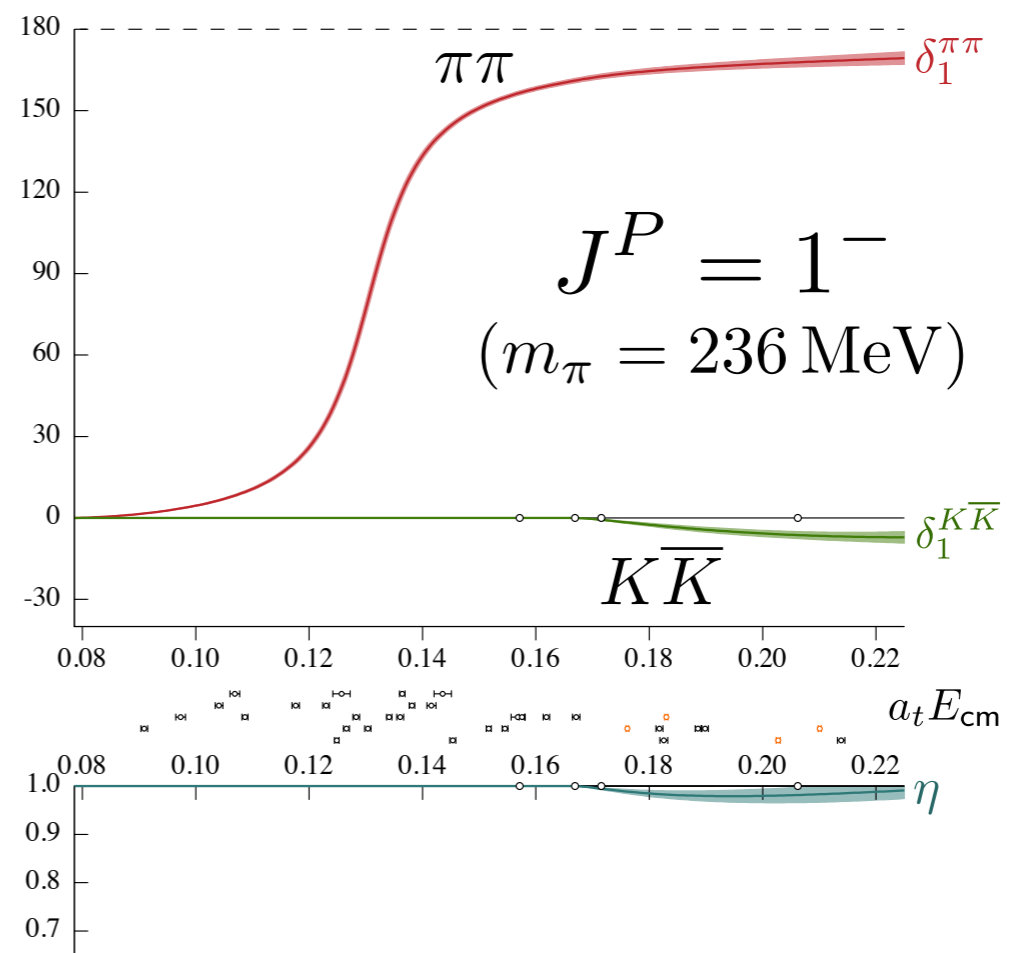
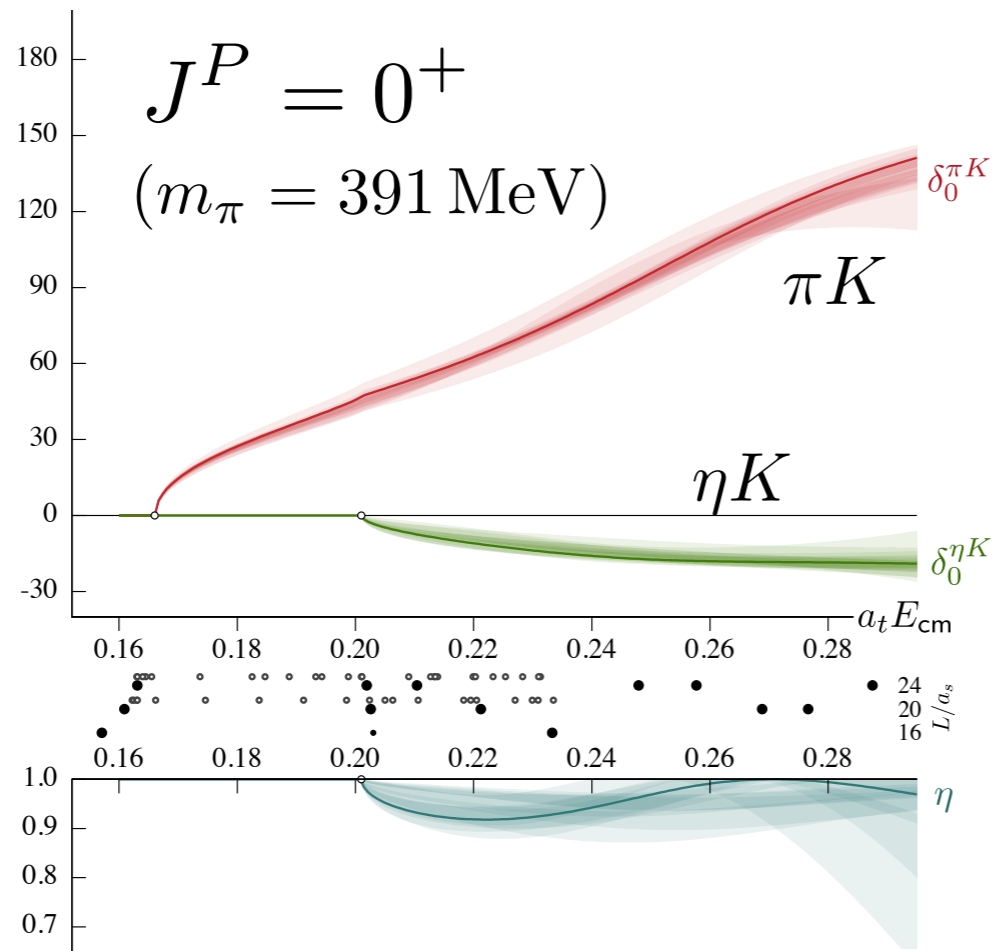
$$\det \left[\begin{pmatrix} \mathcal{M}_{a \rightarrow a} & \mathcal{M}_{a \rightarrow b} \\ \mathcal{M}_{b \rightarrow a} & \mathcal{M}_{b \rightarrow b} \end{pmatrix}^{-1} + \begin{pmatrix} F_a & 0 \\ 0 & F_b \end{pmatrix} \right] = 0$$

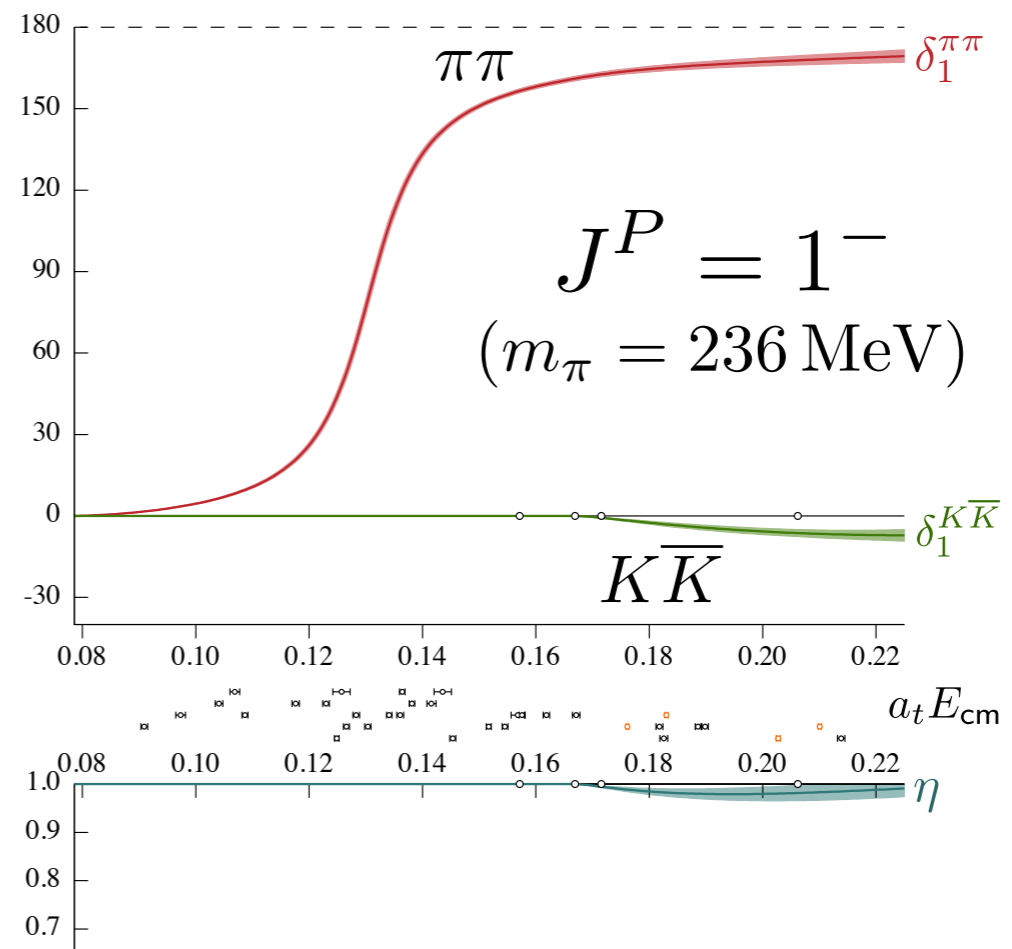
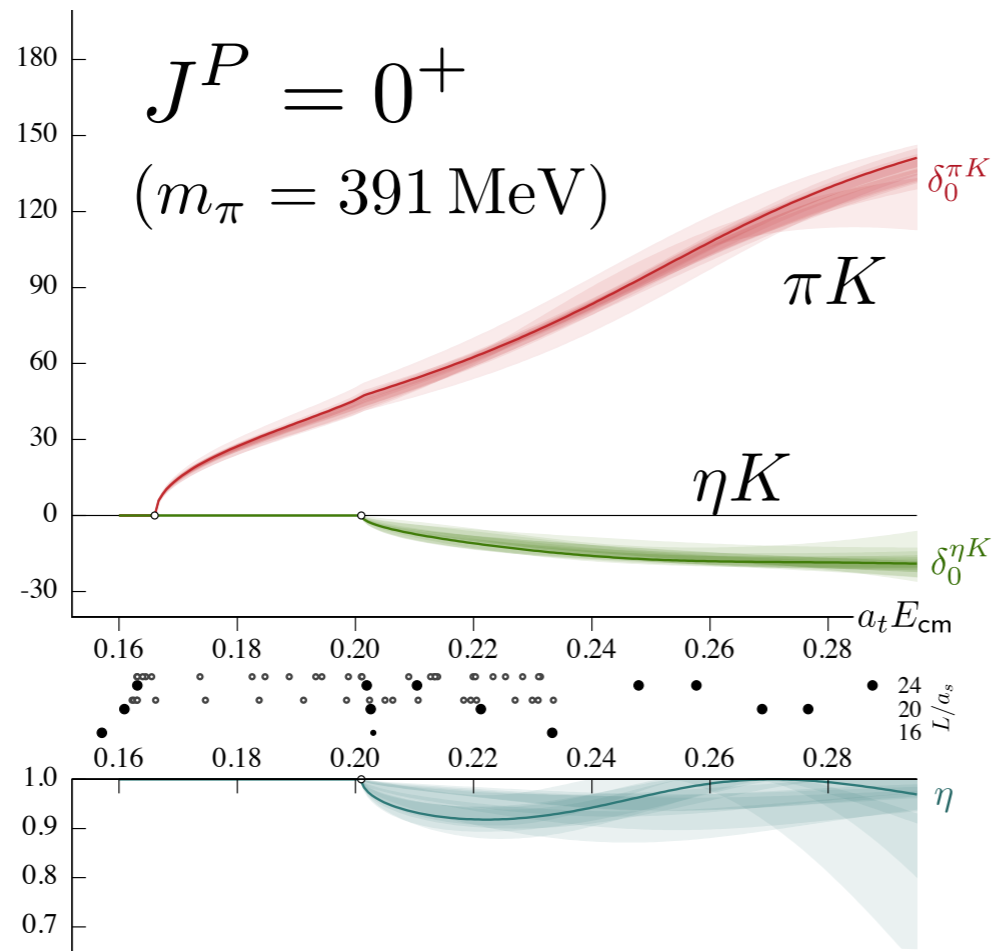
As well as JLab rho study with
 $\pi\pi$, $K\bar{K}$

$$\mathcal{M}(\pi\pi \rightarrow K\bar{K}) \sim \sqrt{1 - \eta^2}$$

Wilson, Briceño, Dudek,
Edwards, Thomas,
arXiv:1507:02599

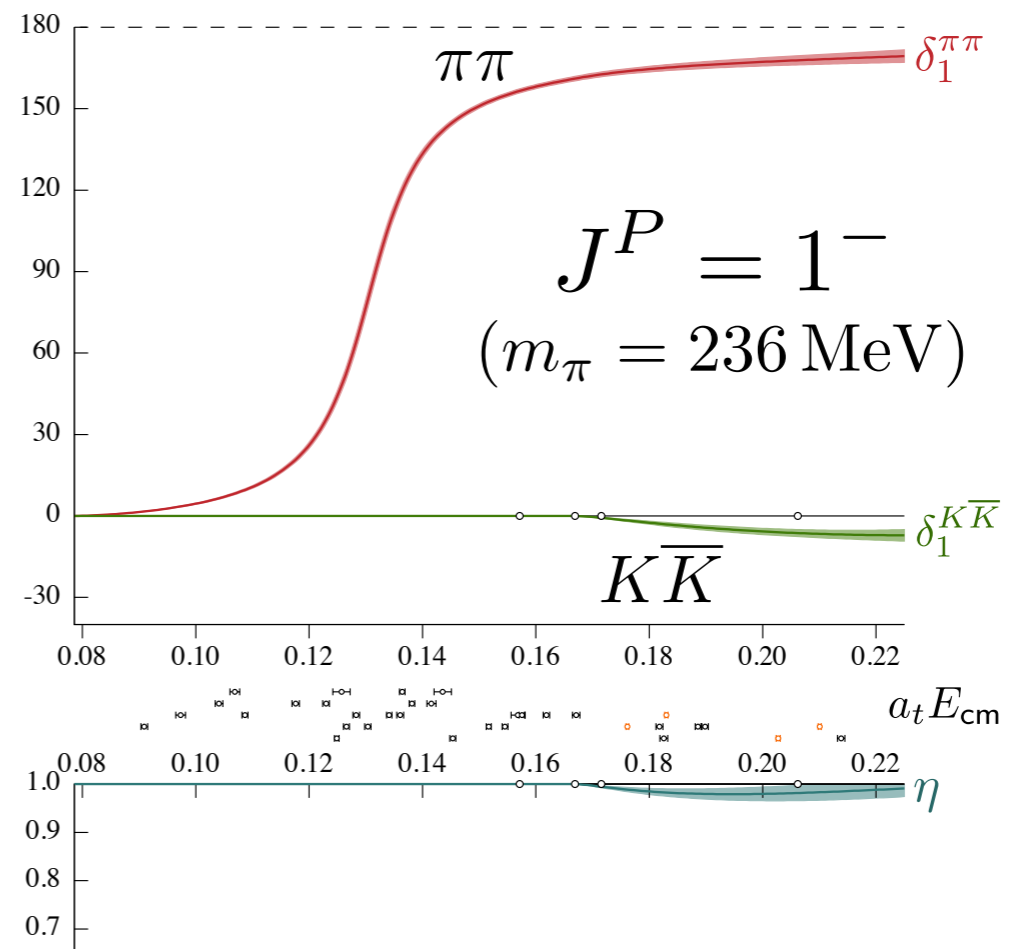
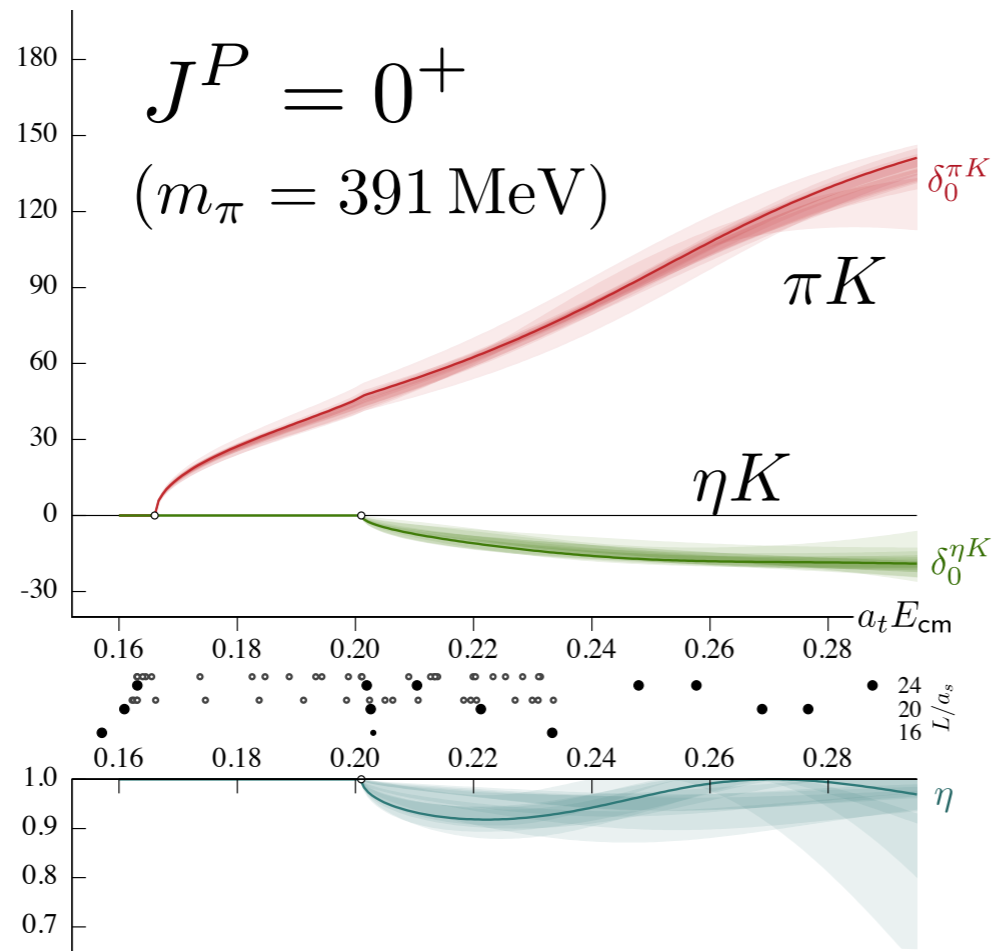






Three volumes are used to calculate many points on phase shift curve

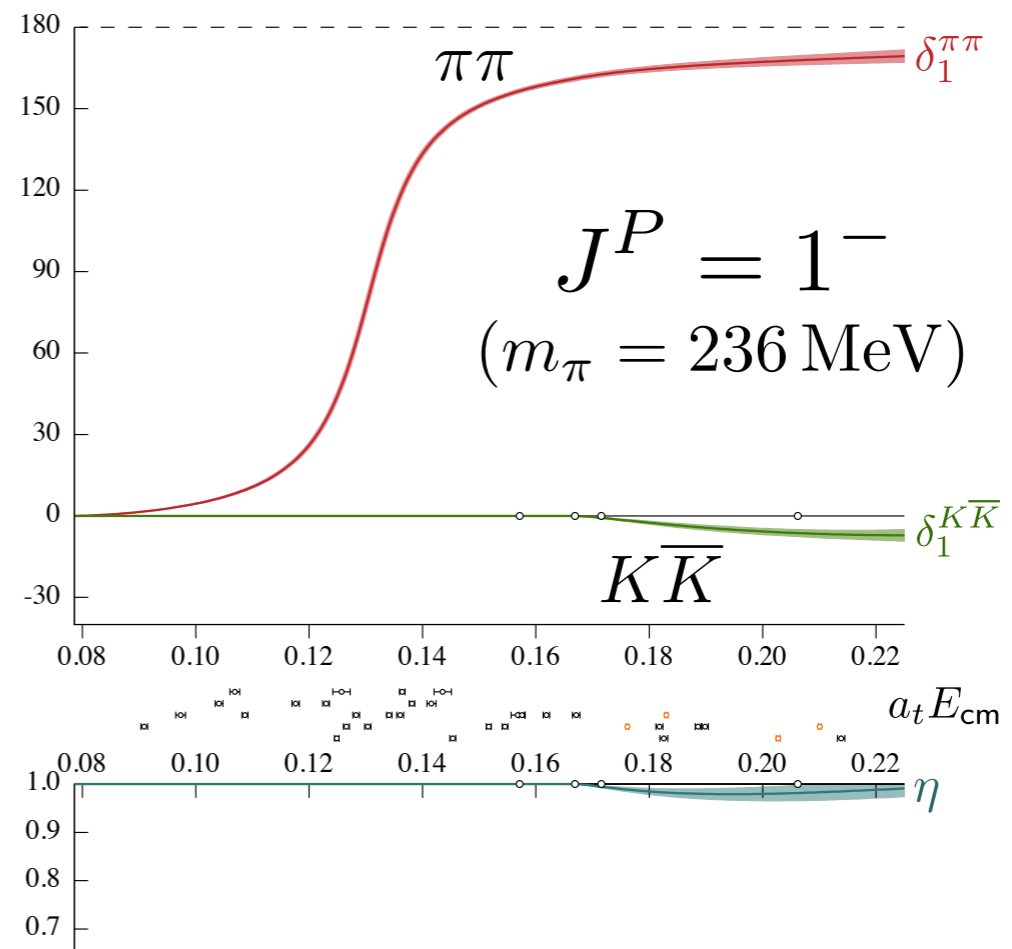
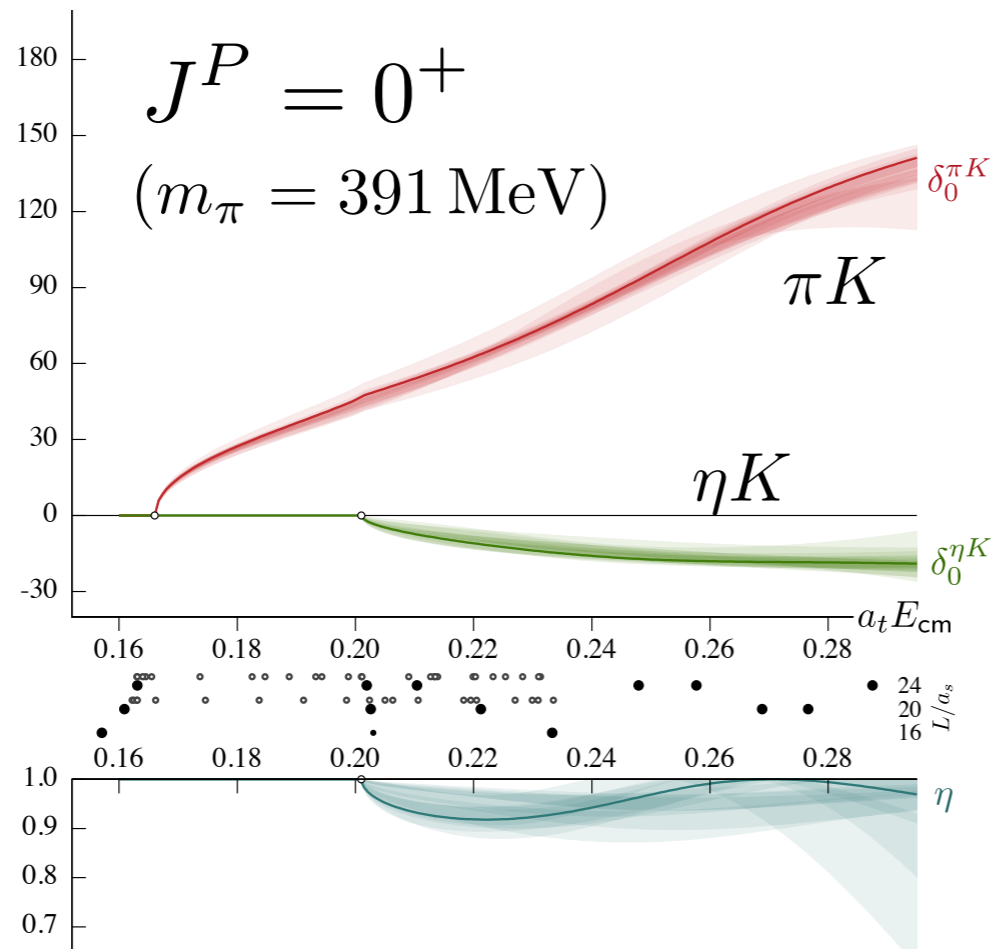
but could still have $e^{-M_\pi L}$ effects



Three volumes are used to calculate many points on phase shift curve

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Three and four-particle thresholds



Three volumes are used to calculate many points on phase shift curve

but could still have $e^{-M_\pi L}$ effects

Three and four-particle thresholds

One lattice spacing

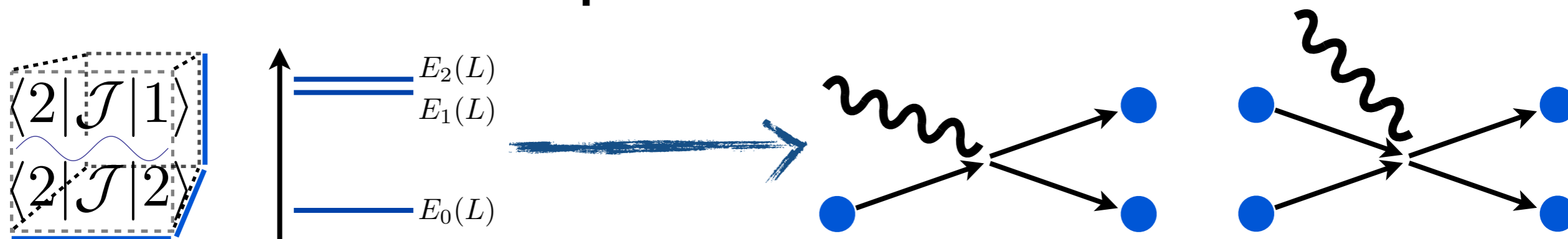
Chiral extrapolation performed in the 1- channel

but not in 0^+

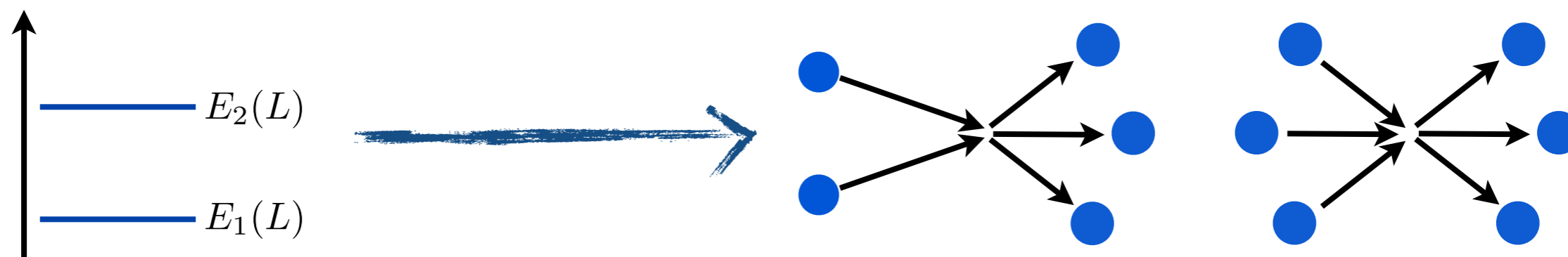
Two-particle scattering



Photo- and electroproduction

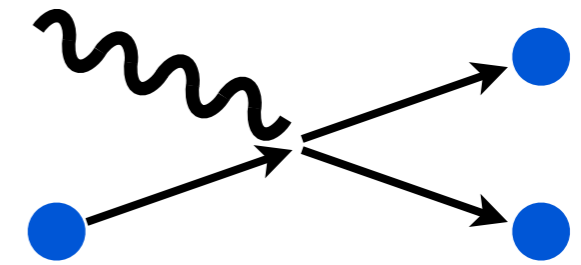


Three-particle scattering



Photoproduction

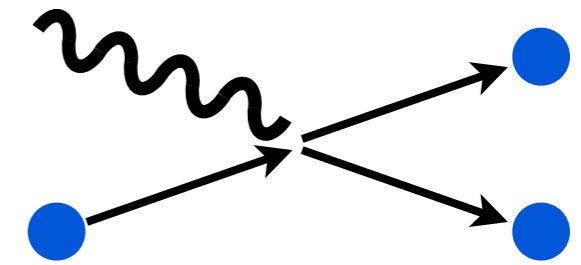
$$\langle \pi\pi, \text{out} | \mathcal{J}_\mu | \pi \rangle \equiv$$



How can we get this from finite-volume observables?

Photoproduction

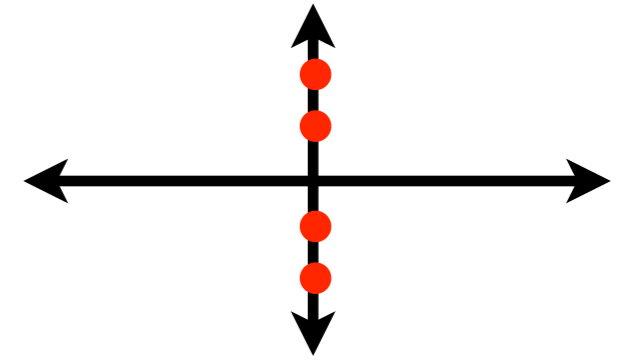
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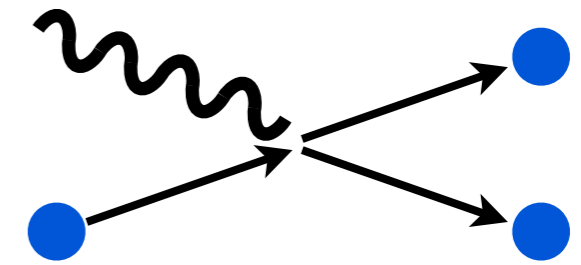
Why did we expect $C_L(P)$ to have poles?

$$C_L(P) \equiv \int_L d^4x e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$



Photoproduction

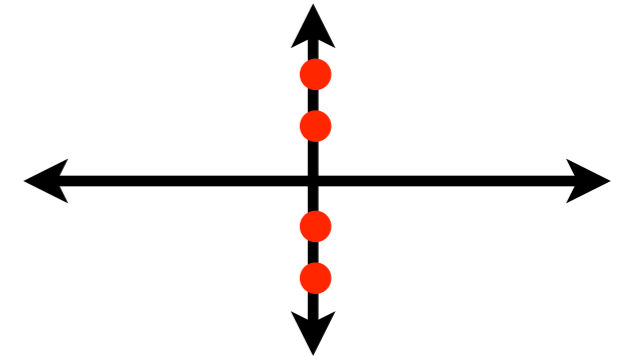
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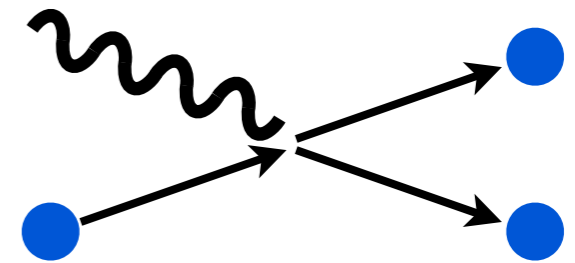
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Insert a complete set finite-volume of states

Photoproduction

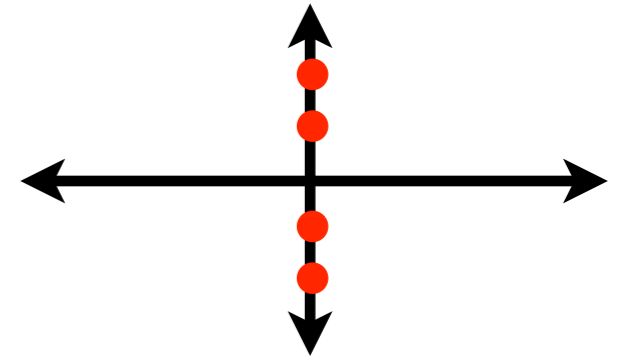
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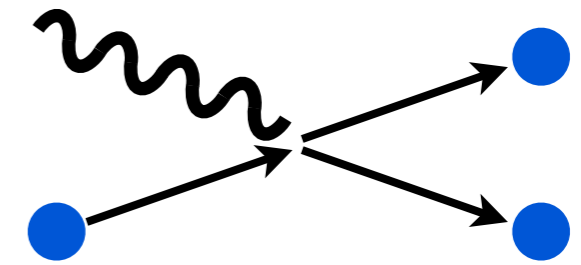


Insert a complete set finite-volume of states

$$C_L(P) \xrightarrow{P_4 \rightarrow iE_n} \frac{L^3 \langle 0 | \mathcal{O}(0) | n, \vec{P}, L \rangle \langle n, \vec{P}, L | \mathcal{O}^\dagger(0) | 0 \rangle}{(E_n + iP_4)}$$

Photoproduction

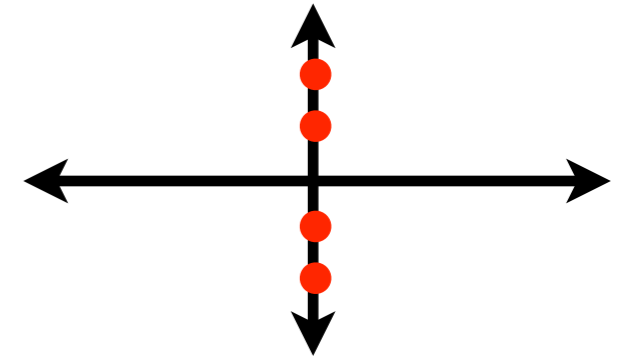
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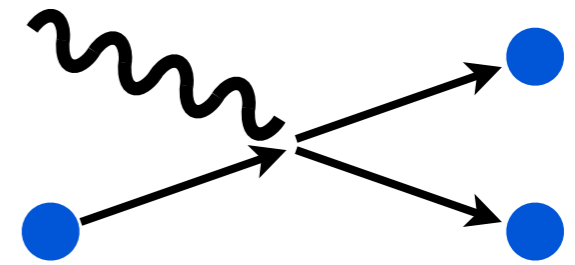
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Now compare this to our factorized result

$$C_L(P) = C_\infty(P) - A' F \frac{1}{1 + \mathcal{M}_{2 \rightarrow 2} F} A$$

Photoproduction

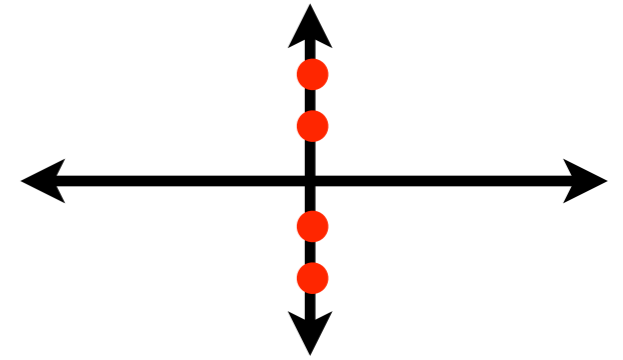
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Insert a complete set finite-volume of states

$$C_L(P) \xrightarrow{P_4 \rightarrow iE_n} \frac{L^3 \langle 0 | \mathcal{O}(0) | n, \vec{P}, L \rangle \langle n, \vec{P}, L | \mathcal{O}^\dagger(0) | 0 \rangle}{(E_n + iP_4)}$$

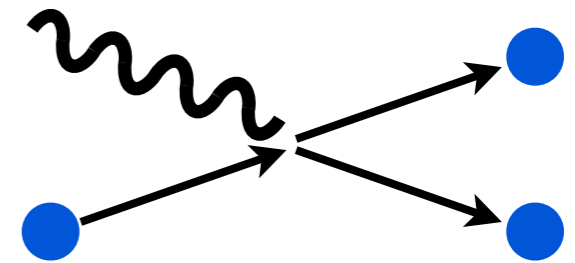
Now compare this to our factorized result

$$C_L(P) = C_\infty(P) - A' F \frac{1}{1 + \mathcal{M}_{2 \rightarrow 2} F} A$$

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Photoproduction

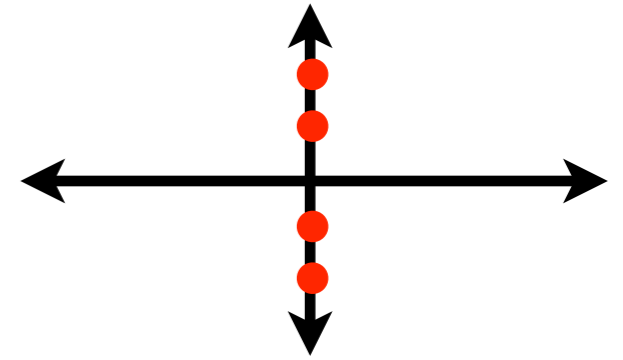
$$\langle \pi\pi, \text{out} | \mathcal{J}_\mu | \pi \rangle \equiv$$



How can we get this from finite-volume observables?

Why did we expect $C_L(P)$ to have poles?

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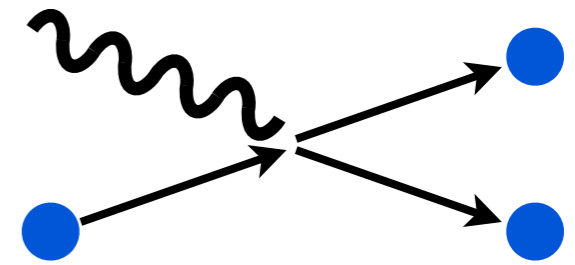
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\mathcal{R} is the residue of this matrix

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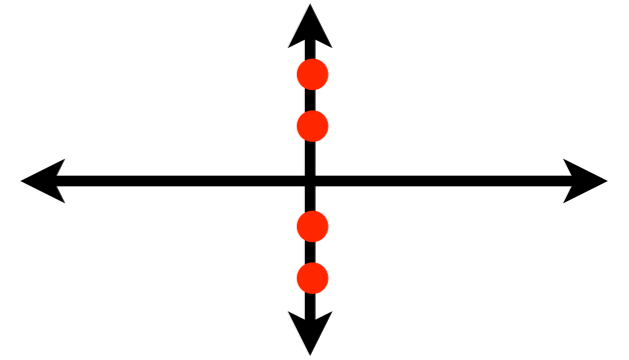
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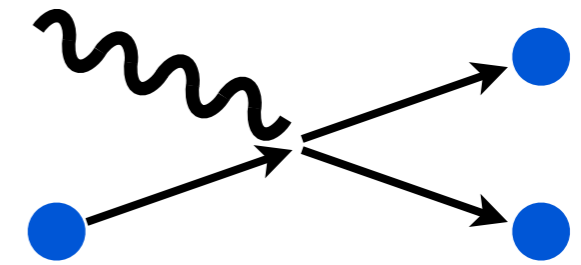
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Photoproduction

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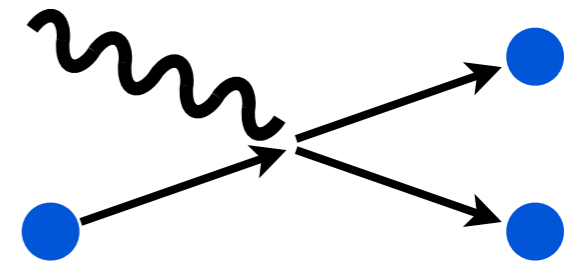
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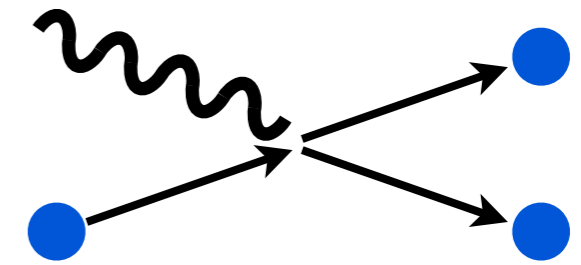
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Photoproduction

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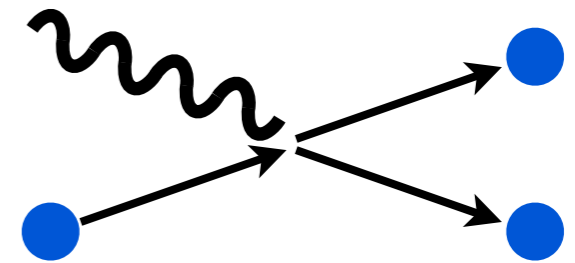
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R. A. Briceño, MTH, A. Walker-Loud, *Phys. Rev. D* **91**, 034501 (2015)

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**experimental
observable**

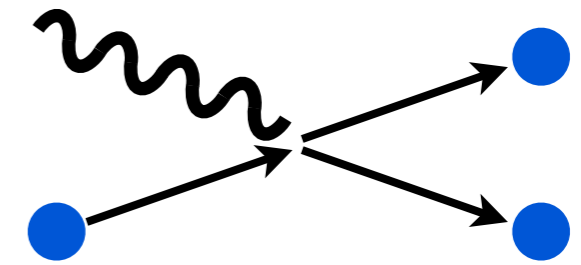
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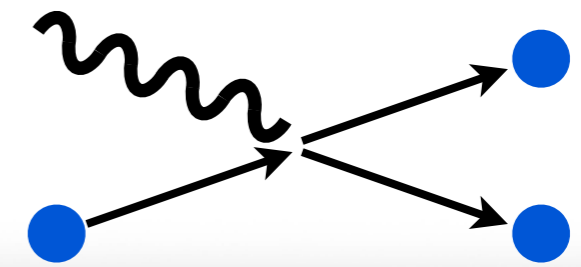
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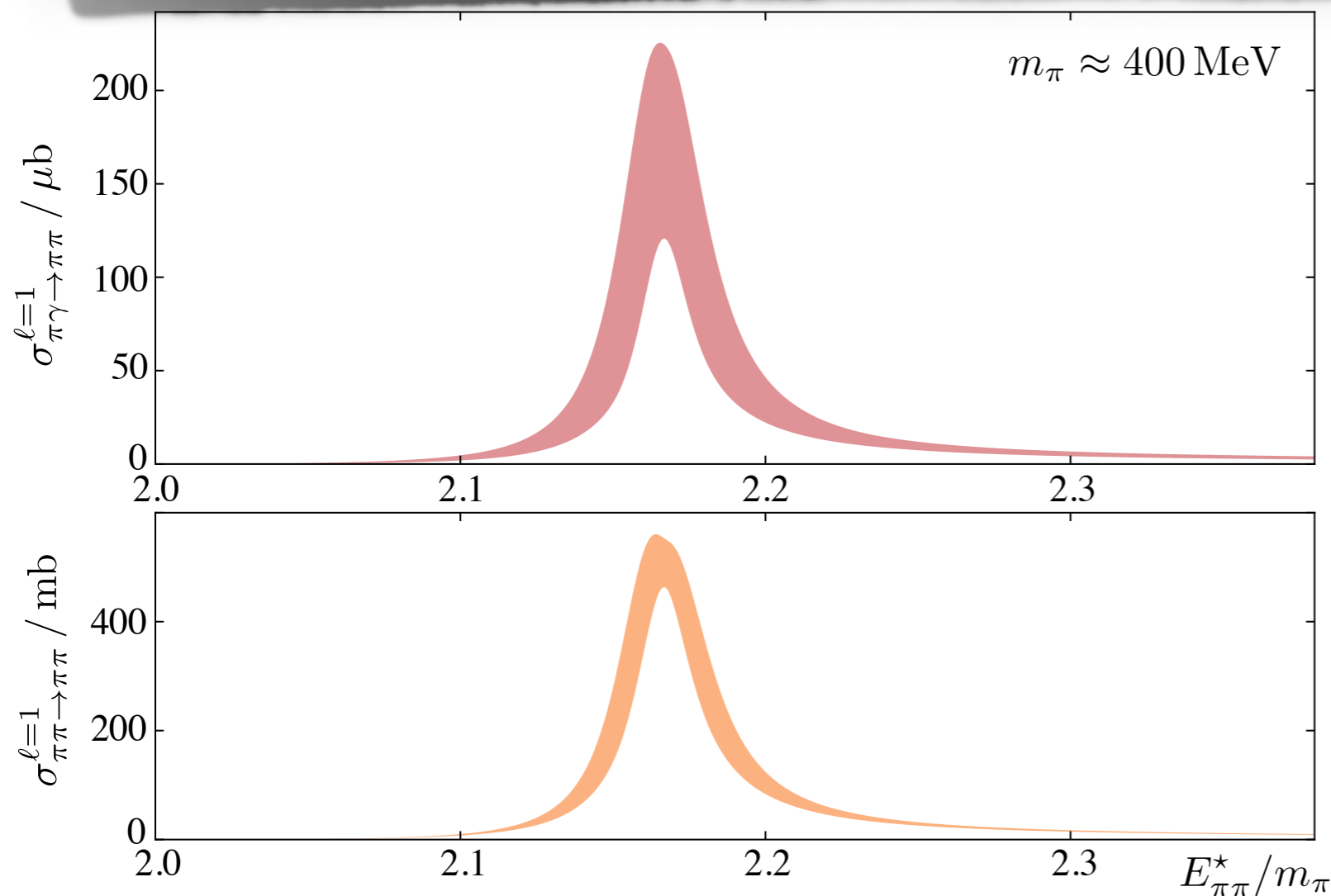
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Briceño, MTH, Walker-Loud/Briceño, MTH

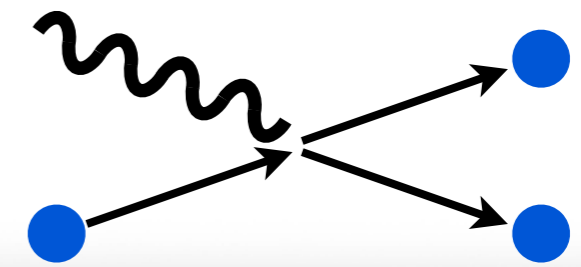


**Photoproduction
in the rho channel**

Briceño, Dudek, Edwards,
Schultz, Thomas, Wilson
arXiv: 1507.6622

Photoproduction

$$\langle \pi\pi, \text{out} | \mathcal{J}_\mu | \pi \rangle \equiv$$



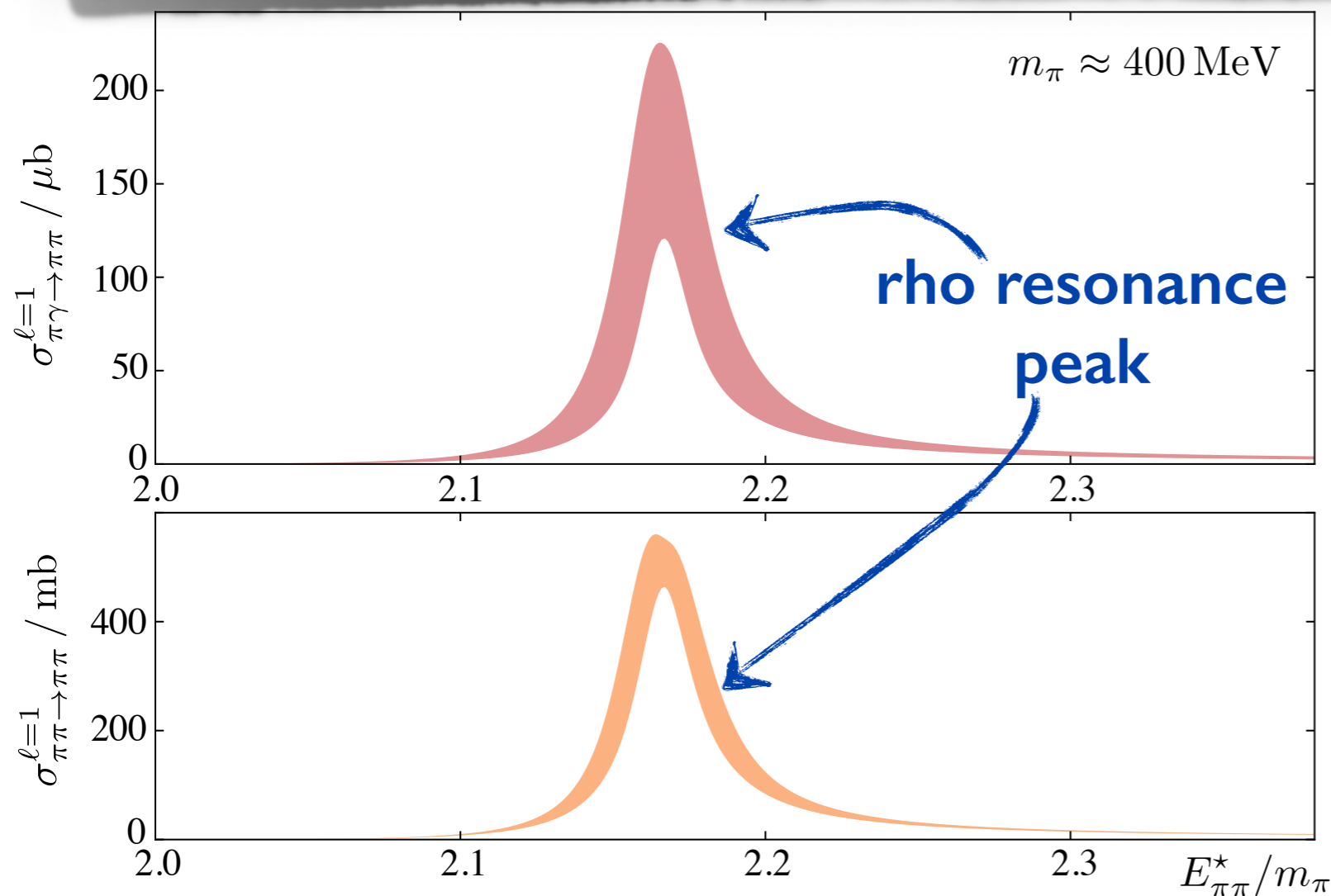
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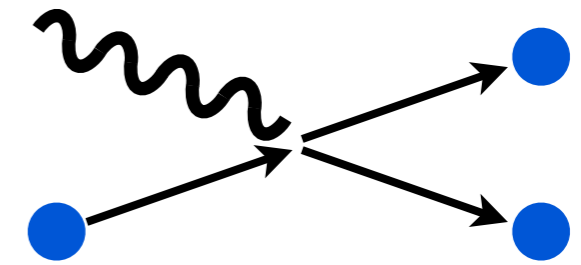


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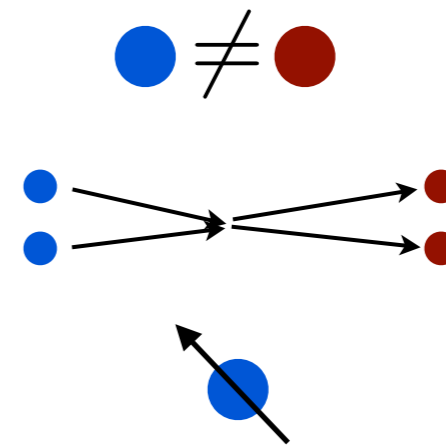


Result is very general

non-identical particles

multiple two-particle channels

particles with spin



$$\bullet \neq \bullet$$



H. B. Meyer, Eur.Phys.J. A49, 84 (2013)

Agadjanov, Bernard, Meißner and Rusetsky, (2014), Nucl.Phys. B886, 1199 (2014).

R. A. Briceño, MTH, A. Walker-Loud, *Phys. Rev. D*91, 034501 (2015)

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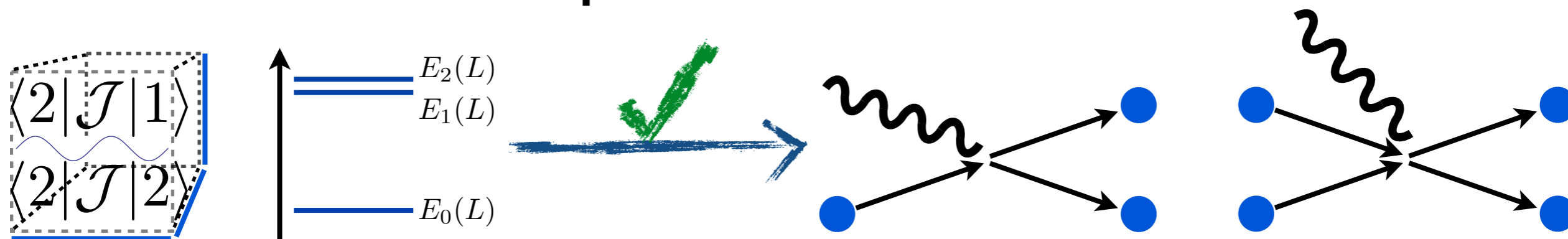
all generalizations of

L. Lellouch and M. Lüscher, *Commun. Math. Phys.* 219, 31 (2001)

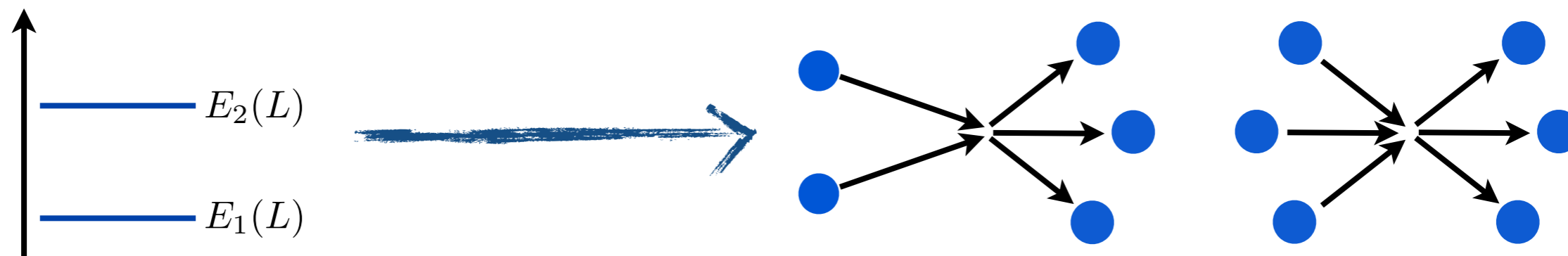
Two-particle scattering



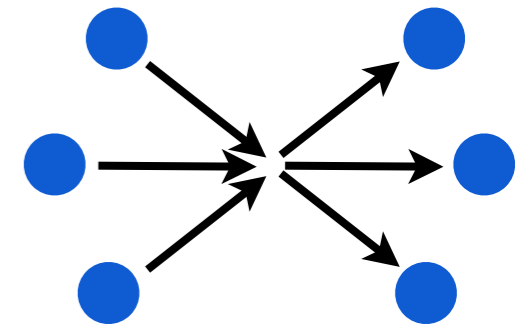
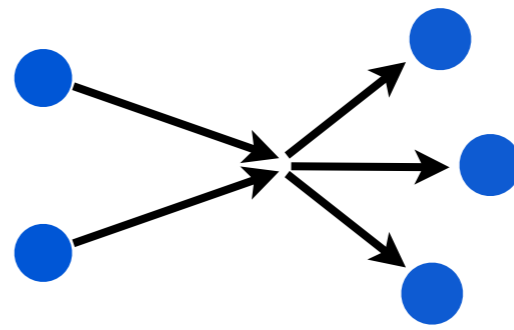
Photo- and electroproduction



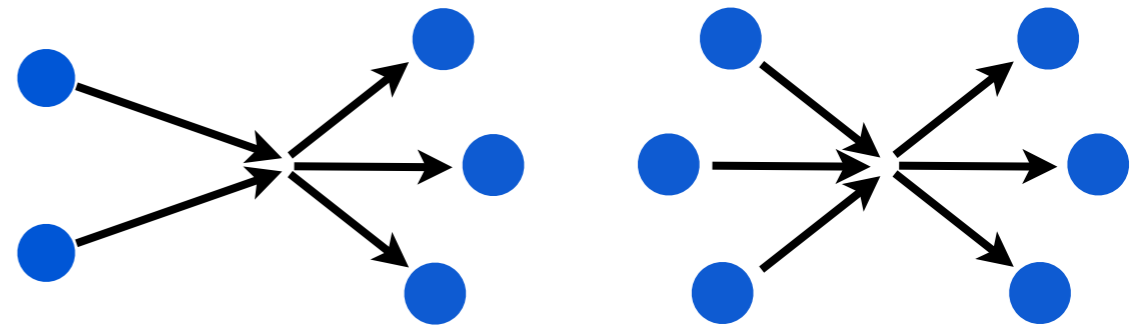
Three-particle scattering



**Begin by considering the
infinite-volume observables**

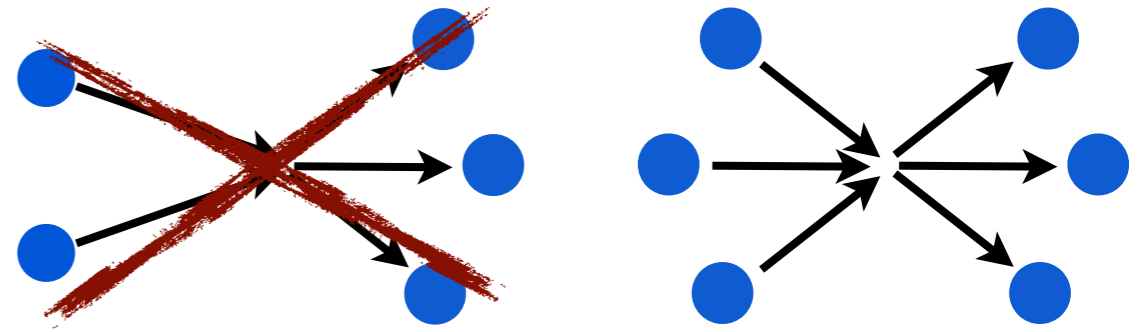


Begin by considering the infinite-volume observables



Because of “finite-volume rescattering” it is not possible to access two-to-three without also accessing three-to-three

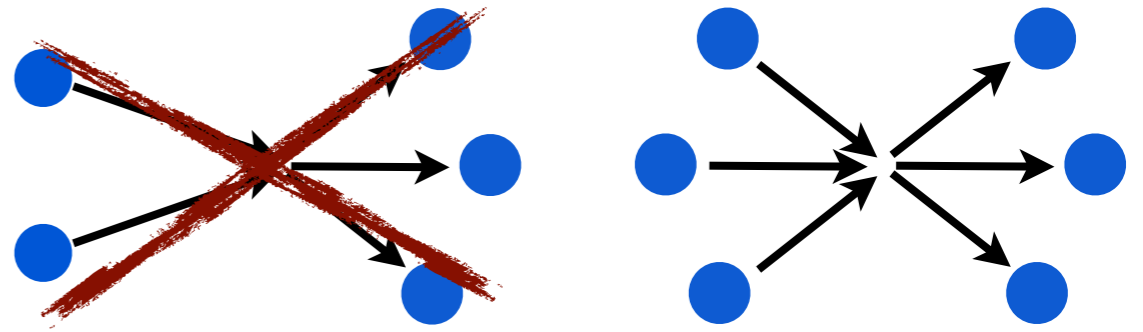
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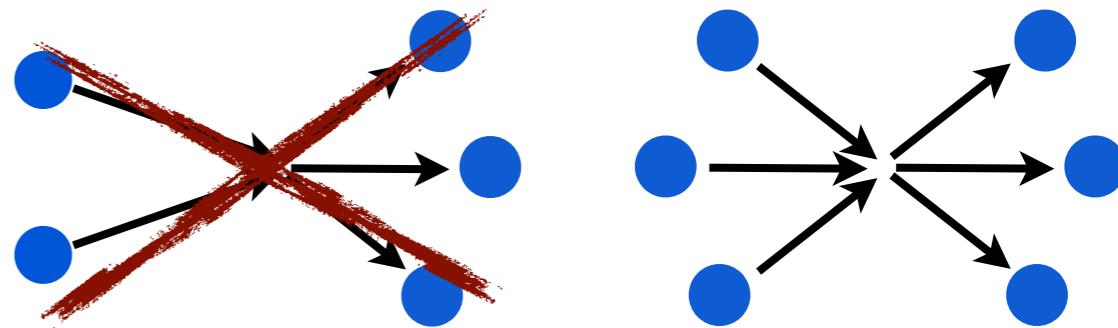
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Three-to-three amplitude has kinematic singularities

$i\mathcal{M}_{3\rightarrow 3} \equiv$ fully connected correlator with
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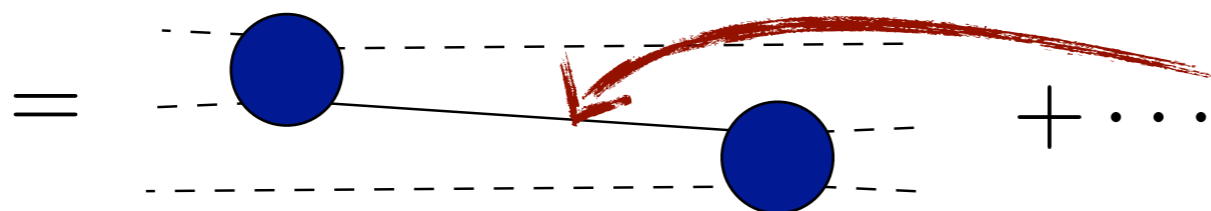


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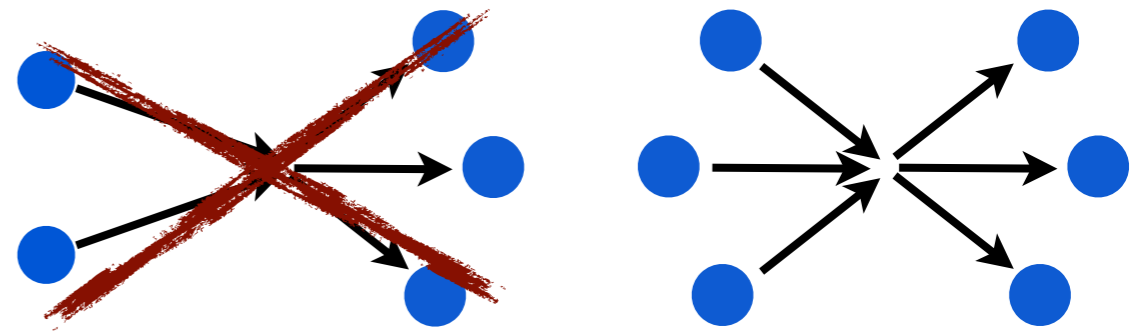
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Certain external momenta put this on-shell!

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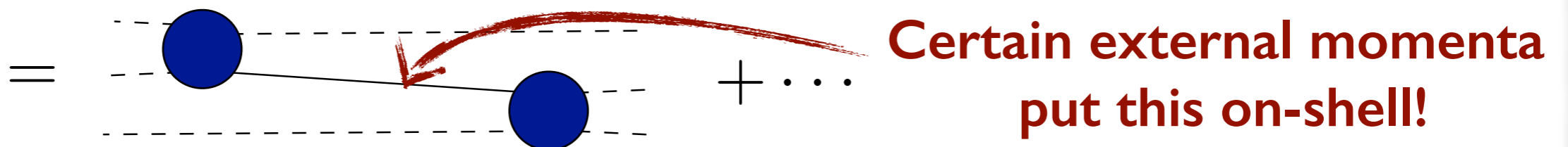


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Three-to-three amplitude has more degrees of freedom

8 degrees of freedom including total energy

Compared with 2 for the two-to-two amplitude

How can we possibly hope to extract a **singular, eight-coordinate function** using finite-volume energies?

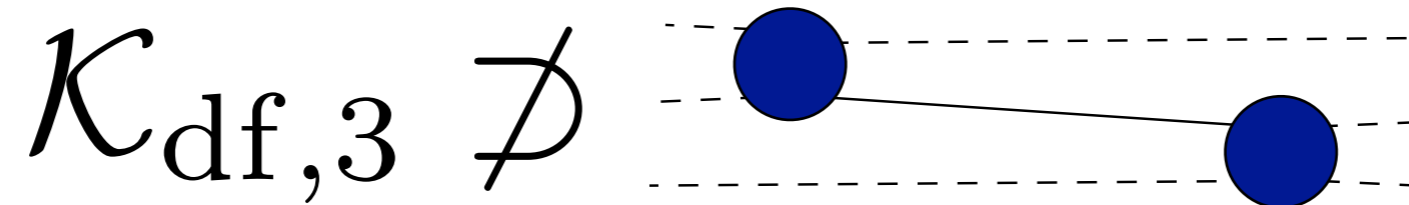
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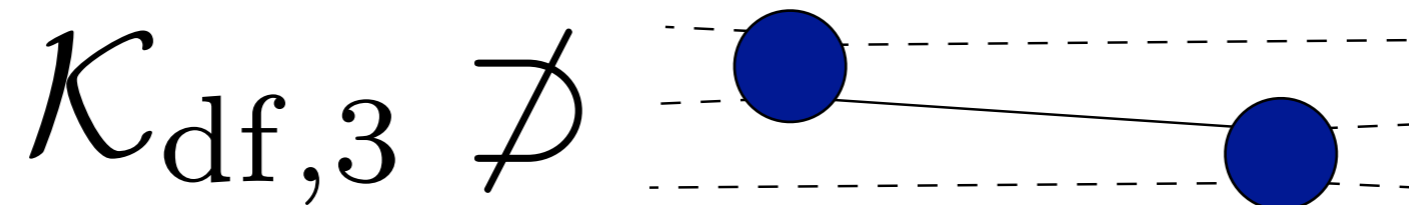
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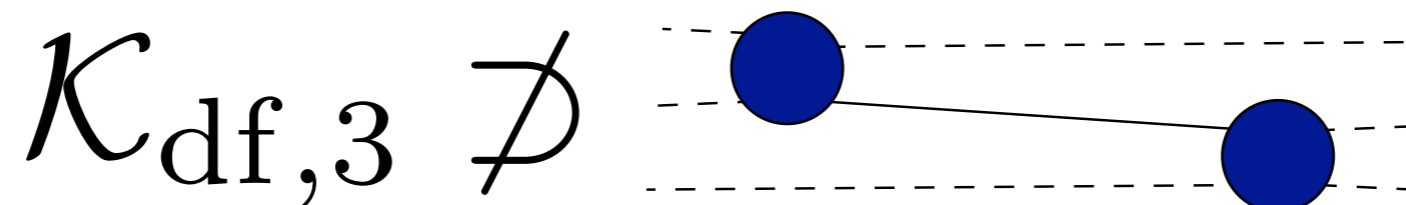
(b) Relation to $\mathcal{M}_{3 \rightarrow 3}$ is known (depends only on on-shell $\mathcal{M}_{2 \rightarrow 2}$)

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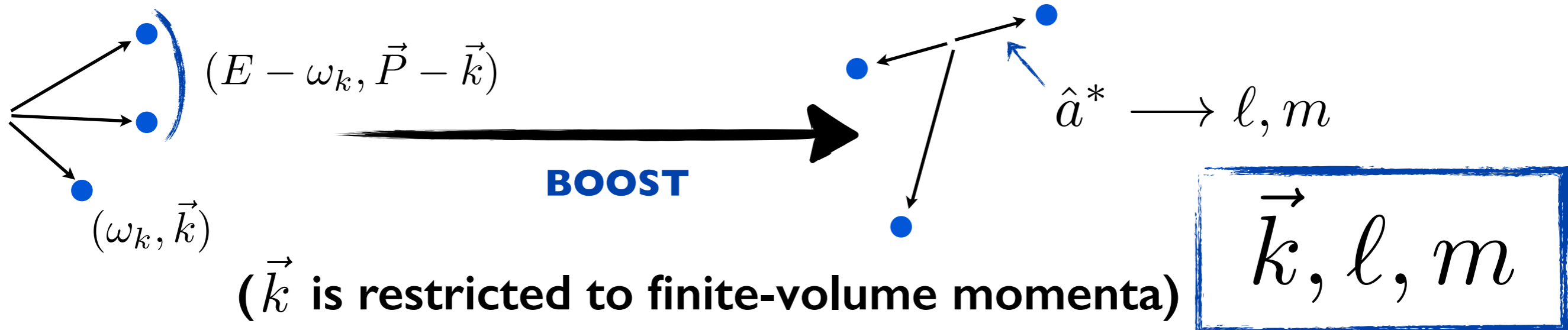


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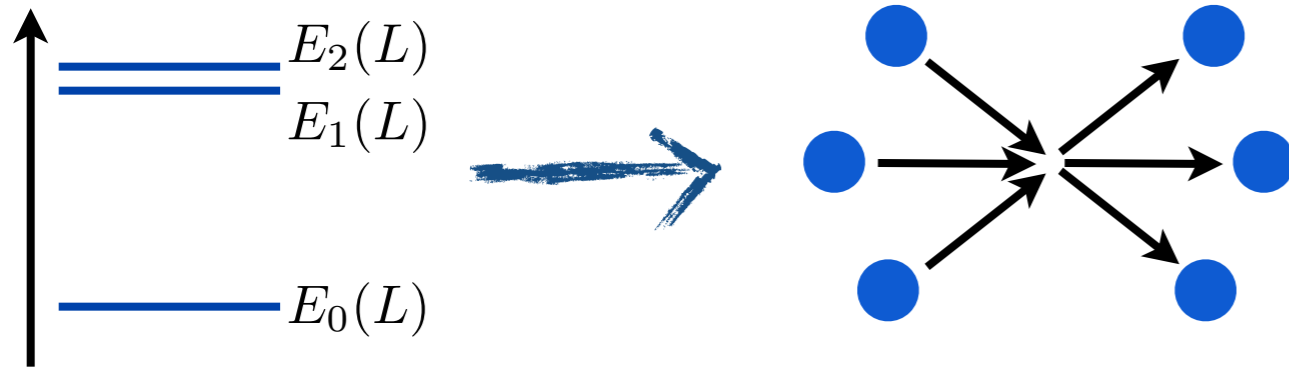
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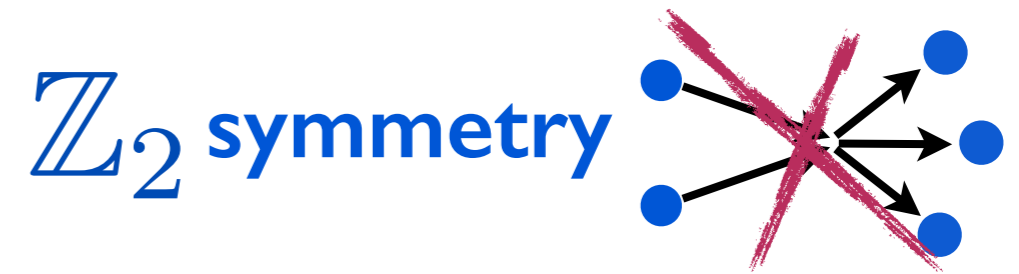
(2). Degrees of freedom encoded in an extended matrix space



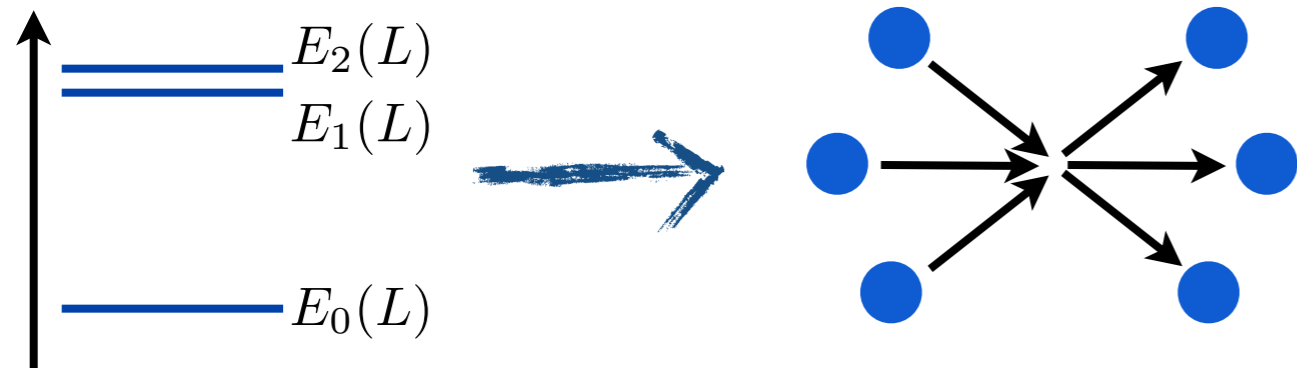
Three-to-three scattering



For now assume...
identical scalars, mass m

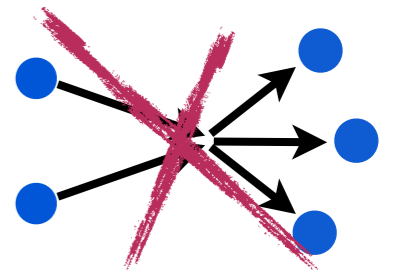


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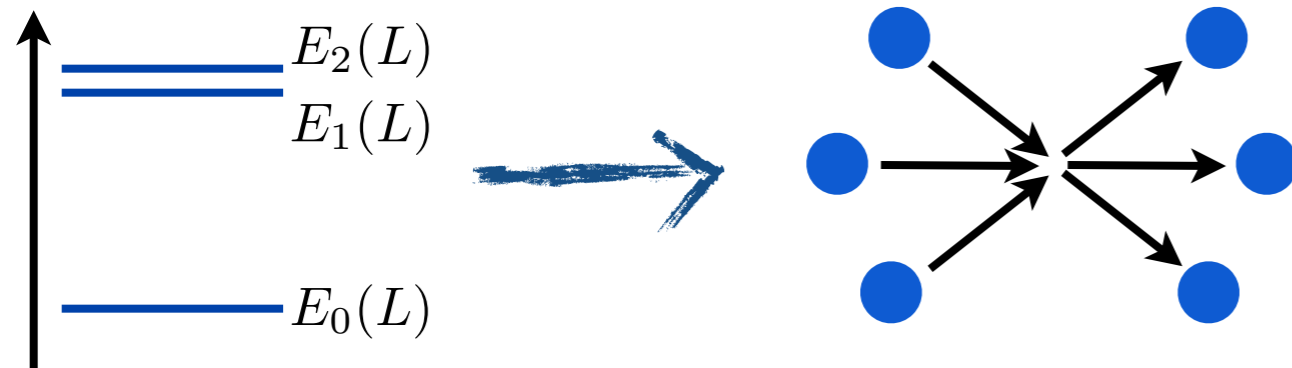
\mathbb{Z}_2 symmetry



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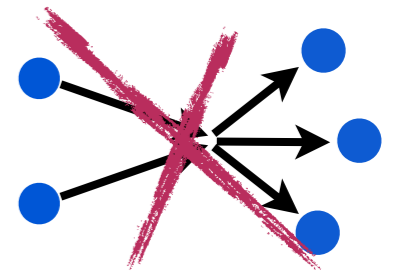
three-pion
interpolator

Three-to-three scattering



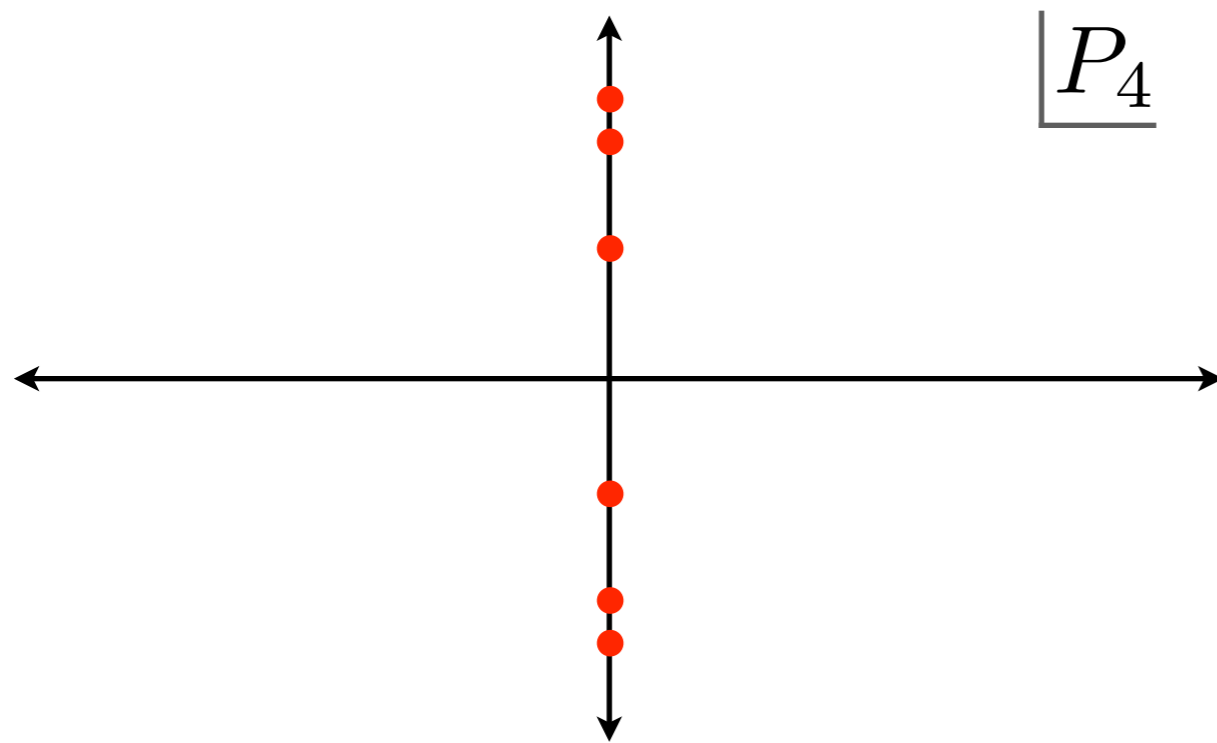
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three-pion
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Calculate $C_L(P)$ to all orders in perturbation theory and determine locations of poles.

Require $m < E^* < 5m$ to isolate three-particle states

Three-particle result

At fixed (L, \vec{P}) , finite-volume energies are solutions to $\det_{k,\ell,m} \left[\mathcal{K}_{\text{df},3}^{-1} + F_3 \right] = 0$

MTH and Sharpe, *Phys. Rev. D* 90, 116003 (2014)

$F_3 \equiv$ matrix that depends on known geometric functions as well as $\mathcal{M}_{2 \rightarrow 2}$.

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 - (2). Use harmonic decomposition + various parametrizations to express $\mathcal{K}_{\text{df},3}(E^*)$ in terms of N unknown parameters

Three-particle result

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- (2). Use harmonic decomposition + various parametrizations to express $\mathcal{K}_{\text{df},3}(E^*)$ in terms of N unknown parameters
- (3). Use quantization condition with lattice (or otherwise) determined energies to determine all parameters

Three-particle result

At fixed (L, \vec{P}) , finite-volume energies are solutions to $\det_{k,\ell,m} \left[\mathcal{K}_{\text{df},3}^{-1} + F_3 \right] = 0$

MTH and Sharpe, *Phys. Rev. D*90, 116003 (2014)

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- (4). Use known relation to recover $\mathcal{M}_{3 \rightarrow 3}$

MTH and Sharpe, *Phys. Rev. D*92, 114509 (2015)

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MTH and Sharpe, *Phys. Rev. D*90, 116003 (2014)

Some nice features...

Matrices automatically truncated in the \vec{k} index

**truncate angular
momentum space**



solvable system

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MTH and Sharpe, *Phys. Rev. D* 90, 116003 (2014)

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Expanding about weak interactions gives an important check

$$E = 3m + \frac{a_3}{L^3} + \frac{a_4}{L^4} + \frac{a_5}{L^5} + \frac{a_6}{L^6} + \mathcal{O}(1/L^7)$$

Our result agrees with existing results for $a_{3 \rightarrow 5}$ and gives a prediction for a_6

K. Huang and C. Yang, *Phys. Rev.* 105 (1957) 767-775

Beane, Detmold, Savage, *Phys. Rev. D* 76 (2007) 074507

MTH and Sharpe, *Phys. Rev. D* 93, 096006 (2016)

Three-particle result $\det_{k,\ell,m} \left[\mathcal{K}_{\text{df},3}^{-1} + F_3 \right] = 0$

Sketch of the derivation...

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Sketch of the derivation...

Recall for two particles we started with a “skeleton expansion”

$$C_L(P) = \begin{array}{c} \bullet \\ \vdots \\ \circ^\dagger \quad \circ \\ \bullet \\ \vdots \end{array} + \begin{array}{c} \bullet \\ \vdots \\ \circ^\dagger \quad \circ \\ \bullet \\ \vdots \end{array} \begin{array}{c} \bullet \\ \vdots \\ iK \\ \bullet \\ \vdots \end{array} \begin{array}{c} \bullet \\ \vdots \\ \circ \\ \bullet \\ \vdots \end{array} + \begin{array}{c} \bullet \\ \vdots \\ \circ^\dagger \quad \circ \\ \bullet \\ \vdots \end{array} \begin{array}{c} \bullet \\ \vdots \\ iK \\ \bullet \\ \vdots \end{array} \begin{array}{c} \bullet \\ \vdots \\ iK \\ \bullet \\ \vdots \end{array} \begin{array}{c} \bullet \\ \vdots \\ \circ \\ \bullet \\ \vdots \end{array} + \dots$$

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So now we need the same for three...

$$C_L(E, \vec{P}) \stackrel{?}{=} \begin{array}{c} \circ \quad \circ \\ \circ \quad \circ \end{array} + \begin{array}{c} \circ \quad \bullet \quad \circ \\ \circ \quad \bullet \quad \circ \end{array} + \begin{array}{c} \circ \quad \bullet \quad \bullet \quad \circ \\ \circ \quad \bullet \quad \bullet \quad \circ \end{array} + \dots$$

Three-particle result $\det_{k,\ell,m} \left[\mathcal{K}_{\text{df},3}^{-1} + F_3 \right] = 0$

Sketch of the derivation...

Recall for two particles we started with a “skeleton expansion”

$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

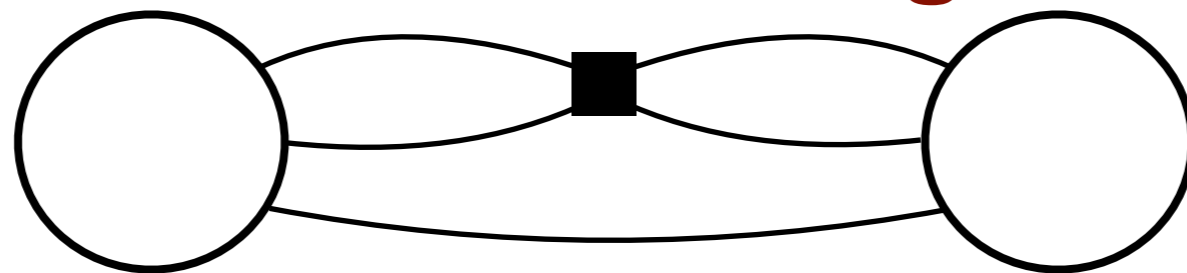
The diagrammatic expansion for $C_L(P)$ consists of a series of terms. Each term is a chain of circles connected by arcs. The first term has two circles, the first labeled \mathcal{O}^\dagger and the second \mathcal{O} . The second term has three circles, with the middle one labeled iK . The third term has four circles, with the two middle ones labeled iK . Each circle has two external legs, and the connections between them are shown as arcs. Dotted lines indicate the continuation of the expansion.

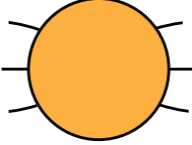
So now we need the same for three...

$$C_L(E, \vec{P}) \stackrel{?}{=} \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

The diagrammatic expansion for $C_L(E, \vec{P})$ is shown with a question mark above the equals sign. It consists of a series of terms. Each term is a chain of circles connected by arcs. The first term has two white circles. The second term has three circles, with the middle one shaded orange. The third term has four circles, with the two middle ones shaded orange. Each circle has two external legs, and the connections between them are shown as arcs. Dotted lines indicate the continuation of the expansion.

No! We also need diagrams like



Disconnected diagrams in  lead to singularities that invalidate the derivation

New skeleton expansion

$$C_L(E, \vec{P}) = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

$$+ \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots$$

The diagrams in the expansion are:

- Diagram 1: Two white circles connected by two arcs, enclosed in a dashed box.
- Diagram 2: A white circle connected to an orange circle, which is connected to another white circle, all enclosed in a dashed box.
- Diagram 3: A white circle connected to an orange circle, which is connected to another orange circle, which is connected to a white circle, all enclosed in a dashed box.
- Diagram 4: A white circle connected to a purple circle, which is connected to another white circle, all enclosed in a dashed box.
- Diagram 5: A white circle connected to a purple circle, which is connected to another purple circle, which is connected to a white circle, all enclosed in a dashed box.
- Diagram 6: A white circle connected to a purple circle, which is connected to another purple circle, which is connected to a third purple circle, which is connected to a white circle, all enclosed in a dashed box.

Kernel definitions:

$$\text{Purple circle} \equiv \text{Diagram A} + \text{Diagram B} + \text{Diagram C} + \dots$$

The diagrams in the kernel definition for the purple circle are:

- Diagram A: A vertex with four external lines.
- Diagram B: A vertex with four external lines and two internal arcs.
- Diagram C: A vertex with four external lines and two internal arcs forming a figure-eight shape.

$$\text{Orange circle} \equiv \text{Diagram D} + \text{Diagram E} + \text{Diagram F} + \dots$$

The diagrams in the kernel definition for the orange circle are:

- Diagram D: A vertex with four external lines.
- Diagram E: A vertex with four external lines and a single internal line.
- Diagram F: A vertex with four external lines and two internal arcs.

New skeleton expansion

$$\begin{aligned}
 C_L(E, \vec{P}) = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\
 & + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots \\
 & + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \dots \\
 & + \text{Diagram 10} + \text{Diagram 11} + \dots
 \end{aligned}$$

The diagrams in the expansion are arranged in four rows. The first row contains three diagrams with two white circles connected by two lines, with one, two, or three orange circles inserted between them. The second row contains three diagrams with two white circles and one, two, or three purple circles inserted between them. The third row contains three diagrams with two white circles and two, three, or four purple circles inserted between them. The fourth row contains two diagrams with two white circles and three purple circles inserted between them. Dashed boxes in each diagram indicate the skeleton structure.

Kernel definitions:

$$\text{Purple circle} \equiv \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

The purple circle kernel is defined as the sum of three diagrams: a single vertex, a vertex with two internal lines forming a loop, and a vertex with two internal lines forming a figure-eight shape.

$$\text{Orange circle} \equiv \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

The orange circle kernel is defined as the sum of three diagrams: a single vertex, a vertex with two internal lines forming a straight line, and a vertex with two internal lines forming a loop.

New skeleton expansion

$$\begin{aligned}
 C_L(E, \vec{P}) = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\
 & + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots \\
 & + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \dots \\
 & + \text{Diagram 10} + \text{Diagram 11} + \dots \\
 & + \dots \\
 & + \text{Diagram 12} + \text{Diagram 13} + \dots
 \end{aligned}$$

The diagrams in the expansion are as follows:

- Row 1: Two white circles connected by two arcs, enclosed in a dashed box. Then a white circle, a dashed box containing an orange circle, another white circle, and another dashed box containing an orange circle, followed by a white circle and an ellipsis.
- Row 2: A white circle, a dashed box containing a purple circle, and another white circle. Then a white circle, a dashed box containing two purple circles, and another white circle. Then a white circle, a dashed box containing three purple circles, and another white circle, followed by an ellipsis.
- Row 3: A white circle, a dashed box containing two purple circles, and another white circle. Then a white circle, a dashed box containing three purple circles, and another white circle. Then a white circle, a dashed box containing four purple circles, and another white circle, followed by an ellipsis.
- Row 4: A white circle, a dashed box containing three purple circles, and another white circle. Then a white circle, a dashed box containing four purple circles, and another white circle, followed by an ellipsis.
- Row 5: An ellipsis.
- Row 6: A white circle, a dashed box containing a purple circle, an orange circle, and another white circle. Then a white circle, a dashed box containing an orange circle, a purple circle, and another white circle, followed by an ellipsis.

Kernel definitions:

$$\text{Purple circle} \equiv \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

The diagrams for the purple circle kernel are:

- Diagram 1: A purple circle with four external lines (two on the left, two on the right).
- Diagram 2: A white circle with four external lines, with two arcs connecting the top and bottom vertices.
- Diagram 3: A white circle with four external lines, with two arcs connecting the top and bottom vertices, and a vertical line segment connecting the two vertices.

$$\text{Orange circle} \equiv \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

The diagrams for the orange circle kernel are:

- Diagram 1: An orange circle with four external lines (two on the left, two on the right).
- Diagram 2: A white circle with four external lines, with a horizontal line segment connecting the two vertices.
- Diagram 3: A white circle with four external lines, with two arcs connecting the top and bottom vertices and a horizontal line segment connecting the two vertices.

Three-to-three scattering



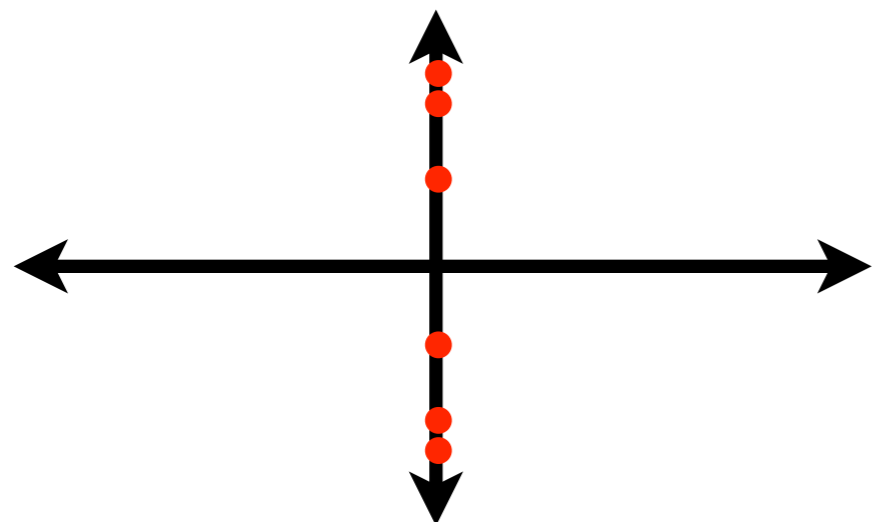
1. Work out the three particle skeleton expansion

$$C_L(E, \vec{P}) = \text{[Diagrammatic expansion of } C_L(E, \vec{P}) \text{ as a sum of skeleton diagrams with orange and purple vertices]} + \dots$$

2. Break diagrams into finite- and infinite-volume parts

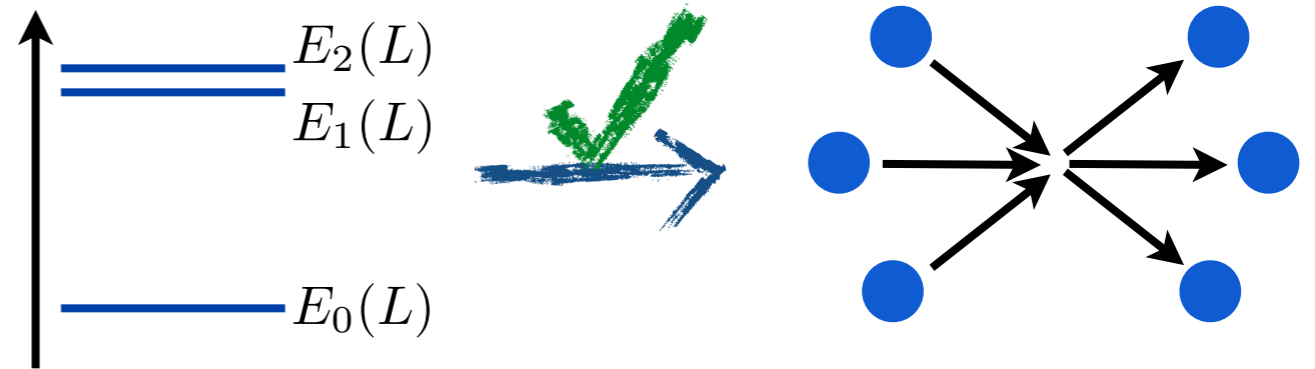
3. Sum subsets of terms to identify *infinite-volume quantities*

4. Relate these to poles in the finite-volume correlator



$$\det_{k,\ell,m} \left[\mathcal{K}_{\text{df},3}^{-1} + F_3 \right] = 0$$

Three-to-three scattering



Current status:

Formalism is complete for the simplest three-scalar system

General, model-independent relation between finite-volume energies and three-to-three scattering amplitude

Derived using a generic relativistic field theory

MTH and Sharpe, *Phys. Rev. D* 90, 116003 (2014)

MTH and Sharpe, *Phys. Rev. D* 92, 114509 (2015)

Important caveats:

Identical particles with no two-to-three transitions

$$\pi\pi\pi \rightarrow \pi\pi\pi$$

Requires that two-particle scattering phase is bounded

$$|\delta_\ell(E)| < \pi/2$$

Currently underway:

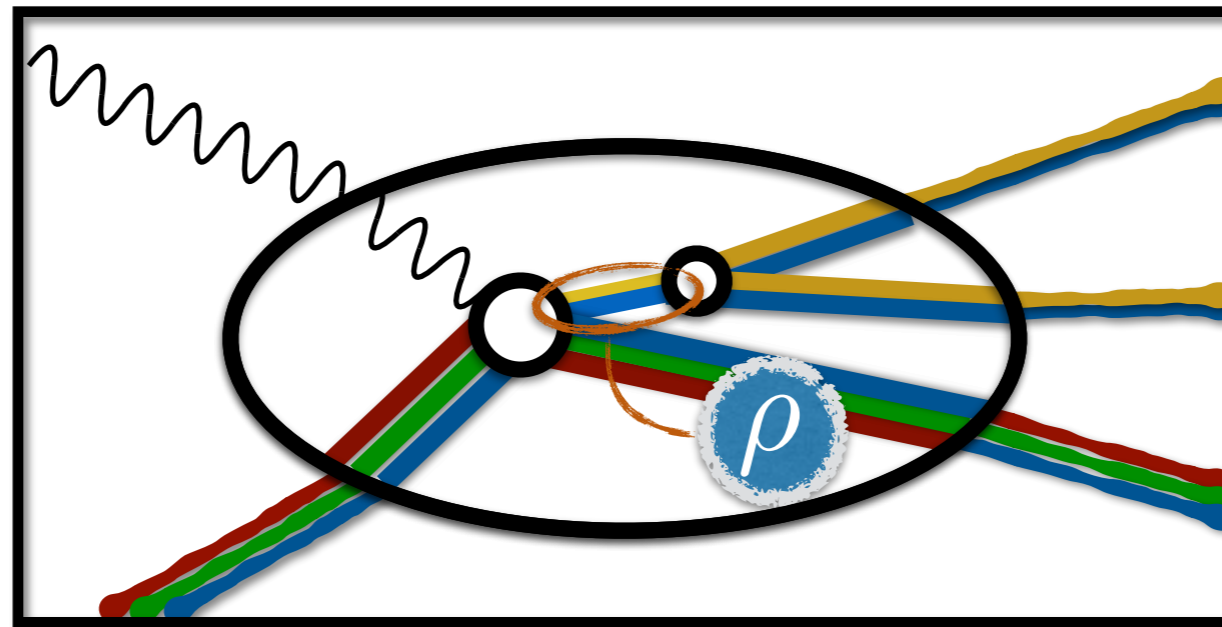
Relax all simplifying assumptions:

**Allow all particle types, allow two-to-three couplings,
remove bound on phase shift**

$$K\pi \rightarrow K\pi\pi \quad N\pi \rightarrow N\pi\pi \quad NNN \rightarrow NNN$$

Briceño, MTH, Sharpe, *in development*

Derive formalism for three-particle transition amplitudes



$$p\gamma \rightarrow N\rho \rightarrow N\pi\pi$$

Also want to make connections to other work...

Polejaeva and Rusetsky, *Eur. Phys. J. A*48, 67 (2012)

Briceño and Davoudi, *Phys. Rev. D*87, 094507 (2013)

Meißner, Rios and Rusektsky. *Phys. Rev. Lett.* 114, 091602 (2015)

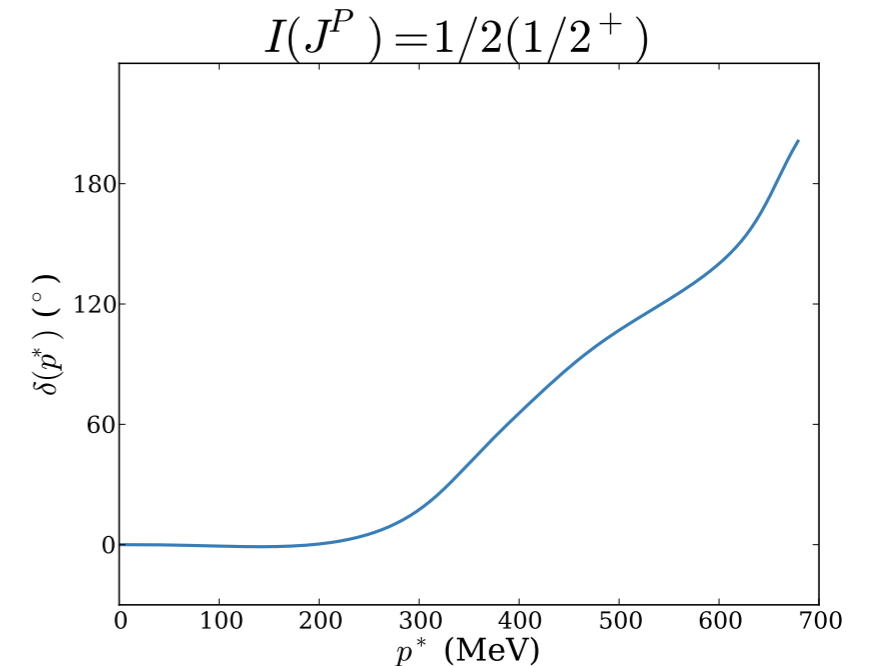
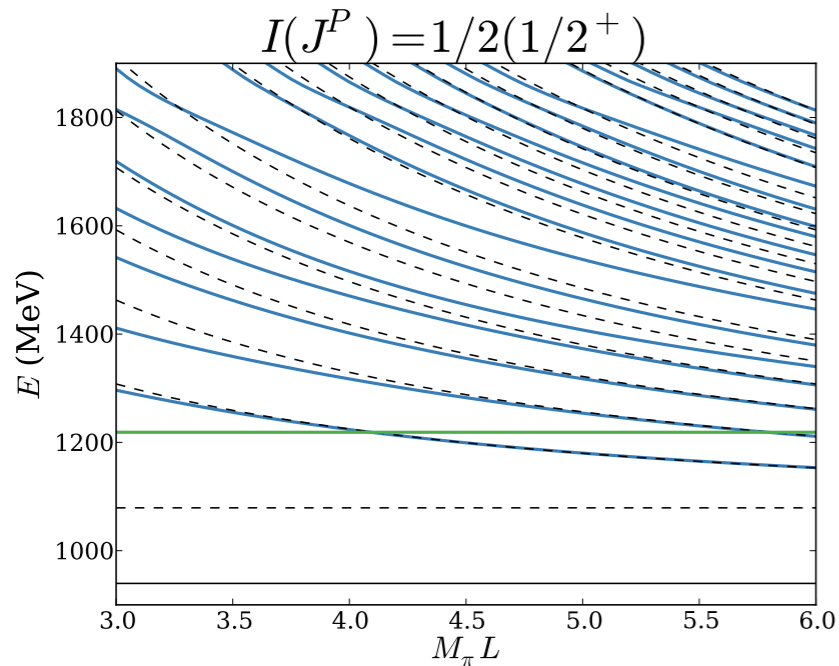
What lattice needs for resonances...

What lattice needs for resonances...

As much information as possible about the finite-volume spectrum

Can functional methods be used to calculate energies in various volumes?

Given energies in one volume can one “bootstrap” to energies in a different volume?

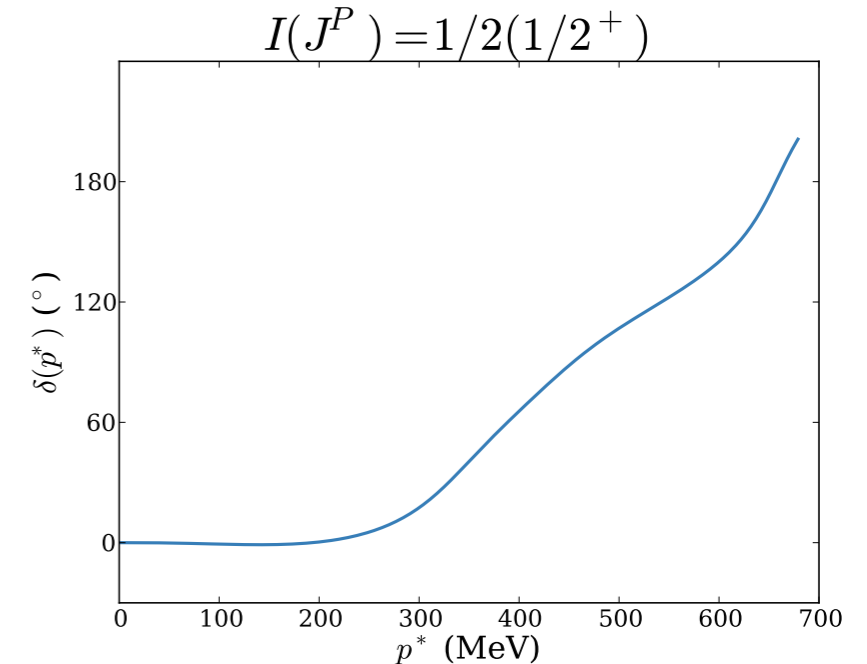
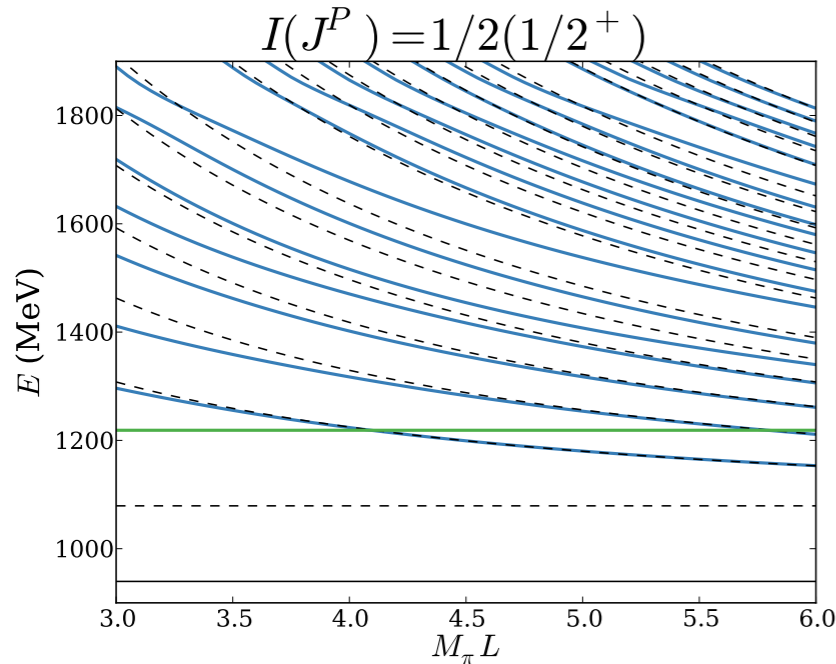


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Better chiral extrapolations

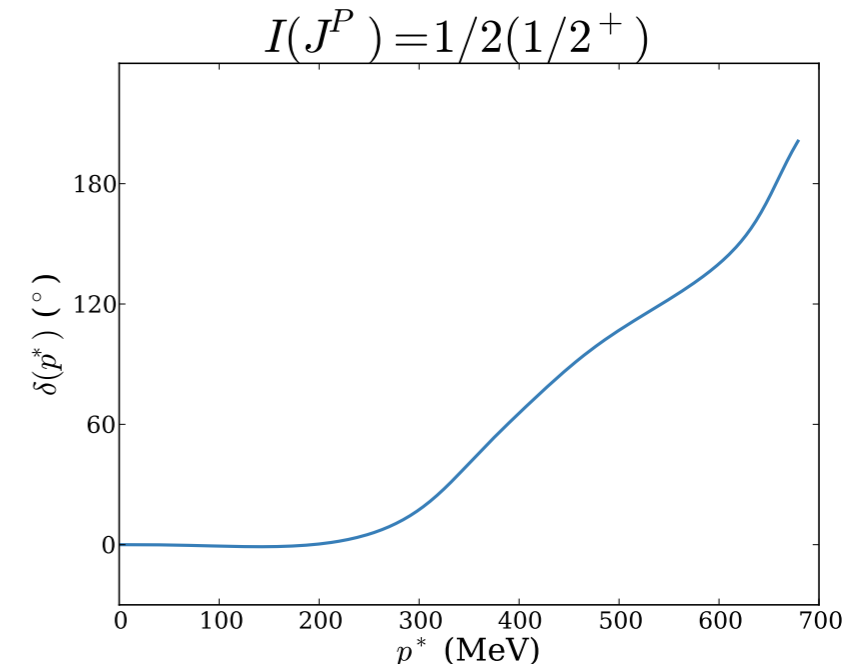
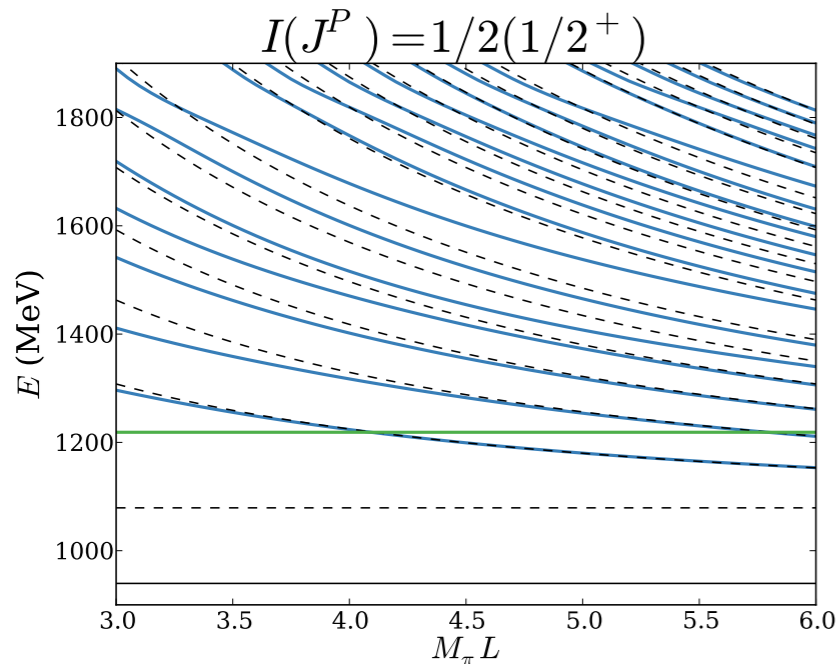
Can functional methods be used to supplement ChPT in interpolating to the physical point?

What lattice needs for resonances...

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Better chiral extrapolations

Can functional methods be used to supplement ChPT in interpolating to the physical point?

Help applying the three-particle formalism

We have a systematic technique for extracting $\mathcal{K}_{\text{df},3}(E^*)$ from the finite-volume spectrum.

Can functional methods help solve the set of Fadeev-like equations that relate it to the scattering amplitude?