Lattice Simulations of two-color QCD at finite density

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The QCD Phase Diagram

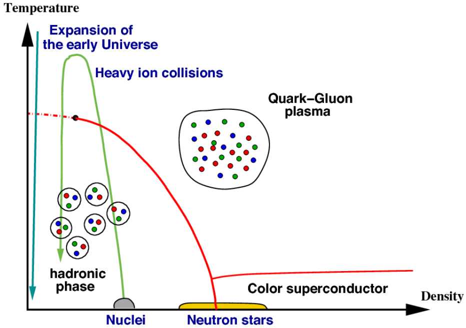

Figure: Edmond Iancu. *QCD in heavy ion collisions*. CERN-2014-003, pp. 197-266, 2012

Previous Studies

- $N_f = 8$ staggered fermions

  Hands et al., Nucl.Phys. B558 (1999) 327-346, arXiv:hep-lat/9902034
- $N_f = 4$ staggered fermions

  Kogut et al., Phys.Rev. D68 (2003) 054507, arXiv:hep-lat/0305003
- $N_f = 2$ Wilson fermions

  Cotter et. al, Phys. Rev. D87 034507 (2013), arXiv:1210.4496 [hep-lat]
- $N_f = 2$ staggered fermions

  Braguta et al., arXiv:1605.04090 [hep-lat]

Motivation

- two-color QCD exhibits the properties of
  - confinement
  - diquark condensation
- testing of χPT predictions
- analogies to QCD at finite isospin density
  - pion condensation

1. Discrete four dimensional Euclidean space-time Lattice

$$\Lambda = \{n = (n_1, n_2, n_3, n_4) \mid$$
$$n_i = 0, 1, 2, .., N_s - 1, i = 1, 2, 3;\;\; n_4 = 0, 1, 2, ..., N_t - 1\}$$

- physical space-time points x obtained by multiplying the lattice spacing a

$$x = a(\beta, m)\; n$$

2. Gluon Fields as Link Variables

- continuum gauge transporters $G(y,x)$
  - $\psi(y)$ and $G(y,x)\psi(x)$ transform the same way under gauge transformations

$$G(x+dx_\mu,x)=\mathbb{I}+i\;A^i_\mu(x)T^i dx_\mu+O(\epsilon^2)$$

$$G(y,x)=\prod_i G_i = P\exp\left(i\int_x^y A_\mu(x)dx_\mu\right)$$

- introduce the link variables on the lattice in analogy

$$U_\mu(n)=\exp\left(ia\;A_\mu\left(n+\frac{\hat{\mu}}{2}\right)\right)$$

$$U_{-\mu}(n)=\exp\left(-ia\;A_\mu\left(n-\frac{\hat{\mu}}{2}\right)\right)=U_\mu(n-\hat{\mu})^\dagger$$

Gauge-invariant Objects

1. Two quark fields connected by a path-ordered product of link variables:

$$\bar{\psi}(n_0)\left[\prod_{(n,\mu)\in\mathscr{P}} U_\mu(n)\right]\psi(n_1)$$

2. The trace of a closed loop of link variables (Wilson loops):

$$\mathrm{tr}\left[\prod_{(n,\mu)\in\mathscr{L}} U_\mu(n)\right]$$

Wilson gauge action

- the 1×1 plaquette

$$
\begin{aligned}
P_{\mu\nu}(n) &= U_\mu(n)\ U_\nu(n+\hat{\mu})\ U_\mu(n+\hat{\nu})^\dagger\ U_\nu(n)^\dagger \\
&= e^{iaA_\mu(n+\frac{\hat{\mu}}{2})}\ e^{iaA_\nu(n+\hat{\mu}+\frac{\hat{\nu}}{2})}\ e^{-iaA_\mu(n+\hat{\nu}+\frac{\hat{\mu}}{2})}\ e^{-iaA_\nu(n+\frac{\hat{\nu}}{2})} \\
&= e^{ia^2 F_{\mu\nu}(n)+O\left(a^3\right)}
\end{aligned}
$$

- lattice gauge action

$$
S_G[U] = \frac{2N_c}{g^2} \sum_{n\in\Lambda} \sum_{\mu>\nu} \left(1 - \frac{1}{N_c} \mathrm{Re\ tr}\left(P_{\mu\nu}\right)\right)
$$

- the inverse coupling $\beta = \frac{2N_c}{g^2}$

3. Discretize the Fermion Action

- free Euclidean continuum fermion action for a single quark flavor

$$S_F^0[\psi,\bar{\psi}] = \int d^4x\ \bar{\psi}(x)\ (\gamma_\mu \partial_\mu + m)\ \psi(x)$$

- $\psi(x) \to \psi(n)$: $4N_c$ dimensional vector at every $n \in \Lambda$
- $\int d^4x \to a^4 \sum_{n\in\Lambda}$
- $\partial_\mu \psi(n) \to \Delta_\mu \psi(n) = \dfrac{\psi(n+\hat{\mu}) - \psi(n-\hat{\mu})}{2a}$

$$\begin{aligned} S_F[\psi,\bar{\psi}] =& a^4 \sum_{n\in\Lambda} \bar{\psi}(n)\ \cdot \\ & \cdot \left(\sum_{\mu=1}^{4} \gamma_\mu \frac{U_\mu(n)\psi(n+\hat{\mu}) - U_\mu(n-\hat{\mu})^\dagger \psi(n-\hat{\mu})}{2a} + m\ \psi(n) \right) \end{aligned}$$

The Staggered Transformation

$$\psi(n) = \gamma_1^{n_1}\gamma_2^{n_2}\gamma_3^{n_3}\gamma_4^{n_4}\ \psi(n)'$$
$$\bar{\psi}(n) = \bar{\psi}(n)'\ \gamma_1^{n_1}\gamma_2^{n_2}\gamma_3^{n_3}\gamma_4^{n_4}$$

- the staggered action

$$S_F[\chi,\bar{\chi}] = a^4 \sum_{n\in\Lambda} \bar{\chi}(n) \cdot \cdot \left(\sum_{\mu=1}^{4} \eta_\mu(n) \frac{U_\mu(n)\chi(n+\hat{\mu}) - U_{-\mu}(n)\chi(n-\hat{\mu})}{2a} + m\chi(n) \right)$$

- staggered phases

$$\eta_1(n) = 1\ ,\ \eta_2(n) = (-1)^{n_1}\ ,\ \eta_3(n) = (-1)^{n_1+n_2}\ ,\ \eta_4(n) = (-1)^{n_1+n_2+n_3}$$

- describe four tastes of continuum quarks
- chiral symmetry

$$\chi(n) \rightarrow e^{i\alpha\eta_5(n)}\, \chi(n)\ , \quad \bar{\chi}(n) \rightarrow \bar{\chi}(n)\, e^{i\alpha\eta_5(n)}$$

- with $\eta_5(n) = (-1)^{n_1+n_2+n_3+n_4}$ the analogue of γ_5

4. Algorithm to calculate Expectation Values

- expectation value in path-integral formalism

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}[U]\mathcal{D}[\psi, \bar{\psi}]\; O[\psi, \bar{\psi}, U]\; e^{-S_F[\psi,\bar{\psi},U]-S_G[U]}$$
$$= \frac{1}{Z} \int \mathcal{D}[U]\; O[D[U]^{-1}, U] \det\left(D[U]\right)\; e^{-S_G[U]}$$

- approximate with Monte Carlo method

$$\langle O \rangle \approx \frac{1}{N} \sum_{i=1}^{N} O[D[U_i]^{-1}, U_i]$$

$$dP(U) = \frac{1}{Z} \det\left(D[U]\right)\; e^{-S_G[U]}\; \mathcal{D}[U]$$

- determinant of staggered dirac operator is real

$$\det\left(D[U]\right) = \det\left(D[U]\right)^*$$

- to ensure positivity

$$D \to DD^\dagger$$

$$dP(U) = \frac{1}{Z} \det\left(D[U]D^\dagger[U]\right) \; e^{-S_G[U]} \; \mathcal{D}[U]$$

- doubles the number of quark flavors!
- create markov chain of gauge field configurations

$$U_0 \to U_1 \to U_2 \to U_3 \to \cdots$$

The HMC Algorithm

- absorb the fermion determinant into the action

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}[U]\; O[D[U]^{-1}, U] \det \Big(D[U] D^\dagger[U] \Big) \; e^{-S_G[U]}$$

$$\propto \frac{1}{Z} \int \mathcal{D}[U]\mathcal{D}[\phi]\mathcal{D}[\phi^\dagger]\; O[D[U]^{-1}, U] \; e^{-S_G[U] - S_F[\phi,\phi^\dagger,U]}$$

$$\text{with} \quad S_F[\phi, \phi^\dagger, U] = \phi^\dagger ((DD^\dagger)[U])^{-1} \phi$$

- molecular dynamics Hamiltonian

$$H[\pi, U] = \sum_{n \in \Lambda} \sum_{\mu} \mathrm{tr} \left[\pi_\mu(n)^2 \right] + S_G[U] + S_F[\phi, \phi^\dagger, U]$$

The HMC Algorithm

- equations of motion

$$\frac{d}{d\tau}U_\mu(n) = U_\mu(n)\left(i\pi_\mu(n)\right)$$
$$\frac{d}{d\tau}\left(i\pi_\mu(n)\right) = U_\mu(n)\frac{\partial S}{\partial U_\mu(n)}\Big|_{TA} \equiv F_\mu(n)\Big|_{TA}$$

- solved numerically with symplectic integrators

$$(U,\pi) \to (U',\pi')$$

- accept/reject-step after the evolution

$$P_{acc}(U',\pi'|U,\pi) = \min\left(1,\exp\left(-\Delta H\right)\right)$$

The HMC Algorithm

- generation of a pseudofermion field
  - treated as a constant background field during evolution

$$\propto \exp\left(-\phi^{\dagger}\left(\left(DD^{\dagger}\right)[U]\right)^{-1}\phi\right)$$

- complex Gaussian distributed η

$$\Rightarrow \phi = D\eta$$

The Rational HMC

- reduce the number of flavors in the simulation
- use roots of the fermion determinant in the HMC

$$\det\left(DD^\dagger\right)^\alpha \approx \frac{1}{\pi^N}\int \mathcal{D}\phi^\dagger\mathcal{D}\phi \; e^{\phi^\dagger r_\alpha(DD^\dagger)\phi} \;,\; |\alpha| < 1$$

- rational approximation in partial fraction form

$$(DD^\dagger)^{-\alpha} \approx r^\alpha(DD^\dagger) = a_0^\alpha + \sum_{n=1}^{N} \frac{a_n^\alpha}{DD^\dagger + b_n^\alpha}$$

Temperature on the Lattice

- Euclidean QFT at non-vanishing temperature T

$$Z(T) = \int \mathcal{D}[\Psi] \; \exp\left(-S_E[\Psi]\right)$$

$$S_E[\Psi] = \int_0^{\frac{1}{T}} dt \int_{\mathbb{R}^3} d\vec{x} \; L_E\left(\Psi(t,\vec{x}), \partial_\mu \Psi(t,\vec{x})\right)$$

- temperature is connected to the time extension

$$\frac{1}{T} = a(\beta, m) \; N_t$$

Chemical Potential on the Lattice

- in continuum theory

$$\mathcal{L} = \bar{\psi}(\gamma_\mu(\partial_\mu + iA_\mu) + m + \mu\gamma_4)\psi$$

- analogue way on the lattice?
  - free energy density diverges in the continuum limit
- introduce additional link variables

$$U_{4,\text{ext}} = e^{ia\tilde{A}_4} = e^{a\mu}$$

$$U_{-4,\text{ext}} = e^{-ia\tilde{A}_4} = e^{-a\mu}$$

The Sign Problem

- fermion determinant is complex at non-vanishing μ

$$\gamma_5 D(\mu)\gamma_5 = D(-\mu)^\dagger \iff \det(D) \in \mathbb{C}$$

- need real fermion determinant for the HMC algorithm
- use SU(2) as the gauge group

$$T_a \in \mathrm{su}(2) \implies (\tau_2)\ T_a\ (\tau_2)^{-1} = -(T_a)^*$$

$$\det\left(D(\mu)\right) = \det\left((C\gamma_5\tau_2)\ D(\mu)\ (C\gamma_5\tau_2)^{-1}\right) = \det\left(D(\mu)\right)^*$$

Introducing a Diquark Source Term

- explicit symmetry breaking term needed to observe spontaneous symmetry breaking
  - breaking U(1) Baryon number conservation
- enlarge the staggered action

$$S_F[\chi, \bar{\chi}] = \bar{\chi}\ D[\mu]\ \chi + \frac{\lambda}{2}\left(\chi^T \tau_2 \chi + \bar{\chi} \tau_2 \bar{\chi}^T\right)$$

- diquarks do not carry color in two-color QCD
  - gauge-invariant diquark source term

- new basis

$$S_F = \frac{1}{2} \begin{pmatrix} \bar{\chi} & \chi^T \tau_2 \end{pmatrix} \underbrace{\begin{pmatrix} \lambda & D[\mu] \\ -D[\mu]^\dagger & \lambda \end{pmatrix}}_{=:A} \begin{pmatrix} \tau_2 \bar{\chi}^T \\ \chi \end{pmatrix}$$

- fermionic part of the partition function

$$Z_F = \det(A) = \det\left(D[\mu]^\dagger D[\mu] + \lambda^2\right)$$

- to obtain 'physical' results
  - extrapolate $\lambda \to 0$

Rational HMC with Diquarks

$$Z_F = \det(A)^\alpha$$

- rational approximation in partial fraction form

$$(D^\dagger D + \lambda^2)^{-\alpha} \approx r^\alpha(D^\dagger D + \lambda^2) = a_0^\alpha + \sum_{n=1}^{N} \frac{a_n^\alpha}{D^\dagger D + \lambda^2 + b_n^\alpha}$$

- absorb λ^2 into coefficients

$$b_j^\alpha \to b_j^\alpha + \lambda^2$$

Symmetry of $N_f = 1$ Staggered Fermion

- at $\mu = m = \lambda = 0$ enlarged symmetry

$$U(1)_V \times U(1)_A \to U(2)$$

- calculating the induced Goldstone modes
  $V_\delta = \mathbb{I} + i\delta\rho \quad , \quad \rho \in \{\mathbb{I}, T_i\}$

$$\begin{array}{cccc} & & \langle\psi\bar{\psi}\rangle & \langle\psi\psi\rangle \\ \mathbb{I} & \Rightarrow & \bar{\chi}\epsilon\chi & \chi^T\tau_2\epsilon\chi + \bar{\chi}\tau_2\epsilon\bar{\chi}^T \\ T_1 & \Rightarrow & \chi^T\tau_2\chi - \bar{\chi}\tau_2\bar{\chi}^T & \mathbb{I} \\ T_2 & \Rightarrow & \chi^T\tau_2\chi + \bar{\chi}\tau_2\bar{\chi}^T & \bar{\chi}\chi \\ T_3 & \Rightarrow & \mathbb{I} & \chi^T\tau_2\chi - \bar{\chi}\tau_2\bar{\chi}^T \end{array}$$

- connected by an explicit global U(2) rotation

$$V = \frac{i}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Symmetry of $N_f = 1$ Staggered Fermion

- at $\mu = m = \lambda = 0$ enlarged symmetry

$$U(1)_V \times U(1)_A \rightarrow U(2)$$

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  $V_\delta = \mathbb{I} + i\delta\rho \ \ , \ \rho \in \{\mathbb{I}, T_i\}$

	$\langle\psi\bar{\psi}\rangle$	$\langle\psi\psi\rangle$
$\mathbb{I} \Rightarrow$	$\bar{\chi}\epsilon\chi$	$\chi^T\tau_2\epsilon\chi + \bar{\chi}\tau_2\epsilon\bar{\chi}^T$
$T_1 \Rightarrow$	$\chi^T\tau_2\chi - \bar{\chi}\tau_2\bar{\chi}^T$	$\mathbb{I}$
$T_2 \Rightarrow$	$\chi^T\tau_2\chi + \bar{\chi}\tau_2\bar{\chi}^T$	$\bar{\chi}\chi$
$T_3 \Rightarrow$	$\mathbb{I}$	$\chi^T\tau_2\chi - \bar{\chi}\tau_2\bar{\chi}^T$

- connection no longer trivial at $\mu \neq 0$!
- phase transition at some μ_c

Symmetry of $N_f = 1$ Staggered Fermion

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$$U(1)_V \times U(1)_A \to U(2)$$
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		$\langle\psi\bar{\psi}\rangle$	$\langle\psi\psi\rangle$
$\mathbb{I}$	$\Rightarrow$	$\bar{\chi}\epsilon\chi$	$\chi^T\tau_2\epsilon\chi + \bar{\chi}\tau_2\epsilon\bar{\chi}^T$
T_1	$\Rightarrow$	$\chi^T\tau_2\chi - \bar{\chi}\tau_2\bar{\chi}^T$	$\mathbb{I}$
T_2	$\Rightarrow$	$\chi^T\tau_2\chi + \bar{\chi}\tau_2\bar{\chi}^T$	$\bar{\chi}\chi$
T_3	$\Rightarrow$	$\mathbb{I}$	$\chi^T\tau_2\chi - \bar{\chi}\tau_2\bar{\chi}^T$

- only one true Goldstone boson in the limit $\lambda \to 0$
  - the scalar diquark of the spontaneous $U(1)_V$ symmetry breaking

General N_f

- enlarged symmetry

$$U(N_f) \times U(N_f) \to U(2N_f)$$

$$U(2N_f) \xrightarrow{\sqrt{\langle\bar{\psi}\psi\rangle^2 + \langle\psi\psi\rangle^2}} O(2N_f)$$

$$U(2N_f) \xrightarrow{\mu \neq 0} U(N_f) \times U(N_f)$$

$$U(N_f) \times U(N_f) \xrightarrow{\langle\bar{\psi}\psi\rangle} U(N_f) \times U(1)_B$$

$$U(N_f) \times U(N_f) \xrightarrow{\langle\psi\psi\rangle} O(N_f)$$

$$U(N_f) \times U(1)_B \xrightarrow{\langle\psi\psi\rangle} O(N_f)$$

Predictions from Chiral Perturbation Theory

- any-color QCD with quarks in the adjoint representation

$$SU(2N_f) \to SO(2N_f)$$

- microscopic Lagrangian at finite quark mass, chemical potential and diquark source

$$\mathcal{L} = \bar{\psi}\gamma_\nu D_\nu \psi + m\bar{\psi}\psi - \mu\bar{\psi}\gamma_0\psi + \frac{\lambda}{2}\left(i\psi^T C\gamma_5\psi + h.c.\right)$$

- construct leading order effective Lagrangian
  Kogut et al. arXiv:hep-ph/0001171v2

$$\mathcal{L}_{\text{eff}}(\Sigma) = \frac{F^2}{2}\left\{\text{tr}\left[\nabla_\nu\Sigma\nabla_\nu\Sigma^\dagger\right] - 2m_\pi^2 \text{Re tr}\left[\hat{M}_\phi\Sigma\right]\right\}$$

- leading order predictions

$$\langle \bar{\psi}\psi \rangle = -\frac{\partial \varepsilon_{vac}}{\partial m} = 2N_f G \cos\alpha(\mu)$$
$$\langle \psi\psi \rangle = -\frac{\partial \varepsilon_{vac}}{\partial \lambda} = 2N_f G \sin\alpha(\mu)$$
$$\langle n_B \rangle = -\frac{\partial \varepsilon_{vac}}{\partial \lambda} = 8N_f F^2 \mu \sin^2\alpha(\mu)$$

- rotation angle $\alpha(\mu)$

$$4\mu^2 \sin\alpha \; \cos\alpha = m_\pi^2 \sin(\alpha - \phi)$$

$$\tan(\phi) = \frac{\lambda}{m}$$

- condensate rotates from chiral to diquark

$$\Sigma_\alpha \equiv \Sigma_c \; \cos\alpha + \Sigma_d \; \sin\alpha$$

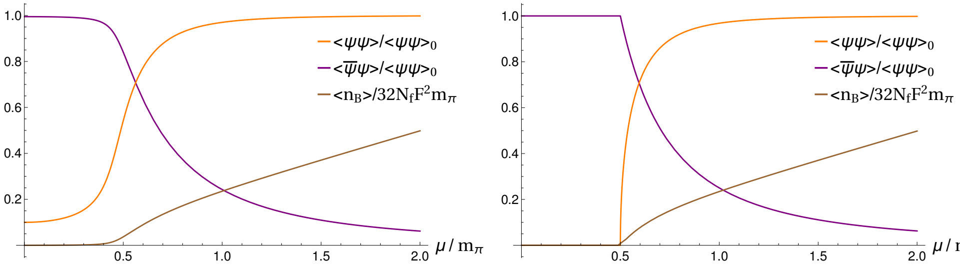


Figure: Predictions from leading order chiral perturbation theory with $\lambda = 0.1m$ (left) and $\lambda = 0$ (right).

- critical chemical potential $\mu_c = \frac{m_\pi}{2}$

The Goldstone Modes

- Goldstone manifold given by the coset $SU(2N_f)/SO(2N_f)$
- parametrized by the rotations

$$\Sigma = V_\alpha U \Sigma_c U^T V_\alpha^T$$

$$U = \exp\left(\frac{i\Pi}{2F}\right) \quad , \quad \Pi = \pi_a \frac{X_a}{\sqrt{2N_f}} \quad , \quad V_\alpha^2 = \exp\left(i\alpha X_2\right)$$

- expand effective Lagrangian up to second order in the Goldstone matrix Π to obtain dispersion laws

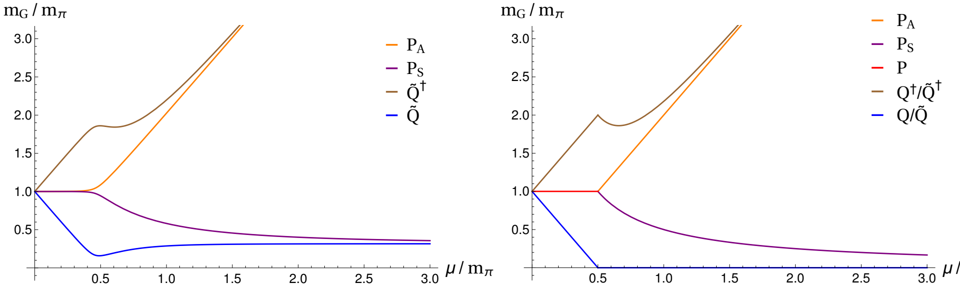


Figure: Predictions from leading order chiral perturbation theory with $\lambda = 0.1m$ (left) and $\lambda = 0$ (right).

- two-color QCD with quarks in the fundamental representation: $SU(2N_f) \to Sp(2N_f)$
  - meson modes interchange: $P_A \leftrightarrow P_S$

The Chiral and Diquark Condensate and the Quark Number Density

- the chiral condensate

$$\langle\bar{\psi}\psi\rangle = \frac{1}{V}\frac{\partial \ln Z}{\partial m} = \frac{1}{2V}\frac{N_f}{4}\left\langle \mathrm{tr}\left(A^{-1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\right\rangle$$

- the diquark condensate

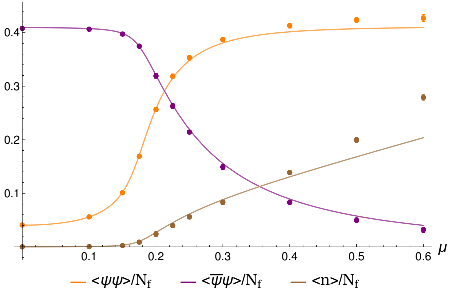
$$\langle\psi\psi\rangle = \frac{1}{V}\frac{\partial \ln Z}{\partial \lambda} = \frac{1}{2V}\frac{N_f}{4}\left\langle \mathrm{tr}\left(A^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\right\rangle$$

- the quark number density

$$\langle n\rangle = \frac{1}{V}\frac{\partial \ln Z}{\partial \mu} = \frac{1}{V}\frac{N_f}{4}\left\langle \mathrm{tr}\left(D^{-1}\frac{\partial D}{\partial \mu}\right)\right\rangle$$

Checking the numerics

- parameters: $12^3 \times 24, \beta = 1.5, m = 0.025, N_f = 2$
  same parameters as Kogut et al. ($N_f = 4$)



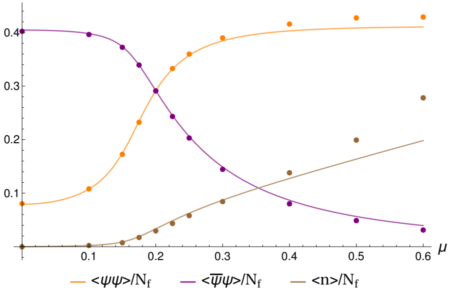


Figure: Predictions from leading order chiral perturbation theory with $\lambda = 0.0025$ (left) and $\lambda = 0.0050$ (right).

- check the prediction $\mu_c = \frac{m_\pi}{2}$

λ	m_π	μ_c	μ_c^{fit}
0.0025	0.3774(03)(11)	0.1887(02)(06)	0.18889(45)
0.0050	0.3791(06)(16)	0.1896(03)(08)	0.18931(47)

The Goldstone Modes on the Lattice

Channel	Operator	J^{PC}	States
1	$\bar{\chi}\chi$	0^{++}	f_0
		0^{-+}	π
2	$\eta_4\bar{\chi}\chi$	0^{+-}	-
		0^{-+}	π

Channel	Operator	States
3	$\frac{1}{2}\left(\chi^T\tau_2\chi - \bar{\chi}\tau_2\bar{\chi}^T\right)$	$qq/\bar{q}\bar{q}$
4	$\eta_5\frac{1}{2}\left(\chi^T\tau_2\chi + \bar{\chi}\tau_2\bar{\chi}^T\right)$	$\varepsilon qq/\varepsilon\bar{q}\bar{q}$

- zero-momentum projected correlation functions

$$C(t) = \sum_{\vec{x}} \left\langle 0 \middle| O(\vec{x}, t)\bar{O}(\vec{0}, 0) \middle| 0 \right\rangle \propto \cosh\left(m\left(t - \frac{N_t}{2}\right)\right)$$

- use Wick's theorem

$$\langle 0|\chi_i(x)\bar{\chi}_j(y)|0\rangle = G_{ij}(x, y)$$

- propagator G obtained from point sources

$$D_{ij}[\mu](x, y)\ w_j(y) = v_i(x) = (1, 0)^T \delta_{x,x_0}$$
$$\Rightarrow \quad w(y) = (G_{11}[\mu](y, x_0), G_{21}[\mu](y, x_0))^T = (a, -b^*)^T[\mu](y, x_0)$$

$$G_{ij} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

$$C_1(t) = -\sum_{\vec{x}} \eta_5(\vec{x},t)\ \mathrm{tr}\left[G^\dagger[-\mu](\vec{x},t;0)G[\mu](\vec{x},t;0)\right]$$

$$C_2(t) = -(-1)^t \sum_{\vec{x}} \mathrm{tr}\left[G^\dagger[-\mu](\vec{x},t;0)G[\mu](\vec{x},t;0)\right]$$

$$C_3(t) = \frac{1}{2}\sum_{\vec{x}} \Big\{\mathrm{tr}\left[G^T[\mu](\vec{x},t;0)\tau_2 G[\mu](\vec{x},t;0)\tau_2\right] + \mathrm{tr}\left[G^\dagger[-\mu](\vec{x},t;0)\tau_2 (G^\dagger)^T[-\mu](\vec{x},t;0)\tau_2\right]\Big\}$$

$$C_4(t) = \frac{1}{2}\sum_{\vec{x}} \eta_5(\vec{x},t)\Big\{\mathrm{tr}\left[G^T[\mu](\vec{x},t;0)\tau_2 G[\mu](\vec{x},t;0)\tau_2\right] + \mathrm{tr}\left[G^\dagger[-\mu](\vec{x},t;0)\tau_2 (G^\dagger)^T[-\mu](\vec{x},t;0)\tau_2\right]\Big\}$$

Enlarged Propagator

$$G_\lambda = A^{-1} = \begin{pmatrix} \left(DD^\dagger + \lambda^2\right)^{-1}\lambda & -\left(DD^\dagger + \lambda^2\right)^{-1} D \\ \left(D^\dagger D + \lambda^2\right)^{-1} D^\dagger & \left(D^\dagger D + \lambda^2\right)^{-1}\lambda \end{pmatrix}$$

- λ-dependence of the $\langle \chi\bar{\chi} \rangle$ contractions

$$G = D^{-1} = \left(D^\dagger D\right)^{-1} D^\dagger \rightarrow \left(D^\dagger D + \lambda^2\right)^{-1} D^\dagger$$

- additional contractions $\langle \chi\chi \rangle$ and $\langle \bar{\chi}\bar{\chi} \rangle$ due to breaking of Baryon number conservation
  - additional terms of $O(\lambda^2)$ in correlation functions

Combined Modes

		$\langle\psi\bar{\psi}\rangle$	$\langle\psi\psi\rangle$
$\mathbb{I}$	$\Rightarrow$	$\bar{\chi}\epsilon\chi$	$\chi^T\tau_2\epsilon\chi + \bar{\chi}\tau_2\epsilon\bar{\chi}^T$
T_1	$\Rightarrow$	$\chi^T\tau_2\chi - \bar{\chi}\tau_2\bar{\chi}^T$	$\mathbb{I}$
T_2	$\Rightarrow$	$\chi^T\tau_2\chi + \bar{\chi}\tau_2\bar{\chi}^T$	$\bar{\chi}\chi$
T_3	$\Rightarrow$	$\mathbb{I}$	$\chi^T\tau_2\chi - \bar{\chi}\tau_2\bar{\chi}^T$

$$\Sigma_\alpha \equiv \Sigma_c \ \cos\alpha + \Sigma_d \ \sin\alpha$$

- to leading order
  $\bar{q}\bar{q}/f_0$: $\frac{1}{2}\left(\chi^T\tau_2\chi + \bar{\chi}\tau_2\bar{\chi}^T\right)\cos\alpha + \bar{\chi}\chi\sin\alpha$
  $\pi/\epsilon qq$: $\bar{\chi}\epsilon\chi\cos\alpha + \frac{1}{2}\left(\chi^T\tau_2\epsilon\chi + \bar{\chi}\tau_2\epsilon\bar{\chi}^T\right)\sin\alpha$

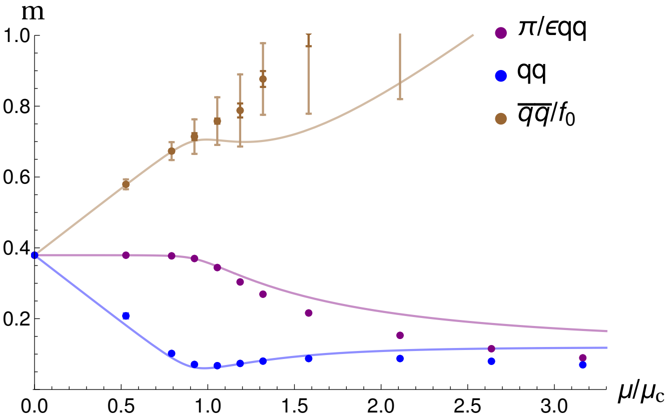


Figure: The Goldstone spectrum on a $12^3 \times 24$ lattice at $\beta = 1.5$ with quark mass $m = 0.025$ and diquark source $\lambda = 0.0025$.

The Z(2) Monopole Density

- bulk phase: artificial phase - lattice artifact - disturbes physics
- order parameter

$$M = 1 - \frac{1}{N_C} \sum_{C} \prod_{P_{\mu\nu} \in \partial C} \text{sign}\left(\text{tr } P_{\mu\nu}\right)$$

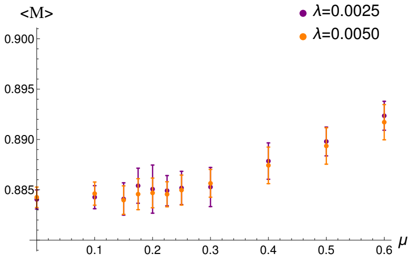


Figure: The Z(2) monopole density on a $12^3 \times 24$ lattice at $\beta = 1.5$ with quark mass $m = 0.025$.

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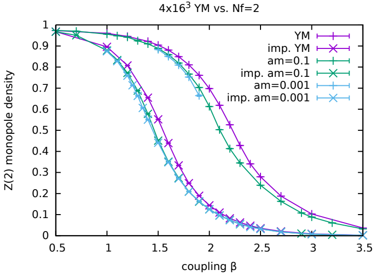


Figure: The Z(2) monopole density (PhD Thesis - David Scheffler).

The Improved Gauge Action

- include additional rectangular loops

$$R_{\mu\nu}(n) = U_\nu(n)\; U_\mu(n+\hat{\nu})\; U_\mu(n+\hat{\mu}+\hat{\nu})\; U_\nu(n+2\hat{\mu})^\dagger\; U_\mu(n+\hat{\mu})^\dagger\; U_\mu(n)^\dagger$$

- reduce discretization error to $O(a^4)$

$$S_G^{imp}[U] = \beta \sum_{n\in\Lambda} \sum_{\mu>\nu} \left(\frac{5\tilde{P}_{\mu\nu}}{3} - \frac{\tilde{R}_{\mu\nu} + \tilde{R}_{\nu\mu}}{12} \right)$$

$$\tilde{P}_{\mu\nu} = 1 - \frac{1}{N_c}\mathrm{Re}\ \mathrm{tr}(P_{\mu\nu}) \quad , \quad \tilde{R}_{\mu\nu} = 1 - \frac{1}{N_c}\mathrm{Re}\ \mathrm{tr}(R_{\mu\nu})$$

Compromise between Bulk Phase Effects and Finite Volume Effects

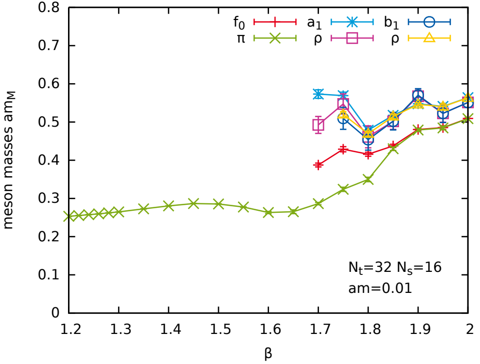


Figure: The meson masses in dependence of the inverse coupling β (PhD Thesis - David Scheffler).

Leaving the Bulk Phase

- new parameters: $16^3 \times 32, \beta = 1.7, m = 0.01, N_f = 2$
- chiral symmetry broken: $\frac{m_\pi}{m_\rho} = 0.5816(27)$
- Z(2) monopole density: ~ 0.27

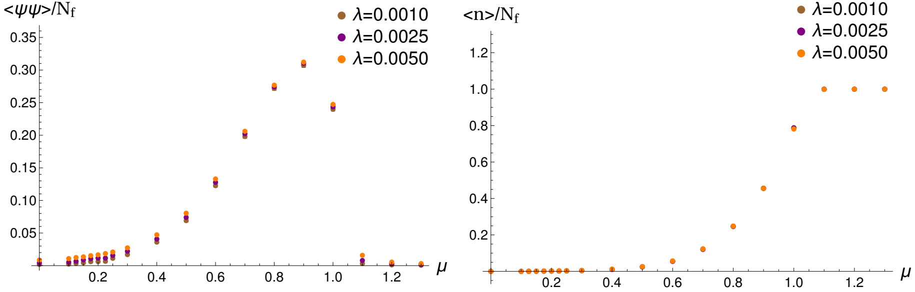


Figure: The diquark condensate (left) and the quark number density (right).

Need for Renormalization

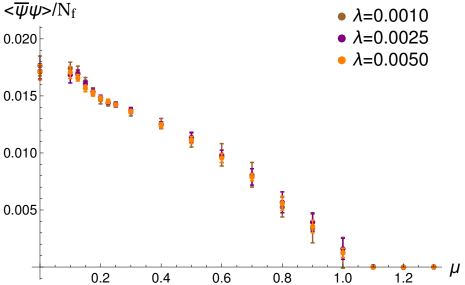


Figure: The chiral condensate.

$$\langle \bar{\psi}\psi \rangle_{m_q} = \langle \bar{\psi}\psi \rangle_0 + c_2 m_q + \frac{c_{UV}}{a^2} m_q + O(m_q^3)$$

$$\chi_{m_q} = c_2 + \frac{c_{UV}}{a^2} + O(m_q^2)$$

PhD Thesis - Wolfang Unger

Connected Susceptibility Subtraction

$$\chi^{con} = \frac{1}{V}\frac{N_f}{4}\ \left\langle \mathrm{tr}\left(D^{-2}\right)\right\rangle \quad \rightarrow \quad \Sigma^{con} = \left\langle \bar{\psi}\psi \right\rangle - m\chi^{con}$$

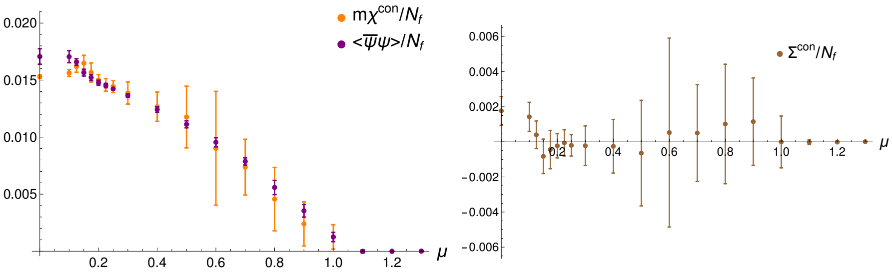


Figure: The chiral condensate and mass times the connected chiral susceptibility (left) and the resulting renormalized chiral condensate (right) for $\lambda = 0.005$.

Singular Part of the Scaling Function

$$\chi \backsim |z|^{\gamma} \quad \text{with} \quad z = \frac{\mu - \mu_c}{\mu_c}$$

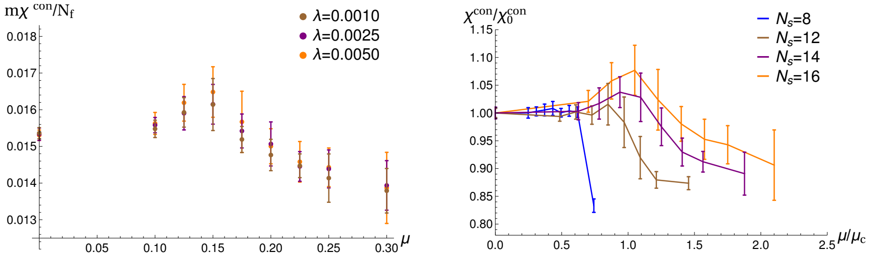


Figure: The connected chiral susceptibility.

- connected chiral susceptibility can not be used for renormalization at non-vanishing chemical potential!

The Critical Chemical Potential μ_c

- linear extrapolation $\lambda \to 0$

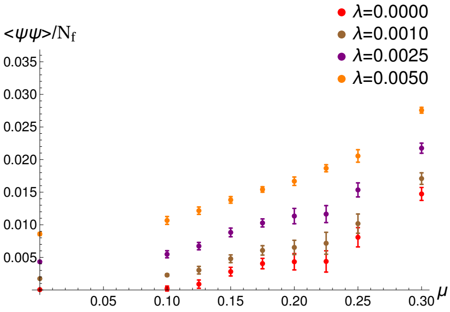
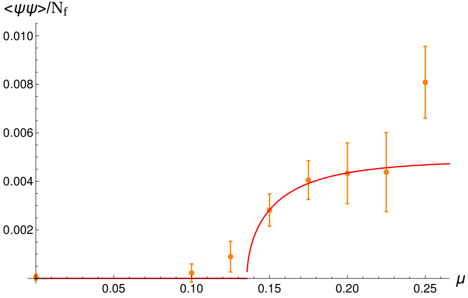


Figure: The diquark condensate (left) and the quark number density (right).

The Critical Chemical Potential

- fitting the $\lambda = 0$ diquark condensate
  - $\langle \psi\psi \rangle = \langle \bar{\psi}\psi \rangle_0 \sqrt{1 - (\mu_c/\mu)^4}$ for $\mu > \mu_c$ and zero otherwise



$$\langle \bar{\psi}\psi \rangle_0 = 0.00490(65)$$
$$\mu_c = 0.1356(86)$$
$$\chi^2/\mathrm{DOF} = 0.48$$

Figure: The resulting fit of the $\lambda = 0$ diquark condensate to the chiral perturbation theory prediction.

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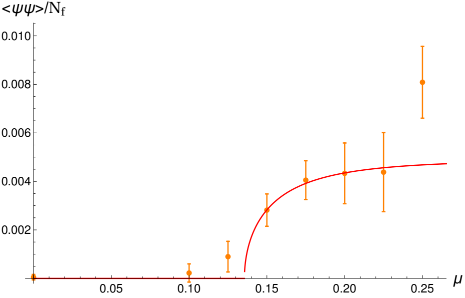

$$\langle \bar{\psi}\psi \rangle_0 = 0.00490(65)$$

$$\mu_c = 0.1356(86) = \frac{m_\pi}{2}$$

$$\chi^2/\mathrm{DOF} = 0.48$$

Figure: The resulting fit of the $\lambda = 0$ diquark condensate to the chiral perturbation theory prediction.

Quenching Effects

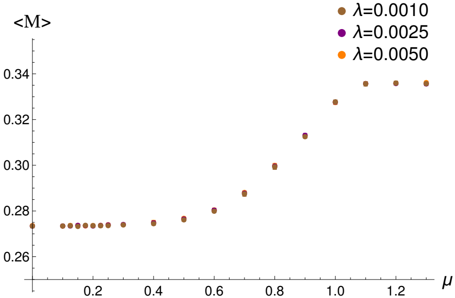

Figure: The Z(2) monopole density.

- The Z(2) monopole density approaches its quenched value in saturation

Quenching Effects in the Polyakov Loop?

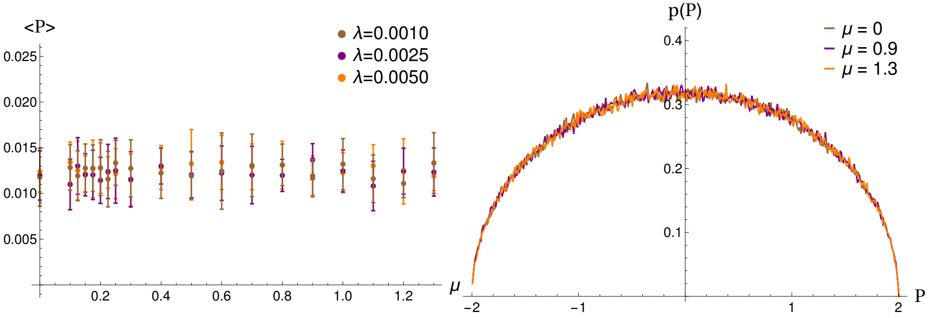

Figure: The Polyakov loop expectation value (left) and local distribution (right).

- value in pure gauge theory: $\langle P \rangle = 0.012896(29)$
  - finite volume effects overshadow the difference

Wilson Fermions

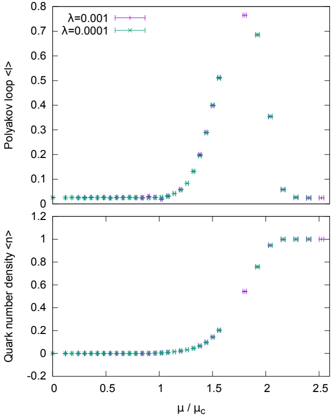

Figure: The Polyakov loop (top) and the quark number density (bottom) for Wilson fermions with $10^3 \times 20, \beta = 1.7, \kappa = 0.124689$ (Lukas Holicki).

The Goldstone Modes

$\bar{q}\bar{q}/f_0$: $\frac{1}{2}\left(\chi^T\tau_2\chi + \bar{\chi}\tau_2\bar{\chi}^T\right)\cos\alpha + \bar{\chi}\chi\sin\alpha$

$\pi/\epsilon qq$: $\bar{\chi}\epsilon\chi\cos\alpha + \frac{1}{2}\left(\chi^T\tau_2\epsilon\chi + \bar{\chi}\tau_2\epsilon\bar{\chi}^T\right)\sin\alpha$

- combined modes not possible, due to different behaviour of f_0 and ϵqq
- meson modes obtain additional oscillatory contribution from opposite parity state

$$C(t) = A\,\cosh\left(m_\pi\left(t - \frac{N_t}{2}\right)\right) + (-1)^t B\,\cosh\left(m_{\pi^\star}\left(t - \frac{N_t}{2}\right)\right)$$

- obtain $m_{\bar{q}\bar{q}}$ from $\bar{\chi}\tau_2\bar{\chi}$ for $\mu < \mu_c$

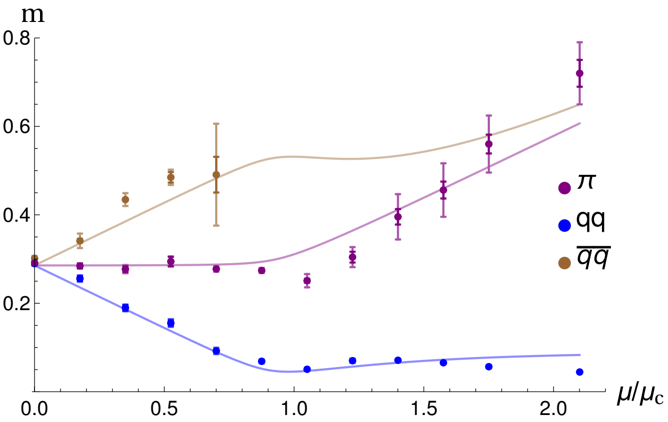

Figure: The Goldstone spectrum on a $16^3 \times 32$ lattice at $\beta = 1.7$ with quark mass $m = 0.01$ and diquark source $\lambda = 0.001$.

Continuum Pattern of Symmetry Breaking

- on the lattice: any-color QCD with quarks in adjoint representation
  - $SU(2N_f) \to SO(2N_f)$
  - Pion mass m_π decreasing for $\mu > \mu_c$
- in the continuum: two-color QCD with quarks in fundamental representation
  - $SU(2N_f) \to Sp(2N_f)$
  - Pion mass m_π increasing for $\mu > \mu_c$

Continuum Pattern of Symmetry Breaking

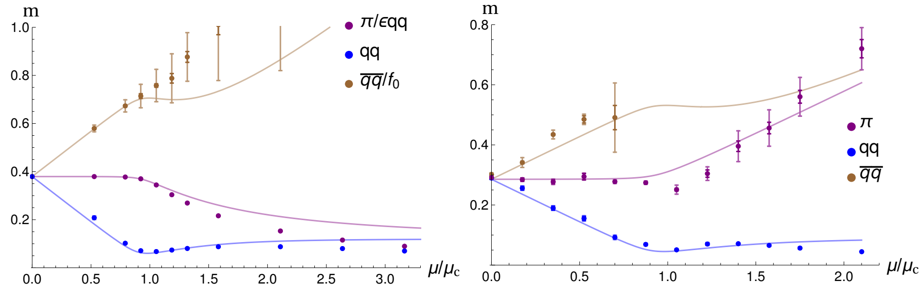

Figure: The Goldstone spectrum for $\beta = 1.5$ (left) and $\beta = 1.7$ (right)

- observed continuum pattern

- λ dependence

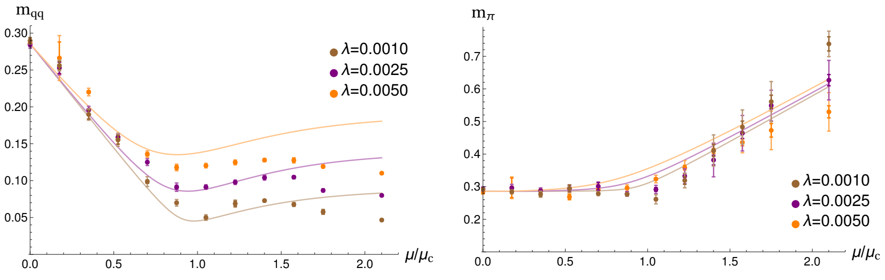

Figure: The scalar diquark mass m_{qq} (left) and the pion mass m_π (right).

Conclusion

- introduced a diquark source λ
- observed same behaviour of observables for $N_f = 2$ as previous study for $N_f = 4$ (Kogut et al.)
- found that bulk phase influences the results
- leaving the bulk phase / closer to continuum
  - connected susceptibility subtraction not usable for renormalization of the chiral condensate
  - pattern of symmetry breaking (Goldstone spectrum) changes to its according continuum theory pattern
- observed quenching effects in Z(2) monopole density but not in Polyakov loop

Outlook

- develop different renormalization
  - Wilson flow (no additive renormalization needed)
- Z(2) monopole suppression
- improve spectroscopy
  - disconnected contributions, noisy sources, higher statistics
- at which β does the pattern of symmetry breaking change?

Thank you for your attention!

Backup Slides

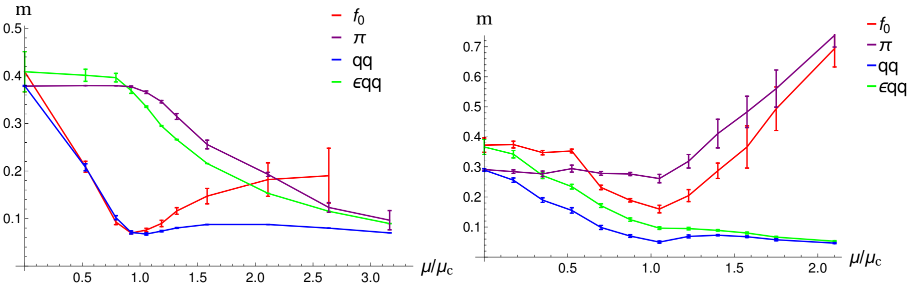

Figure: The Goldstone modes for $\beta = 1.5$ (left) and $\beta = 1.7$ (right).

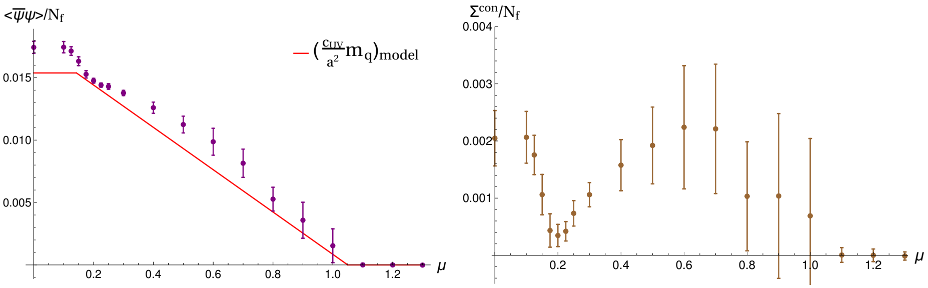

Figure: Naive model of the UV-divergent contribution.

$$\begin{aligned}
\mathcal{L} &= \bar{\psi}(\gamma_\mu D_\mu + \mu\gamma_4 + m)\psi \\
&= \bar{\psi}(\gamma_i(\partial_i + iA_i) + \gamma_4(\partial_4 + iA_4 + \mu) + m)\psi \\
&= \bar{\psi}(\gamma_i(\partial_i + iA_i) + \gamma_4(\partial_4 + iA_4 + i\tilde{A}_4) + m)\psi \\
&= \bar{\psi}(\gamma_\mu \tilde{D}_\mu + m)\psi
\end{aligned}$$

$$
\begin{aligned}
F_{F,\mu}(n,\mu)\Big|_{TA} =& \frac{\eta_\mu(n)}{2a}\Big[U_\mu(n)\Big((D^\dagger\chi)(n+\hat{\mu})\chi^\dagger(n)e^{a\mu\delta_{\mu,4}} \\
&-\chi(n+\hat{\mu})(D^\dagger\chi)^\dagger(n)e^{-a\mu\delta_{\mu,4}}\Big) \\
&+\Big((D^\dagger\chi)(n)\chi^\dagger(n+\hat{\mu})e^{-a\mu\delta_{\mu,4}} \\
&-\chi(n)(D^\dagger\chi)^\dagger(n+\hat{\mu})e^{a\mu\delta_{\mu,4}}\Big)U_\mu^\dagger(n)\Big]\Big|_{TA}
\end{aligned}
$$

with $\chi = (DD^\dagger)^{-1}\phi$

$$\begin{aligned} F_\mu &= U_\mu \phi^\dagger \frac{\partial \bar{r}(DD^\dagger)}{\partial U_\mu}\phi \\ &= \sum_{n=1}^{N} a_n U_\mu \phi^\dagger \frac{\partial (DD^\dagger + b_n)^{-1}}{\partial U_\mu}\phi \\ &= \sum_{n=1}^{N} a_n U_\mu \phi^\dagger (DD^\dagger + b_n)^{-1} (\frac{\partial D}{\partial U_\mu} D^\dagger + D \frac{\partial D^\dagger}{\partial U_\mu})(DD^\dagger + b_n)^{-1}\phi \end{aligned}$$

$$(DD^\dagger + b_n)\chi_n = \phi$$

$$\Lambda' = \left\{ y = (y_1, y_2, y_3, y_4) \ \middle| \ y_\mu = 0, 1, 2, ..., \frac{N_\mu}{2} - 1 \right\} \ , \quad s_\mu = 0, 1$$

$$\Gamma_s = \gamma_1^{s_1} \gamma_2^{s_2} \gamma_3^{s_3} \gamma_4^{s_4}$$

$$q(y)_{\alpha a} = \frac{1}{8} \sum_s \Gamma_{s,\alpha a} \ \chi(2y + s)$$

$$\bar{q}(y)_{a\alpha} = \frac{1}{8} \sum_s \bar{\chi}(2y + s) \ \Gamma^\dagger_{s,a\alpha}$$

$$S_F[q, \bar{q}] = b^4 \sum_y \bar{q}(y) \left\{ m \, (\mathbb{I} \otimes \mathbb{I}) + \sum_\mu \left[(\gamma_\mu \otimes \mathbb{I}) \, \nabla_\mu - \frac{b}{2} \, (\gamma_5 \otimes t_5 t_\mu) \ \Delta_\mu \right] \right\} q(y)$$

$$\delta_{ij} = \langle \chi_i \chi_j \rangle \simeq \frac{1}{K} \sum_{k=1}^{K} \chi_i^{(k)} \chi_j^{(k)*}$$

$$\mathrm{tr}\left[D^{-1}\right] = \sum_{i,j} \left(D^{-1}\right)_{ij} \delta_{ij} \approx \frac{1}{K} \sum_{k=1}^{K} \chi^{(k)\dagger} D^{-1} \chi^{(k)}$$