Gauge covariance of the Schwinger-Dyson equations for QED propagators in Minkowski space

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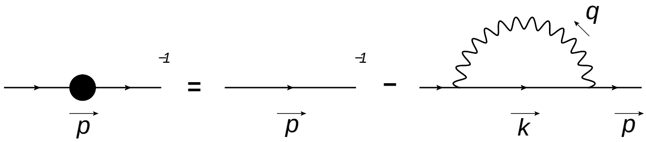
Table of Contents



$$S_F^{-1}(p) = \not{p} - m + \not{p}\frac{\alpha\xi}{4\pi}\left\{\tilde{C} + \left(1+\frac{m^2}{p^2}\right)\left[1+\left(1-\frac{m^2}{p^2}\right)\ln\frac{m^2}{m^2-p^2}\right]\right\}$$
$$- m\left\{\frac{\alpha(\xi+3)}{4\pi}\left[\tilde{C}+\frac{4}{3}+\left(1-\frac{m^2}{p^2}\right)\ln\frac{m^2}{m^2-p^2}\right]+\frac{\alpha\xi}{6\pi}\right\}, \quad (1)$$

where

$$\tilde{C} = \frac{1}{\epsilon} - \gamma_E + \ln\frac{4\pi\mu^2}{m^2}, \quad (2)$$

and $d = 4 - 2\epsilon$.

QCD in the confinement region ($\alpha_s \sim 1$), perturbation expansions fail.

$\Rightarrow$ nonperturbative approaches

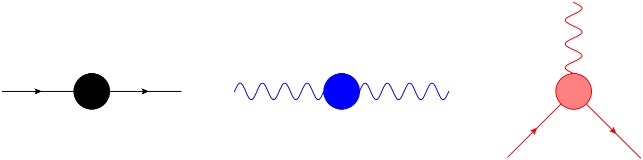
Strongly coupled QED as modeling of QCD. They share common features:

- gauge covariance,
- gauge invariance,
- renormalizability.

Keeping $d = 4 - 2\epsilon$ explicit:

super-renormalizable $QED_{2+1} \longrightarrow$ renormalizable QED_{3+1}.

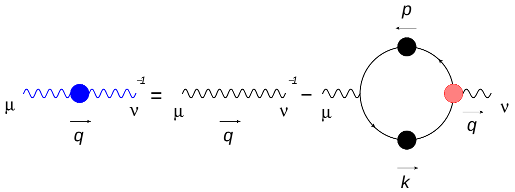
Primitive divergent diagrams $S_F(p)$, $D^{\mu\nu}(q)$, $\Gamma^{\mu}(k, p)$:



SDE for the photon propagator

The photon propagator depends on the gauge fixing conditions (ξ).

$$D_{\mu\nu}(q;\xi) = \Delta_{\mu\nu}(q) + \xi \frac{q^\mu q^\nu}{q^4 + i\varepsilon}, \quad \Delta_{\mu\nu}(q) = \frac{G(q^2)}{q^2 + i\varepsilon}\left(g_{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right). \qquad (3)$$



- $q = k - p, \quad \int d\underline{k} \equiv \int d^d k/(2\pi)^d$.
- Physical observables are gauge independent, while theory (Green's functions) depends on the gauge.
- One-loop integral only

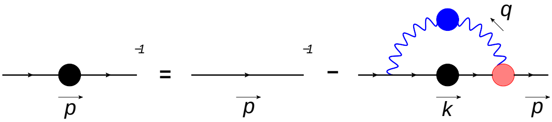
$$\frac{1}{G(q^2)}(q^2 g^{\mu\nu} - q^\mu q^\nu) = (q^2 g^{\mu\nu} - q^\mu q^\nu)$$
$$- ie^2 \mathrm{Tr} \int d\underline{k}\gamma^\nu S_F(k;\xi)\Gamma^\mu(k,p;\xi)S_F(p;\xi). \qquad (4)$$

SDE for the QED fermion propagator

The momentum space fermion propagator written as Schwinger functions

$$S_F(p;\xi) = \frac{\mathcal{F}(p^2;\xi)}{\not{p} - \mathcal{M}(p^2;\xi)} = \frac{1}{A(p^2)\not{p} + B(p^2)}. \quad (5)$$

The SDE for the fermion propagator



$$S_F^{-1}(p;\xi) = (\not{p} - m_B) + ie^2 \int d\underline{k}\gamma^\nu S_F(k;\xi)\Gamma^\mu(k,p;\xi)D_{\mu\nu}(q;\xi). \quad (6)$$

A truncation scheme is required. From the Ward-Green-Takahashi identity: $\Gamma^\mu(k,p) = \Gamma^\mu_L(k,p) + \Gamma^\mu_T(k,p)$. The longitudinal part $\Gamma^\mu_L(k,p)$ is given by the Ball-Chiu vertex [Phys.Rev.D,22:2542(1980)]. Transverse vertices are constructed to respect, renormalizability, gauge covariance, analytic structures.

Bethe-Salpeter equation, Faddeev equation

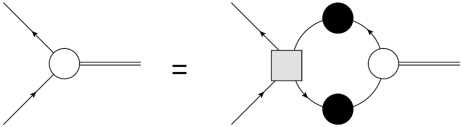


Figure 1 : Bethe-Salpeter equation

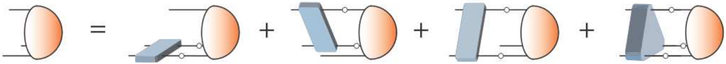


Figure 2 : Faddeev equation[1]

Here the fermion propagator works as input conditions for both equations.

[1] Eichmann, Fischer, and Sanchis-Alepuz, Phys.Rev. D94, 094033 (2016)

SDE for the quark propagators

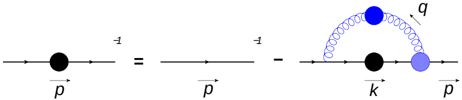


Figure 3 : SDE for the quark propagator in QCD

$$\Gamma^a_\mu(k,p)D^{\mu\nu}(q) \rightarrow \frac{\lambda^a}{2}\gamma_\mu D^{\mu\nu}(q)f(q^2)$$

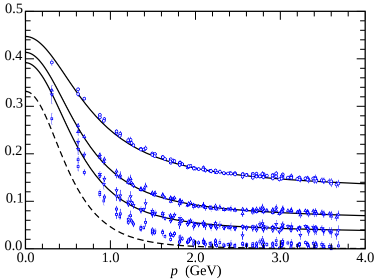


Figure 4 : The mass function of the fermion propagator from SDE (lines) and lattice QCD (data points) from [Bhagwat, Pichowsky, and C.D. Roberts, Phys.Rev. C68 015203 (2003)]

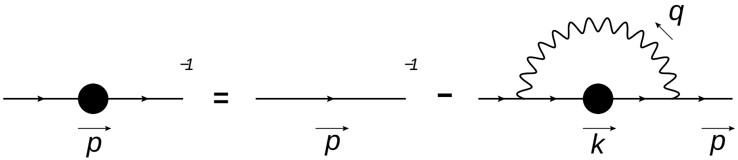
Violation of QED gauge covariance

Rainbow-Ladder trunction

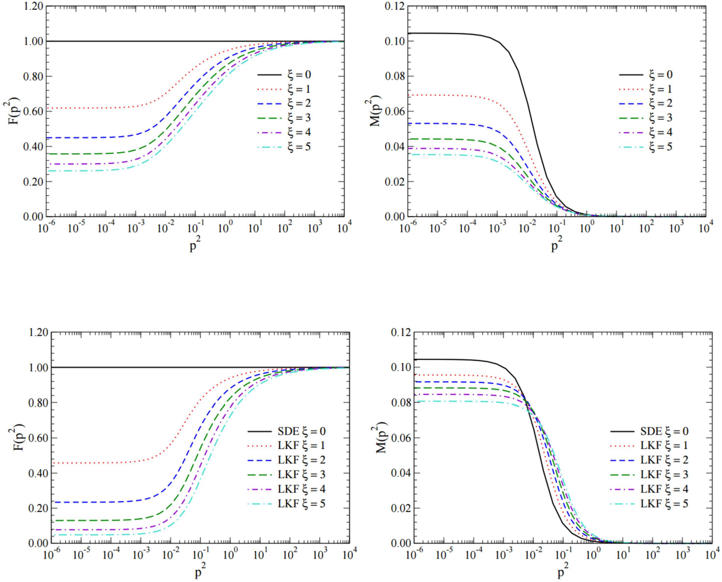
$$\Gamma^\mu(k,p;\xi) \to \gamma^\mu$$

Quenched approximation

$$G(q^2) \to 1$$

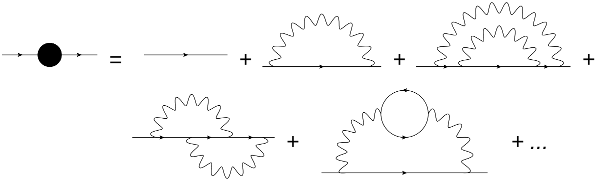


$$\frac{\not{p} - \mathcal{M}(p^2;\xi)}{\mathcal{F}(p^2;\xi)} = (\not{p} - m_B) + ie^2 \int d\underline{k}\gamma^\nu \frac{\mathcal{F}(k^2;\xi)}{\not{k} - \mathcal{M}(k^2;\xi)} \frac{\gamma^\mu}{q^2} \left[g^{\mu\nu} + (\xi - 1)\frac{q^\mu q^\nu}{q^2} \right] \tag{7}$$

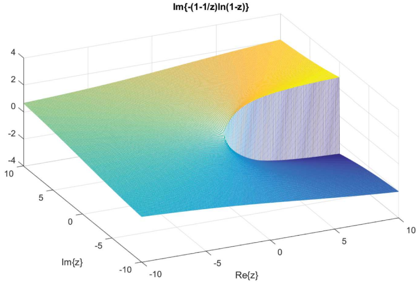


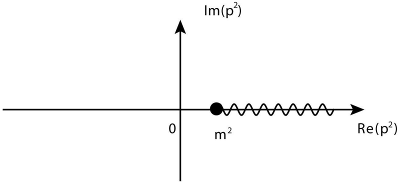
[R. Williams, Ph.D. Thesis (2007)] Results are presented in 3-dimensions.

Analytic structures of the fermion propagator



$$S_F(p) \sim \left(\prod_{i=1}^{n} \int dx_i \right) \frac{N(x_i, p)}{[p^2 - \Delta(x_i, \mu^2) + i\varepsilon]^\alpha}$$





Spectral representation for the fermion propagator

Dirac components

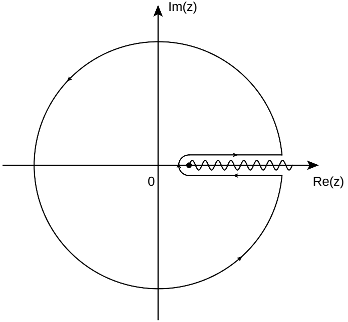
two spectral functions

$$S_F(p;\xi) = \not{p} S_1(p^2;\xi) + \mathbf{1} S_2(p^2;\xi)$$

$$S_j(p^2;\xi) = \int_{m^2}^{+\infty} ds \frac{\rho_j(s;\xi)}{p^2 - s + i\varepsilon}, \ (j = 1, 2)$$

For the fermion propagator with poles and branch cuts along the positive real axis,

$$\rho_j(s;\xi) = -\frac{1}{\pi}\mathrm{Im}\{S_j(s + i\varepsilon;\xi)\}.$$

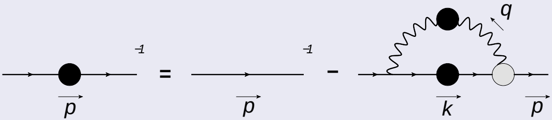


One-to-one correspondence:

$$\{S_j(p^2,\xi)\} \leftrightarrow \{\rho_j(s;\xi)\}$$

SDE for the fermion propagator spectral functions

The nonlinear equation

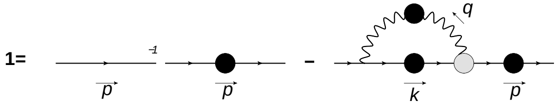


$$S_F^{-1}(p;\xi) = (\not p - m_B) + ie^2 \int d\underline{k}\gamma^\nu S_F(k;\xi)\Gamma^\mu(k,p;\xi)D_{\mu\nu}(q;\xi).$$

The Ward identity $Z_1 = Z_2$ indicates $S_F(k)\Gamma^\mu(k,p)S_F(p)$ is linear in $\rho_j(s)$. One example is the Gauge Technique [Delbourgo, Salam, and Strathdee (1964)],

$$S_F(k)\Gamma^\mu(k,p)S_F(p) = \int dW \frac{1}{\not k - W}\gamma^\mu \frac{1}{\not p - W}\rho(W), \tag{8}$$

where $\rho(W) = \mathrm{sign}(W)[\rho_2(W^2) + W\rho_1(W^2)]$.



$$\begin{cases} p^2 S_1(p^2) - mS_2(p^2) + \sigma_1(p^2) = 1 \\ S_2(p^2) - mS_1(p^2) + \sigma_2(p^2) = 0 \end{cases} \tag{9}$$

After taking the imaginary part of Eq. (9),

$$\begin{cases} s\rho_1(s;\xi) - m_B\rho_2(s;\xi) - \frac{1}{\pi}\mathrm{Im}\{\sigma_1(s+i\varepsilon;\xi)\} = 0 \\ \rho_2(s;\xi) - m_B\rho_1(s;\xi) - \frac{1}{\pi}\mathrm{Im}\{\sigma_2(s+i\varepsilon;\xi)\} = 0. \end{cases} \tag{10}$$

The distributions Ω_{ij}

encode all required linear operations on the spectral functions $\rho_j(s;\xi)$, depends on the ansatz.

$$\Omega(s,s') = -\frac{\delta}{\delta\rho(s')}\frac{1}{\pi}\mathrm{Im}\{\sigma(s+i\varepsilon)\}. \tag{11}$$

$$\begin{pmatrix} \rho_1(s;\xi) \\ \rho_2(s;\xi) \end{pmatrix} + \int ds' \begin{pmatrix} \Omega_{11}(s,s';\xi) & \Omega_{12}(s,s';\xi) \\ \Omega_{21}(s,s';\xi) & \Omega_{22}(s,s';\xi) \end{pmatrix} \begin{pmatrix} \rho_1(s';\xi) \\ \rho_2(s';\xi) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{12}$$

The Gauge Technique in the quenched approximation, close to 4 dimensions

The loop integral that defines Ω becomes

$$\sigma_1(p^2) + \not{p}\sigma_2(p^2) = ie^2 \int d\underline{k} \int dW\, \rho(W)\gamma^\nu \frac{1}{\not{k} - W}\gamma^\mu \frac{1}{\not{p} - W} D_{\mu\nu}(q). \qquad (13)$$

$$\begin{aligned}
\Omega_{11}(s, s'; \xi) &= -\frac{3\alpha}{4\pi}\Bigg\{ \left(\frac{1}{\epsilon} - \gamma_E + \ln 4\pi + \frac{4}{3} + \ln\frac{\mu^2}{s}\right) \\
&\quad \times \delta(s - s') - \frac{s'}{s^2}\theta(s - s')\Bigg\} - \frac{\alpha\xi}{4\pi}\frac{1}{s}\theta(s - s'), \\
\Omega_{12}(s, s'; \xi) &= -\frac{m_B}{s}\delta(s - s'), \quad \Omega_{21}(s, s'; \xi) = -m_B\delta(s - s'), \\
\Omega_{22}(s, s'; \xi) &= -\frac{3\alpha}{4\pi}\Bigg\{ \left(\frac{1}{\epsilon} - \gamma_E + \ln 4\pi + \frac{4}{3} + \ln\frac{\mu^2}{s}\right) \\
&\quad \times \delta(s - s') - \frac{1}{s}\theta(s - s')\Bigg\} - \frac{\alpha\xi}{4\pi}\frac{s'}{s^2}\theta(s - s'), \qquad (14)
\end{aligned}$$

The Landau gauge solution

With on-shell renormalization conditions,

$$\rho_1(s) = \delta(s - m^2) + r_1(s), \quad \rho_2(s) = m\,\delta(s - m^2) + r_2(s). \tag{15}$$

In the Landau gauge

$$\begin{cases} \rho_1(s;0) = \delta(s-m^2) + \dfrac{2a\,\theta(s-m^2)}{(a+1)s}\Big\{1 + \dfrac{a^2}{(2a+1)} \\ \qquad\qquad \times\, {}_2F_1\left(a+1, a+1; 2a+2; -\dfrac{s-m^2}{m^2}\right)\Big\} \\ \rho_2(s;0) = m\,\delta(s-m^2) - \dfrac{2a^2\,\theta(s-m^2)}{(2a+1)m}\, {}_2F_1\left(a+1, a+2; 2a+2; -\dfrac{s-m^2}{m^2}\right), \end{cases} \tag{16}$$

where $a = \dfrac{3\alpha}{(4\pi)(1-\alpha/\pi)}$.

LKFT for coordinate space propagators

LKFT in coordinate space for covariant gauges

$$S_F(x-y;\xi) = \exp\left\{ie^2(\xi-\xi')\left[M(x-y)-M(0)\right]\right\}S_F(x-y;\xi'), \tag{17}$$

$$M(x-y) = -\int d\underline{l}\,\frac{e^{-il\cdot(x-y)}}{l^4+i\varepsilon}. \tag{18}$$

$$\begin{aligned}&\frac{\partial}{\partial\xi}S_F(x-y;\xi)\\ &= ie^2[M(x-y)-M(0)]\exp\left\{ie^2(\xi-\xi')\left[M(x-y)-M(0)\right]\right\}S_F(x-y;\xi')\\ &= ie^2[M(x-y)-M(0)]S_F(x-y;\xi).\end{aligned} \tag{19}$$

After Fourier transform (effective one-loop integral)

$$\frac{\partial}{\partial\xi}S_F(p;\xi) = ie^2\int d\underline{l}\,\frac{1}{l^4+i\varepsilon}[S_F(p;\xi)-S_F(p-l;\xi)]. \tag{20}$$

LKFT for $\rho_j(s;\xi)$ in the differential form

$$\begin{cases} \frac{\partial}{\partial\xi} S_F(p;\xi) = ie^2 \int d\underline{l} \frac{1}{l^4 + i\varepsilon} [S_F(p;\xi) - S_F(p-l;\xi)] \\ S_j(p^2;\xi) = \int_{m^2}^{+\infty} ds \frac{\rho_j(s;\xi)}{p^2 - s + i\varepsilon} \end{cases}$$

$$\Rightarrow \quad \frac{\partial}{\partial\xi} \int ds \, \frac{\rho_j(s;\xi)}{p^2 - s + i\varepsilon} = \frac{-\alpha}{4\pi} \int ds \frac{\Xi_j(p^2, s)}{p^2 - s + i\varepsilon} \rho_j(s;\xi), \tag{21}$$

with functions $\Xi_j(p^2, s)$ given by

$$\begin{cases} \frac{\Xi_1(p^2,s)}{p^2 - s} = \frac{\Gamma(\epsilon)}{s} \left(\frac{4\pi\mu^2}{s}\right)^{\epsilon} \frac{-2}{(1-\epsilon)(2-\epsilon)} \; {}_2F_1(\epsilon+1, 3; 3-\epsilon; z) \\ \frac{\Xi_2(p^2,s)}{p^2 - s} = \frac{\Gamma(\epsilon)}{s} \left(\frac{4\pi\mu^2}{s}\right)^{\epsilon} \frac{-1}{1-\epsilon} \; {}_2F_1(\epsilon+1, 2; 2-\epsilon; z), \end{cases} \tag{22}$$

where $z = p^2/s$ and $d = 4 - 2\epsilon$.

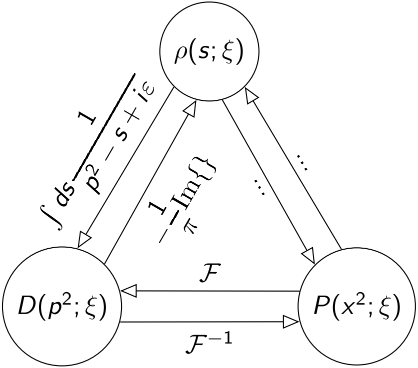
$$S_F(x-y;\xi) = \exp\left\{ie^2\xi\,[M(x-y)-M(0)]\,\right\} \times S_F(x-y;0).$$

LKFT in coordinate space: a continuous group parameterized by ξ.

With the spectral representation, in general,

$$\rho_j(s;\xi) = \int ds'\,\mathcal{K}_j(s,s';\xi)\,\rho_j(s';0),$$

where $\mathcal{K}_j(s,s';\xi)$ are distributions.



Isomorphic representations of the same group

$$\left\{\exp\left\{ie^2\xi\,[M(x-y)-M(0)]\,\right\}\right\} \Leftrightarrow \{\mathcal{K}(s,s';\xi)\} = \mathbf{K}$$

1. Closure $\int ds'\mathcal{K}(s,s';\xi)\mathcal{K}(s',s'';\xi')$ is also an element of $\mathbf{K}$;
2. Associativity

$$\int ds'\mathcal{K}(s,s';\xi)\int ds''\mathcal{K}(s',s'';\xi')\mathcal{K}(s'',s''';\xi'')$$
$$=\int ds''\left[\int ds'\mathcal{K}(s,s';\xi)\mathcal{K}(s',s'';\xi')\right]\mathcal{K}(s'',s''';\xi'');$$

3. Identity Element $\exists\ \mathcal{K}_I(s,s')\in\mathbf{K}$ such that

$$\int ds'\mathcal{K}_I(s,s')\mathcal{K}(s',s'';\xi)$$
$$=\int ds'\mathcal{K}(s,s';\xi)\mathcal{K}_I(s',s'')=\mathcal{K}(s,s'';\xi);$$

4. Inverse Element $\exists\ \mathcal{K}_{inv}(s,s';\xi)$ such that

$$\int ds'\,\mathcal{K}_{inv}(s,s';\xi)\,\mathcal{K}(s',s'';\xi)$$
$$=\int ds'\,\mathcal{K}(s,s';\xi)\,\mathcal{K}_{inv}(s',s'';\xi)\;=\;\mathcal{K}_I(s,s'').$$

Solution utilizing inverse elements

$$\frac{\partial}{\partial \xi} \int ds \, \frac{\rho_j(s;\xi)}{p^2 - s + i\varepsilon} = \frac{-\alpha}{4\pi} \int ds \frac{\Xi_j(p^2, s)}{p^2 - s + i\varepsilon} \rho_j(s;\xi)$$

$$\rho_j(s;\xi) = \int ds' \, \mathcal{K}_j(s, s';\xi) \, \rho_j(s';0).$$

$$\Rightarrow \quad \frac{\partial}{\partial \xi} \int ds \, \frac{\mathcal{K}_j(s, s';\xi)}{p^2 - s + i\varepsilon} = -\frac{\alpha}{4\pi} \int ds \, \frac{\Xi_j(p^2, s)}{p^2 - s + i\varepsilon} \, \mathcal{K}_j(s, s';\xi).$$

Define the distribution exponential as

$$\exp\left\{\lambda \Phi\right\} = \sum_{n=0}^{+\infty} \frac{\lambda^n}{n!} \Phi^n = \delta(s - s') + \lambda \Phi + \frac{\lambda^2}{2!} \Phi^2 + \frac{\lambda^3}{3!} \Phi^3 + \ldots,$$

$$\Phi^n(s, s') = \int ds'' \Phi(s, s'') \Phi^{n-1}(s'', s') \quad \text{for } n \geq 1, \quad \text{and } \Phi^0(s, s') = \delta(s - s').$$

$$\frac{\partial}{\partial \xi} \int ds \, \frac{\mathcal{K}_j(s, s'; \xi)}{p^2 - s + i\varepsilon} = -\frac{\alpha}{4\pi} \int ds \, \frac{\Xi_j(p^2, s)}{p^2 - s + i\varepsilon} \, \mathcal{K}_j(s, s'; \xi)$$

$$\Rightarrow \quad \int ds \int ds' \frac{1}{p^2 - s + i\varepsilon} \left[\frac{\partial}{\partial \xi} \mathcal{K}(s, s'; \xi)\right] \mathcal{K}(s', s''; -\xi)$$

$$= -\frac{\alpha}{4\pi} \int ds \int ds' \frac{\Xi(p^2, s)}{p^2 - s + i\varepsilon} \mathcal{K}(s, s'; \xi) \mathcal{K}(s', s''; -\xi),$$

$$\Rightarrow \quad \int ds \frac{1}{p^2 - s + i\varepsilon} \frac{\partial}{\partial \xi} \ln \, \mathcal{K}(s, s''; \xi) = -\frac{\alpha}{4\pi} \frac{\Xi(p^2, s'')}{p^2 - s'' + i\varepsilon}.$$

Therefore $\partial_\xi \ln \, \mathcal{K}(s, s''; \xi) = -\frac{\alpha}{4\pi} \Phi(s, s''), \quad \mathcal{K}_j = \exp\left(-\frac{\alpha\xi}{4\pi} \Phi_j\right).$

$$\boxed{\int ds \, \frac{\Phi_j(s, s')}{p^2 - s + i\varepsilon} \; = \; \frac{\Xi_j(p^2, s')}{p^2 - s' + i\varepsilon}}. \tag{23}$$

When $\Xi = 1$, Φ becomes a delta function. For other p^2 dependences, other linear operations are required.

Solving LKFT with fractional calculus

We have reduced LKFT for the fermion propagator spectral functions into solving distributions Φ_j from

$$\int ds \frac{\Phi_j(s,s')}{p^2 - s + i\varepsilon} = \frac{\Xi_j(p^2,s')}{p^2 - s' + i\varepsilon},$$

with

$$\frac{\Xi_1}{p^2 - s} = \frac{\Gamma(\epsilon)}{s}\left(\frac{4\pi\mu^2}{s}\right)^{\epsilon} \frac{(-2)\ {}_2F_1(\epsilon+1,3;3-\epsilon;z)}{(1-\epsilon)(2-\epsilon)}$$

$$\frac{\Xi_2}{p^2 - s} = \frac{\Gamma(\epsilon)}{s}\left(\frac{4\pi\mu^2}{s}\right)^{\epsilon} \frac{-1}{1-\epsilon}\ {}_2F_1(\epsilon+1,2;2-\epsilon;z).$$

Hypergeometric series

$$\frac{-s}{p^2 - s} = \frac{1}{1-z} = {}_2F_1(1,n;n;z) = \sum_{n=0}^{+\infty} z^n, \qquad {}_2F_1(a,b;c;z) = \sum_{n=0}^{+\infty} \frac{(a)_n (b)_n}{(c)_n\, n!} z^n,$$

with the Pochhammer symbol defined as $(a)_n = \Gamma(a+n)/\Gamma(n)$.

Differentiation Formulae (integer orders) [Abramowitz and Stegun]

$$(15.2.3)\quad \frac{d^n}{dz^n}[z^{a+n-1}F(a,b;c;z)] = (a)_n z^{a-1}F(a+n,b;c;z)$$
$$(15.2.4)\quad \frac{d^n}{dz^n}[z^{c-1}F(a,b;c;z)] = (c-n)_n z^{c-n-1}F(a,b;c-n;z)$$

Riemann-Liouville fractional calculus

$$I^\alpha f(z) = \frac{1}{\Gamma(\alpha)}\int_0^z dz'(z-z')^{\alpha-1}f(z'), \quad D^\alpha f(z) = \left(\frac{d}{dz}\right)^{\lceil\alpha\rceil} I^{\lceil\alpha\rceil-\alpha}f(z),$$

where $\lceil\alpha\rceil$ is the ceiling function. Specifically for $\alpha \in (0,1)$, $\lceil\alpha\rceil = 1$ and

$$D^\alpha f(z) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dz}\int_0^z dz'(z-z')^{-\alpha}f(z'). \tag{24}$$

Differentiation formulae in fractional orders

$$D^\alpha z^{a+\alpha-1}\ {}_2F_1(a,b;c;z) = (a)_\alpha z^{a-1}\ {}_2F_1(a+\alpha,b;c;z), \tag{25}$$
$$D^\alpha z^{c-1}\ {}_2F_1(a,b;c;z) = (c-\alpha)_\alpha z^{c-\alpha-1}\ {}_2F_1(a,b;c-\alpha;z), \tag{26}$$

Distribution identities for Φ_j

$$\int ds \frac{\Phi_j(s,s')}{p^2 - s + i\varepsilon} = \frac{\Xi_j(p^2,s')}{p^2 - s' + i\varepsilon}.$$

$$\frac{\Xi_1}{p^2 - s} = \frac{\Gamma(\epsilon)}{s}\left(\frac{4\pi\mu^2}{s}\right)^{\epsilon} \frac{(-2)\ {}_2F_1(\epsilon+1,3;3-\epsilon;z)}{(1-\epsilon)(2-\epsilon)}$$

$$\frac{\Xi_2}{p^2 - s} = \frac{\Gamma(\epsilon)}{s}\left(\frac{4\pi\mu^2}{s}\right)^{\epsilon} \frac{-1}{1-\epsilon}\ {}_2F_1(\epsilon+1,2;2-\epsilon;z).$$

At the operator level, define $\int ds'\Phi = \phi$. Then

$$z = p^2/s, \qquad \phi \frac{z}{z - 1 + i\varepsilon} = \frac{p^2\,\Xi}{p^2 - s + i\varepsilon}. \tag{27}$$

Applying Eqs. (25, 26) produces

$$\phi_n = \Gamma(\epsilon)\left(\frac{4\pi\mu^2}{p^2}\right)^{\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1+\epsilon)} z^{2\epsilon+2-n} D^{\epsilon} z^{n-1} D^{\epsilon} z^{\epsilon-1}, \tag{28}$$

with $j = 1, 2$ for $n = 3, 2$ respectively.

$j = 1, 2$ for $n = 3, 2$. With Φ_j solved,

$$\mathcal{K}_j z^\beta = \exp\left(-\frac{\alpha\xi}{4\pi}\phi_n\right) z^\beta = \exp\left(-\overline{\alpha}\overline{\phi}_n\right) z^\beta = \sum_{m=0}^{+\infty} \frac{(-\overline{\alpha})^m}{m!}\overline{\phi}_n^m z^\beta$$
$$= \sum_{m=0}^{+\infty} \frac{(-\overline{\alpha})^m}{m!} \frac{\Gamma(n+\beta+(m-1)\epsilon-1)\Gamma(\beta+m\epsilon)}{\Gamma(n+\beta-\epsilon-1)\Gamma(\beta)} z^{\beta+m\epsilon}, \qquad (29)$$

where $\overline{\alpha} \equiv \frac{\alpha\xi}{4\pi}\frac{\Gamma(\epsilon)\Gamma(1-\epsilon)}{\Gamma(1+\epsilon)}\left(\frac{4\pi\mu^2}{p^2}\right)^\epsilon$. Combined with the spectral representation for the fermion propagator, we obtain

the gauge dependence of fermion propagator in momentum space

$$S_j(p^2;\xi) = \int ds \int ds' \frac{1}{p^2 - s + i\varepsilon} \mathcal{K}_j(s, s';\xi)\, \rho_j(s';0)$$
$$= -\int ds \frac{1}{p^2} \sum_{\beta=1}^{+\infty}\sum_{m=0}^{+\infty} \frac{(-\overline{\alpha})^m}{m!} \frac{\Gamma(n+\beta+(m-1)\epsilon-1)\Gamma(\beta+m\epsilon)}{\Gamma(n+\beta-\epsilon-1)\Gamma(\beta)} z^{\beta+m\epsilon}\rho_j(s;0). \qquad (30)$$

Example 1: $d = 3$

$$\begin{cases} \mathcal{K}_1(s,s';\xi) = \dfrac{\sqrt{s'}}{\sqrt{s'}+\dfrac{\alpha\mu\xi}{2}}\,\delta\left(s-\left(\sqrt{s'}+\dfrac{\alpha\mu\xi}{2}\right)^2\right) \\ \qquad\qquad\qquad + \dfrac{\alpha\mu\xi}{4s^{3/2}}\theta\left(s-\left(\sqrt{s'}+\dfrac{\alpha\mu\xi}{2}\right)^2\right), \\ \mathcal{K}_2(s,s';\xi) = \delta\left(s-\left(\sqrt{s'}+\alpha\mu\xi/2\right)^2\right). \end{cases} \tag{31}$$

Example 2: $d = 4-2\epsilon,\ \epsilon \to 0$

$$\mathcal{K}_j(\xi) = \left(\frac{\mu^2 z}{p^2}\right)^{-\nu} \exp\left\{-\nu\left[\frac{1}{\epsilon}+\gamma_E+\ln 4\pi+\mathcal{O}(\epsilon^1)\right]\right\} z^{2-n} I^{\nu} z^{n-1-\nu} I^{\nu} z^{-\nu-1}, \tag{32}$$

where $\nu = \alpha\xi/(4\pi)$.

Gauge covariance of Ω

Substituting $\rho_j(\xi) = \mathcal{K}_j(\xi)\rho_j(0)$ into

$$\begin{pmatrix} \rho_1(\xi) \\ \rho_2(\xi) \end{pmatrix} + \begin{pmatrix} \Omega_{11}(\xi) & \Omega_{12}(\xi) \\ \Omega_{21}(\xi) & \Omega_{22}(\xi) \end{pmatrix} \begin{pmatrix} \rho_1(\xi) \\ \rho_2(\xi) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{33}$$

gives

$$\begin{pmatrix} \rho_1(0) \\ \rho_2(0) \end{pmatrix} + \begin{pmatrix} \mathcal{K}_1(-\xi) & \\ & \mathcal{K}_2(-\xi) \end{pmatrix} \begin{pmatrix} \Omega_{11}(\xi) & \Omega_{12}(\xi) \\ \Omega_{21}(\xi) & \Omega_{22}(\xi) \end{pmatrix} \begin{pmatrix} \mathcal{K}_1(\xi) & \\ & \mathcal{K}_2(\xi) \end{pmatrix} \begin{pmatrix} \rho_1(0) \\ \rho_2(0) \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{34}$$

Meanwhile, Eq. (33) in the Landau gauge,

$$\begin{pmatrix} \rho_1(0) \\ \rho_2(0) \end{pmatrix} + \begin{pmatrix} \Omega_{11}(0) & \Omega_{12}(0) \\ \Omega_{21}(0) & \Omega_{22}(0) \end{pmatrix} \begin{pmatrix} \rho_1(0) \\ \rho_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{35}$$

Comparing Eq. (34) with Eq. (35), we obtain

the consistency requirement on Ω_{ij} from the gauge covariance

$$\begin{pmatrix} \Omega_{11}(0) & \Omega_{12}(0) \\ \Omega_{21}(0) & \Omega_{22}(0) \end{pmatrix} = \begin{pmatrix} \mathcal{K}_1(-\xi) & \\ & \mathcal{K}_2(-\xi) \end{pmatrix} \begin{pmatrix} \Omega_{11}(\xi) & \Omega_{12}(\xi) \\ \Omega_{21}(\xi) & \Omega_{22}(\xi) \end{pmatrix} \begin{pmatrix} \mathcal{K}_1(\xi) & \\ & \mathcal{K}_2(\xi) \end{pmatrix}. \tag{36}$$

Meanwhile, substituting Eq. (36) back into Eq. (35) gives

$$\begin{pmatrix} \mathcal{K}_1(\xi)\rho_1(0) \\ \mathcal{K}_2(\xi)\rho_2(0) \end{pmatrix} + \begin{pmatrix} \Omega_{11}(\xi) & \Omega_{12}(\xi) \\ \Omega_{21}(\xi) & \Omega_{22}(\xi) \end{pmatrix} \begin{pmatrix} \mathcal{K}_1(\xi)\rho_1(0) \\ \mathcal{K}_2(\xi)\rho_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{37}$$

as the equations for $\mathcal{K}_j(\xi)\rho_j(0)$, which is identical to Eq. (33). Therefore $\rho_j(\xi) = \mathcal{K}_j(\xi)\rho_j(0)$.
Eq. (36) is the necessary and sufficient condition for the solutions of fermion propagator SDE to be consistent with LKFT.

When an ansatz is expected to be valid in one gauge

The consistency requirement is valid in the neighborhood of ξ.

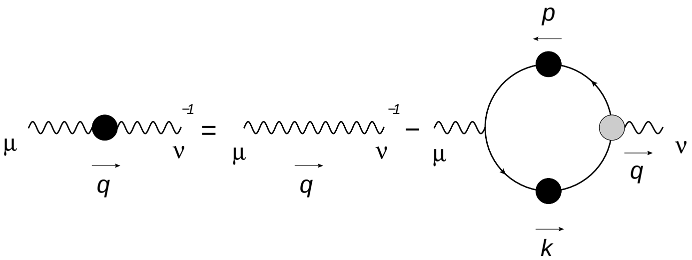
The differential version of Eq. (36),

$$\frac{\partial}{\partial\xi}\begin{pmatrix}\Omega_{11}^{\xi} & \Omega_{12}^{\xi}\\ \Omega_{21}^{\xi} & \Omega_{22}^{\xi}\end{pmatrix} = \frac{\alpha}{4\pi}\left[\begin{pmatrix}\Omega_{11}^{\xi} & \Omega_{12}^{\xi}\\ \Omega_{21}^{\xi} & \Omega_{22}^{\xi}\end{pmatrix}, \begin{pmatrix}\Phi_1 & \\ & \Phi_2\end{pmatrix}\right], \tag{38}$$

is expected to hold in that gauge.

For LKFT, the Landau gauge is not special. Shifts in ξ do not modify the LKFT. While from the renormalization point of view, the Landau gauge may be the simplest.

Gauge invariance of the vacuum polarization



$$D_{\mu\nu}(q;\xi) = \frac{G(q^2)}{q^2 + i\varepsilon}\left(g_{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) + \xi\frac{q^\mu q^\nu}{q^4 + i\varepsilon}, \quad \frac{1}{G(q^2)} = 1 + \Pi(q^2). \tag{39}$$

Because $S_F(k)\Gamma^\mu(k,p)S_F(p)$ is linear in ρ_j,

$$\Pi(q^2) = \int ds\,(\Omega_1^\gamma(q^2,s;\xi),\ \Omega_2^\gamma(q^2,s;\xi))\begin{pmatrix}\rho_1(s;\xi)\\ \rho_2(s;\xi)\end{pmatrix}. \tag{40}$$

The ξ independence of $\Pi(q^2)$ specifies

$$\boxed{\Omega_j^\gamma(q^2,s;\xi) = \int ds'\,\Omega_j^\gamma(q^2,s';0)\,\exp\left[\frac{\alpha\xi}{4\pi}\Phi_j(s',s)\right]}. \tag{41}$$

Summary

1. Analytic structure of the fermion propagator, the spectral representation
2. Gauge covariance of the fermion propagator in momentum space
3. Consistency requirements on truncation schemes

Future perspective

1. The construction of a gauge covariant ansatz to meet Eqs. (36, 41)

References

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Example: Explicit calculation shows that in three dimensions, for the Diarc scalar component of the LKFT,

$$\lim_{\epsilon\to 1/2} \Xi_2(p^2, s) = -\frac{4\pi}{z-1}\sqrt{\frac{\mu^2}{s}}. \tag{42}$$

In this case, for $\Phi_2(s, s')$

$$\int ds \frac{\Phi_2(s, s')}{p^2 - s + i\varepsilon} = -\frac{4\pi\mu\sqrt{s'}}{(p^2 - s')^2}. \tag{43}$$

Then with $\phi_2 = \int ds'\Phi$,

$$\phi_2 = -2\pi\mu \frac{d}{ds^{1/2}}. \tag{44}$$

We then have

$$\mathcal{K}_2 = \exp\left(-\frac{\alpha\xi}{4\pi}\phi_2\right) = \exp\left(\frac{\alpha\xi\mu}{2}\frac{d}{ds^{1/2}}\right). \tag{45}$$

Consequently, the gauge dependence of $\rho_2(s;\xi)$ is given by

$$\rho_2(s;\xi) = \int ds' \left(1 + \frac{\alpha\mu\xi}{2\sqrt{s'}}\right)^{-1} \delta\left(s' - \left(\sqrt{s} - \frac{\alpha\mu\xi}{2}\right)^2\right) \rho_2(s';0). \tag{46}$$

Example: the Gauge Technique in four dimensions

$$\Omega_{11}(\xi) = -\frac{3\alpha}{4\pi}\left[\tilde{C} + 4/3 + \ln(z) - z^{-1}I\right] - \frac{\alpha\xi}{4\pi}Iz^{-1},$$
$$\Omega_{12} = -\frac{m_B}{p^2}z,$$
$$\Omega_{21} = -m_B,$$
$$\Omega_{22}(\xi) = -\frac{3\alpha}{4\pi}\left[\tilde{C} + 4/3 + \ln(z) - Iz^{-1}\right] - \frac{\alpha\xi}{4\pi}z^{-1}I, \quad (47)$$

where $\tilde{C} = 1/\epsilon - \gamma_E + \ln(4\pi\mu^2/p^2)$. Therefore

$$z^{\beta}\Omega_{11}(\xi) = \left\{ -\frac{3\alpha}{4\pi}\left[\tilde{C} + 4/3 - \frac{1}{\beta+1} + \ln z\right] - \frac{\nu}{\beta}\right\}z^{\beta}, \quad (48)$$
$$z^{\beta}\Omega_{12}(\xi) = -\frac{m_B}{p^2}z^{\beta+1}, \quad (49)$$
$$z^{\beta}\Omega_{21}(\xi) = -m_B z^{\beta}, \quad (50)$$
$$z^{\beta}\Omega_{22}(\xi) = \left\{ -\frac{3\alpha}{4\pi}\left[\tilde{C} + 4/3 - \frac{1}{\beta} + \ln z\right] - \frac{\nu}{\beta+1}\right\}z^{\beta}. \quad (51)$$

Example: the Gauge Technique in 4D (continued)

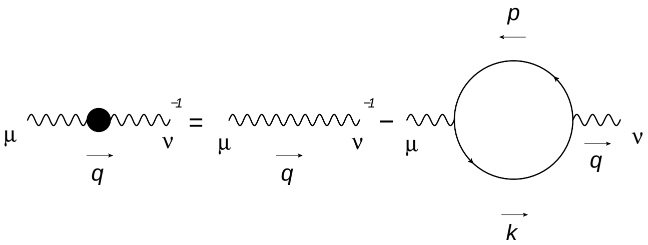
However

$$z^{\beta}\mathcal{K}_1(\xi)\Omega_{11}(0)\mathcal{K}_1(-\xi)$$
$$= -\frac{3\alpha}{4\pi}\Big\{\tilde{C} + 4/3 - \frac{1}{\beta-\nu+1} + \psi(\beta) - \psi(\beta-\nu) +$$
$$\psi(\beta+2) - \psi(\beta+2-\nu) + \ln z\Big\} z^{\beta}, \quad (52)$$

$$z^{\beta}\mathcal{K}_1(\xi)\Omega_{12}(0)\mathcal{K}_2(-\xi) = -\frac{m_B}{p^2}\frac{\beta}{\beta-\nu} z^{\beta+1}, \quad (53)$$

$$z^{\beta}\mathcal{K}_2(\xi)\Omega_{21}(0)\mathcal{K}_1(-\xi) = -m_B\frac{\beta+1}{\beta+1-\nu} z^{\beta}, \quad (54)$$

$$z^{\beta}\mathcal{K}_2(\xi)\Omega_{22}(0)\mathcal{K}_2(-\xi)$$
$$= -\frac{3\alpha}{4\pi}\Big\{\tilde{C} + 4/3 - \frac{1}{\beta-\nu} + \psi(\beta) - \psi(\beta-\nu) +$$
$$\psi(\beta+1) - \psi(\beta+1-\nu) + \ln z\Big\} z^{\beta}. \quad (55)$$



$$D^{-1}_{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu) + \frac{1}{\xi} q^\mu q^\nu - \frac{\alpha}{3\pi}(q^2 g^{\mu\nu} - q^\mu q^\nu)\Bigg\{ \tilde{C} + \frac{5}{3} + \frac{4m^2}{q^2} + \frac{2(q^2+2m^2)}{q^2}\sqrt{\frac{q^2-4m^2}{q^2}}\,\mathrm{arctanh}\left(\sqrt{\frac{q^2}{q^2-4m^2}}\right)\Bigg\}, \quad (56)$$

again with

$$\tilde{C} = \frac{1}{\epsilon} - \gamma_E + \ln\frac{4\pi\mu^2}{m^2},$$

and $d = 4 - 2\epsilon$.