Generalized Spin Representations

Robin Lautenbacher

Justus-Liebig University Giessen

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Overview

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- Treatment on the level of Kac-Moody algebras.

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- Basis of $\mathfrak{sl}(3, \mathbb{R})$:

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \right.$$
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- Any element of $\mathfrak{sl}(3, \mathbb{R})$ can be obtained as a linear combination of commutators of the generators:

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Kac-Moody algebras in general

A general Kac-Moody algebra $\mathfrak{g}(A)\,(\mathbb{K})$ has generators $e_1, \ldots, e_n$, $f_1, \ldots, f_n$ and $h_1, \ldots, h_n$ which are subject to the relations

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For $\mathfrak{sl}(n, \mathbb{R})$ these generators are $(a_{ii} = 2\,,\; a_{i,i+1} = -1\,,\; a_{ij} = 0)$

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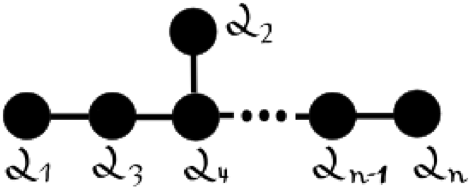

Generalized Dynkin diagram of E_n

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- Towards E_{10}: infinite dimensional but finitely many generators

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Generalized spin representations of $\mathfrak{k}(E_n)$ pt 2.

The representation matrices $\rho : \mathfrak{k}(E_n)(\mathbb{R}) \mapsto \mathbb{C}^{s\times s}$ given by

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- Given any simply laced $\mathfrak{k}(A)$ a set of matrices which satisfy the above relations define a representation of $\mathfrak{k}(A)$ (see [Köhl and others...])

Timeline

- The above extensions of were first performed in quantum gravity (see Thibault Damour, Axel Kleinschmidt and Hermann Nicolai, “Hidden symmetries and the fermionic sector of eleven-dimensional supergravity ”, arXiv:hep-th/0512163 10 February 2006.)

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- A mathematical treatment and their extension to arbitrary symmetrizable Kac-Moody algebras were done later (see Guntram Hainke, Ralf Köhl and Paul Levy, “Generalized Spin Representations”, Münster Journal of Mathematics (2015).)

Higher Spin Representations?

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- $\mathfrak{h}^*$ has Lorentzian signature
- Generators $\to$ Root system $\subset \mathfrak{h}^*$ $\to$ Tensor product rep. via roots

Roots and root spaces

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- Call $0 \neq \mathfrak{g}_\alpha := \{x \in \mathfrak{g}(A) \mid [h, x] = \alpha(h) x\ \forall\, h \in \mathfrak{h}\}$ for $\alpha \in \mathfrak{h}^*$ root space.

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- There is a decomposition of $\mathfrak{g}(A)$ as
$$\mathfrak{g}(A) = \left(\bigoplus_{0 \neq \alpha \in Q_+} \mathfrak{g}_{-\alpha} \right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{0 \neq \alpha \in Q_+} \mathfrak{g}_{\alpha} \right) .$$
where $Q_+ = \sum_{i=1}^n \mathbb{Z}_{\geq 0} \alpha_i$.

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- For E_{10}, $\mathfrak{h}^*$ is ten-dimensional and the bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}^*$ has signature $(-, +, +, \ldots, +)$.

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- This defines a rep if

$$\begin{aligned} [X(\alpha), X(\beta)] &= 0 && \text{if } (\alpha|\beta) = 0 \\ \{X(\alpha), X(\beta)\} &= X(\alpha \pm \beta) && \text{if } (\alpha|\beta) = \mp 1. \end{aligned}$$

for all $\alpha, \beta \in \Lambda$.

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- One chooses $V = \mathfrak{h}^*$ or symmetric powers of $\mathfrak{h}^*$. For $V = \mathfrak{h}^*$,

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- Transformed as mixtures of $\frac{1}{2}-$ and $\frac{3}{2}-$spin w.r.t- $\mathfrak{so}(10)$.

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Further results and open questions

- As Paul Levy pointed out the $\frac{3}{2}-$ and $\frac{5}{2}-$representations take the form of natural reflection actions on $\mathfrak{h}^*$ resp. $Sym^2\left(\mathfrak{h}^*\right)$.

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- What is the ismomorphism type of these representations?

Thank you for your attention

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