Local Quantum Field Theory and Confinement in QCD

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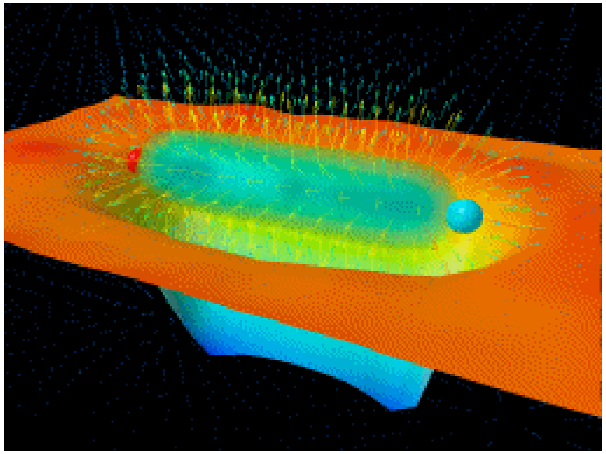
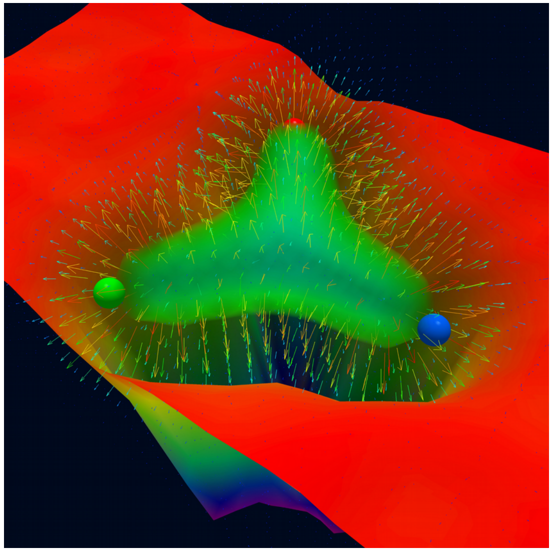
SLAC NATIONAL ACCELERATOR LABORATORY

Outline

1. LQFT: an axiomatic approach to QFT
2. Consequences of LQFT
3. Confinement and the cluster decomposition property
4. Summary and outlook



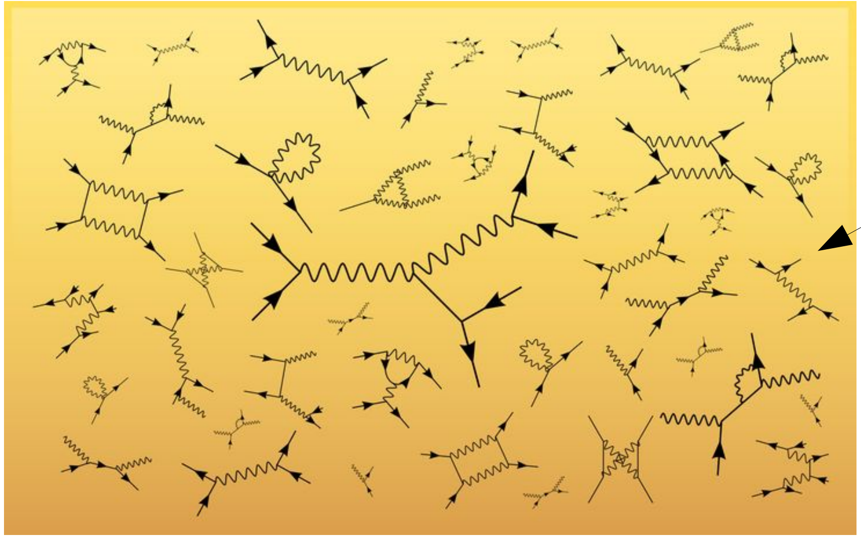
[The University of Adelaide (2015)]



[The University of Adelaide (2015)]

1. LQFT: an axiomatic approach to QFT

- Perturbation theory has proven to be an extremely successful tool for investigating problems in particle physics



But by definition this procedure is only valid in a *weakly interacting regime*

– Form factors?
– Parton distribution functions?
– Convergence of perturbative series?

- This emphasises the need for a non-perturbative approach!

→ **Local quantum field theory** (LQFT) is one such approach

1. LQFT: an axiomatic approach to QFT

- LQFT approaches are defined by a core set of axioms:

Axiom 1 (Hilbert space structure). *The states of the theory are rays in a Hilbert space* $\mathcal{H}$ *which possesses a continuous unitary representation* $U(a, \alpha)$ *of the Poincaré spinor group* $\overline{\mathscr{P}}^{\uparrow}_{+}$.

Axiom 2 (Spectral condition). *The spectrum of the energy-momentum operator* P^{μ} *is confined to the closed forward light cone* $\overline{V}^{+} = \{p^{\mu} \mid p^2 \geq 0,\, p^0 \geq 0\}$, *where* $U(a, 1) = e^{iP^{\mu}a_{\mu}}$.

Axiom 3 (Uniqueness of the vacuum). *There exists a unit state vector* $|0\rangle$ *(the vacuum state) which is a unique translationally invariant state in* $\mathcal{H}$.

Axiom 4 (Field operators). *The theory consists of fields* $\varphi^{(\kappa)}(x)$ *(of type* (κ)*) which have components* $\varphi^{(\kappa)}_{l}(x)$ *that are operator-valued tempered distributions in* $\mathcal{H}$, *and the vacuum state* $|0\rangle$ *is a cyclic vector for the fields.*

Axiom 5 (Relativistic covariance). *The fields* $\varphi^{(\kappa)}_{l}(x)$ *transform covariantly under the action of* $\overline{\mathscr{P}}^{\uparrow}_{+}$:

$$U(a, \alpha)\varphi^{(\kappa)}_{i}(x)U(a, \alpha)^{-1} = S^{(\kappa)}_{ij}(\alpha^{-1})\varphi^{(\kappa)}_{j}(\Lambda(\alpha)x + a)$$

where $S(\alpha)$ *is a finite dimensional matrix representation of the Lorentz spinor group* $\overline{\mathscr{L}}^{\uparrow}_{+}$, *and* $\Lambda(\alpha)$ *is the Lorentz transformation corresponding to* $\alpha \in \overline{\mathscr{L}}^{\uparrow}_{+}$.

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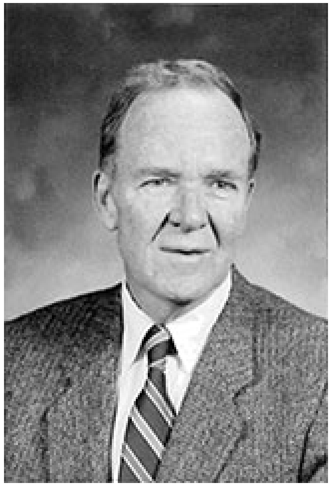
$$[\varphi^{(\kappa)}_{l}(f), \varphi^{(\kappa')}_{m}(g)]_{\pm} = \varphi^{(\kappa)}_{l}(f)\varphi^{(\kappa')}_{m}(g) \pm \varphi^{(\kappa')}_{m}(g)\varphi^{(\kappa)}_{l}(f) = 0$$

when applied to any state in $\mathcal{H}$, *for any fields* $\varphi^{(\kappa)}_{l}, \varphi^{(\kappa')}_{m}$.



A. Wightman

[R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and all that* (1964).]

R. Haag

[R. Haag, *Local Quantum Physics*, Springer-Verlag (1996).]

1. LQFT: an axiomatic approach to QFT

- The central idea with LQFT is that these axioms are <u>physically motivated</u>

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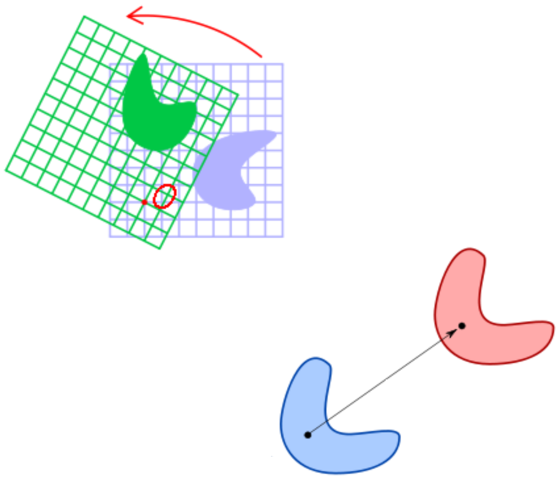
when applied to any state in $\mathcal{H}$, for any fields $\varphi^{(\kappa)}_{l}, \varphi^{(\kappa')}_{m}$.

"The theory is invariant under Poincaré transformations"



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when applied to any state in $\mathcal{H}$, for any fields $\varphi_l^{(\kappa)}, \varphi_m^{(\kappa')}$.

"Energy is bounded from below– the theory is stable"

$$H \geq 0$$

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"The vacuum state is unique and looks the same to all observers"

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"Quantum fields φ are distributions, not functions"

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1. LQFT: an axiomatic approach to QFT

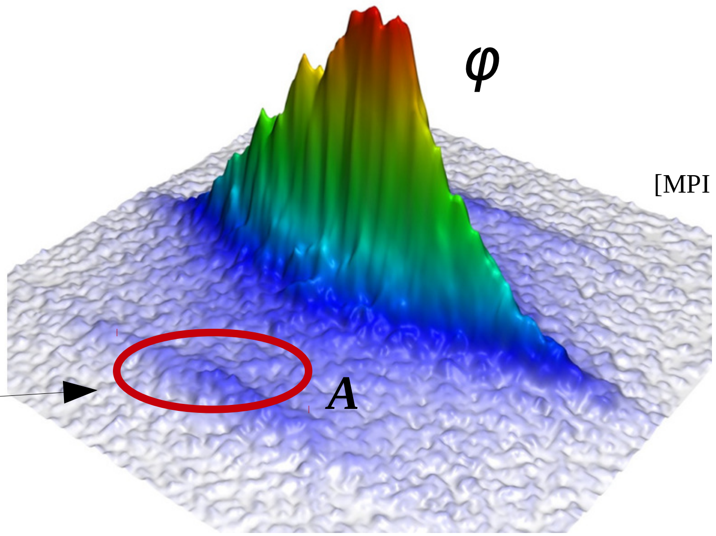
- Quantum fields $\boldsymbol{\varphi}(x)$ are distributions – *what difference does this make?*

→ This means that they cannot be evaluated at a single point (e.g. think of the Dirac delta $\delta(x)$ at x=0)

→ Need to 'average them out' over some spacetime region ***A***

$$\mathcal{M}_\varphi := \int_A d^4x\, \varphi(x) f(x)$$

Can think of this as the performance of a measurement M_φ in the region ***A*** where $f(x)$ is non-zero



[MPI Munich (2004)]

- But why? – Heisenberg's uncertainty principle! $\Delta x \Delta p \sim \frac{\hbar}{2}$

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"This connects the permitted physical states and the field degrees of freedom which define them"

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when applied to any state in $\mathcal{H}$, for any fields $\varphi^{(\kappa)}_{l}, \varphi^{(\kappa')}_{m}$.

"Measurements performed in the future cannot affect measurements performed in the past – causality!"

2. Consequences of LQFT

- Axioms 1-6 imply many structural QFT properties, including:

  – Correlation functions $\langle 0|\varphi_{l_1}^{(\kappa_1)}(x_1)\cdots\varphi_{l_n}^{(\kappa_n)}(x_n)|0\rangle$ are (tempered) distributions

  – ***The Reconstruction Theorem*** – a QFT which satisfies Axioms 1-6 can be uniquely reconstructed from knowledge of all the correlation functions

    → This result justifies why the vacuum expectation values of products of fields are of central importance in QFT!

  – Correlation functions can be analytically continued in a unique manner

    → In particular, these distributions are the boundary values of complex analytic functions

E.g. Free massive scalar field

$$\langle 0|\phi(x)\phi(y)|0\rangle = \lim_{\epsilon\to 0,\epsilon\in V^+}\frac{m^2}{4\pi^2}\frac{K_1(\sqrt{-m^2(\xi-i\epsilon)^2})}{\sqrt{-m^2(\xi-i\epsilon)^2}}$$

2. Consequences of LQFT

- Why is LQFT useful?

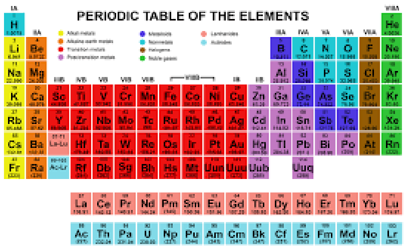
  → It has the potential to make profound non-perturbative predictions

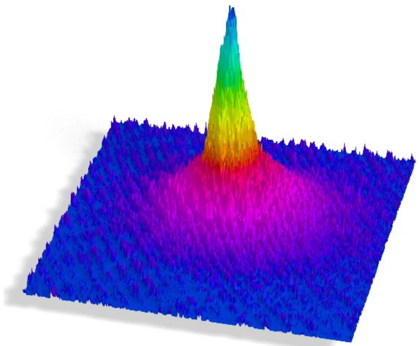
- Important examples of these include:

  → **Spin-statistics theorem**

  $'Boson' \leftrightarrow [\varphi(x),\varphi(y)]=0$

  $'Fermion' \leftrightarrow \{\psi(x),\psi(y)\}=0$

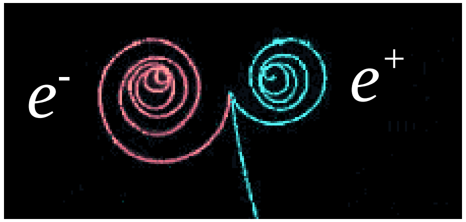




[The University of Toronto (2004)]

  → ***CPT* theorem**

  CPT is a symmetry of any QFT



  → **Free theory implies canonical quantisation**

  → **Connection of Minkowski and Euclidean QFTs ($t \to i\tau$)**

  → **Non-locality of charged states**

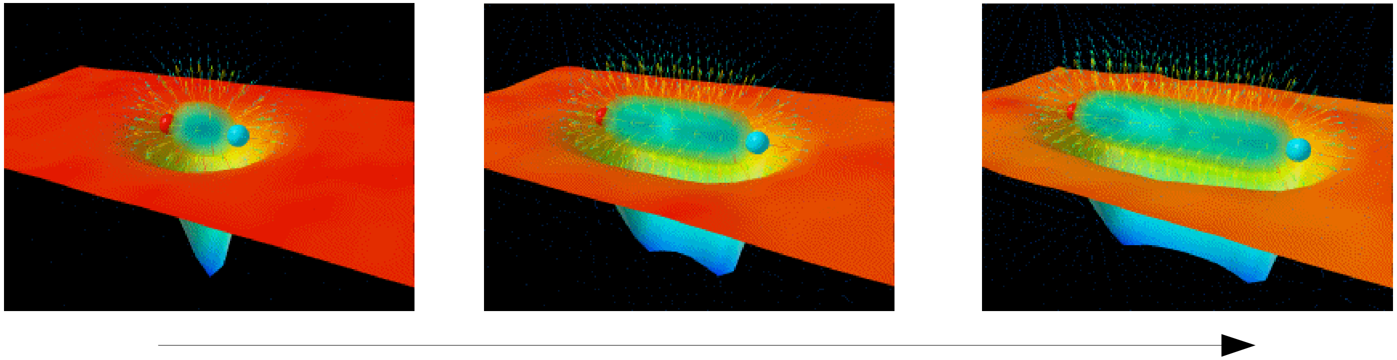
3. Confinement and the CDP

- Although QCD has been experimentally verified to extraordinary precision, there remains an important unresolved question: *why have coloured states never been experimentally observed?*

  → It is believed that these states must be **confined**

But how does confinement occur?

- One approach to understanding confinement is to analyse how the correlation strength between clusters of states depends on the distance between the clusters



Increasing separation

3. Confinement and the CDP

- For QFTs that satisfy the Wightman axioms one can prove that the correlation strength between clusters of fields always **decreases** with separation [Araki (1960), Araki, Hepp, Ruelle (1962)]

  → this is called the **cluster decomposition property** (CDP)

- Therefore, if QCD did indeed satisfy the Wightman axioms, one would be permitted to 'pull apart' coloured states

- However, it turns out that gauge theories <u>do not</u> satisfy the Wightman Axioms – *charged fields are non-local!*

- There are two approaches for defining a quantised gauge theory:

  (1) One preserves positivity of the Hilbert space, but loses locality (e.g. Coulomb gauge)

  (2) One preserves locality, but loses positivity (e.g. BRST quantised gauge theories)

[H. Araki, *Ann. Phys.* **11**, 260 (1960).]
[H. Araki, K. Hepp and D. Ruelle, *Helv. Phys. Acta* **35**, 164 (1962).]

3. Confinement and the CDP

- By choosing option (2), and preserving locality, one maintains many of the features satisfied by Wightman QFTs, but now the space of states $\mathcal{V}$ can contain negative norm states (e.g. ghosts)
- QFTs of this form satisfy a modified version of the Wightman axioms called the *Pseudo-Wightman* (PW) axioms [Strocchi (1978), Bogolubov et al. (1990)]
- After identifying the physical subspace of states $\mathcal{V}_{\rm phys} \subset \mathcal{V}$ the Hilbert space is defined by: $\mathcal{H} = \mathcal{V}_{\rm phys}/\mathcal{V}_0$
- In the specific case of BRST quantised gauge theories one defines the physical subspace via the subsidiary condition: $Q_B \mathcal{V}_{\rm phys} = 0$
- Given a QFT that satisfies the PW axioms an interesting question to ask is whether the CDP continues to always hold?

→ *Remarkably, it doesn't!*

[F. Strocchi, *Phys. Rev. D* **17**, 2010 (1978).]
[N. N. Bogolubov, A. A. Logunov and A. I. Oksak, General Principles of Quantum Field Theory, (1990).]

3. Confinement and the CDP

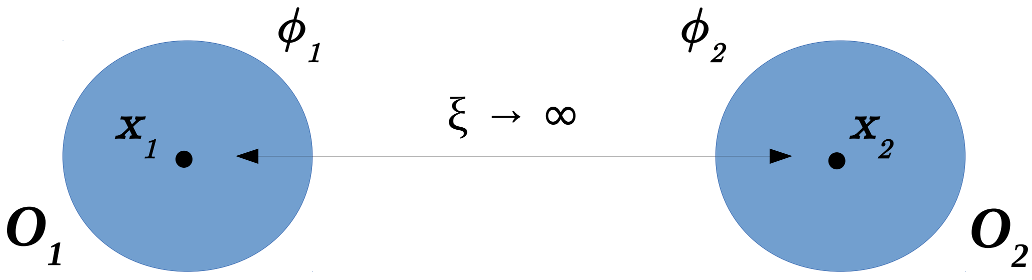
- For a QFT that satisfies the PW axioms, the behaviour of the correlation strength is described by the following theorem [Strocchi (1976)]:

Theorem (Cluster Decomposition).

$$\left|\langle 0|\mathcal{B}_1(x_1)\mathcal{B}_2(x_2)|0\rangle^T\right| \leq \begin{cases} C_{1,2}[\xi]^{2N-\frac{3}{2}} e^{-M[\xi]}\left(1+\frac{|\xi_0|}{[\xi]}\right), & \textit{with a mass gap } (0,M) \textit{ in } \mathcal{V} \\ \widetilde{C}_{1,2}[\xi]^{2N-2}\left(1+\frac{|\xi_0|}{[\xi]^2}\right), & \textit{without a mass gap in } \mathcal{V} \end{cases}$$

where: $\langle 0|\mathcal{B}_1(x_1)\mathcal{B}_2(x_2)|0\rangle^T = \langle 0|\mathcal{B}_1(x_1)\mathcal{B}_2(x_2)|0\rangle - \langle 0|\mathcal{B}_1(x_1)|0\rangle\langle 0|\mathcal{B}_2(x_2)|0\rangle$, $N \in \mathbb{Z}_{\geq 0}$, $\xi = x_1 - x_2$ *is large and space-like, and* $C_{1,2}, \widetilde{C}_{1,2}$ *are constants independent of* ξ *and* N.

$$\mathcal{B}_i(x_i) := \int_{\mathcal{O}_i} d^4y\, \phi_i(y) f(y - x_i), \quad f \in \mathcal{D}(\mathbb{R}^{1,3})$$



N*=0** → the correlation strength ***decreases with distance

***N*>0** → the behaviour depends on whether the space of states $\mathcal{V}$ has a mass gap or not

[F. Strocchi, *Phys. Lett.* B **62**, 60 (1976).]

3. Confinement and the CDP

- An important feature of this theorem is that, under certain conditions, the CDP can be **violated**
- A violation of the CDP between clusters of states implies that the correlations between these states are not damped, no matter how far they are separated

  → The measurement of one state therefore cannot be performed independently of the other, and this prevents the formation of physical (asymptotic) coloured states → **confinement**

- From the Cluster Decomposition Theorem one can see that whether or not the CDP is preserved depends crucially on whether the parameter N vanishes.

→ **Can one establish a condition for when N=0 ?**

3. Confinement and the CDP

- It turns out that one has the following necessary and sufficient condition [PL 1511.02780]:

Theorem. *Let* $\widehat{\mathcal{T}}^{T}_{(1,2)}(p) = \mathcal{F}\left[\langle 0|\mathcal{B}_1(x_1)\mathcal{B}_2(x_2)|0\rangle^{T}\right]$, *then:*

$$N = 0 \quad \Longleftrightarrow \quad \widehat{\mathcal{T}}^{T}_{(1,2)} \text{ defines a measure}$$

- Conversely, this means that if the Fourier transform of the correlator does not define a measure, then $N > 0$
- For a distribution $D(p)$ to define a measure this requires that the integral of $D(p)$ with any continuous function $f(p)$ (of compact support) must be well defined
- Example: (i) $D(p) = g(p)$ → *D defines a measure for g cont & bounded*

  (ii) $D(p) = \delta(p)$ → *D defines a measure*

  (iii) $D(p) = \delta'(p)$ → *D does not define a measure, since if* $f(p)$ *were not differentiable at* $p=0$, *then* $\int \delta'(p)f(p) = -f'(0)$ *would be ill defined*

3. Confinement and the CDP

- *But what effect does this condition have on the structure of correlators in general?*
- Using LQFT [Bogolubov et al. (1990)] the Fourier transform of any correlator $\widehat{T}(p) = \mathcal{F}\left[\langle 0|\phi_1(x_1)\phi_2(x_2)|0\rangle\right]$ can be written in the following form:

Lorentz covariance (Axiom 5) →

$$\widehat{T}_{(1,2)}(p) = \sum_{\alpha=1}^{\mathcal{N}} Q_\alpha(p)\,\widehat{T}_{\alpha(1,2)}(p)$$

Spectral condition (Axiom 2) →

$$\widehat{T}_\alpha(p) = P(\partial^2)\delta(p) + \int_0^\infty ds\,\theta(p^0)\delta(p^2-s)\rho_\alpha(s)$$

"Spectral density"

- <u>Example:</u> vector field correlator

$$\widehat{D}_{\mu\nu}(p) = \mathcal{F}\left[\langle 0|A_\mu(x)A_\nu(y)|0\rangle\right] = g_{\mu\nu}\,\widehat{D}_1(p) + p_\mu p_\nu\,\widehat{D}_2(p)$$

Two possible Lorentz covariant polynomial functions of p

[N. N. Bogolubov, A. A. Logunov and A. I. Oksak, General Principles of Quantum Field Theory, (1990).]

3. Confinement and the CDP

- These general relations demonstrate that the spectral densities $\rho_\alpha(s)$ are central to determining the structure of *any* correlator, and hence the value of N
- It is complicated to directly relate N and $\rho_\alpha(s)$, but it follows from the previous theorem and the definition of a measure that [PL 1511.02780]:

$$\rightarrow \textit{If } \rho_\alpha(s) \sim \delta(s-s_0) \ \textit{then } N=0$$

$$\rightarrow \textit{If } \rho_\alpha(s) \sim \delta'(s-s_1) \ \textit{then } N>0$$

- In order for the CDP to be violated for a (cluster) correlator in QCD one requires that:

  (1) the correlator in question has $N > 0$

  (2) the space of states $\mathcal{V}_{QCD}$ has no mass gap

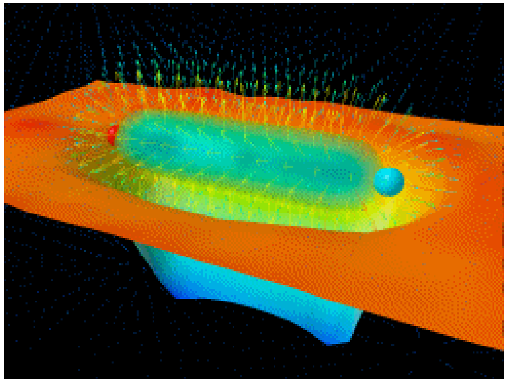
Note: this is the full (indefinite inner product) space of states

3. Confinement and the CDP

- If both conditions (1) and (2) are satisfied, then the correlation strength between clusters of fields ϕ_1 and ϕ_2 behaves like:

$$F^{\phi_1\phi_2}(r) \sim r^{2N-2}, \quad \text{for } r \to \infty$$

- In the case where ϕ_1 and ϕ_2 are quark fields one can effectively think of $F(r)$ as describing the "force" between the quarks in the two spacelike separated clusters O_1 and O_2

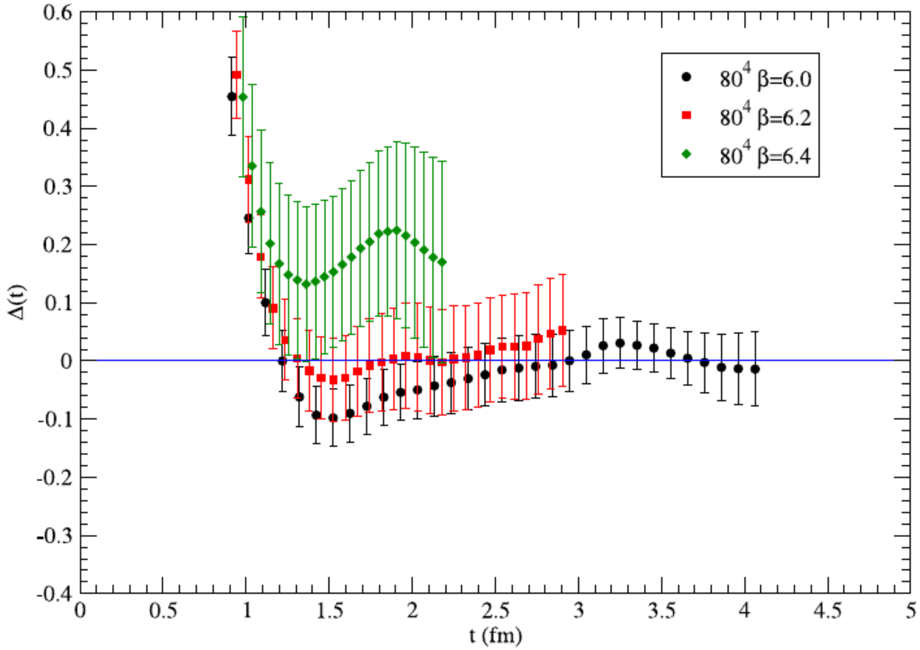


- But is there a way to test whether the CDP is violated in QCD?

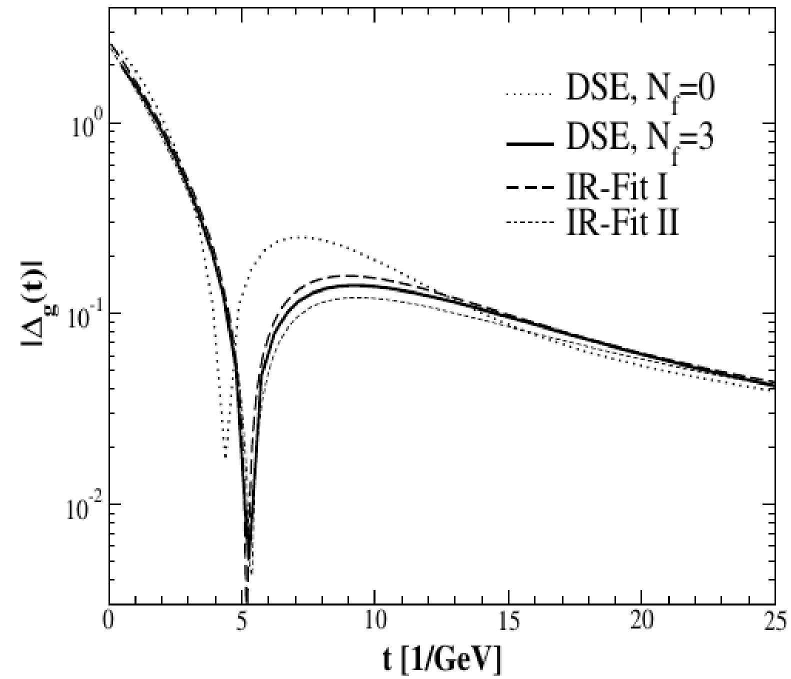
→ Yes, one can calculate the Schwinger function $\Delta_\alpha(t)$ for coloured correlators using non-perturbative techniques

$$\Delta_\alpha(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dp_0\, e^{ip_0 t} D_\alpha(p^2)|_{\mathbf{p}=0} = \int_0^{\infty} ds\, \rho_\alpha(s)\frac{e^{-\sqrt{s}\,t}}{2\sqrt{s}}$$

- There is evidence to suggest that $\Delta_\alpha(t)$ becomes *negative* at some value of t for both the quark and gluon spectral densities



[Oliveira, Dudal, Silva 1210.7794]



[Alkofer, Detmold, Fischer, Maris hep-ph/0309077]

- This behaviour is sometimes interpreted as evidence that $\rho_\alpha(s)$ is negative over some continuous range of s

→ *But this is not necessarily the case!*

3. Confinement and the CDP

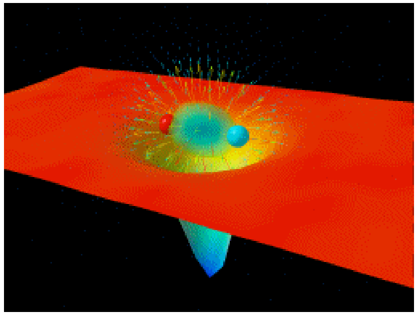
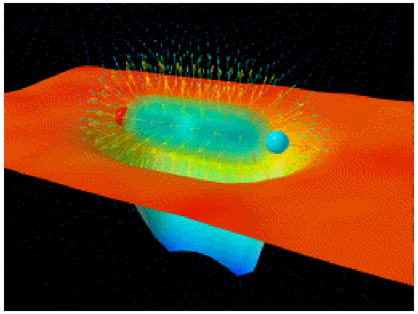
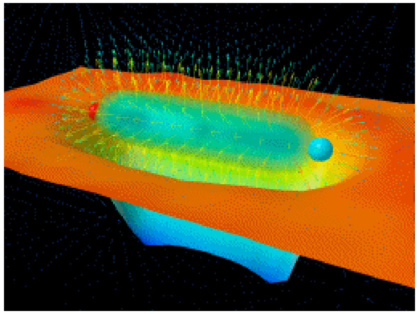
- The spectral densities $\rho_\alpha(s)$ are (tempered) distributions, not functions, and so they do not necessarily depend continuously on s (i.e. they may not be regular distributions)
- By including a singular non-measure term like $\delta'(s-s_1)$ in the spectral density, this can cause $\Delta_\alpha(t)$ to become negative
- In fact, a purely singular ansatz of the form: $\rho_\alpha(s) \sim A\delta(s-s_0) + B\delta'(s-s_1)$ (with $A>0$, $B<0$) can also reproduce the qualitative behaviour of $\Delta_\alpha(t)$
- Hence, $\Delta_\alpha(t)$ violating non-negativity can be interpreted as evidence that $\rho_\alpha(s)$ contains a non-measure component, and thus $N > 0$

→ *Suggestive that confinement occurs due a violation of the CDP*

Key point: confinement arising from a violation of the CDP depends crucially on whether $\rho_\alpha(s)$ defines a measure or not
→ $\rho_\alpha(s)$ being non-negative is sufficient but not necessary for $\rho_\alpha(s)$ to define a measure

<u>The confinement puzzle in QCD</u> → *how does confinement occur?*

- An increase in the correlation strength between coloured fields provides a mechanism by which coloured states can be confined
- Whether or not this occurs depends on the structure of the spectral densities $\rho_{\alpha}(s)$ of correlators involving coloured fields
- There is evidence from lattice QCD and Schwinger-Dyson calculations to suggest that this does indeed occur for clusters of quark and gluon fields

Outstanding questions

- To what extent are the calculations performed using non-perturbative techniques sensitive to the distributional nature of QFT quantities?
- Can non-measure defining distributions like $\delta'(s\text{-}a)$ be *directly* tested on the lattice or using Schwinger-Dyson?
- Is it possible to generalise the CDP confinement scenario to non locally quantised gauge theories (e.g. in Coulomb gauge)? What are the characteristic features?
- Are there quantities other than $\Delta_{\alpha}(t)$ that would be sensitive to non-measure contributions?

Backup

- The parameter *N* is characterised by the following theorem:

Theorem 2 (Bros-Epstein-Glaser). *Let $\mathscr{T} \in \mathcal{S}'(\mathbb{R}^{1,3})$ be a tempered distribution with support in $\overline{V}^+$. Then there exists a non-negative integer $N \in \mathbb{Z}_{\geq 0}$, and finite constant $C > 0$, such that:*

$$|\mathscr{T}(f)| \leq C \sum_{|\alpha| \leq N} \sup_{p \in \mathbb{R}^{1,3}} (1 + \|p\|)^N |D^\alpha f(p)|, \qquad \forall f \in \mathcal{S}(\mathbb{R}^{1,3})$$

where $\|p\|^2 = \sum_{\mu=0}^{3} |p_\mu|^2$, $D^\alpha = \frac{\partial^{|\alpha|}}{(\partial p_0)^{\alpha_0} \cdots (\partial p_3)^{\alpha_3}}$, *and:* $|\alpha| = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$

- In the context of the cluster decomposition theorem, *N* corresponds to the *order* of the distribution $\widehat{\mathcal{T}}^T_{(1,2)}(p)$
- In principle, if one knows the structure of $\widehat{\mathcal{T}}^T_{(1,2)}(p)$, then it is possible to put an upper bound on the value of *N*
- It is important to note that the *cluster decomposition theorem* refers to the space of states $\mathcal{V}_{QCD}$, and so it is possible for $\mathcal{H}_{\text{QCD}} := \mathcal{V}_{\text{QCD}}^{\text{phys}}/\mathcal{V}_{\text{QCD}}^0$ to have a mass gap, but for $\mathcal{V}_{QCD}$ to not

Backup

- In order to better understand confinement it is important to analyse the non-perturbative structure of the gluon propagator. After taking into account the constraints from the ETCRs and equations of motion, one has [PL 1702.02954]:

$$\widehat{D}_{\mu\nu}^{ab\,F}(p) = i\int_0^\infty \frac{ds}{2\pi}\,\frac{\left[g_{\mu\nu}\rho_1^{ab}(s) + p_\mu p_\nu \rho_2^{ab}(s)\right]}{p^2 - s + i\epsilon} + \sum_{n=0}^{N+1}\left[c_n^{ab}\, g_{\mu\nu}(\partial^2)^n + d_n^{ab}\partial_\mu\partial_\nu(\partial^2)^{n-1}\right]\delta(p)$$

By contrast to the photon propagator, the gluon propagator is permitted to contain explicit polynomials in derivatives of $\delta(p)$

- It is therefore possible that a violation of the CDP can arise from these non spectral density terms!
- This behaviour can perhaps be looked for in lattice calculations of the Schwinger function $\Delta_\alpha(t)$ since it would lead to additional t polynomial contributions