The Variational Approach to Gauge Field Theory

Markus Quandt

Institut für Theoretische Physik
Eberhard-Karls-Universität Tübingen

Justus-Liebig-Universität Gießen
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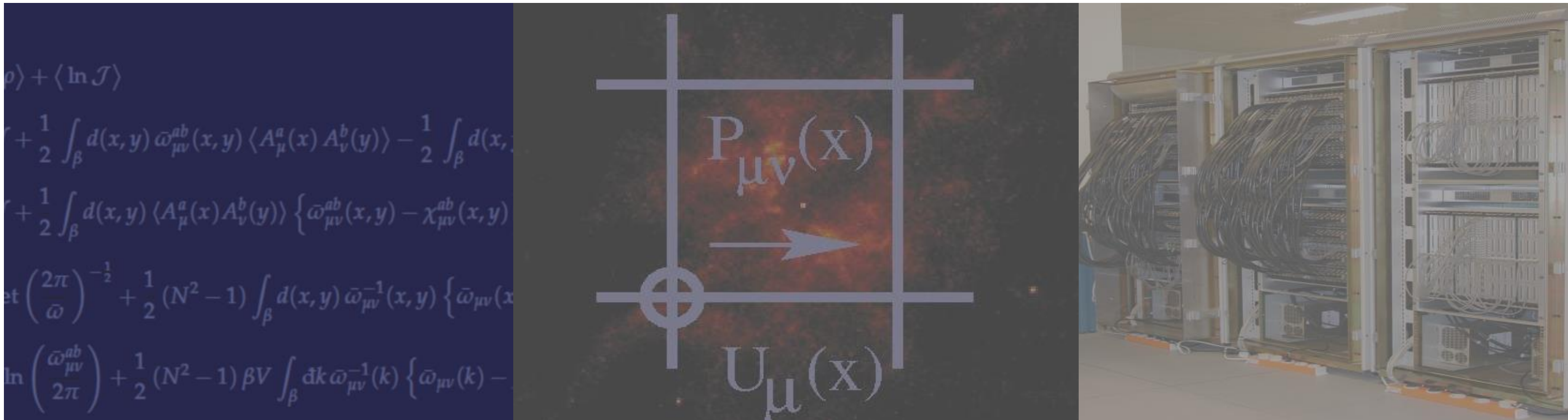
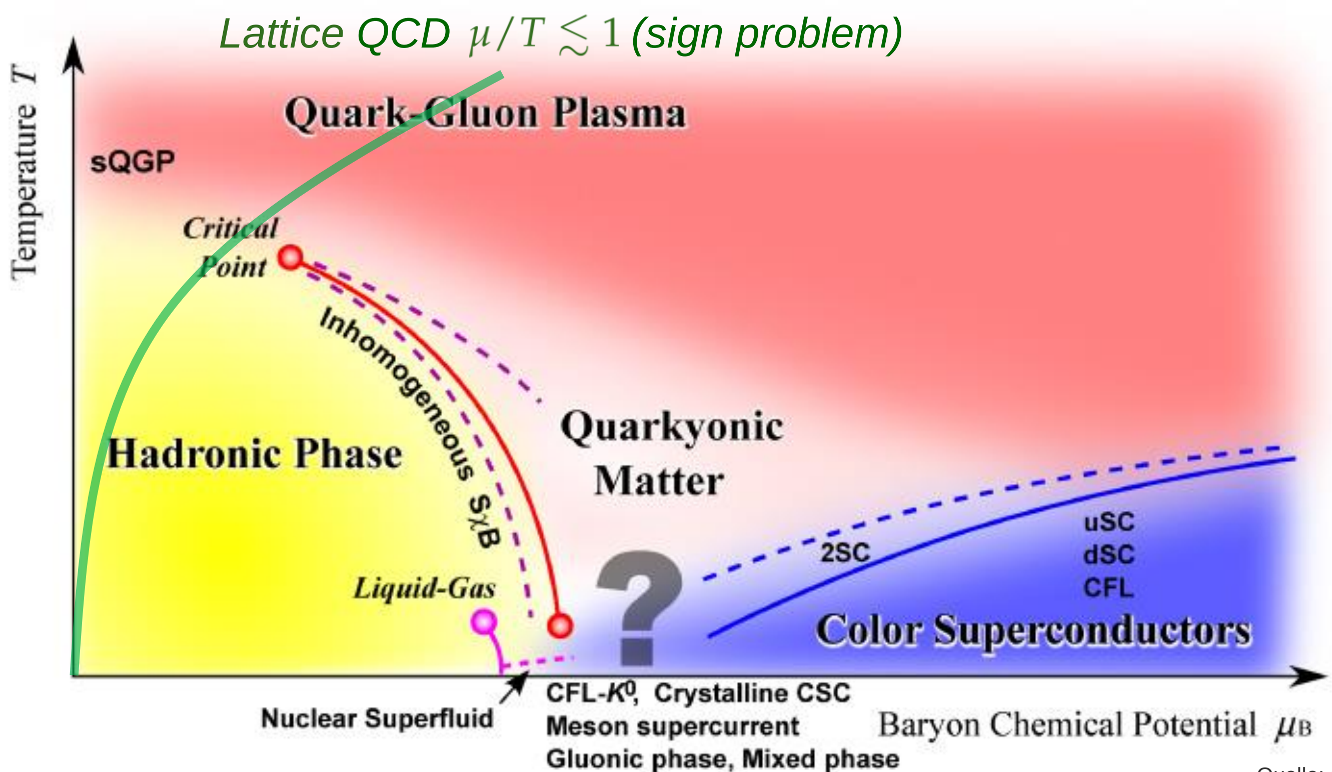
Overview

1. The uses of functional methods
2. Two flavours of variational approaches
3. Results for Yang-Mills theory
4. Beyond the Gaussian ansatz
5. Results for fermions
6. Summary and outlook



The Uses of Functional Methods

QCD Phase Diagram



Quelle: CBM @ FAIR

Functional methods

- Dyson-Schwinger equations (DSE)
- Functional renormalization group flow equations (FRG)
- Variational methods (VAR)

Comparison

- DSE ⟺ VAR: can use DSE technique in VAR (⟹ auto-tuned DSE)
- FRG ⟺ VAR: many small steps vs. one giant leap

Features of functional methods

- Ab initio calculation of QCD @ $\mu > 0$
- Numerical effort modest
- Unquenching penalty (relatively) small
- Traditional field theory , partially analytic
- Yields Green‘s functions, but also Polykov loop, thermodynamics…

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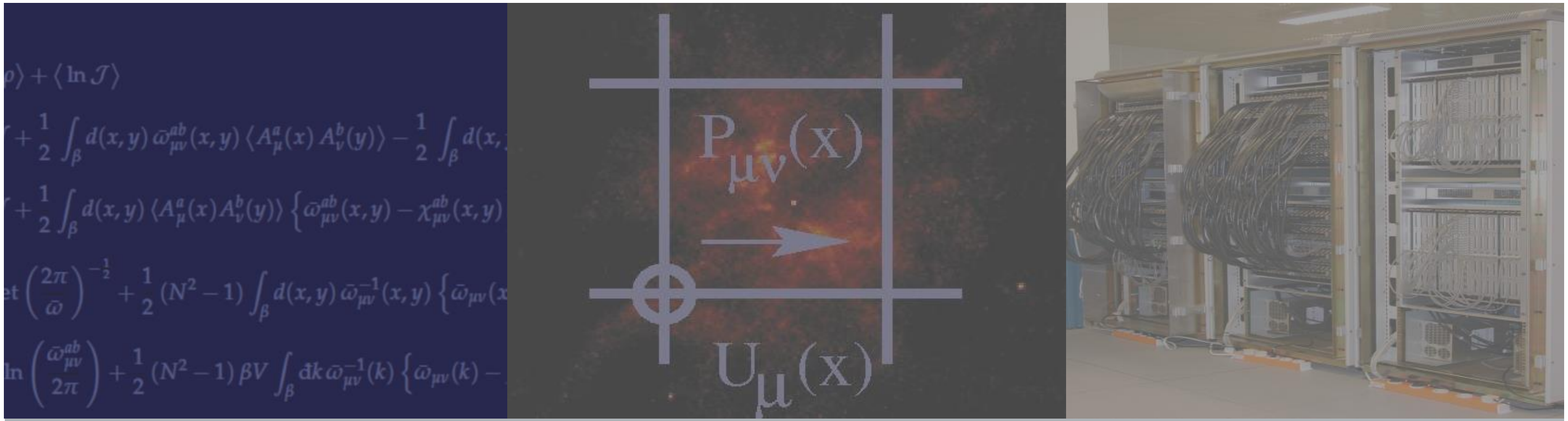


Two flavours of variational approaches

Hamiltonian Approach	Covariant Variational Approach
non-covariant (Coulomb gauge)	fully covariant (Landau gauge)
intuitive	intuitive
complicated Hamiltonian	simple dynamics
renormalization unclear	standard renormalization
Gauss' law resolved exactly	BRST symmetry broken
extension to T>0 complicated (but $S^1 \times \mathbb{R}^2$)	extension to T>0 very simple & natural

Hamiltonian approach

- Weyl gauge $A_0 = 0$ and Coulomb gauge $\nabla \vec{A} = 0$
- Resolve Gauß' law by elimination of $\vec{\Pi}^{\parallel}$

$$H = \frac{1}{2} \int \left[\mathcal{J}_A^{-1} \vec{\Pi} \cdot \mathcal{J}_A \vec{\Pi} + \vec{B}^2 \right] + \frac{g^2}{2} \int \mathcal{J}_A^{-1} \rho \mathcal{J}_A \cdot F_A \cdot \rho$$

- Electric field $\vec{\Pi}(\vec{x}) = \frac{1}{i} \frac{\delta}{\delta \vec{A}(\vec{x})}$
- Magnetic field $\vec{B} = \nabla \times \vec{A} + g\,[\vec{A}, \vec{A}]$
- FP determinant $\mathcal{J}_A = \det(-\nabla \cdot \vec{D})$
- Coulomb kernel $F_A = (-\nabla \cdot \vec{D})^{-1}) \cdot (-\nabla^2) \cdot (-\nabla \cdot \vec{D})^{-1})$

Hamiltonian approach

- Schrödinger picture + static ground state wave functional $\Psi_0[\vec{A}(\vec{x})] = \langle \vec{A} | \Psi_0 \rangle$

$$\frac{\langle \Psi_0 | H | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle} = \min$$

D. Schütte 1984
A. Szczepaniak, E. Swanson 2002
C. Feuchter, H. Reinhardt 2004

- Observables

$$\langle \Omega \rangle = \int \mathcal{D}\vec{A}\, \mathcal{J}_A \cdot \Psi_0[\vec{A}]^* \Omega[\vec{A}]\, \Psi_0[\vec{A}]$$

variational kernel =
static gluon energy =
static gluon propagator
$\langle A(\vec{x})\, A(\vec{y}) \rangle = \left[2\omega(\vec{x},\vec{y})\right]^{-1}$

- Trial wave functional

$$\Psi_0[\vec{A}] \sim \frac{1}{\sqrt{\mathcal{J}_A}} \cdot \exp\left[-\frac{1}{2} \int d(\vec{x},\vec{y}) \vec{A}(\vec{x}) \omega(\vec{x},\vec{y}) \vec{A}(\vec{y}) + \cdots \right]$$

C. Feuchter, H. Reinhardt 2004

Greensite, Matevosyan, Olejinik, M.Q., Reinhardt, Sczcepaniak Phys. Rev. D**83**, 114509 (2011)

Covariant variation principle

- Variation principle for functional probability measure $d\mu = dA \cdot \rho(A)$

$$\langle \hat{A}(x_1) \cdots \hat{A}(x_n) \rangle = \int d\mu(A)\, A(x_1) \cdots A(x_n)$$

- Free action $F(\mu) = \langle S(A) \rangle_\mu - \hbar\, \mathcal{W}(\mu)$

euclidean action

entropy $\mathcal{W} = -\langle \ln \rho \rangle_\mu$

- Variation principle I

$$F(\mu) \stackrel{!}{=} \min \quad \Longrightarrow \quad d\mu_0(A) = Z^{-1} \exp\left[\hbar^{-1} S(A)\right] dA$$

Gibbs measure

$$\langle A(x_1) \cdots A(x_n) \rangle_{\mu_0}$$

Schwinger functions

- Variation principle II (Quantum effective action)

$$F(\mu,\omega) = \min_{\mu} \left\{ F(\mu) \,\middle|\, \langle\, \Omega \,\rangle_\mu = \omega \right\} \quad \stackrel{\min}{\Longrightarrow} \quad d\mu = d\mu_\omega$$

$$\Gamma(\omega) = F(\mu_\omega, \omega) \stackrel{!}{=} \min$$

Note: Usually $\Omega = A \quad (\omega = \mathcal{A})$ and proper functions $\dfrac{\delta\Gamma(\mathcal{A})}{\delta\mathcal{A}(x_1)\cdots\delta\mathcal{A}(x_n)}$

- Variation principle III (Yang-Mills theory)

$$d\mu_0(A) = dA\, \mathcal{J}(A)\, \exp\left[-\hbar^{-1} S_{\mathrm{gf}}(A) \right]$$

FP determinant

⇒ $\bar{\mathcal{W}}(\mu) = \mathcal{W}(\mu) + \langle \ln \mathcal{J} \rangle = -\langle \ln(\rho/\mathcal{J}) \rangle$ relative entropy

- trial measure: $$d\mu(A) = dA \cdot \mathcal{N} \exp\Big[- R(A) \Big]$$

- Variation principle: Effective action for operator Ω with classical value ω

$$F(\mu, \omega) \equiv \Big\{ \langle S_{\mathrm{gf}} \rangle_\mu - \hbar \Big[\langle R \rangle_\mu + \langle \mathcal{J} \rangle_\mu + \ln \mathcal{N} \Big] \,\Big|\, \langle \Omega \rangle_\mu = \omega \Big\} \stackrel{!}{=} \min$$

⇒ 1. Determines kernels in $R(A) = \frac{1}{2} A \omega A + \frac{1}{3!} \gamma A^3 + \cdots$

2. Minimal value $\Gamma(\omega) = F(\mu_{\min}, \omega)$ is effective action (1PI)

A word of caution

Minimal free action $\stackrel{?}{=}$ generating functional of 1PI functions for solution $d\mu$

⇒ only for exact solution

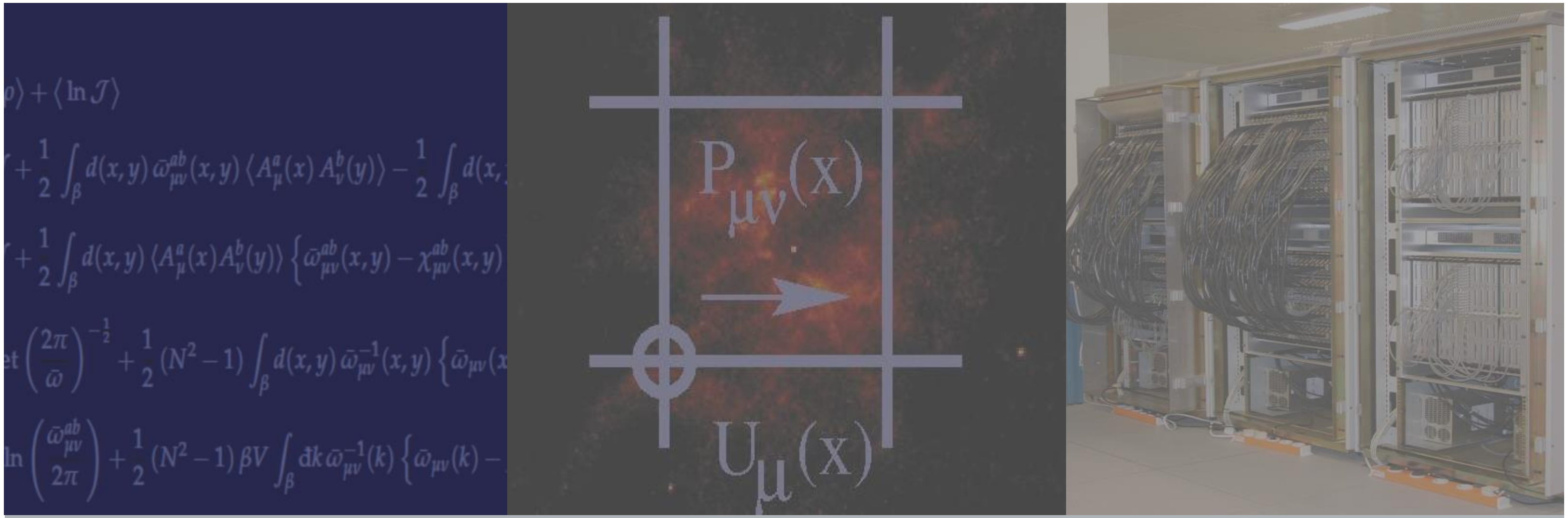
Example: ϕ^4 theory with Gaussian trial measure

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi + \frac{m^2}{2}\,\phi^2 + \frac{\lambda}{4!}\,\phi^4 \qquad d\mu(\phi) = \det\omega^{\frac{1}{2}}\cdot\exp\left[-\frac{1}{2}\int\phi\,\omega\,\phi\right]$$

⇒

- 1PI generating functional $\quad \Gamma = \int dx\,\left[\frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi + \frac{M^2}{2}\,\phi^2\right]$
- minimal free action $\quad \Gamma = \int dx\,\left[\frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi + \frac{M^2}{2}\,\phi^2 + \frac{\lambda}{4!}\,\left[1+\cdots\right]\phi^4 + \mathcal{O}(\phi^6)\right]$

dynamical mass (in both cases) $\quad M^2 = m^2 + \frac{\hbar\lambda}{2}\int\frac{d^dk}{(2\pi)^d}\,\frac{1}{k^2+M^2}$



Yang-Mills Theory at T=0

Gaussian ansatz

- UV : gluons weakly interacting
- IR : ghost dominance near Gribov horizon, self-interaction subdominant

$$d\mu(A) = \mathcal{N}(\omega) \cdot \mathcal{J}(A)^{-1} \cdot \exp\left[-\frac{1}{2}\int d(x,y)\, A_\mu^c(x)\, \delta^{ab}\, \omega_{\mu\nu}(x,y)\, A_\nu^b(y)\right]$$

Curvature Approximation

$$\ln \mathcal{J}(A) = -\frac{1}{2}\int d(x,y)\, A_\mu^a(x)\, \chi_{\mu\nu}^{ab}(x,y)\, A_\nu^b(y) + \cdots$$

$$\chi_{\mu\nu}^{ab}(x,y) = -\left\langle \frac{\delta^2 \ln \mathcal{J}}{\delta A_\mu^a(x)\, \delta A_\nu^b(y)} \right\rangle \longrightarrow \delta^{ab}\, t_{\mu\nu}(k)\, \chi(k)$$

curvature

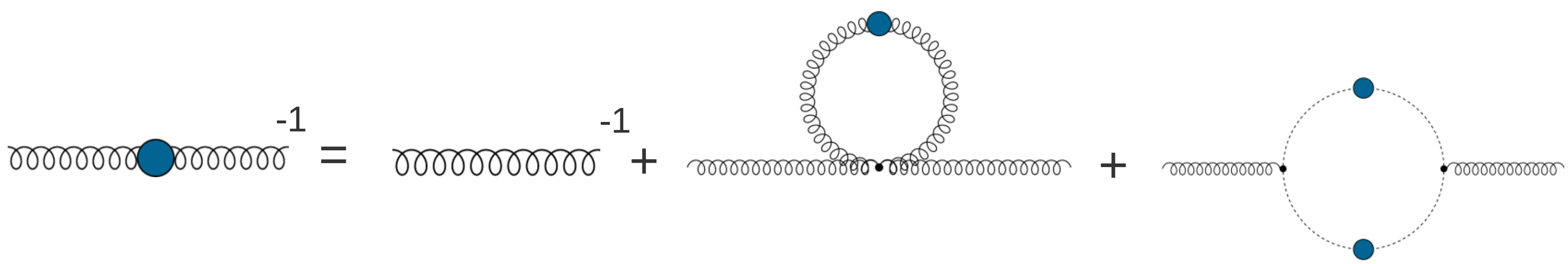
Free action

- evaluation of $F(\omega) = \langle S_{\mathrm{gf}} \rangle_\omega - \bar{\mathcal{W}}(\omega)$: only Wick‘s theorem

Gap Equation

$$\frac{\delta}{\delta\omega(k)} F(\omega) = 0$$

$$\omega(k) = k^2 + M^2 + \chi(k)$$



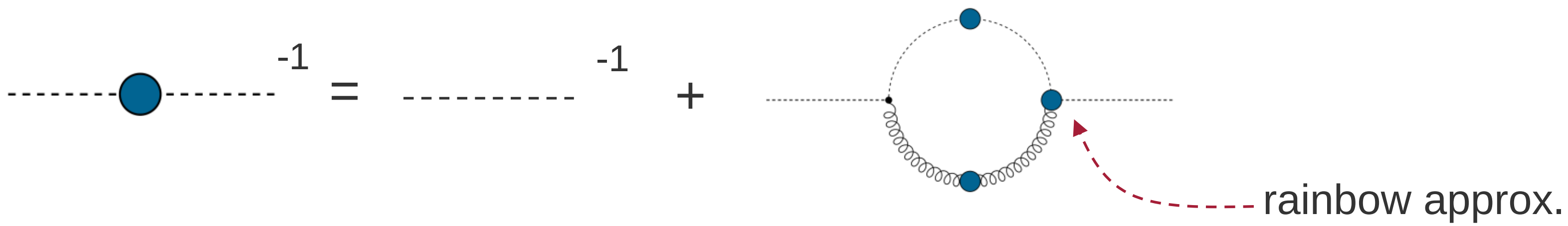
Ghost sector

Use resolvent identity on FP operator $G^{-1} = \partial_\mu \hat{D}^\mu = G_0^{-1} + h$

$$G_0 \langle G^{-1} \rangle = 1 + G_0 \langle hG \rangle \langle G^{-1} \rangle$$

in terms of ghost form factor $G(k) = \frac{\eta(k)}{k^2}$

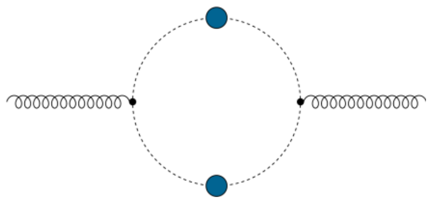
$$\eta(k)^{-1} = 1 - Ng^2 \int \frac{d^4q}{(2\pi)^4} \frac{\eta(k-q)}{(k-q)^2} \frac{1-(\hat{k}\cdot\hat{q})}{\bar{\omega}(q)}$$



Curvature Equation

To given loop order

$$\chi(k) = \mathrm{Tr}\left\langle G\,\frac{\delta(-\partial D)}{\delta A(2)}\,G\,\frac{\delta(-\partial D)}{\delta A(1)}\right\rangle \approx \mathrm{Tr}\langle G\rangle\,\Gamma_0\,\langle G\rangle\,\Gamma_0$$

$=$ 

$$\chi(k) = \frac{1}{3}\,N g^2 \int \frac{d^4 q}{(2\pi)^4}\,\frac{\eta(k-q)\,\eta(q)}{(k-q)^2}\,\left[1 - (\hat{k}\cdot\hat{q})\right]$$

Counterterms

$$\mathcal{L}_{\text{ct}} = \delta Z_A \cdot \frac{1}{4} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 + \delta M^2 \cdot \frac{1}{2} \left(A_\mu^a\right)^2 + \delta Z_c \cdot \partial_\mu \bar{c}\, \partial^\mu c$$

gluon field

gluon mass

ghost field

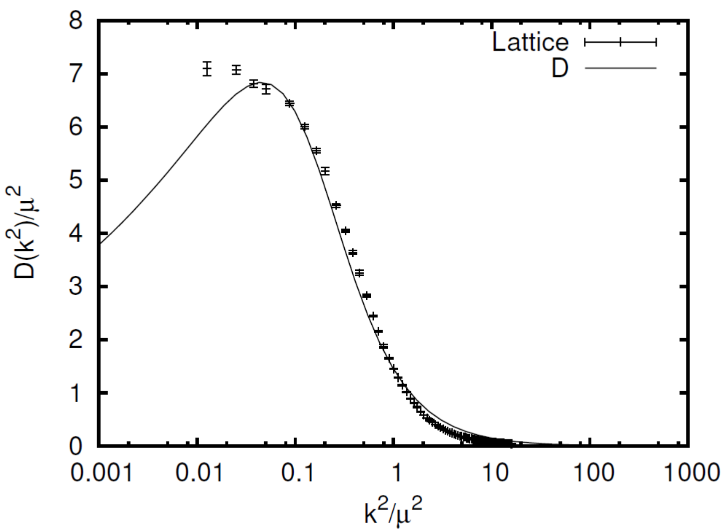
Renormalization conditions (3 scales $0 \le \mu_c \le \mu_0 \ll \mu$)

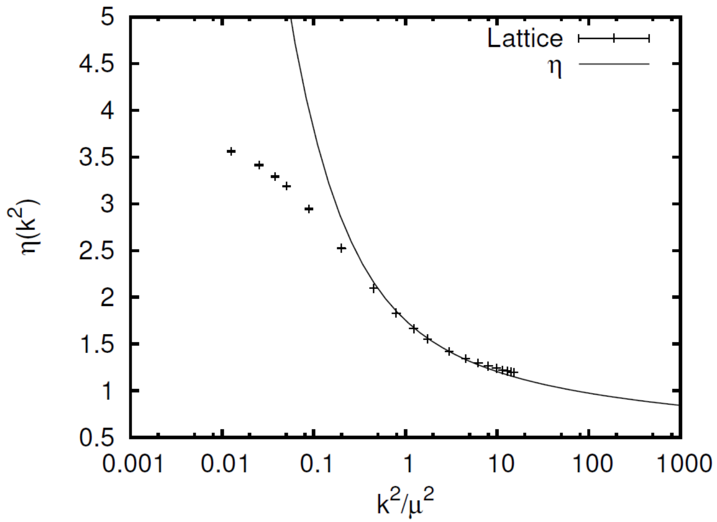
- fix $\eta(\mu_c) = Z_c$ scaling/decoupling
- fix $\omega(\mu) = Z\,\mu^2$
- fix $\omega(\mu_0) = Z\,M_A^2$ constitutent *mass parameter*

Scaling Solution (G=SU(2))

MQ, H. Reinhardt, J. Heffner, Phys. Rev. **D89** 035037 (2014)

Lattice data from Bogolubsky et al., Phys. Lett. **B676** 69 (2009)





IR exponents: $\omega(k) \sim (k^2)^{\alpha}$ $\quad \alpha = \frac{1}{49}\left(44 - \sqrt{1201}\right) \approx 0.1907$

$\eta(k) \sim (k^2)^{-\beta}$ $\quad \beta = \frac{1}{98}\left(93 - \sqrt{1201}\right) \approx 0.5953$

Lerche, v. Smekal PRD **65**

numerical: $\alpha = 0.191(1)$ $\quad \beta = 0.595(3)$

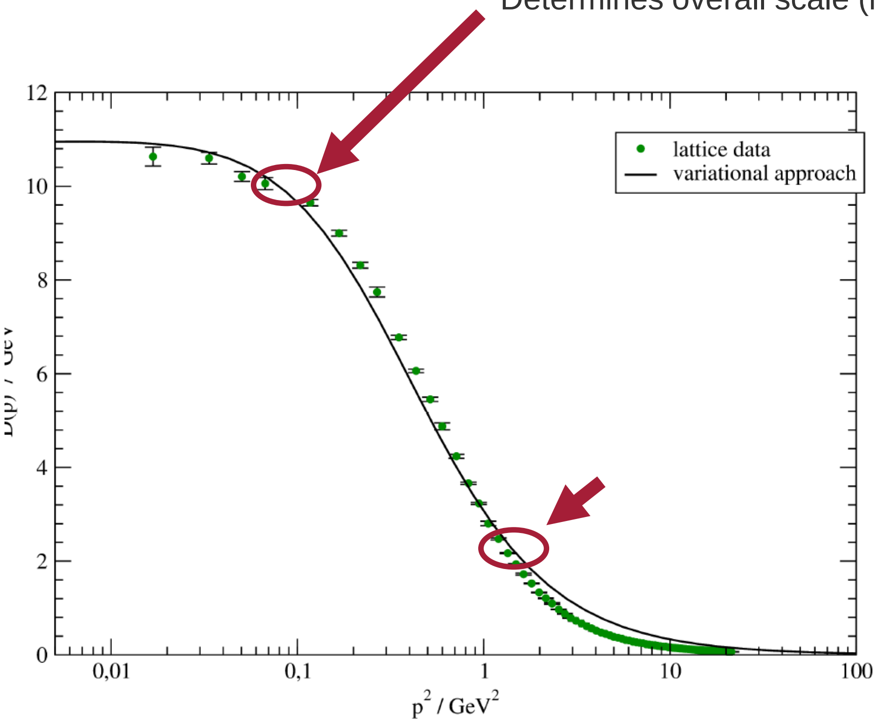
$$\alpha - 2\beta + \left(\frac{d}{2} - 1\right) < 10^{-3}$$

Decoupling Solution (G=SU(2))

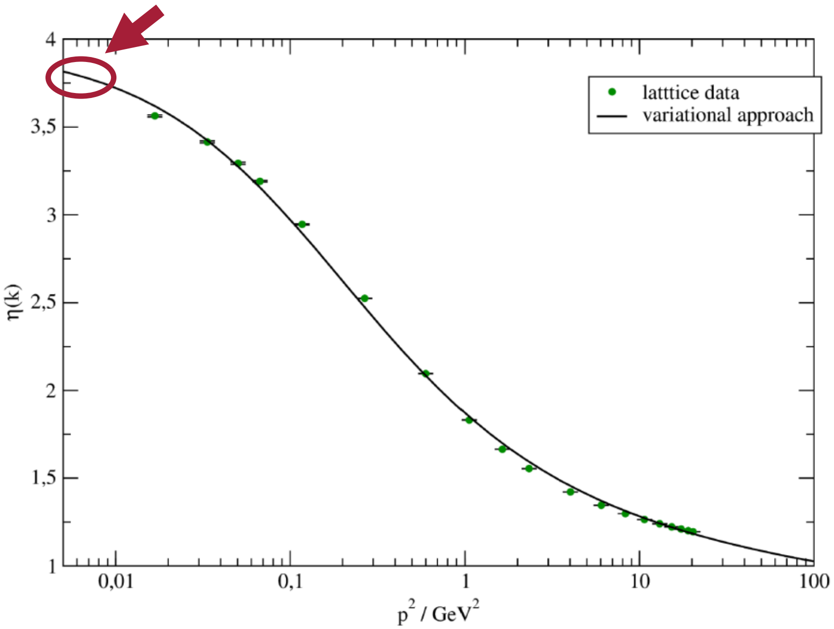
MQ, H. Reinhardt, J. Heffner, Phys. Rev. **D89** 035037 (2014)

Lattice data from Bogolubsky et al., Phys. Lett. **B676** 69 (2009)

Determines overall scale (not very precise !)

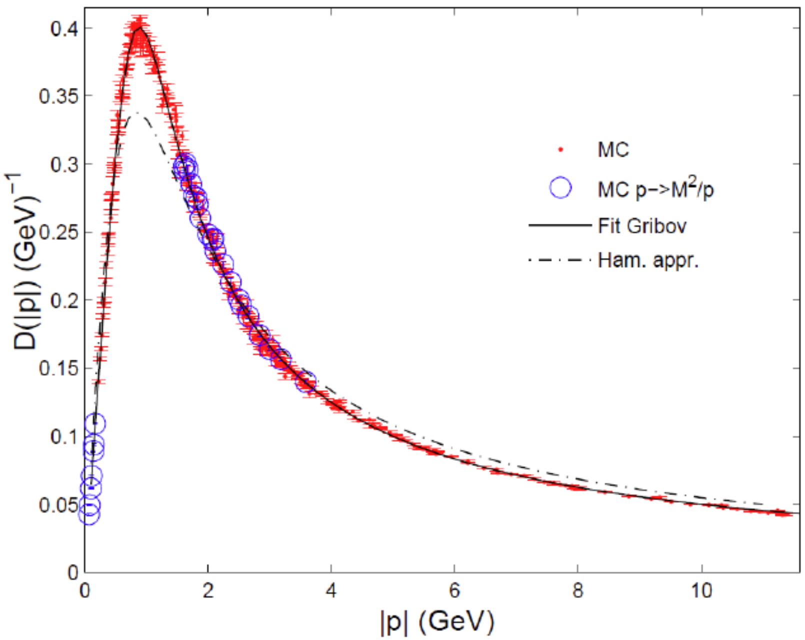


gluon propagator



ghost form factor

Hamiltonian Approach: Static gluon propagator



G. Burgio, M.Q., H.Reinhardt PRL **102** 032002 (2009)

Gribov‘s formula

$$D(k) = \sqrt{k^2 + \frac{M^4}{k^2}}$$

Gribov mass $M \approx 0.88\,\mathrm{GeV}$

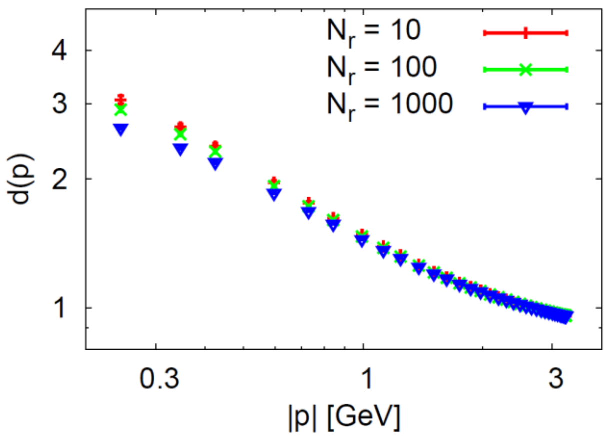
Determines overall scale

Lattice data: scaling violations

Removed by carefully isolating discretization artifacts in the time direction

Hamiltonian Approach: the ghost puzzle

ghost form factor (bc gauge fixing)



G. Burgio, M.Q., H. Reinhardt, H. Vogt PRD *95* 14503 (2017)

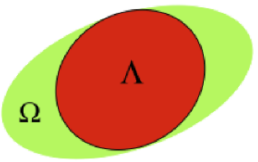
<u>IR behaviour</u>

- lattice: $d(k) \sim k^{-\frac{1}{2}}$
- Hamilton: $d(k) \sim k^{-\beta}$ $\beta \approx 1$
- gluon: $\omega(k) \sim k^{-\alpha}$ $\alpha = 1$

Sum rule $\alpha = 2\beta + 2 - d$

apparently *violated* on the lattice

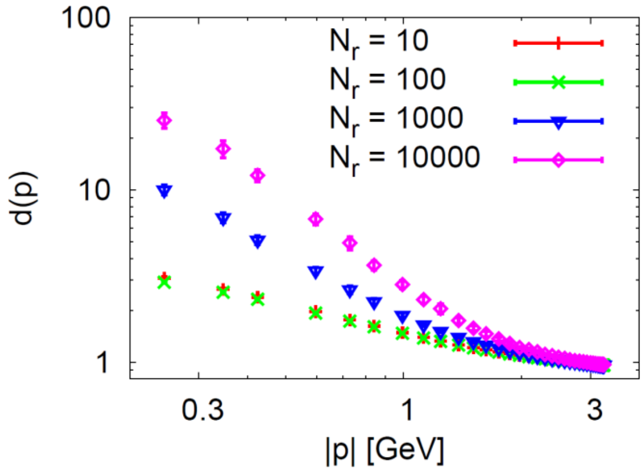
Schleifenbaum, Leder, Reinhardt PRD **73**



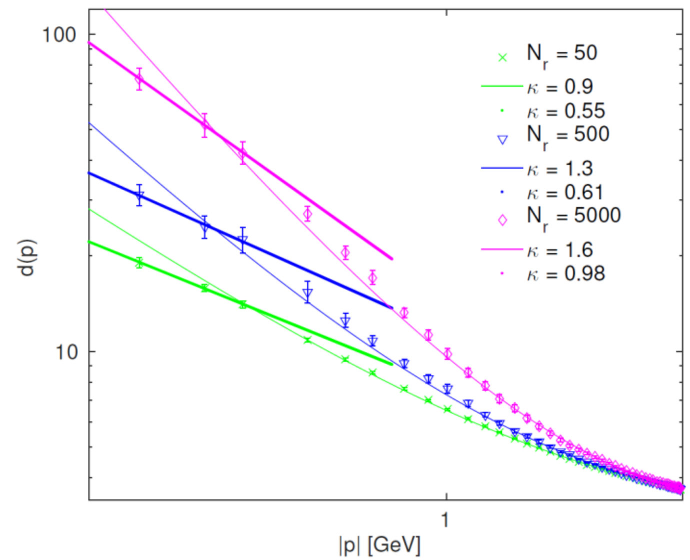
- *Sternbeck, Müller-Preusker, Maas* (Landau gauge) :
  gauge fix to configuration with lowest eigenvalue of FP operator (*lc*)
- *Zwanziger & Cooper* (Coulomb gauge):
  should be closer to continuum

Hamiltonian Approach: the ghost puzzle

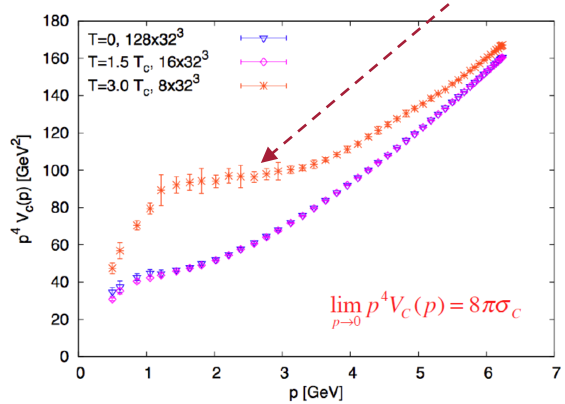
ghost form factor (lc gauge fixing)



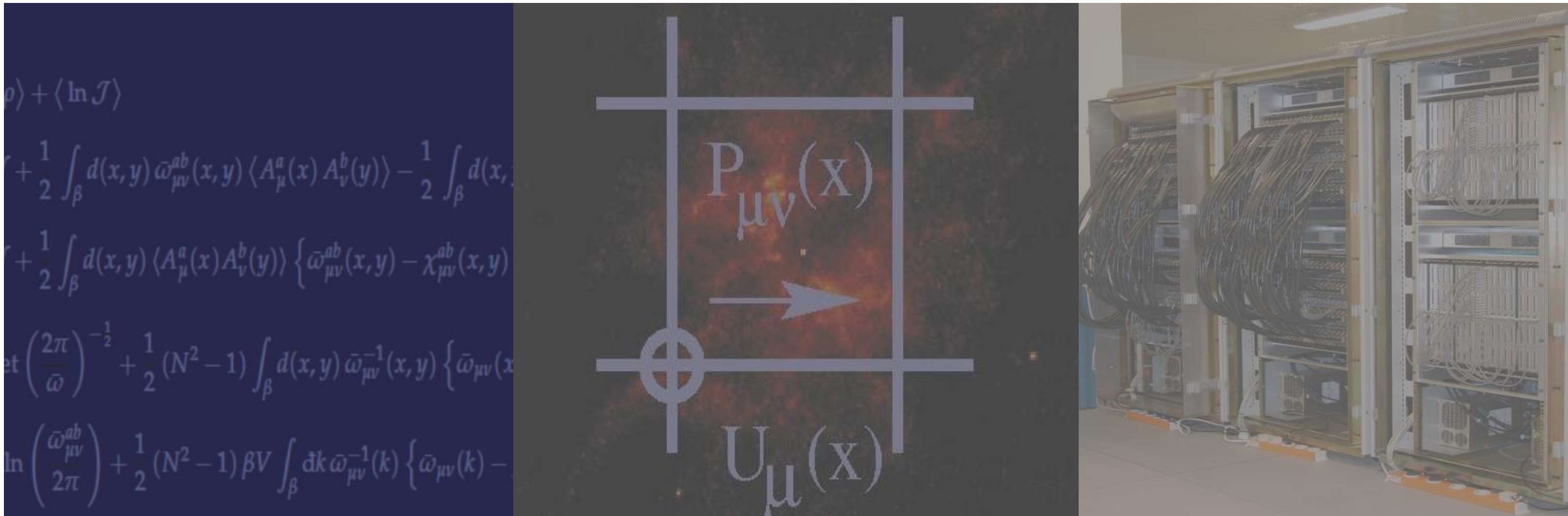
G. Burgio, M.Q., H. Reinhardt, H. Vogt PRD *95* 14503 (2017)



Toki et. al (2006)



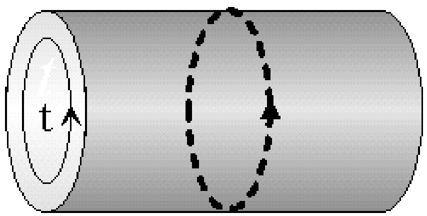
- Ghost IR enhanced in *lc*
- might be compatible with $\beta \approx 1$ (saturation unclear)
- Gluon propagator unchanged between *bc* and *lc*



Yang-Mills Theory at Finite Temperature

Extension to Finite Temperature

- imaginary time formalism



compactify euclidean time $t \in [0, \beta]$

periodic b.c. for gluons (up to center twists)

periodic b.c. for ghosts (even though fermions) !

$$A(t,\mathbf{x}) = \beta^{-1} \sum_{n\in\mathbb{Z}} \int \frac{d^3k}{(2\pi)^3}\, e^{i\,(\nu_n t + \mathbf{k}\mathbf{x})}\, A_n(\mathbf{k})$$

$$\nu_n = \frac{2\pi}{\beta}\, n \qquad\qquad (n \in \mathbb{Z})$$

⇒ Extension to T>0 straightforward

$$\int \frac{d^4k}{(2\pi)^4} \longrightarrow \int_\beta đq \equiv \beta^{-1} \sum_{n\in\mathbb{Z}} \int \frac{d^3k}{(2\pi)^3}$$

- Lorentz structure of propagator

heat bath singles out restframe (1,0,0,0) ⇨ breaks Lorentz invariance

two different 4-transversal projectors

!

$$\mathcal{P}^T_{\mu\nu}(k) = (1-\delta_{\mu 0})\,(1-\delta_{\nu 0})\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{\mathbf{k}^2}\right)$$ ← 3-transversal

$$\mathcal{P}^L_{\mu\nu}(k) = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) - \mathcal{P}^T_{\mu\nu}(k)$$ ← 3-longitudinal

⇨ Two Lorentz structures for kernel and curvature

$$\omega_{\mu\nu}(k) = \omega_\perp(k)\cdot\mathcal{P}^T_{\mu\nu}(k) + \omega_{||}(k)\cdot\mathcal{P}^L_{\mu\nu}(k) + \text{long.}$$

$$\chi_{\mu\nu}(k) = \chi_\perp(k)\cdot\mathcal{P}^T_{\mu\nu}(k) + \chi_{||}(k)\cdot\mathcal{P}^L_{\mu\nu}(k) + \text{long.}$$

Gap Equations

$$\omega_\perp(k) = k_0^2 + \mathbf{k}^2 + \chi_\perp(k) + M^2(\beta)$$

$$\omega_\parallel(k) = k_0^2 + \mathbf{k}^2 + \chi_\parallel(k) + M^2(\beta) + \frac{\mathbf{k}^2}{k_0^2 + \mathbf{k}^2}\,\widetilde{M}^2(\beta)$$

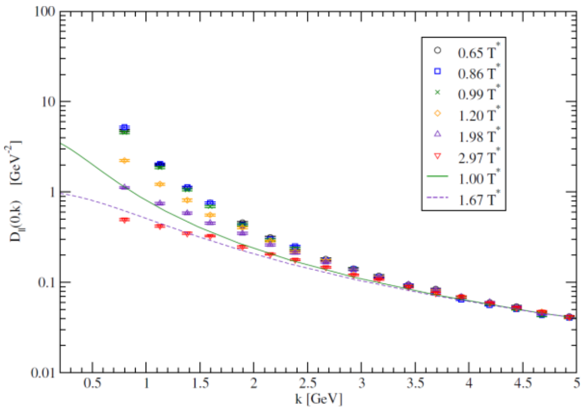
induced gluon masses now temperature-dependent

$$M^2(\beta) = \frac{1}{2} N g^2 \int_\beta đq \left[\frac{A}{\omega_\perp(q)} + \frac{B(q)}{\omega_\parallel(q)}\right]$$

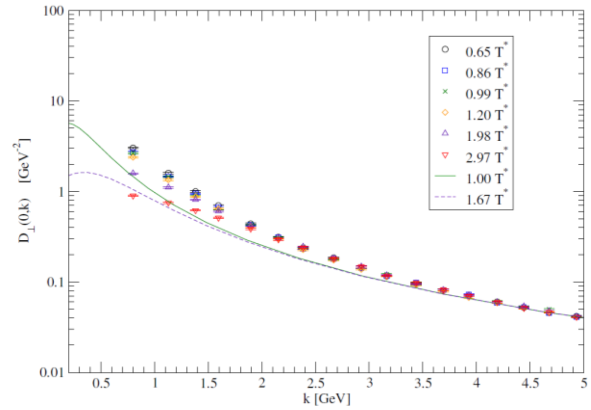
$$\widetilde{M}^2(\beta) = \frac{1}{3} N g^2 \int_\beta đq \left[\frac{2}{\omega_\perp(q)} + \left(\frac{q_0^2 - 3\mathbf{q}^2}{q_0^2 + \mathbf{q}^2}\right)\frac{1}{\omega_\parallel(q)}\right]$$

renormalization by T=0 counter terms (in principle …)

MQ, H. Reinhardt, Phys. Rev. **D92** 025051 (2015)



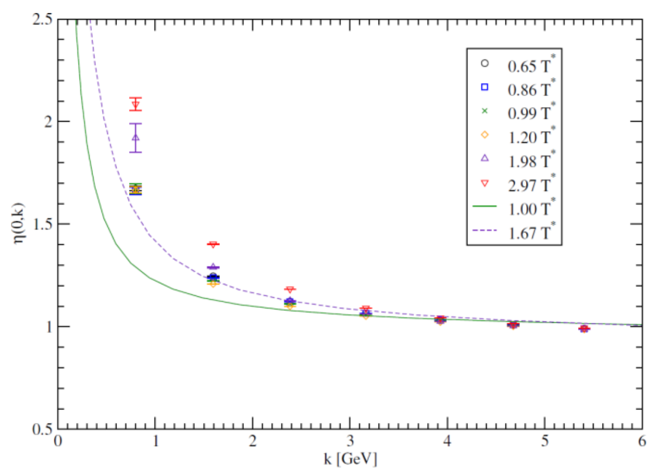
longitudinal gluon $D_{||}(0,p)$



transversal gluon $D_{\perp}(0,p)$

ghost formfactor

$\eta(0,p)$

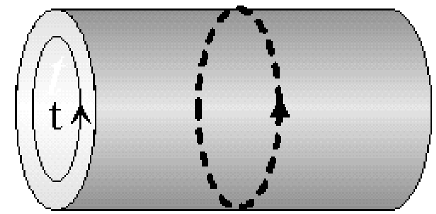


With increasing T

- Slight ghost enhancement
- Mild gluon suppression
- Temperature sensitivity larger in direction longitudinal to the heat bath

Polyakov loop

$$\mathsf{L}(\mathbf{x}) = P \exp\left[- \int_0^\beta dt\, A_0(t,\mathbf{x}) \right]$$



Interpretation: free static quark energy

$$\langle \mathrm{tr}\, \mathsf{L}(\mathbf{x}) \rangle = \exp\left[-\beta F_q(\mathbf{x}) \right]$$

$$\langle \mathrm{tr}\, \mathsf{L}(\mathbf{x})\, \mathsf{L}(\mathbf{y})^\dagger \rangle = \exp\left[-\beta\, F_{q\bar{q}}(\mathbf{x}-\mathbf{y}) \right]$$

Center symmetry

- maps $\mathsf{L} \to z \cdot \mathsf{L}$
- If unbroken $\langle \mathsf{L} \rangle = 0$ [**confinement**]
- If broken $\langle \mathsf{L} \rangle \neq 0$ [**deconfinement**]

Alternative order parameter

$$x \equiv \frac{\beta \langle A_0^3 \rangle}{2\pi} = \frac{\beta \mathsf{a}}{2\pi} \in [0,1]$$

G=SU(2)
Polyakov gauge $[\, \partial_0 A_0 = A_0^{\text{ch}} = 0 \,]$
Background gauge $[\, \partial_0 \mathsf{a} = 0 \,]$

Background gauge

$$A_\mu = \mathsf{A}_\mu + Q_\mu = \mathsf{a}\, \delta_{\mu 0} + Q_\mu$$

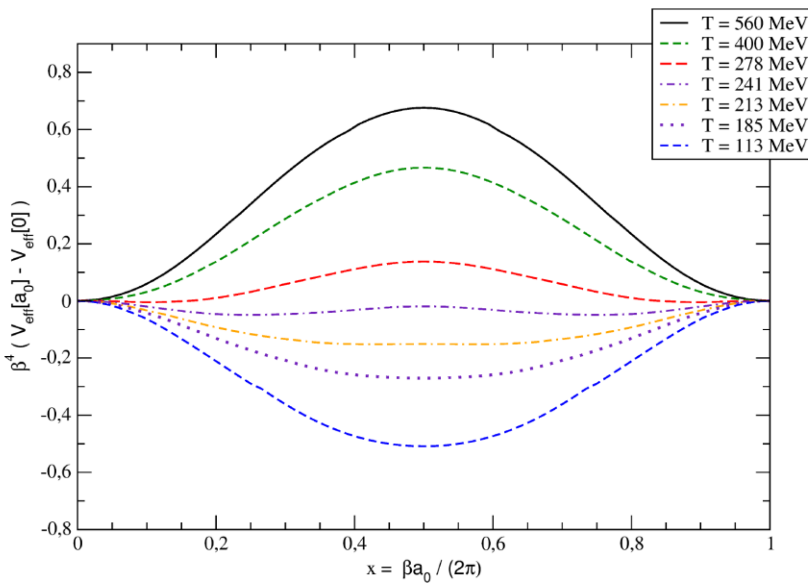
$$[D_\mu(\mathsf{a}), Q_\mu] = [\mathsf{d}_\mu, Q_\mu] = 0$$

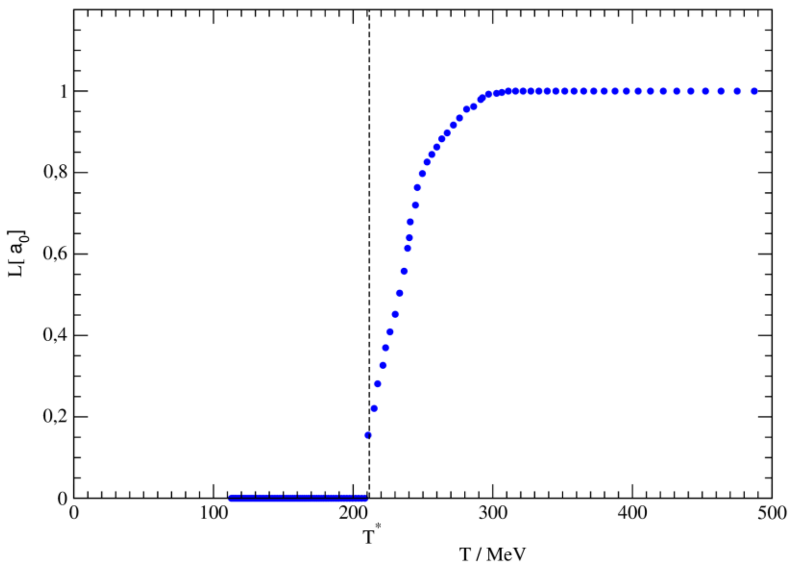
Transfer Landau -- Background

- replace $\partial_\mu \, \delta^{ab} \mapsto \hat{\mathsf{d}}_\mu^{ab}$ in basis where rhs is diagonal
- replace $p_\mu \mapsto p_\mu - \sigma\, \mathsf{a}\, \delta_{\mu 0}$ where σ are the simple roots
- replace $N^2 - 1 = 3 \mapsto \sum_{\sigma = 0, \pm 1}$ sum over simple roots

Phase transition for G=SU(2)

MQ, H. Reinhardt, Phys. Rev. **D,** in press (2016)





Eff. Potential for Polyakov loop

ghost dominance

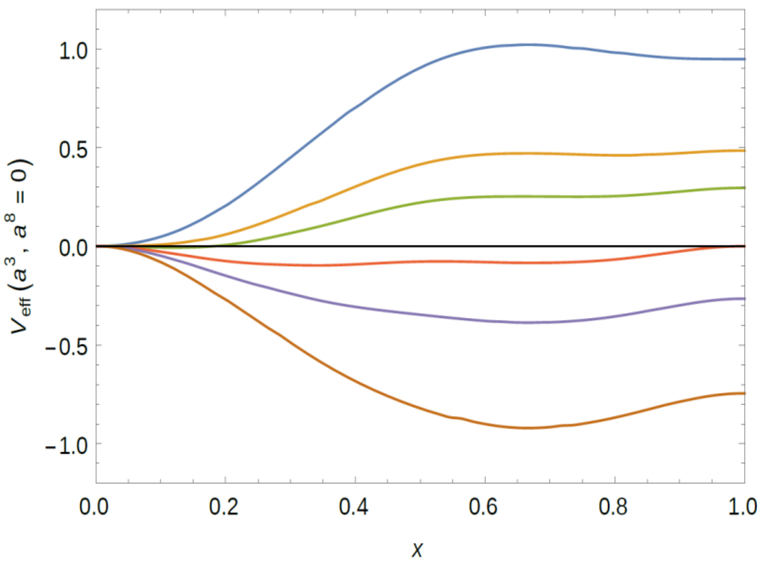
- 2nd order transition
- critical temperature $T^* = 216\,\mathrm{MeV}$

Lattice $T^* = 306\,\mathrm{MeV}$

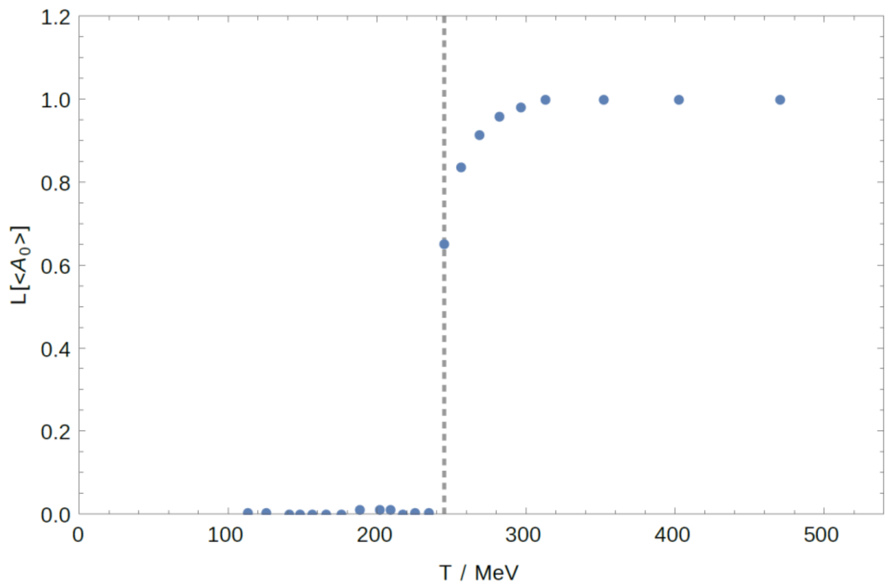
Lucini, Teper, Wenger, JHEP **01** (2004) 061

Phase transition for G=SU(3)

MQ, H. Reinhardt, Phys. Rev. **D,** in press (2016)



slice of eff. Potential for Polyakov loop



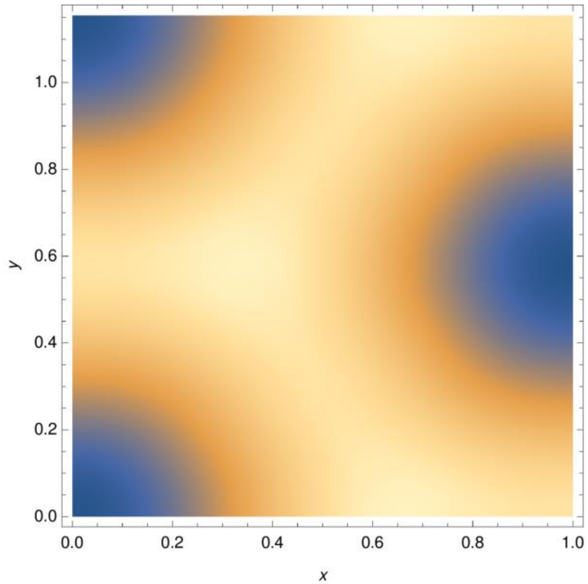
- 1st order transition
- critical temperature $T^* = 245\,\mathrm{MeV}$

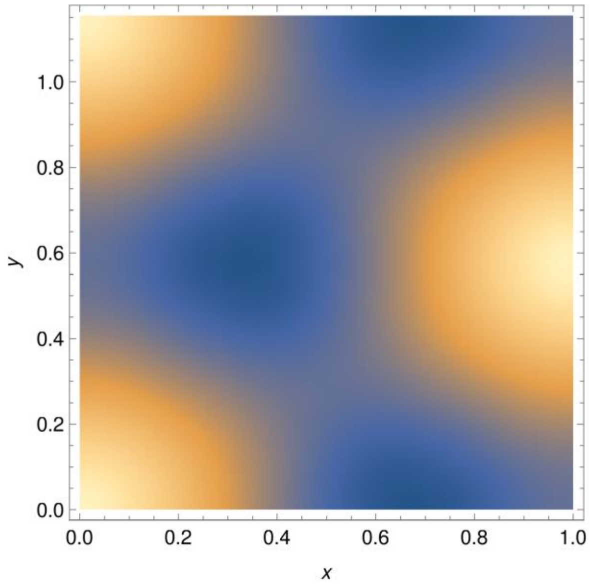
Lattice $T^* = 284\,\mathrm{MeV}$

Lucini, Teper, Wenger, JHEP **01** (2004) 061

Effective potential for Polakov loop in G=SU(3)

MQ, H. Reinhardt, Phys. Rev. **D,** in press (2016)





Deconfined phase

V(x,y) maximal at center symmetrc points

$$T = 400\,\mathrm{MeV}$$

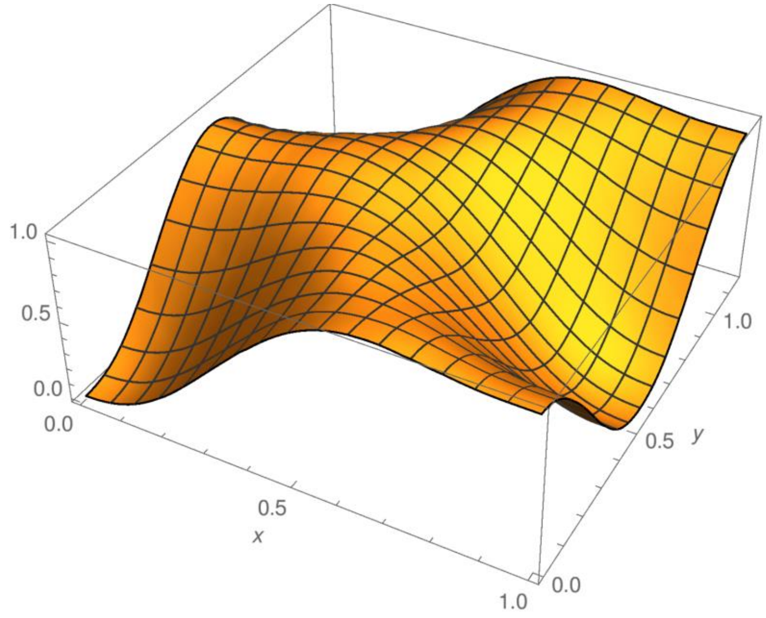
Confined phase

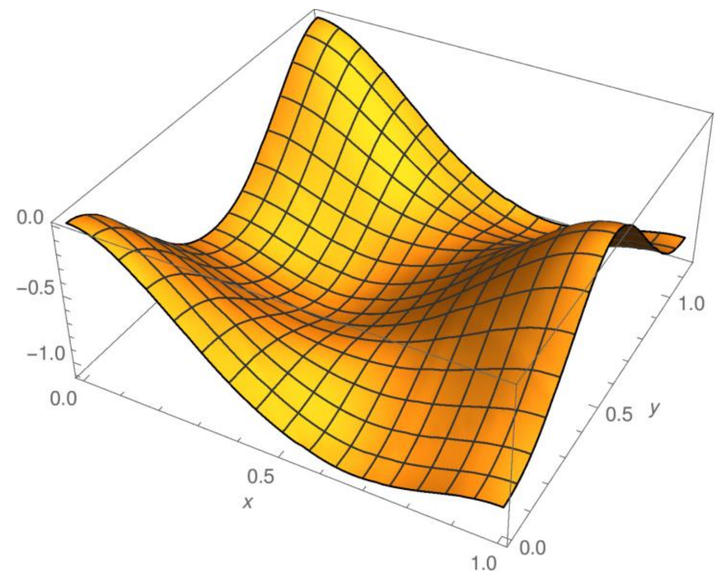
V(x,y) minimal at center symmetrc points

$$T = 141\,\mathrm{MeV}$$

Effective potential for Polakov loop in G=SU(3)

MQ, H. Reinhardt, Phys. Rev. **D,** in press (2016)





Deconfined phase

V(x,y) maximal at center symmetrc points

$$T = 400\,\mathrm{MeV}$$

Confined phase

V(x,y) minimal at center symmetrc points

$$T = 141\,\mathrm{MeV}$$

Hamiltonian Approach: finite temperature

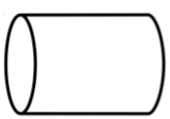
- grand canonical ensemble: Quasi-particle ansatz for density

- Cannot compactify time direction and Polyakov loops unavailable $A_0 = 0$

  - Compactify a spatial axis to a circle of length L and interprete 1/L as temperature
  - Hamiltonian approach on space manifold $\mathbb{R}^2 \times S^1(L)$

H. Reinhardt, Phys. Rev. D**94** 045016 (2016)

$T^{-1} = L$

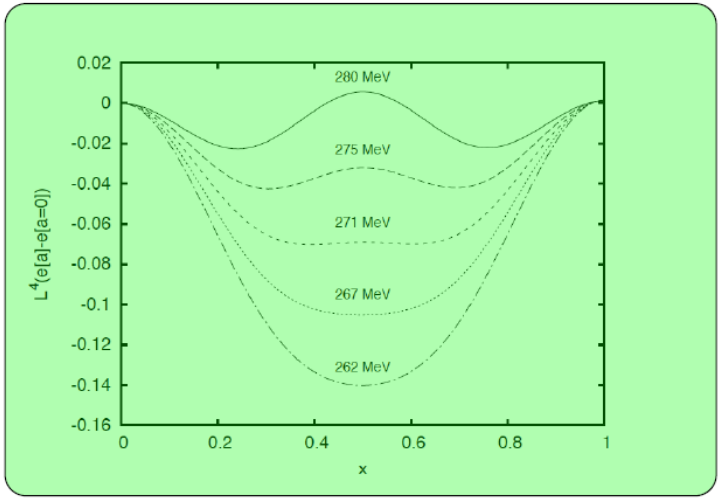


- *Casimir pressure* as free energy (O(4) invariance)

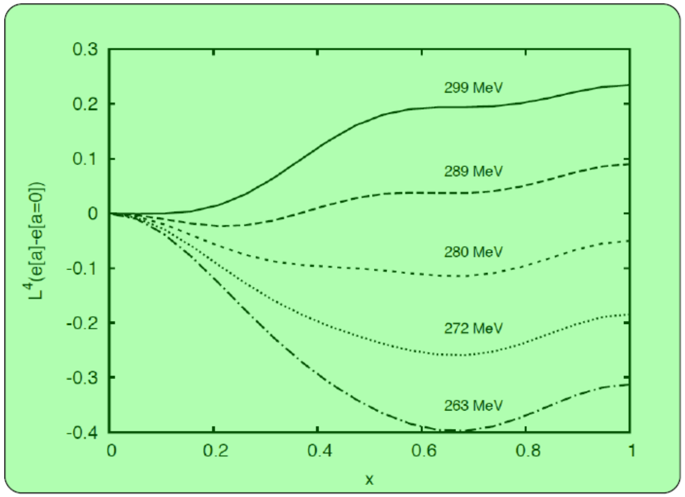
  Works for free bose and fermion gas with and without chemical potential

Hamilton Approach: Polyakov Loop

G= SU(2)



G= SU(3)



- Input: $M = 880\,\mathrm{MeV}$
- Second order
- criticality $T^* = 269\,\mathrm{MeV}$

- Input: $M = 880\,\mathrm{MeV}$
- first order
- criticality $T^* = 283\,\mathrm{MeV}$

H. Reinhardt, J. Heffner, PRD **88** (2013)

Thermodynamics of the YM plasma

M.Q., H. Reinhardt (in preparation))

- Free energy density:

$$F(\beta) = \min_{\mu} F_{\beta}(\mu) = \min_{a} \Gamma_{\beta}[a] = -\ln Z(\beta) = V_3 \cdot \beta f(\beta)$$

- pressure: $p(\beta) = -f(\beta)$

- energy density: $\epsilon(\beta) = f(\beta) + \beta \, \partial f / \partial \beta$

- Interaction strength: $\Delta(\beta) = \beta^{-3} \partial(p \beta^4) / \partial \beta$

Free relativstic gas: $p \sim T^4 \quad \Longrightarrow \quad \Delta = 0$

From variational solution of gluon and curvature

$$\beta^4 \bar{f}_\beta(x) = 12\pi \int_0^\infty dq\, q^2 \sum_{n\in\mathbb{Z}} \left[\ln \omega(k_n(x)) - \frac{\chi_R(k_n(x))}{\omega(k_n(x))} - \frac{1}{3} \ln k_n(x)^2 \right]$$

$$k_n(x) \equiv \frac{2\pi}{\beta} \sqrt{(n+x)^2 + q^2} \qquad x \equiv \frac{\beta \mathsf{a}_0}{2\pi} \in [0,1]$$

Divergent → Need to subtract the vacuum energy ($\beta = 0$)

$$f_\beta(x) = \bar{f}_\beta(x) - \bar{f}_\infty(x)$$

Gives wrong pressure at T=0

Poisson resummation

T=0 subtracted

$$u(x,\beta) \equiv \beta^4 \bar{f}_\beta(x) - \left[\beta^4 \bar{f}_\beta(x)\right]_\infty = -\frac{2}{\pi^2} \sum_{\nu=1}^{\infty} \frac{\cos(2\pi\nu x)}{\nu^4} h(\beta\nu)$$

$$h(\lambda) = -\frac{1}{4} \int_0^\infty d\tau\, \tau^2 J_1(\tau)\, \phi(\tau/\lambda)$$

$$\phi(k) = 3 \ln \omega(k) - 3 \frac{\chi(k)}{\omega(k)} - \ln k^2$$

- Massless modes due to gauge fixing and ghost dominance ruin IR limit
  (only in continuum approach)

- Free (massive) boson $\phi_M(k) = \ln(k^2 + M^2)$

$\Rightarrow$ $$h_M(\lambda) = \frac{1}{2} (\lambda M)^2 K_2(\lambda M)$$

Bessel function

$$\lim_{\lambda\to\infty} h_M(\lambda) = \begin{cases} 0 & : \quad M > 0 \\ 1 & : \quad M = 0 \end{cases}$$

- Massless modes: must set $M \to 0$ before $\beta \to \infty$

$$\beta^4 f_\beta(x) = \beta^4 \bar{f}_\beta(x) - \lim_{\beta\to\infty} \beta^4 \bar{f}_\beta(x) = u(x,\beta) - \lim_{\beta\to\infty} u(x,\beta)$$

does not vanish! $\lim_{\beta\to\infty} \beta^4 \bar{f}_\beta(x) - \left[\beta^4 \bar{f}_\beta\right]_{\beta=\infty} \neq 0$

- Limiting values

$$u(x,\beta) = \begin{cases} W(x) - \frac{2\pi^2}{45} & : \quad \beta \to 0 \\ \frac{\pi^2}{45} - \frac{1}{2} W(x) & : \quad \beta \to \infty \end{cases}$$

4 massless gluons + 2 massless ghosts

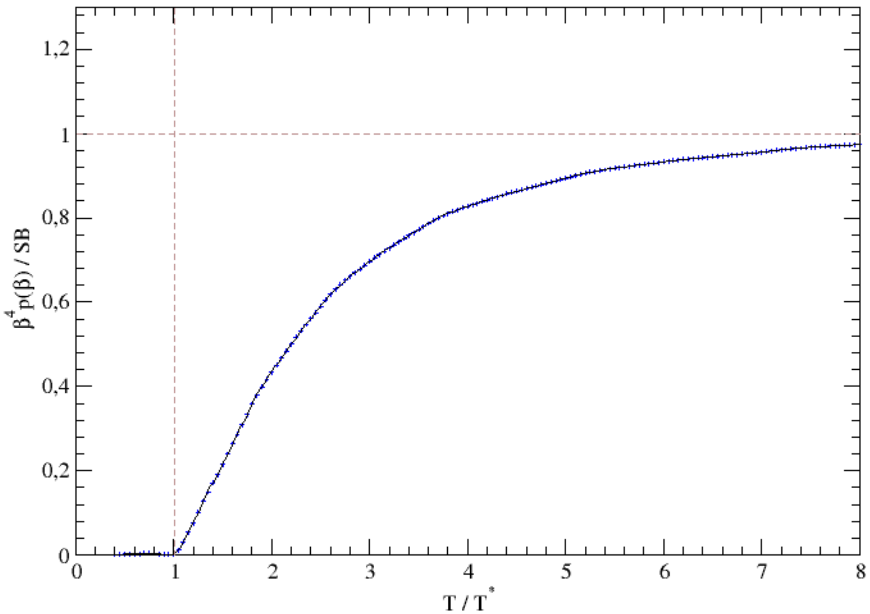
Transversal gluons mass-like, ghosts enhanced
1 massless gluon + 2 massless ghosts
(ghost dominance)

$$W(x) = \frac{4}{3}\pi^2 x^2 (1-x)^2$$ -- Weiss potential

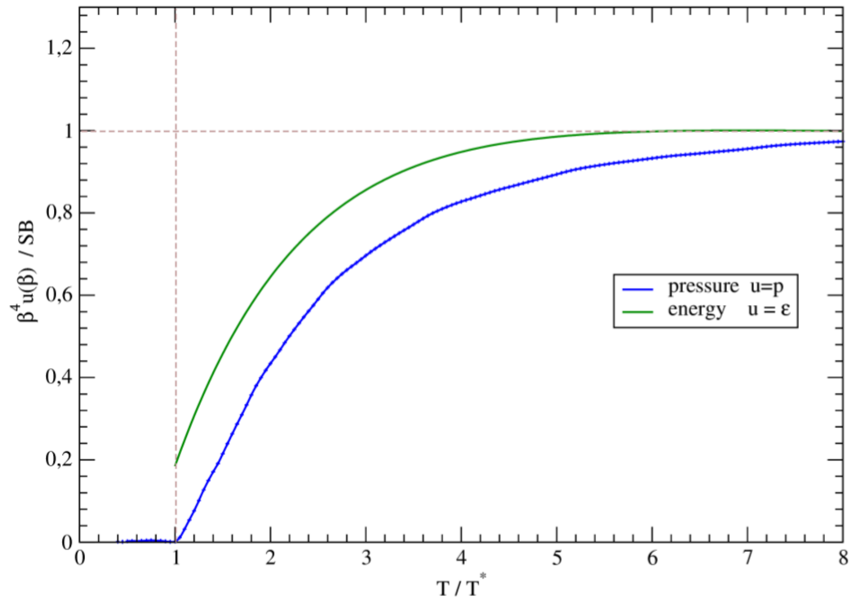
See Polyakov loop earlier

Preliminary results

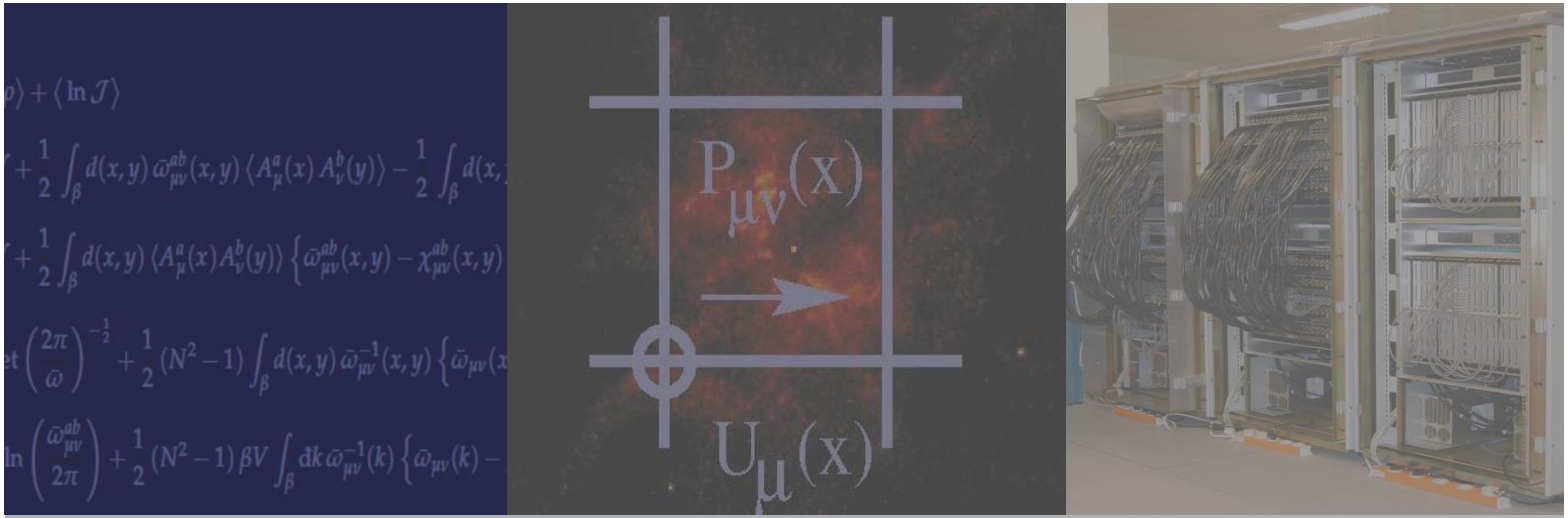
pressure



energy density



- Correct SB limit
- Energy density $\beta^4 \epsilon$ „steeper“ than pressure $\beta^4 p \quad \Longrightarrow \quad \Delta > 0$
- Lattice data slightly steeper than variational results



Beyond the Gaussian ansatz

Using DSE in the variational approach

- general trial measure $d\mu \sim \mathcal{J}_A \cdot \exp\big(-R[A,\bar{q},q]\big)$

$$R = \frac{1}{2}\gamma_2 A^2 + \frac{1}{3!}\gamma_3 A^3 + \frac{1}{4!}\gamma_4 A^4 + \bar{q}(\Gamma_0 + \Gamma_1 A)q + \cdots$$

1. Use DSE on R to evaluate $\langle \cdots \rangle$ in free energy *F*
2. Minimize *F* to find optimal kernels

- Similar to Hamiltonian approach   D. Campagniari, H. Reinhardt PRD **82** (2010), **92** (2015)

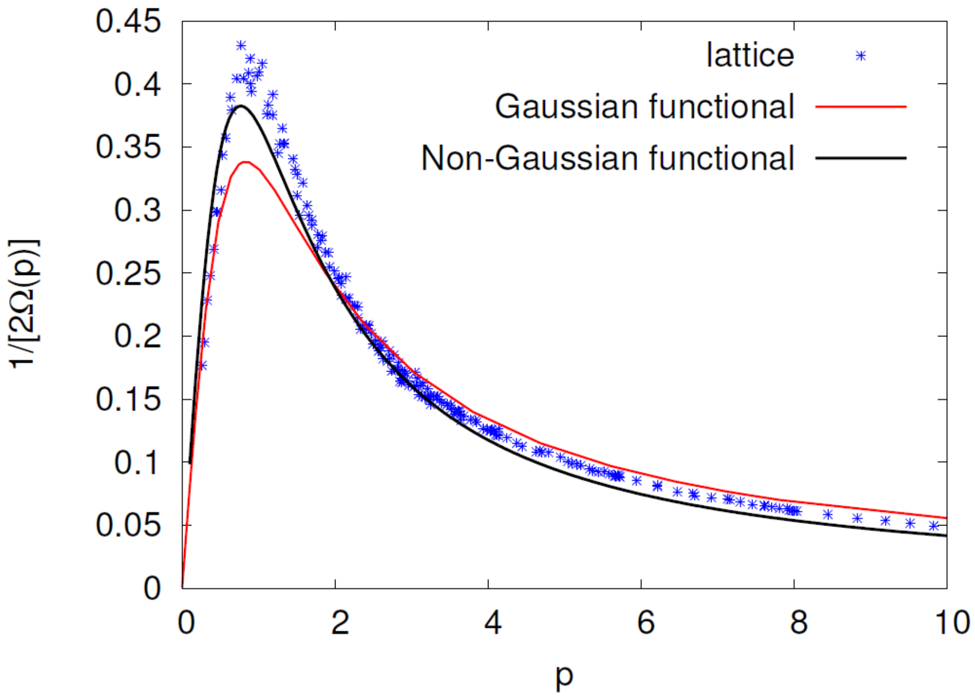
DSE exact and $S \subseteq R$ ⇒ all kernels bare

DSE truncated ($R_{\text{trunc}} \neq R$) ⇒ gap equation tries to „make up" for truncation

(Auto-tuned DSE)

Does not apply to Hamiltonian approach, since exact vacuum wave functional unknown

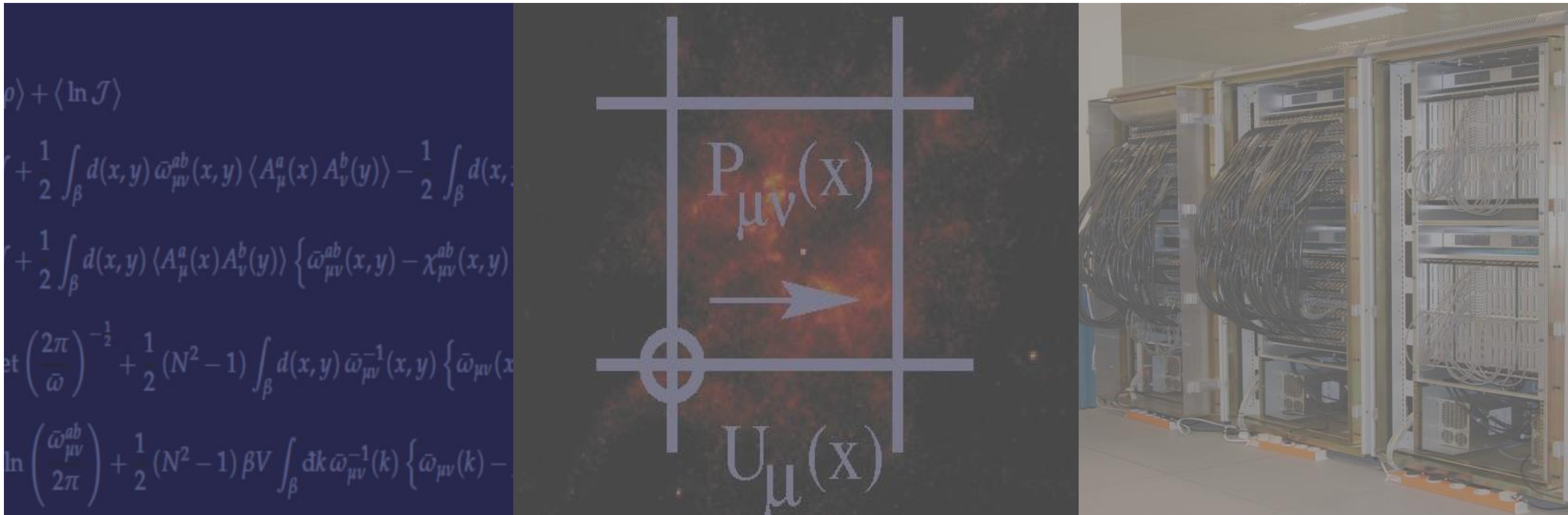
Example: static gluon propagator in Hamiltonian approach



D. Campagniari, H. Reinhardt PRD **82** (2010)

- Enlarging the search space can only improve the results!

→ Hard work always pays off (for once) !



Results for QCD

Hamiltonian Approach

Quark part of vacuum wave functional

$$\Psi_0 \sim \exp\left[\int \xi^\dagger\left(\beta s + \vec{\alpha}\cdot\vec{A}\,v + \beta\vec{\alpha}\cdot\vec{A}\,w\right)\right]$$

- Scalar coupling: $v = w = 0$

  Adler & Davis, Alkofer & Amundsen, Finger & Mandula

  chiral symmetry breaking, condensate too small

  (has no effect in covariant variational approach)

- Vector coupling: $v = w = 0$

  Pak & Reinhardt

  Chiral condensate & constitutent mass too small

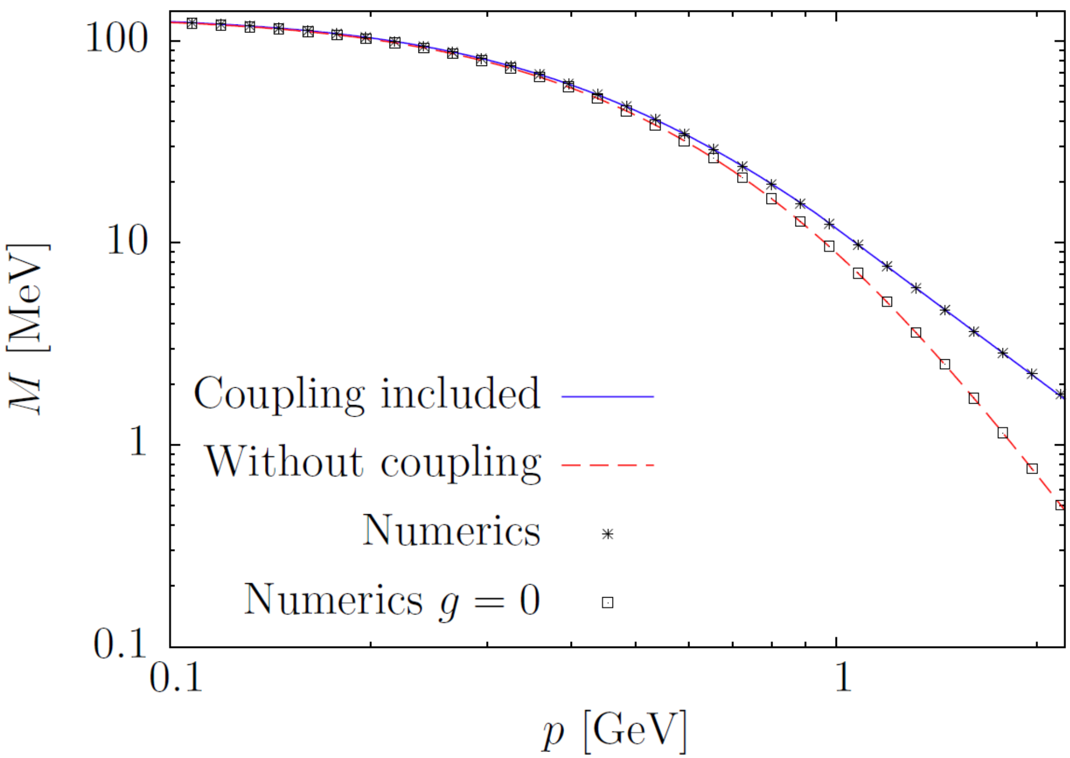
  linear divergences (*unrenormalizable*)

- Ext. vector coupling $v = w = 0$

  Campagniari, Ebadati, Reinhardt, Vastag

  All linear divergences cancel (non-trivial !)

D.Campagniari, E.Ebadati, H. Reinhardt, P.Vastag, PRD **94** 074027 (2016)



Quark condensate

$$\langle \bar{q}q \rangle = (-236\,\mathrm{Mev})^3$$

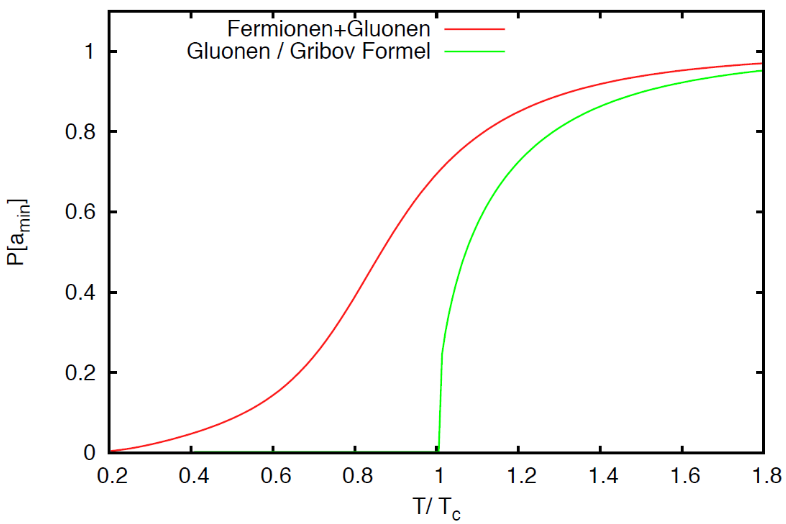
$$g = 2.1$$

Adler-Davis:

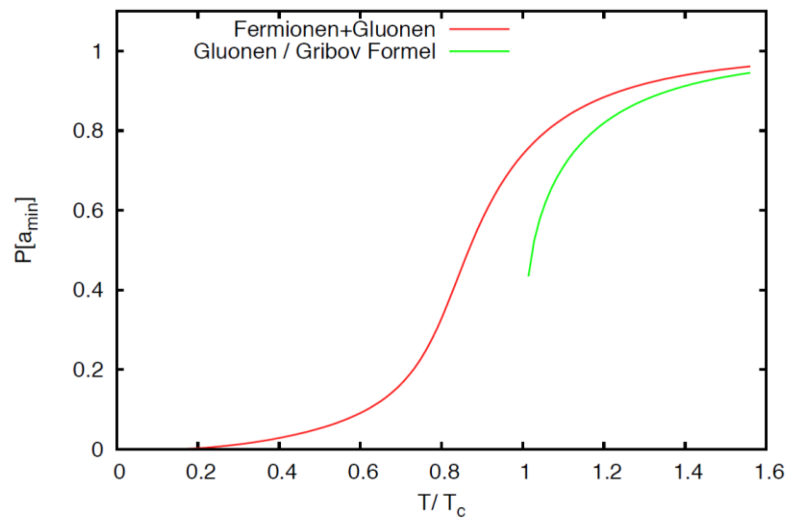
$$\langle \bar{q}q \rangle = (-185\,\mathrm{Mev})^3$$

Polyakov loop with dynamical fermions

J. Heffner, H.Reinhardt, P.Vastag, to be published



G=SU(2)



G=SU(3)

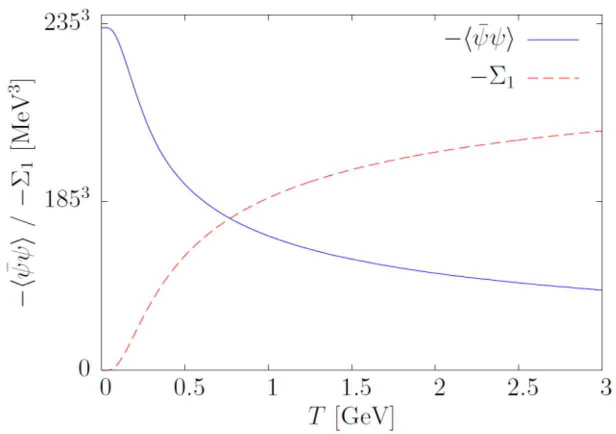
Deconfinement phase transition becomes cross-over at smaller T

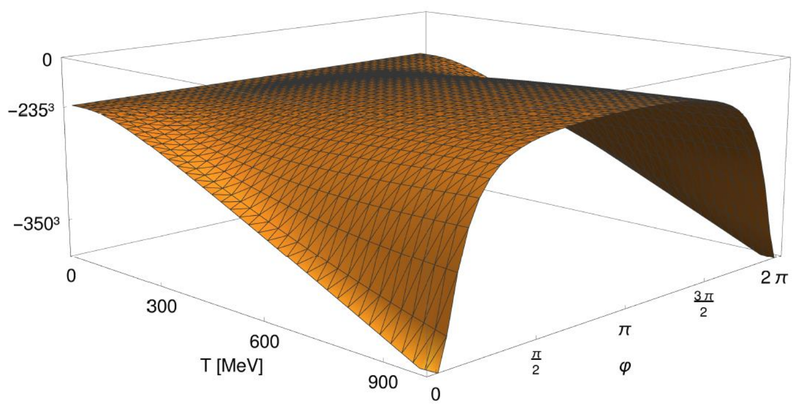
Chiral and dual condensate

Gattringer, PRL**97** (2006)

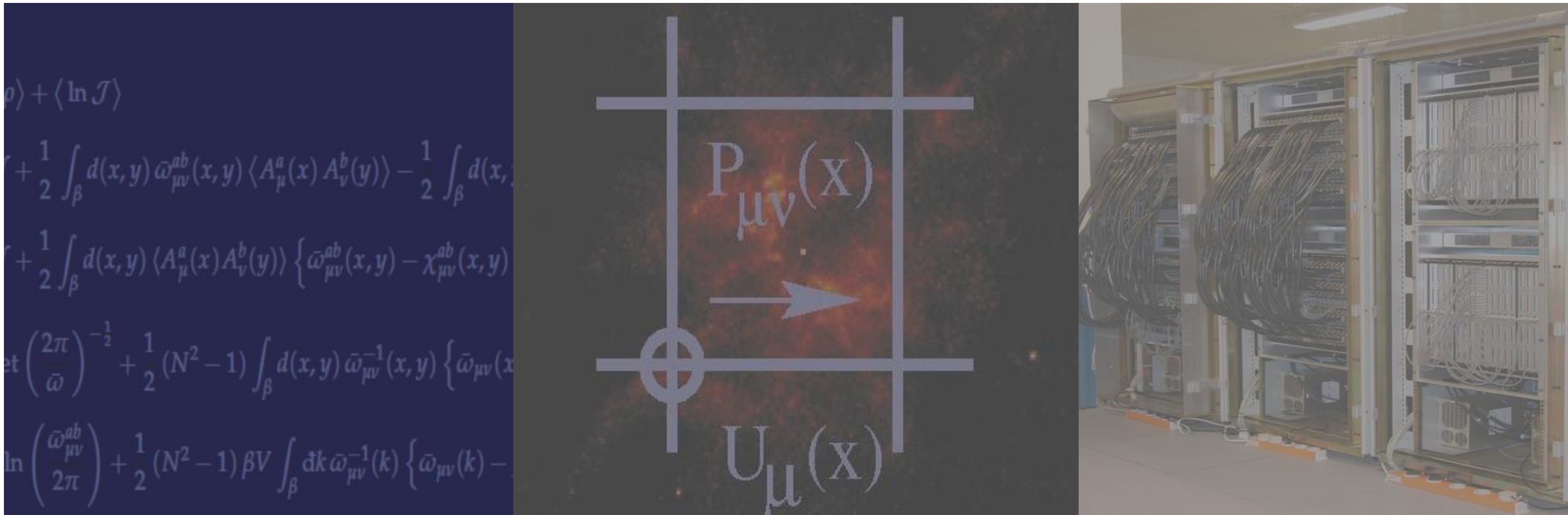
$$\Sigma_n \equiv \int_0^{2\pi} \frac{d\varphi}{2\pi}\, e^{in\varphi}\, \langle (\bar{q}q)_\varphi \rangle \qquad\qquad q(\beta) = e^{i\varphi}\, q(0)$$

- Loops winding n-times around the compactified time
- Σ_1 dressed Polyakov loop
- Imaginary chemical potential $\mu = i\,\dfrac{\pi - \varphi}{\beta}$





D.Campagniari, E.Ebadati, H. Reinhardt, P.Vastag, PRD **94** 074027 (2016)



Sumary and Outlook

Summary

- Variational Principles
  - Valuable tools to describe YM theory at T=0 and T>0
  - Propagators, deconfinement and thermodynamics (ghost dominance)
  - Deconfinement phase transition and thermodynamics from ghost dominance
- Extension to non-Gaussian measure ⇒ *DSE* (tuned kernels)
- Inclusion of *fermions*
  - Realistic chiral condensate and constituent mass
  - Realistic deconfinement transition (cross-over)
  - Dual condensate (imaginary chemical potential)

Outlook

- Fermions with *real* chemical potentials