Dynamic Critical Behaviour of φ^4

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Motivation

- QCD has critical point at end of CT line
- At 2OPT, things happen
  - Scale invariance
  - Universality
  - ⇒ Strong predictions about static quantities
- Phase diagram exploration in collision experiment: non-static

⇒ Finding signatures of a critical point in dynamic observables

Real Scalar Field Theory

$$\mathcal{L}(\varphi) = \frac{1}{2}\left(\partial^\mu \varphi \partial_\mu \varphi - m^2 \varphi^2\right) - \frac{\lambda}{4!}\varphi^4 \tag{1}$$

- free field with additional self-interaction term φ^4
- $m^2 < 0$: Has 2nd order phase transition
- Well-known (see e.g. Montvay-Münster)
- Easy to model on a lattice
  - Lattice of real numbers
  - Hamiltonian Dynamics with conjugate momentum field

Real Scalar Field Theory

$$\mathcal{L}(\varphi) = \frac{1}{2}\left(\partial^\mu \varphi \partial_\mu \varphi - m^2 \varphi^2\right) - \frac{\lambda}{4!}\varphi^4 - J\varphi \qquad (1)$$

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Second Order Phase Transitions

- Phase transition of nth order $\Leftrightarrow$ discontinuity in nth derivative of G (Ehrenfest classification)
- More modern: *Continuous* phase transition
  - Infinite correlation length ξ
  - Divergent susceptibility χ
  - Power law decays of correlators near critical point
- Examples: CO_2, QCD, ϕ^4

Example I

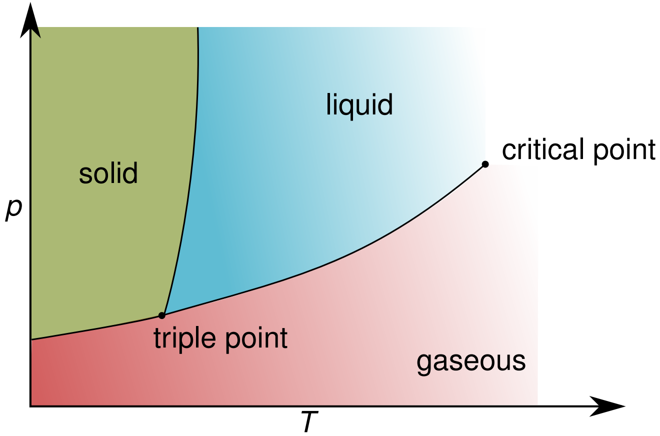

Figure: Phase diagram of CO_2. The line between liquid and gas phase ends in a critical point.

Example II

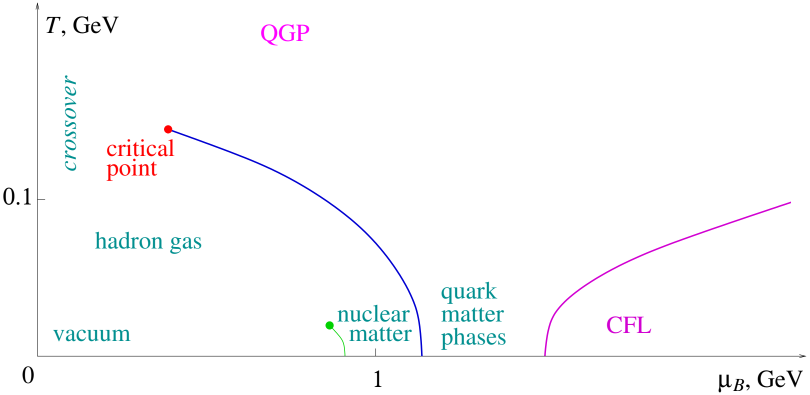

Figure: Semi-quantitative phase diagram of QCD, from: M. A. Stephanov: *QCD phase diagram: an overview*; PoS LAT2006:024,2006.

Example III

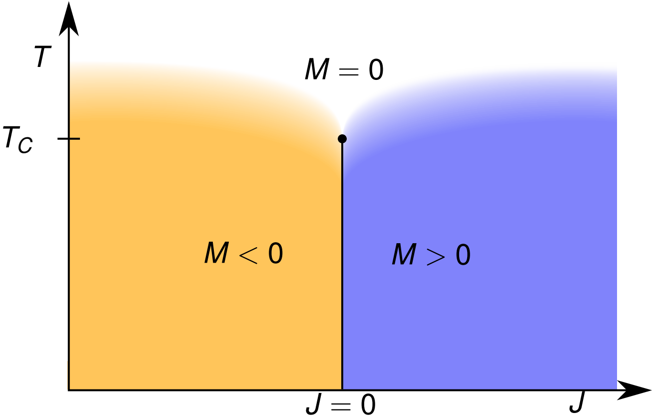

Figure: Simplified phase diagram of a ferromagnet/ϕ^4. The line between between opposing magnetization phases ends in a critical point at the curie temperature.

Quantification of Critical Behaviour

- Thermodynamic potential:

$$G = U - TS \tag{2}$$

- Antagonizing minimization processes
  - Low $T \Leftrightarrow$ minimize U by maximizing order
  - High $T \Leftrightarrow$ maximize S by maximizing disorder
  - $T = T_C \Leftrightarrow$ draw

$\Rightarrow$ Critical fluctuations, diverging correlation length

Correlation length ξ

- Defined by $\langle \varphi(x)\varphi(y)\rangle \propto \exp\left(-\left(x-y\right)/\xi\right)$
- Diverges at $T = T_c \Rightarrow$ *Clusterization*
  - Strongly correlated regions
  - Behave like a single field variable or Ising spin
  - Clusters themself form larger clusters

  $\Rightarrow$ *Scale invariance*

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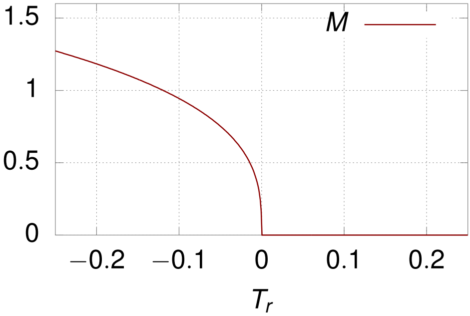
  $\Rightarrow$ *Scale invariance*
- From scale invariance: universality, power laws

Order Parameter

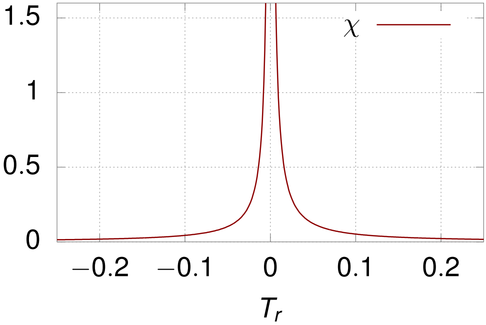
- $M = \langle\phi\rangle = \langle\sum\varphi(x)\rangle$
- $\langle\phi\rangle \propto \frac{\partial G}{\partial J}$
- a.k.a.
  magnetization, chiral condensate, Δn



$$\langle\phi\rangle(T) \propto \begin{cases} (T_c - T)^\beta & T < T_c \\ 0 & T \geq T_c \end{cases} \qquad (3)$$

Susceptibility

- $\chi = \frac{\partial \langle \phi \rangle}{\partial J} = \frac{\partial^2 G}{\partial J^2} = \frac{1}{T}\left(\langle \phi^2 \rangle - \langle \phi \rangle^2\right)$
- Magnetic/chiral susceptibility, compressibility
- Diverges at $T = T_c$



$$\chi \propto |T - T_c|^{-\gamma} \tag{4}$$

Universality Classes

- More power laws for ξ, $G(x-y, T, J)$, C
- Numerical values for exponents depend on:
  - spatial dimensions
  - spin-like degress of freedom
  - range of the interaction (long vs. short)

$\Rightarrow$ Universality classes

- one-component ϕ^4: 3D Ising UC

β	.326
γ	1.24
ν	.630
η	.036

Visualization I

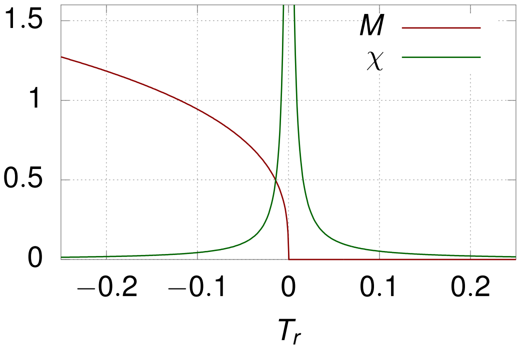


Figure: Power laws of order parameter and susceptibility

Visualization II

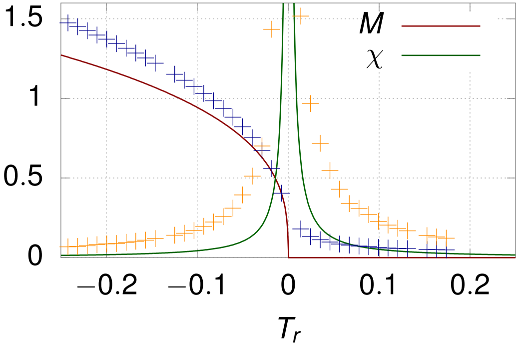


Figure: Power laws compared to data

Visualization III

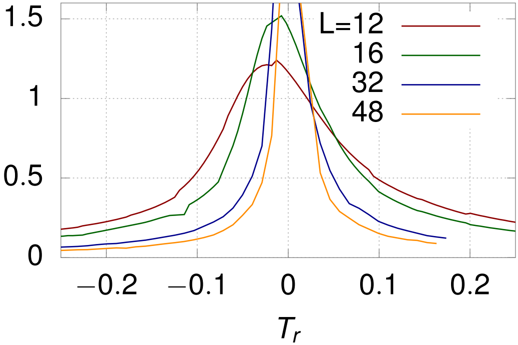


Figure: Finite Size Scaling of χ

Finite Size Scaling

- Correlation length ξ bound by L
- Observables depend on ξ/L
- $S \sim T_r^{\sigma}$ in inf. volume $\Rightarrow S_L(T = T_c) \sim L^{\sigma/\nu}$

$$\begin{aligned} S\left(T_r, L\right) &\approx L^{\frac{\sigma}{\nu}} f_S\left(\xi_\infty / L\right) \\ &= L^{\frac{\sigma}{\nu}} \bar{f}_S\left(\xi\left(T_r, L\right) / L\right) \\ S(T_c, L) &= L^{\frac{\sigma}{\nu}} \bar{f}_S(1) \sim L^{\frac{\sigma}{\nu}} \end{aligned}$$

What happens in real time?

- Remember: huge (infinite) susceptibility and correlation length at T_c
- Known phenomenon: Critical Slowing-Down (of MC algorithms)
- Signatures of crit. point in dynamic properties?

$\Rightarrow$ Look at real time observables, e.g. spectral function!

Definition

$$G(x,y) = \langle T\,\varphi(x)\varphi(y)\rangle \tag{5}$$

$$= F(x,y) - \frac{i}{2}\rho(x,y)\,sgn\left(x^0 - y^0\right) \tag{6}$$

$$F(x,y) = \frac{1}{2}\,\langle\{\varphi(x),\varphi(y)\}\rangle \tag{7}$$

$$\rho(x,y) = i\,\langle[\phi\,(x)\,,\phi\,(y)]\rangle \tag{8}$$

- Decomposition of the time-ordered propagator
- Contains information on dynamic properties
- Critical behaviour predicted

Meaning

- Statistical two-point function F: occupation numbers
- Spectral function ρ: available states
- Example: MFT spectral function

$$\rho(\omega, \vec{p}, T) = 2\pi i \, sgn(\omega)\delta\left(\omega^2 - \vec{p}^2 - M^2(T)\right) \tag{9}$$

Critical Behaviour of ρ

We know that

$$\rho(\omega, 0, 0) \sim \omega^{-\frac{2-\eta}{z}}, \tag{10}$$

$$\rho(t, 0, T_r) \sim t^{\frac{2-\eta}{z}-1} g\left(\frac{t}{\xi_t(T_r)}\right), \tag{11}$$

$$g(t) = \exp(-t), \quad \xi_t \sim \xi_L^z \sim T_r^{-(z\nu)} \tag{12}$$

in infinite volume.

- Introducing *dynamic critical exponent* z, *correlation time* ξ_t
- z determined by a "dynamic universality class"

Dynamic “Universality Classes”

- Additional influcences on z:
  - Conserved densities
  - Poisson brackets
- Classification scheme by Halperin/Hohenberg
  - “Models”, ordered by conserved fields and non-vanishing Poisson brackets
  - φ^4 w. Hamiltonian dynamics: Model C $\rightarrow z = 2.17$
  - Second closest match: Model A $\rightarrow z = 2.03$

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Challenge: Extract z from data, confirm Model C!

Classical Approximation I

- Spectral function defined via commutator $\leftrightarrow$ Poisson brackets, hard to calculate directly

$\Rightarrow$ Use fluctuation-dissipation theorem:

$$F(\omega, \vec{p}) = -i \left(\frac{1}{2} + n_T(\omega) \right) \rho(\omega, \vec{p}) \tag{13}$$

- Approximate BE distribution $n_T(\omega) \approx \frac{T}{\omega}$ for small ω:

$$F(\omega, \vec{p}, T) \approx -i \frac{T}{\omega} \rho(\omega, \vec{p}, T) \tag{14}$$

$$\Rightarrow \quad \rho(t, \vec{p}, T) \approx -\frac{1}{T} \frac{\partial}{\partial t} F(t, \vec{p}, T) \tag{15}$$

Classical Approximation II

- Use that $\pi(x) = \partial_t \varphi(x)$:

$$\rho(t, \vec{p}, T) = -\frac{1}{T}\partial_t \langle \varphi(t, \vec{p})\varphi(0,0)\rangle = -\frac{1}{T}\langle \pi(t, \vec{p})\varphi(0,0)\rangle \tag{16}$$

- Only look at $\vec{p} = 0$:

$$\rho(t, 0, T) = -\frac{1}{T}\left\langle \left(\int \mathrm{d}^d x \pi(t, x)\right)\left(\int \mathrm{d}^d y \varphi(0, y)\right)\right\rangle \tag{17}$$

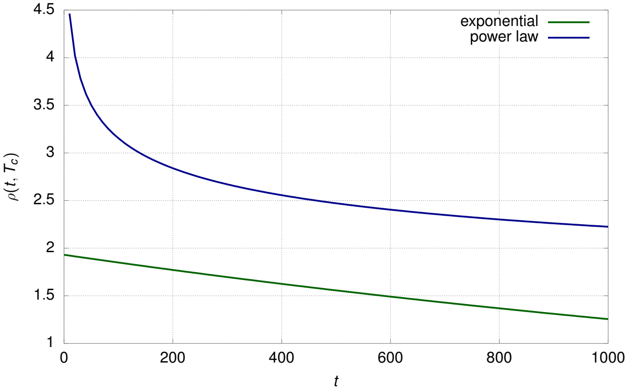
$\Rightarrow$ Use this approx. to find z!

Lattice Data

1. Generate equilibrium state (e.g. using HMC)
2. Evolve in RT using Hamiltonian Dynamics
3. Compute ρ using approximation
4. Repeat for N uncorrelated ensembles, take average

Extracting ξ_t, z

$$\rho(t, \vec{p} = 0, T_r) \sim t^{\frac{2-\eta}{z} - 1} \cdot \exp\left(-\frac{t}{\xi_t(T_r)}\right)$$



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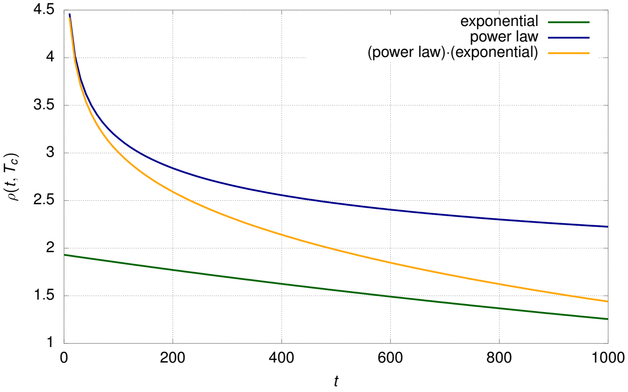

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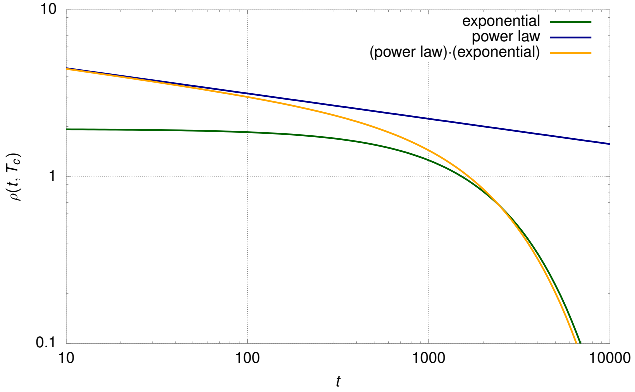

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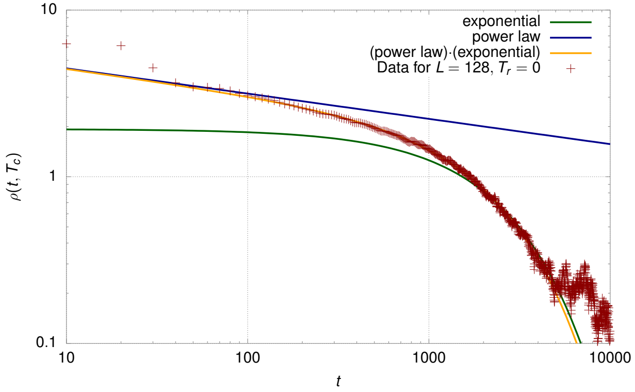
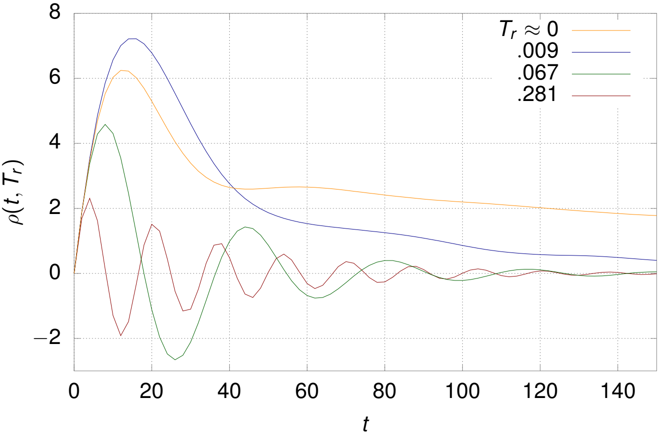


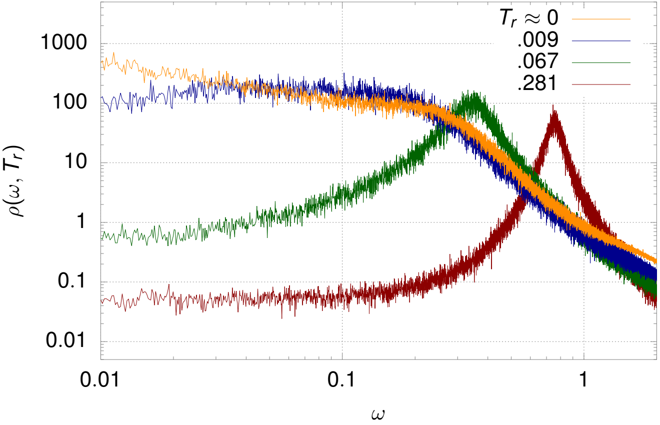
Caveats

- Simultaneous fit of power law and exponential
  - Power law exponent small (≈ -0.1)
  - Exponential decay time large ($\sim 10^3$)
- Power law determined by early times ($t < 200$)
- Exponential determined by late times ($t > 1000$)
- Strong correlation in intermediate times

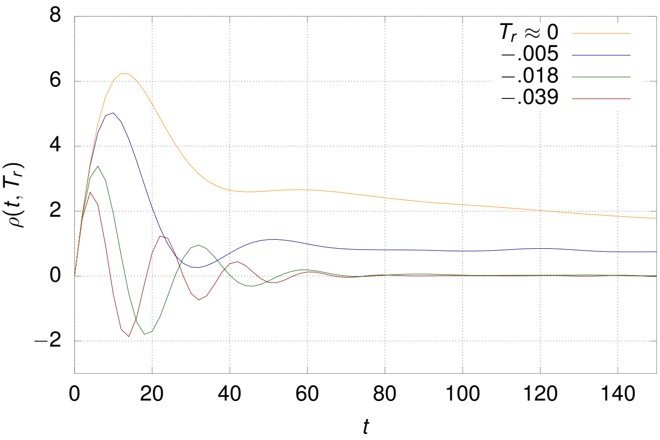
$T \geq T_c$ I



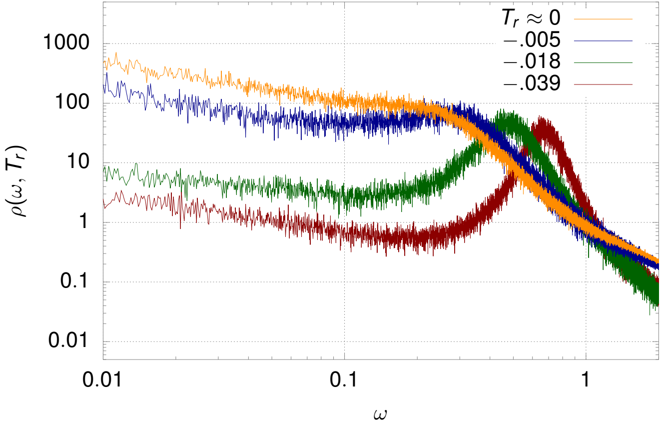
$T \geq T_c$ II


$T_r \approx 0$
.009
.067
.281
1000
100
10
1
0.1
0.01
0.01
0.1
1
$\rho(\omega, T_r)$
ω

$T \leq T_c$ I



$T \leq T_c$ II


$T_r \approx 0$
$-.005$
$-.018$
$-.039$
1000
100
10
1
0.1
0.01
$\rho(\omega, T_r)$
0.01
0.1
1
ω

Spectral Function

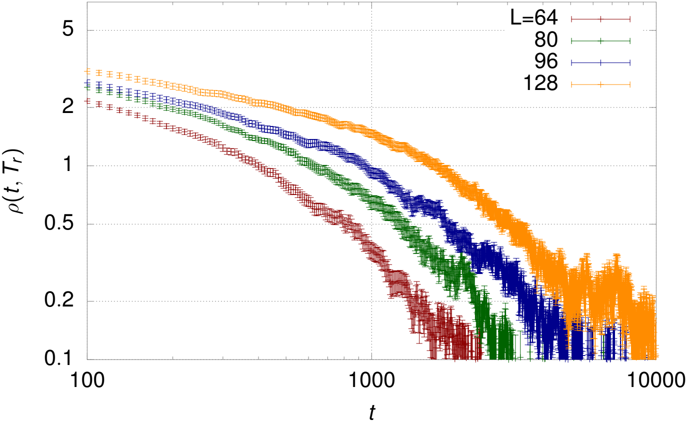


Figure: Finite size scaling of the spectral function. $\xi_t \sim L^z$

Power Law of ξ_t I

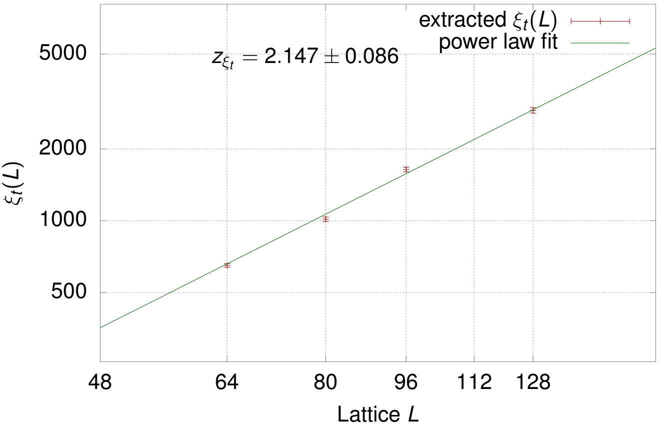


Figure: Finite size scaling of $\xi_t \sim L^z$. Model C predicts $z = 2.17$.

Power Law of ξ_t II

- ξ_t increases with volume as expected
- Good result of:

$$\boxed{z = 2.15(9)}$$

- Reminder: $z(\text{Model C}) = 2.17$, $z(\text{Model A}) = 2.03$

Power Law above T_c I

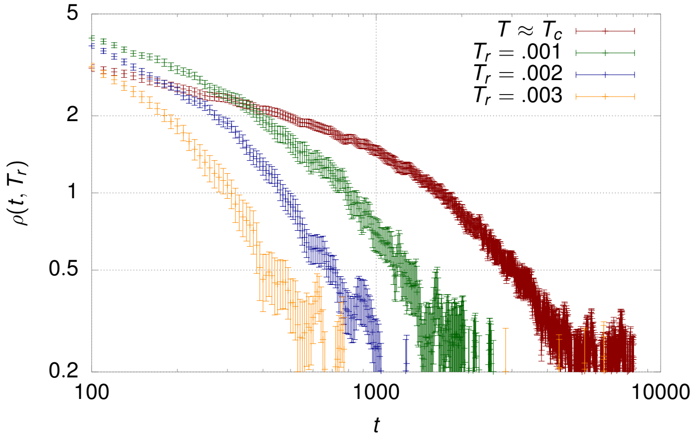


Figure: Increasing correlation time when approaching T_c. $\xi_t \sim T_r^{-(z\nu)}$

Power Law above T_c II

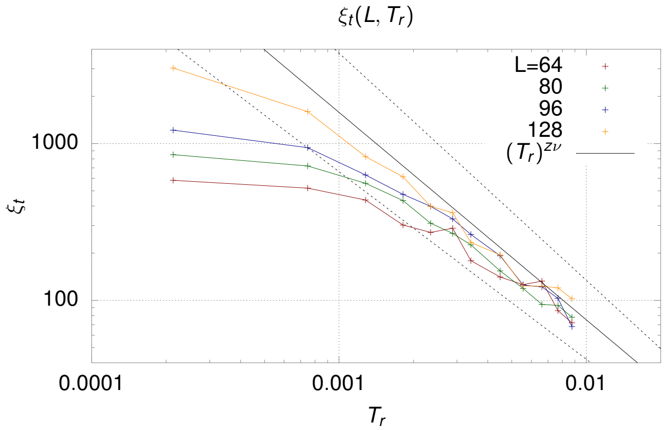
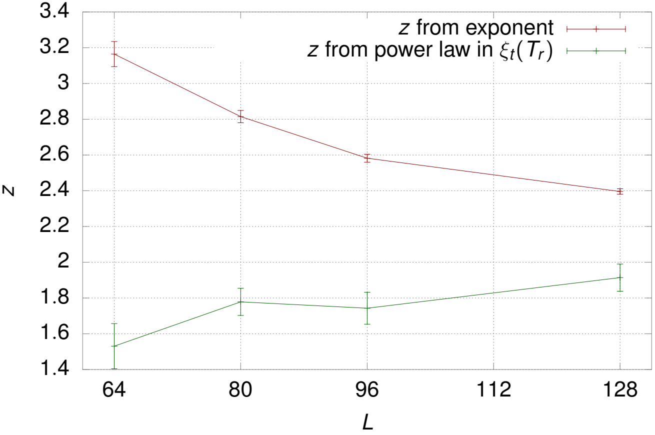


Figure: Scaling of $\xi_t \sim T_r^{-(z\nu)}$

Power Law above T_c III

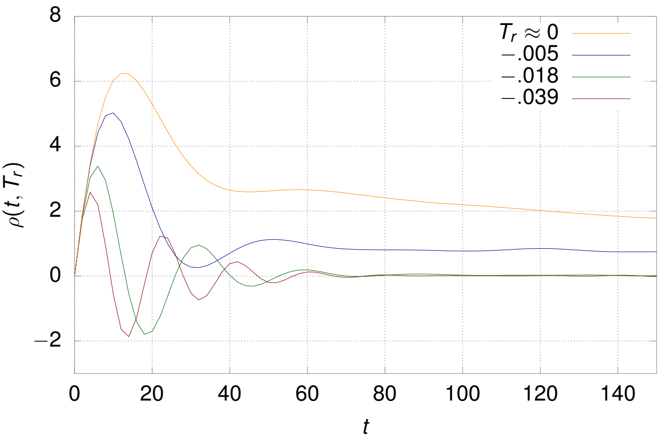


Power Law above T_c IV

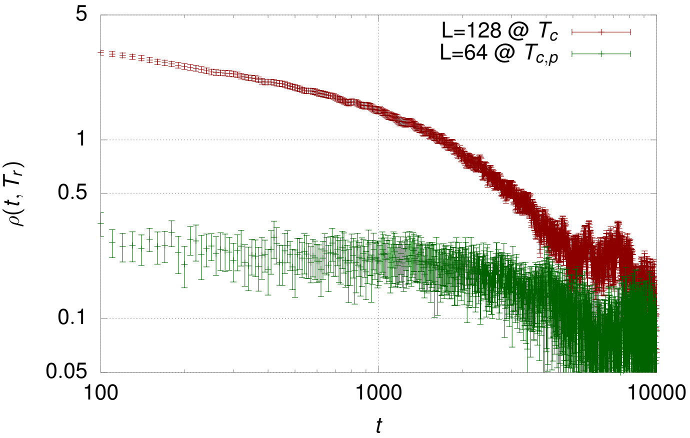
- Direct power law consistently overestimates z
- Edge of peak of $\xi_t(T_r)$ underestimates z
- Underwhelming precision, but increases with lattice volume
- Conservative estimate:

$$\boxed{z = 2.2(2)}$$

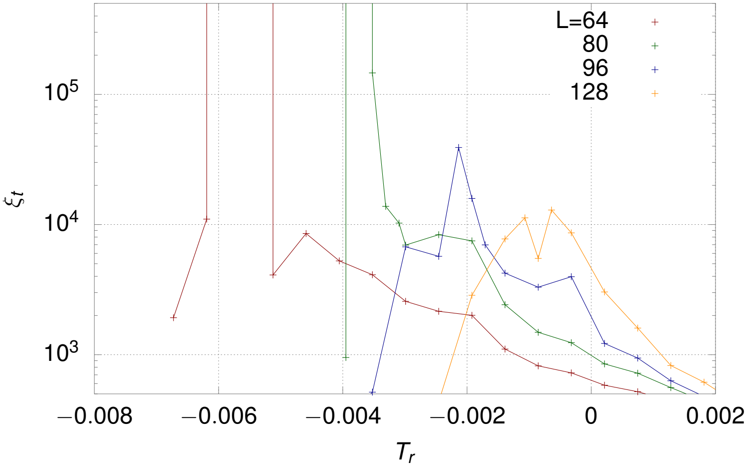
Pseudo-Critical Points I



Pseudo-Critical Points II



Pseudo-Critical Points III



Pseudo-Critical Points IV

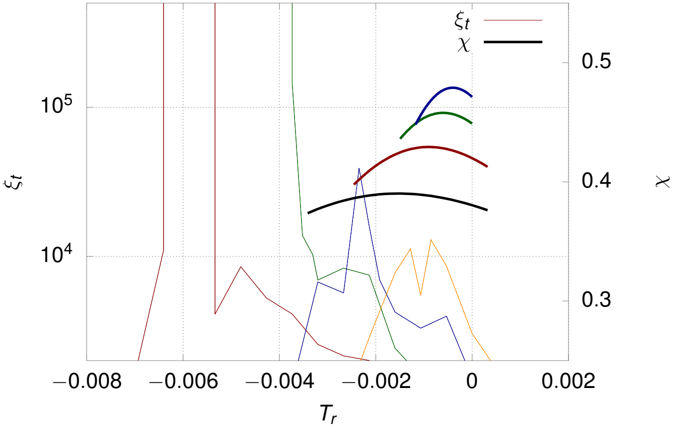


Figure: Comparison between ξ_t and χ. The peaks hardly overlap for identical lattice volumes.

Pseudo-Critical Points V

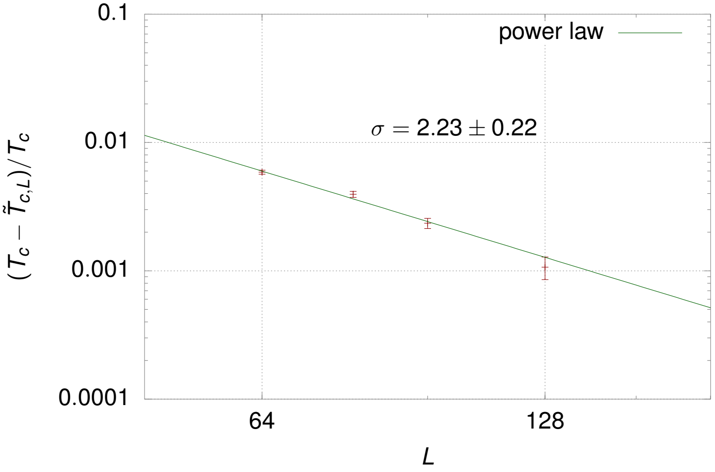
- Maximum of ξ_t below T_c
- Increasingly hard to fit
- Further away from therm. limit than χ

Summary

- Result of $z = 2.15(9)$ consistent **Model C**
- Exact calculation of z difficult, best results with FSS at T_c
- Huge lattices needed for good precision

- Outlook
  - Increasing field components
  - Different evolution algorithms
  - Different models/theories

Pseudo-Critical Points I



Pseudo-Critical Points II

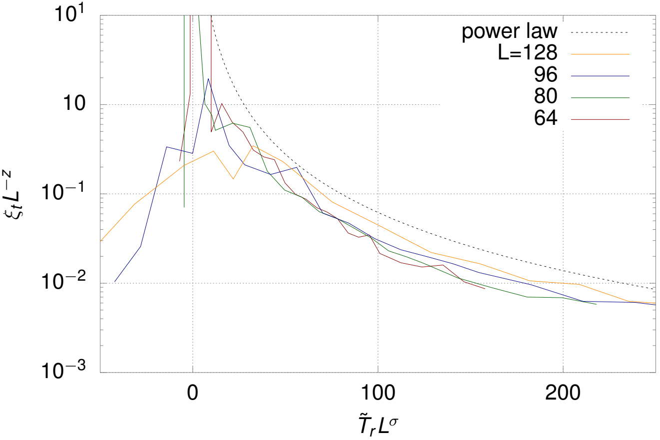


Figure: The ξ_t-peaks “rescaled”. The edge resembles a power law.

Pseudo-Critical Points III

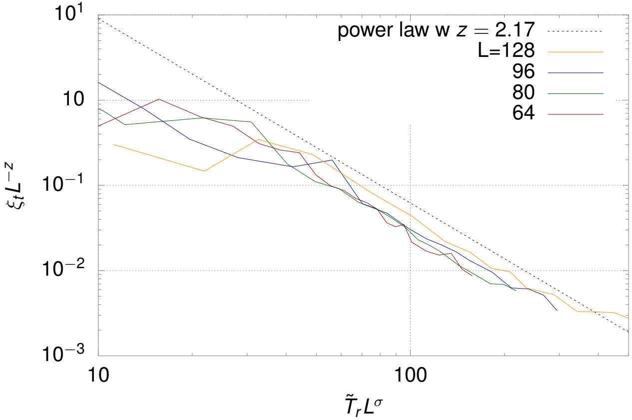


Figure: The ξ_t-peaks “rescaled”. The edge resembles a power law with exponent $z = 2.17$