Cooling Stochastic Quantization with Colored Noise

Scale-controlled Gradient Flows

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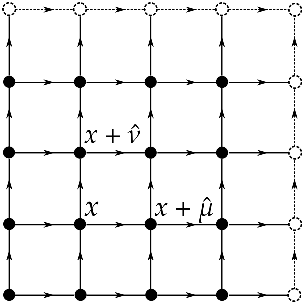
Motivation

Lattice Field Theory

- **Idea:** Regularize path-integral on a space-time lattice

[Wilson, "Confinement of Quarks", 1974]



Two-dimensional lattice with periodic b.c.

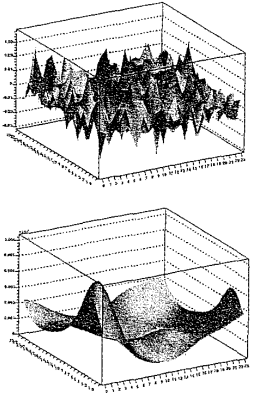
- Powerful non-perturbative framework
- Ab-initio calculations of hadron masses
- **Interest:** quark confinement, chiral symmetry breaking, QCD phase diagram

UV-fluctuations and the need for smoothing

- **Interest:** QCD phase diagram and finite density
  → **sign problem**
- **Tackling** the sign problem: Stochastic quantization promising candidate
- **Problems:** Convergence, UV-fluctuations
  → **cooling methods required**
  →scale separation, improvement of signal-to-noise ratio without altering the physics of interest

Cooling methods

- **Context:** QCD vacuum and topology in Yang-Mills theory (e.g. topological charge density)
- **Problem:** short-distance fluctuations of order of the lattice spacing blur the underlying classical structure
- **Solution:** cooling methods reveal physical properties on the lattice



[Rothe, "Lattice Gauge Theories", 2005: Wantz, 2003]

Cooling methods

- **Wilson flow** [Lüscher, 2010]

$$\boxed{\frac{\partial \varphi(x, t_{\mathrm{F}})}{\partial t_{\mathrm{F}}} = -\frac{\delta S_E}{\delta \varphi(x, t_{\mathrm{F}})}}$$

- **Procedure:** Smooth field configurations in a damping equation such as the Wilson flow or use action minimization methods [Garcia-Perez, Philipsen, Stamatescu, 1999]
- **Problem:** When does one **stop cooling**?
  $\rightarrow$ physical scale of the system

Idea: Combine Langevin equation with gradient flow

Langevin

$$\frac{\partial\phi(x,\tau)}{\partial\tau} = -\frac{\delta S_E}{\delta\phi(x,\tau)} + \eta(x,\tau)$$

Generation of field configurations
(full fluctuation spectrum)

Wilson flow

$$\frac{\partial\varphi(x,t_\mathrm{F})}{\partial t_\mathrm{F}} = -\frac{\delta S_E}{\delta\varphi(x,t_\mathrm{F})}$$

Cooling of configurations
$\varphi(x,t_\mathrm{F}=0) = \phi(x,\tau=\infty)$

Related cooling methods?
→ modify noise spectrum in the Langevin equation by introducing momentum cutoff
→ sample smooth fields directly
→ no stopping criterion needed

Stochastic Quantization with Colored Noise

- Analogy between a Euclidean quantum field theory and a classical statistical mechanical system in thermal equilibrium with a heat reservoir. [Parisi, Wu, 1981]
- **Stochastic process** – evolution of fields in fictitious time τ described by the **Langevin equation**

$$\boxed{\frac{\partial\phi(x,\tau)}{\partial\tau} = -\frac{\delta S_E}{\delta\phi(x,\tau)} + \eta(x,\tau)}$$

$$\langle\eta(x,\tau)\rangle_\eta = 0, \quad \langle\eta(x,\tau)\eta(y,\tau')\rangle_\eta = 2\delta^{(d)}(x-y)\delta(\tau-\tau')$$

- Quantum fluctuations encoded in Gaussian white noise
$\Rightarrow$ **Aim:** attack UV fluctuations here!

Stochastic quantization: Fokker-Planck equation

$$\boxed{\frac{\partial P(\phi,\tau)}{\partial\tau} = \int \mathrm{d}^d x \frac{\delta}{\delta\phi(x,\tau)}\left(\frac{\delta S_E}{\delta\phi(x,\tau)} + \frac{\delta}{\delta\phi(x,\tau)}\right)P(\phi,\tau)\,.}$$

Stationary distribution $\exp(-S_E[\phi])$ is the Boltzmann weight in the partition function.

Stochastic regularization in the continuum

- Modify the noise term using a regulator $r_\Lambda(\Delta_x) \to$ control the UV-fluctuations by the cutoff parameter Λ [Bern et. al. , 1987]
- White noise limit: $r_\Lambda(\Delta_x) \to 1,\ \Lambda \to \infty$
- Regularized Langevin and Fokker-Planck equation

$$\frac{\partial\phi(x,\tau)}{\partial\tau} = -\frac{\delta S_E}{\delta\phi(x,\tau)} + r_\Lambda(\Delta_x)\,\eta(x,\tau)\,,$$

$$\frac{\partial P[\phi,\tau]}{\partial\tau} = \int \mathrm{d}^d x \frac{\delta}{\delta\phi_x}\left(\frac{\delta S_E}{\delta\phi_x} + r_\Lambda^2(\Delta_x)\frac{\delta}{\delta\phi_x}\right)P[\phi,\tau]\,.$$

Lattice QFT with Colored Noise

Colored noise using a sharp regulator on the lattice

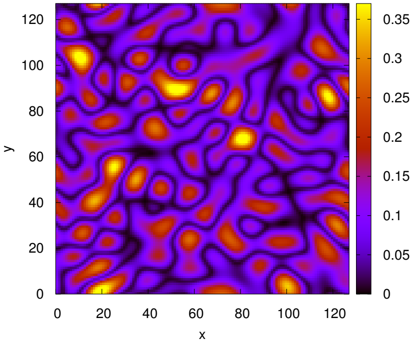
- Consider d-dimensional lattice of volume $\Omega = (a\, N)^d$
- Lattice momenta:
  $k_\mu = \frac{2\pi}{a} n_\mu \quad , \mu = 1, \ldots, d \,,$
  where $n_\mu = -N/2 + 1, \ldots, N/2$
- Momentum cutoff:

$$p_\Lambda(s_q) := \left(d \left(\frac{2\pi}{aN} \right)^2 {s_q}^2 \right)^{1/2} ,$$

$$s_q = 0, 1, \ldots, N/2$$

- Sharp regulator:

$$\boxed{r_{p_\Lambda}(k^2) = \theta\left(p_\Lambda^2(s_q) - k^2\right)}$$

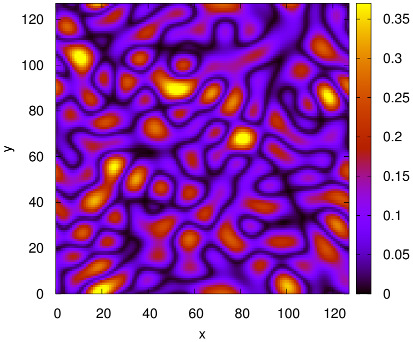


- Regularized noise field

$$\eta_{p_\Lambda}(x) = \frac{1}{\Omega^{1/2}} \sum_k e^{ik\cdot x}\, r_{p_\Lambda}(k^2)\, \eta(k)$$

Colored noise using a sharp regulator on the lattice

- Implementation of $r_{p_\Lambda}(k^2) = \theta\left(p_\Lambda^2(s_q) - k^2\right)$: Retain momentum modes with $n_\mu n_\mu \leq s_q^2$ and remove larger modes with $n_\mu n_\mu > s_q^2$.
- Limits:
  - $s_q = N/2 \leftrightarrow$ white noise
  - $s_q = 0$: only zero-momentum mode contributes



Implications of using a sharp cutoff

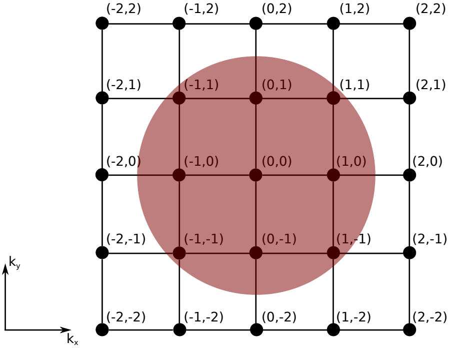
Colored noise Langevin equation

$$\phi(x,\tau_{n+1}) = \phi(x,\tau_n) - \frac{\delta S_E}{\delta\phi(x,\tau_n)}\Delta\tau + \sqrt{2\Delta\tau}\,\eta_{p_\Lambda}(x,\tau_n)$$

- Split field into classical and quantum contribution

$$\phi(k) = \phi_{\mathrm{cl}}(k) + \delta\phi_{\mathrm{qu}}(k)$$

- $\delta\phi_{\mathrm{qu}}(k)$ non-zero on momentum sublattice where $k^2 \leq p_\Lambda^2(s_q) \rightarrow$ **quantum lattice**
- Classical field ϕ_{cl} lives on the full momentum lattice $\rightarrow$ **classical lattice**



A simple model to probe colored noise

Real scalar field theory on the lattice

$$S = \sum_x a^d \left\{ \frac{1}{2} \sum_{\mu=1}^{d} \frac{(\phi_0(x + a\,\hat{\mu}) - \phi_0(x))^2}{a^2} + \frac{m_0^2}{2}\phi_0(x)^2 + \frac{g_0}{4!}\phi_0(x)^4 \right\}$$

More convenient for <u>numerical simulations</u>:

$$\boxed{S = \sum_x \left\{ -2\kappa \sum_{\mu=1}^{d} \phi(x)\phi(x+\hat{\mu}) + \phi(x)^2 + \lambda[\phi(x)^2 - 1]^2 - \lambda \right\}}$$

Parameter relations

$$\boxed{a^{\frac{d-2}{2}}\phi_0 = (2\kappa)^{\frac{1}{2}}\phi, \quad (am_0)^2 = \frac{1-2\lambda}{\kappa} - 2d, \quad a^{-4\frac{d-2}{2}+d} g_0 = \frac{6\lambda}{\kappa^2}}$$

Observables

- Magnetization (order parameter)

$$M := \frac{1}{\Omega}\sum_x \phi(x)$$

- Connected two-point correlation function

$$G_c(x,y) := \langle\phi(x)\phi(y)\rangle - \langle\phi(x)\rangle\langle\phi(y)\rangle$$

- Correlation function of time slices

$$G_c(t) = \frac{1}{V}\sum_{\vec{x}} G_c(x,0)$$

- **Connected susceptibility**

$$\chi_2 = \sum_x G_c(x,0) = \Omega\left(\langle M^2\rangle - \langle M\rangle^2\right)$$

- **Fourth order cumulant** [Binder, 1981]

$$U_L = 1 - \frac{1}{3}\frac{\langle M^4\rangle}{\langle M^2\rangle^2}$$

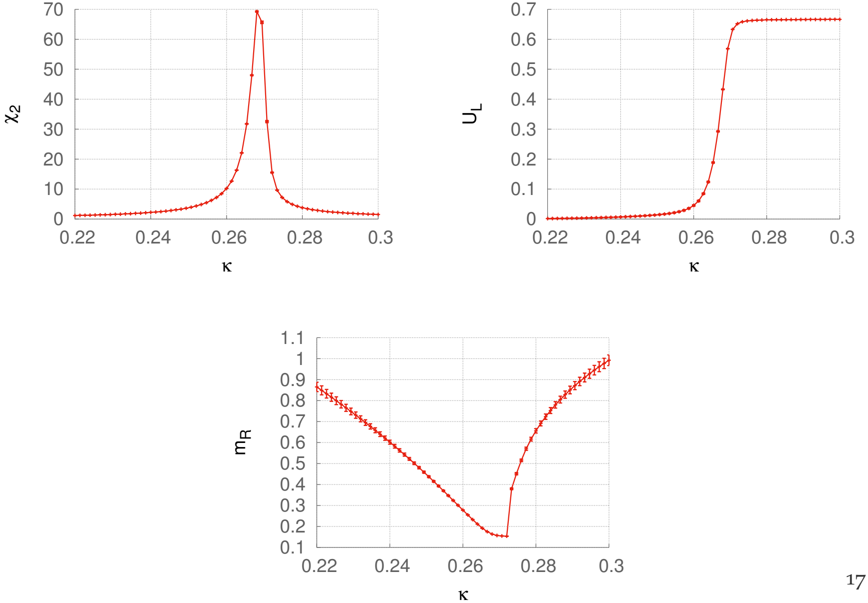
- Second moment of the correlator

$$\mu_2 := 2\,d\,V\sum_t t^2\,G(t)_c$$

- **Renormalized mass**

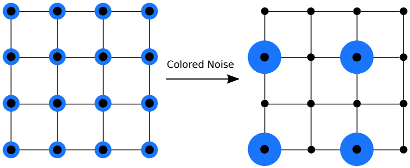
$$m_R^2 = \frac{2\,d\,\chi_2}{\mu_2}$$

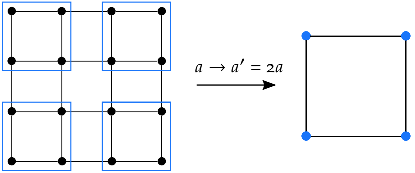
Phase structure of the theory in $d = 2$



Colored noise and the Renormalization group

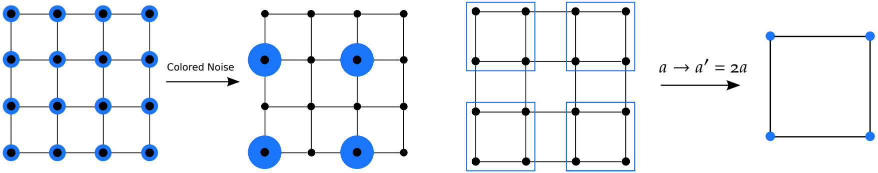
Classical and quantum lattice





- Infinitesimal change of Λ governed by the RG $\to$ changes couplings (κ, λ).
- Colored noise leads to separation into classical and quantum lattice.
- Real space: quantum lattice coarser than the classical lattice.
- CN on the lattice locally smears fields with smoothing scale given by the cutoff s_q.
- Compensate decrease of cutoff s_q by "block spinning" step.
- Thereto, define the theory on a finer lattice but with physical volume fixed.

Block spin transformation



- To define theory on finer lattice introduce scale factors

$$a \to a' = s^{-1}a\,,\ N \to N' = sN\,,\ k \to k' = sk\,,\ s \geq 1$$

- Cutoff transformation:

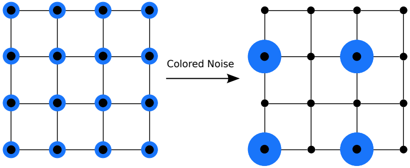
$$p_\Lambda(a,N) \to p'_\Lambda(a',N') = s^{-1}p_\Lambda(a',N')\,,$$

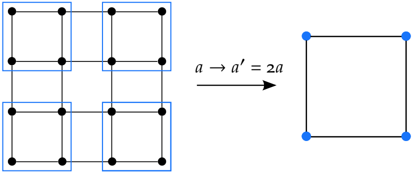
- Bare parameters:

$$(am_0)^2 \to s^{-2}\,(am_0)^2\,,$$

$$a^2 g_0 \to s^{-2}a^2 g_0\,.$$

Block spin transformation





- Block-spinning equations:

$$\boxed{\begin{aligned} s^{-2}\left[\frac{1-2\lambda}{\kappa} - 4\right] &= \frac{1-2\lambda'}{\kappa'} - 4 \\ s^{-2}\frac{6\lambda}{\kappa^2} &= \frac{6\lambda'}{\kappa'^2} \end{aligned}}$$

- Recall in $d = 2$:

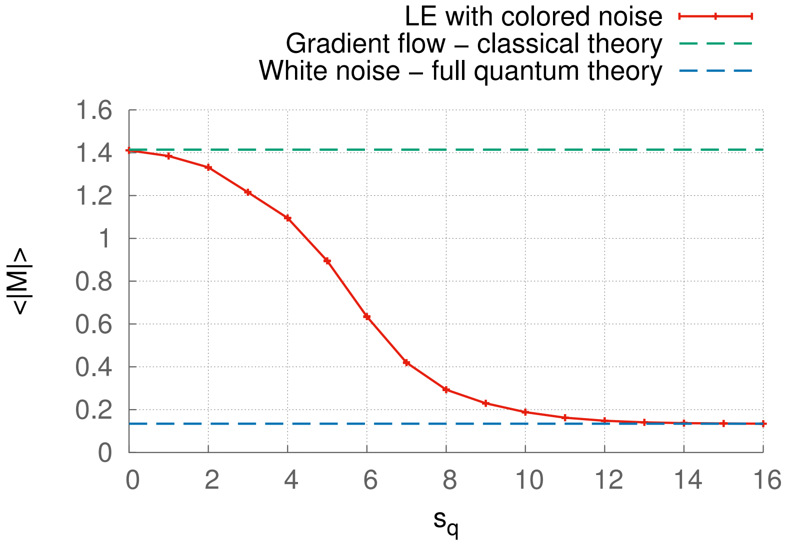
$$(am_0)^2 = \frac{1-2\lambda}{\kappa} - 4$$

$$a^2 g_0 = \frac{6\lambda}{\kappa^2}$$

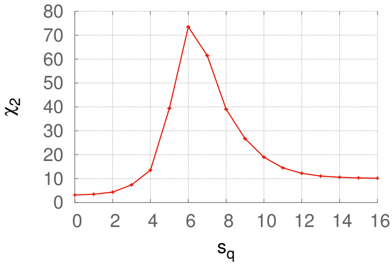
- Solve for (κ', λ') to be used in the colored noise simulation.

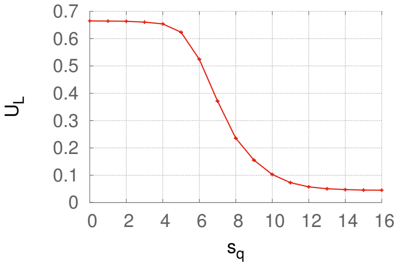
Colored noise: incomplete blocking

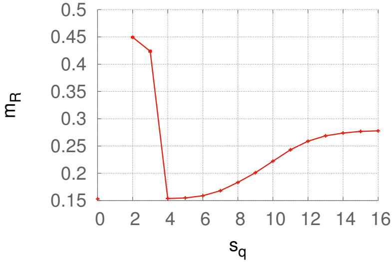
Colored noise effects for fixed (κ, λ)



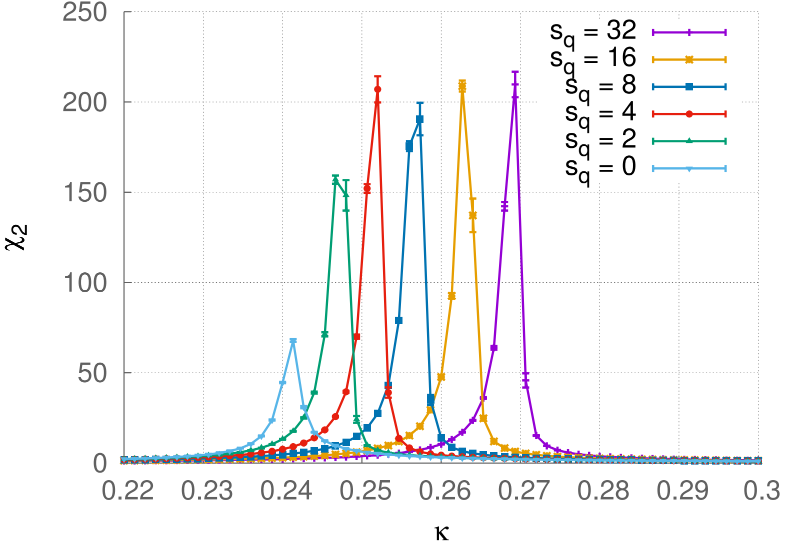
Colored noise effects for fixed (κ, λ)





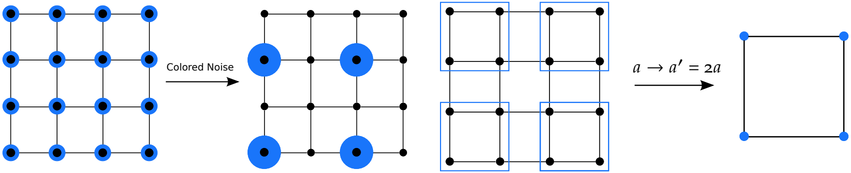


Parameter scan in κ for fixed λ

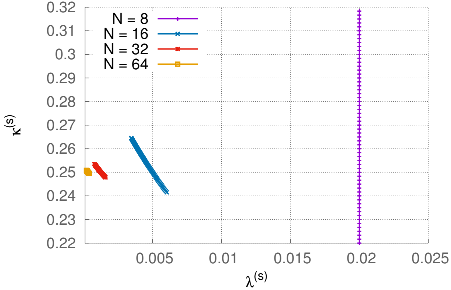


Colored noise: complete blocking

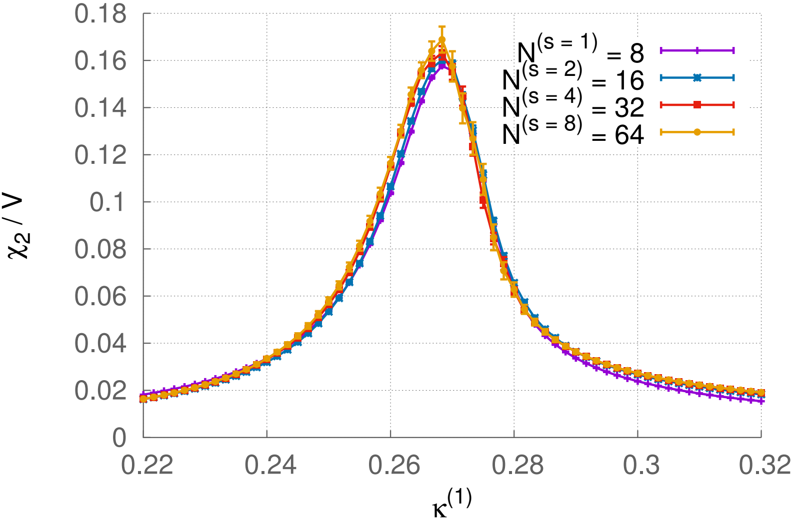
Colored noise: Complete blocking



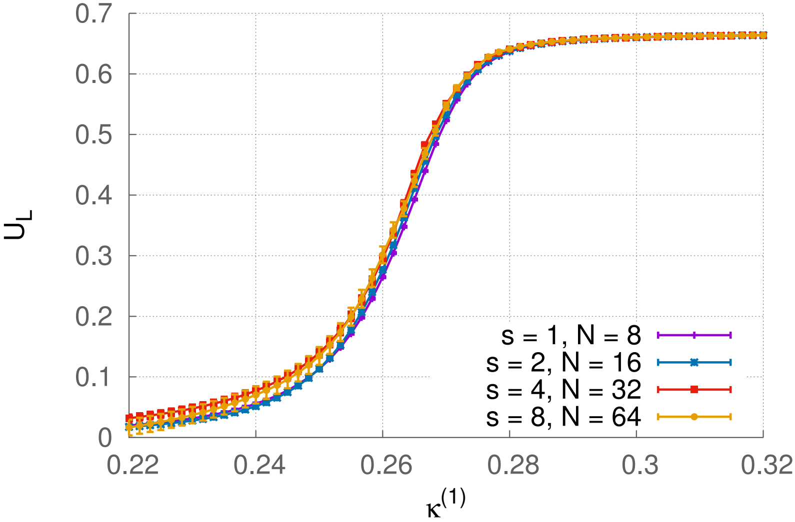
- **Reference simulation with white noise** $(s = 1)$, $0.22 \leq \kappa^{(s=1)} \leq 0.32$, $\lambda^{(s=1)} = 0.02 = \mathrm{const}$, $p_\Lambda = p_{\Lambda,\mathrm{max}}$, $a = a^{(s=1)}$, $N = N^{(s=1)} = 8$
- **Compare with colored noise simulations on finer lattices** with $a^{(s)} = s^{-1} a^{(1)}$, $s = 2, 4, 8$, $N^{(s)} = s\,N$, (κ^s, λ^s) $\to$ **cutoff** $p_\Lambda = s^{-1} p_{\Lambda,\mathrm{max}}$



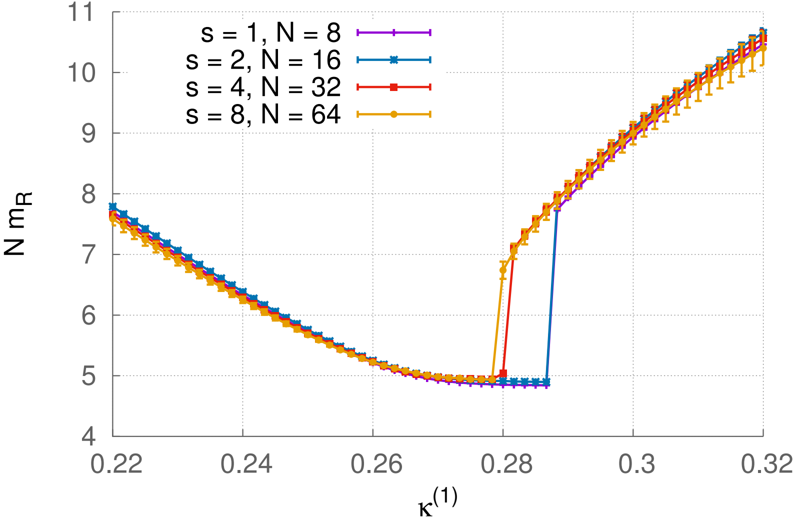
Colored noise: Complete blocking



Colored noise: Complete blocking



Colored noise: Complete blocking



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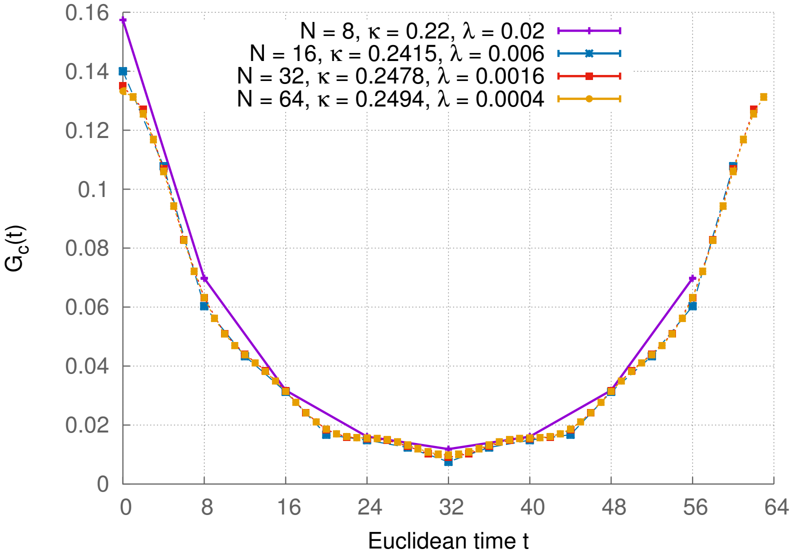


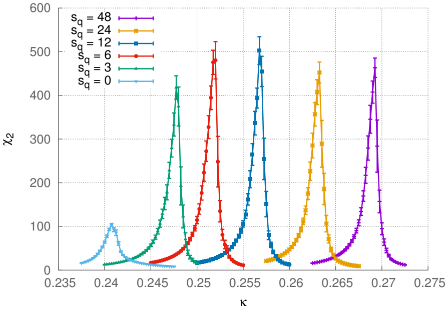
Summary so far

- Same physics in white noise and colored noise simulations by rescaling the lattice spacing → compensate for colored noise effect by adjusting the parameters (κ, λ).
- Equivalent to standard blocking transformation at tree-level.
- Procedure fails if lattice is too small → limited number of blocking steps.
- **Problem:** Costly procedure since decreasing the lattice spacing requires a simultaneous increase of the number of lattice points.

Applying colored noise cooling

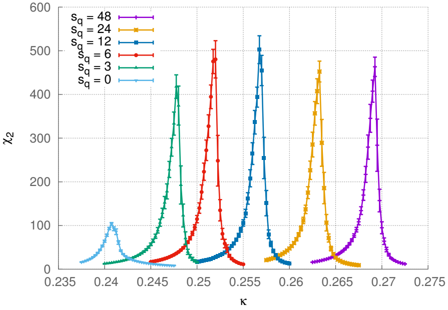
Applicability of colored noise cooling

- Colored noise simulation: physics content of the theory unchanged if $s_q \gg s_q^{\mathrm{phys}}$.
- Bare couplings in the classical action must be changed due to RG running.
- At low cutoffs $s_q \ll s_q^{\mathrm{phys}}$ relevant physical fluctuations are removed from the theory.



Applicability of colored noise cooling

- **Goal:** Simulate at fixed lattice size and given lattice spacing a
- Check in which range of s_q the use of the classical action in the colored noise simulation is valid.
  → Investigate this in terms of **(lattice) functional renormalization group**



Approximation: Use continuum flow equations for lattice parameters!

Functional renormalization group in the continuum

- Dimensionful one-loop FRG flow equations for the mass m and the coupling g in $d = 2$

$$\boxed{\begin{aligned} \Lambda\,\partial_\Lambda m^2 &= -\frac{g}{4\pi}\frac{1}{1+m^2/\Lambda^2} \\ \Lambda\,\partial_\Lambda g &= \frac{3}{4\pi}\frac{g^2}{\Lambda^2}\frac{1}{(1+m^2/\Lambda^2)^2}\,. \end{aligned}}$$

- Render flow equations dimensionless by rescaling with constant lattice spacing a
  $\rightarrow m^2 \rightarrow (am)^2,\, g \rightarrow a^2 g,\, \Lambda \rightarrow a\Lambda.$

- Dimensionless flow equations

$$\Lambda\,\partial_\Lambda (a\,m)^2 = -\frac{a^2 g}{4\pi}\,\frac{1}{1+(am)^2/(a\Lambda)^2}$$

$$\Lambda\,\partial_\Lambda a^2 g = \frac{3}{4\pi}\,\frac{(a^2 g)^2}{(a\Lambda)^2}\,\frac{1}{(1+(am)^2/(a\Lambda)^2)^2}\,.$$

- translate into flow equation for the lattice parameters (κ, λ)

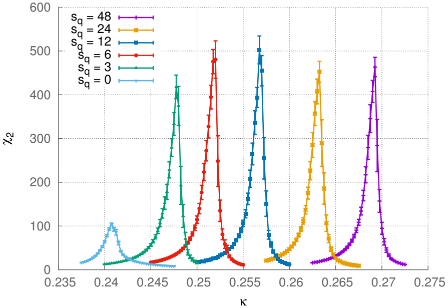
$$\partial_t\,\kappa(t) = \frac{3}{2}\,\frac{\lambda(t)}{\pi}\,\kappa(t)\,\frac{e^{2t}}{1+2\lambda(t)}\,\frac{\kappa(t)\left(e^{2t}-4\right)-8\lambda(t)+1}{\left[\kappa(t)\,(e^{2t}-4)-2\lambda(t)+1\right]^2}$$

$$\partial_t\,\lambda(t) = \frac{3}{2}\,\frac{\lambda(t)^2}{\pi}\,\frac{e^{2t}}{1+2\lambda(t)}\,\frac{2\,\kappa(t)\left(e^{2t}-4\right)-10\lambda(t)+5}{\left[\kappa(t)\;(e^{2t}-4)-2\lambda(t)+1\right]^2}\,.$$

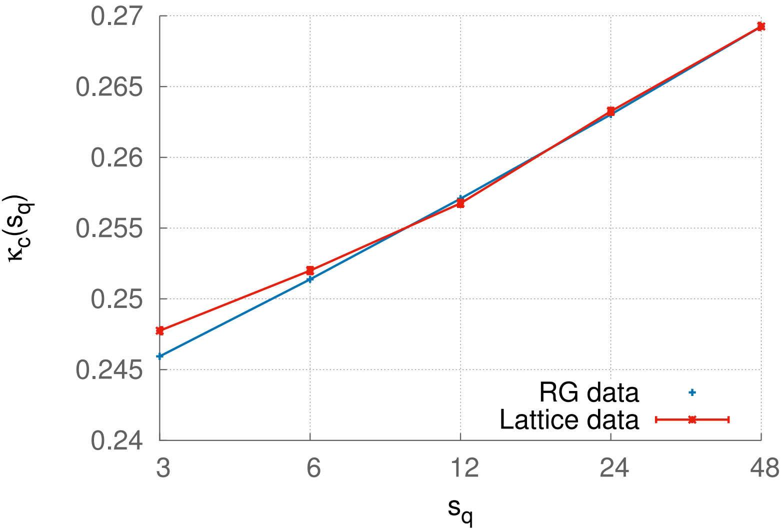
with the flow time $t = \log(a\,\Lambda)$.

Solutions to the RG equations

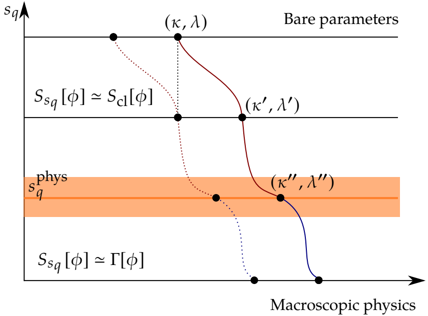
- Take **peak position** κ_c of white noise simulation as **input** for flow equation at scale $t_{\max} = \log(a\Lambda_{\max}) = \log(C\sqrt{2}\pi s_q)$ with a constant parameter C.
- Decrease cutoff $t_{\max} \to t_{\max} - n\ \log(2)\,,\ n = 1, 2, 3, 4$ and compare resulting values for κ with the peak positions depending on s_q.
- Caveat: In the lattice simulations λ is hold fixed.

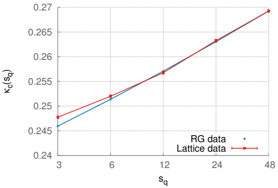


Comparison of lattice and FRG data



Applicability of the classical action





- $s_q \gg s_q^{\text{phys}}$: use classical action with adjusted couplings (κ, λ) according to RG.
- Higher couplings generated by the RG flow negligible
- Map out intermediate regime $s_q \approx s_q^{\text{phys}}$: Results from flow equation as lattice simulation input.

Conclusions and Outlook

Conclusions

- If $s_q \gg s_q^{\text{phys}}$ no relevant physics is removed.
- At low s_q RG transformations of bare parameters in the classical action not sufficient to keep physics constant.
- Effective action on the lattice needed → (Symanzik) improved actions
- **Our goal:** Improvement to standard methods by simulating at lowest possible s_q → **sample smooth fields efficiently**
- Error sources:
  - Constant λ in lattice simulations
  - Approximation of continuum FRG equations to calculate lattice couplings
  - Colored noise using the classical action is an approximation, too.

Outlook

- Further tests of validity range of classical action approximation, parameter scans using the FRG results as lattice input → work in progress
- Application of colored noise cooling to finite desity models and complex Langevin → work in progress

Thank you very much for your attention!