Pion-pole contribution to hadronic light-by-light scattering in a dispersive approach

in collab. with M. Hoferichter, B. Kubis, S. Leupold and S. P. Schneider

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Outline

Muon anomalous magnetic moment a_μ

- Standard Model contributions
- Hadronic vacuum polarization (HVP)
- Hadronic light-by-light (HLbL) scattering

Pion-pole contribution to a_μ

Pion transition form factor

- Definition and dispersion relations (DR)
- Double-spectral representation
- Matching to the asymptotic behavior

Numerical results

Conclusions and outlook

Standard Model contributions

Contributions	$a_\mu \times 10^{11}$	$\Delta a_\mu \times 10^{11}$
QED (5 loops)	116584718.95	0.08
Electroweak (2 loops)	153.6	1.0
HVP (3 loops)	6819.6	47.2
HLbL (3 loops)	108	26
a_μ^{SM}	116591800	54
a_μ^{Exp}	116592089	63
$a_\mu^{\mathrm{Exp}} - a_\mu^{\mathrm{SM}}$	289	83

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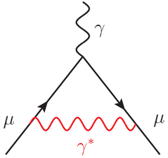
QED $\mathcal{O}(\alpha)$:



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Uncertainties mainly stem form hadronic contributions

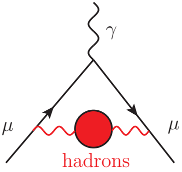
A discrepancy of $3.5\,\sigma$

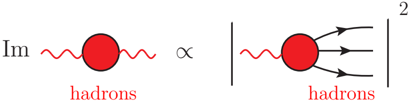
Future experiments aim for $\Delta a_\mu \approx 16 \times 10^{-11}$

$\Rightarrow$ Require more precise theoretical predictions

Hadronic vacuum polarization

- m_μ as characteristic scale
  $\Rightarrow$ Not a perturbative QCD problem!
- Optical theorem from probability conservation
- Dispersion relations to relate to the observable





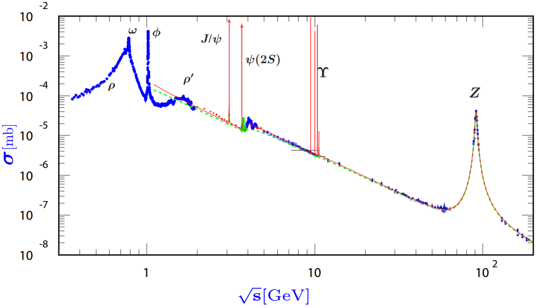
$\propto \quad \sigma(e^+e^- \to \gamma^* \to \text{hadrons})$

$$a_\mu^{\text{HVP}} = \int_{4M_\pi^2}^{\infty} \mathrm{d}s K(s) \sigma(e^+e^- \to \text{hadrons})$$

- Kinematical function $K(s)$: $\quad K(s) \propto 1/s$ for large s
- $\sigma(e^+e^- \to \text{hadrons}) \propto 1/s$ for large s

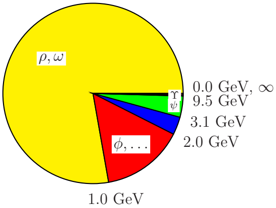
Hadronic vacuum polarization

- Experimental error induces theoretical uncertainty



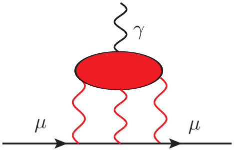
C. Patrignani et al., 2016

- Low energy contributions enhanced: 75% from $4M_\pi^2 \leq s \leq M_\phi^2$



Hadronic light-by-light scattering

- Estimates based on hadronic models so far except for lattice
- New model-independent initiatives: employ DR to relate dominant contributions to observables like form factors — Colangelo et al., 2014, Pauk, Vanderhaeghen, 2014



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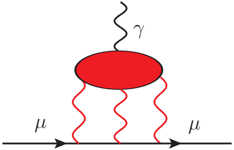
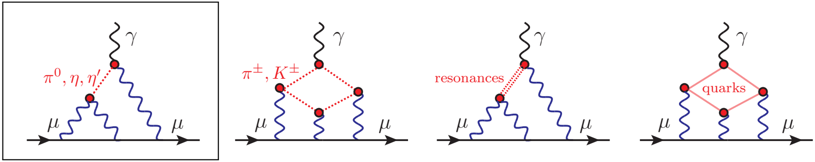




- π^0-pole term is the largest individual contribution to HLbL

Hadronic light-by-light scattering

A general master formula for the complete HLbL contributions:

Colangelo et al., 2015

$$a_\mu^{\text{HLbL}} = \frac{2\alpha^3}{3\pi^2} \int_0^\infty \mathrm{d}Q_1 \int_0^\infty \mathrm{d}Q_2 \int_{-1}^{1} \mathrm{d}\tau \sqrt{1-\tau^2}\, Q_1^3 Q_2^3 \sum_{i=1}^{12} T_i(Q_1,Q_2,\tau)\bar{\Pi}_i(Q_1,Q_2,\tau)$$

- $T_i(Q_1,Q_2,\tau)$: kernel functions
- $\bar{\Pi}_i(Q_1,Q_2,\tau)$: hadronic scalar functions

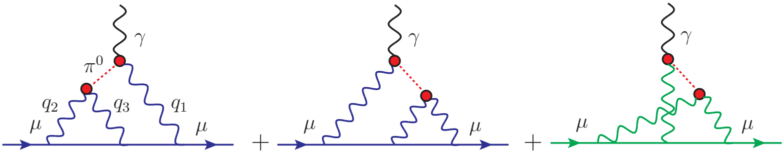
The pion pole easily identified with the hadronic functions $\bar{\Pi}_i$:

$$\bar{\Pi}_1^{\pi^0\text{-pole}}(Q_1,Q_2,\tau) = -\frac{F_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_2^2)F_{\pi^0\gamma^*\gamma^*}\big(-(Q_1+Q_2)^2,0\big)}{(Q_1+Q_2)^2+M_{\pi^0}^2}$$

$$\bar{\Pi}_2^{\pi^0\text{-pole}}(Q_1,Q_2,\tau) = -\frac{F_{\pi^0\gamma^*\gamma^*}\big(-Q_1^2,-(Q_1+Q_2)^2\big)F_{\pi^0\gamma^*\gamma^*}(-Q_2^2,0)}{Q_2^2+M_{\pi^0}^2}$$

Pion-pole contribution to a_μ

Pion-pole contribution to the muon anomaly a_μ: Knecht, Nyffeler, 2002



$$a_\mu^{\mathrm{HLbL};\pi^0} = -e^6 \int \frac{\mathrm{d}^4 q_1}{(2\pi)^4} \int \frac{\mathrm{d}^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 (q_1+q_2)^2 [(p+q_1)^2 - m_\mu^2][(p-q_2)^2 - m_\mu^2]}$$

$$\times \Big[\frac{F_{\pi^0\gamma^*\gamma^*}(q_1^2,(q_1+q_2)^2)\, F_{\pi^0\gamma^*\gamma^*}(q_2^2,0)}{q_2^2 - M_\pi^2} \hat{T}_1(q_1,q_2;p)$$

$$+ \frac{F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2)\, F_{\pi^0\gamma^*\gamma^*}((q_1+q_2)^2,0)}{(q_1+q_2)^2 - M_\pi^2} \hat{T}_2(q_1,q_2;p) \Big]$$

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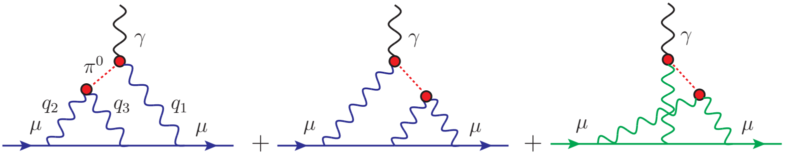

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$$\times \Bigg[\frac{F_{\pi^0\gamma^*\gamma^*}(q_1^2,(q_1+q_2)^2)\, F_{\pi^0\gamma^*\gamma^*}(q_2^2,0)}{q_2^2 - M_\pi^2} \hat{T}_1(q_1,q_2;p)$$
$$+ \frac{F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2)\, F_{\pi^0\gamma^*\gamma^*}((q_1+q_2)^2,0)}{(q_1+q_2)^2 - M_\pi^2} \hat{T}_2(q_1,q_2;p) \Bigg]$$

- Residues are given by the on-shell pion transition form factor $F_{\pi^0\gamma^*\gamma^*}$
- First and second diagrams collected in $\hat{T}_1(q_1,q_2;p)$; the last diagram contributes to $\hat{T}_2(q_1,q_2;p)$

Pion-pole contribution to a_μ

Performing 5 angular integrals using the Gegenbauer polynomials after Wick rotations leads to a 3 dimensional representation: Jegerlehner, Nyffeler, 2009

$$a_\mu^{\mathrm{HLbL};\pi^0} = \left(\frac{\alpha}{\pi}\right)^3 \int_0^\infty \mathrm{d}Q_1 \int_0^\infty \mathrm{d}Q_2 \int_{-1}^1 \mathrm{d}\tau$$
$$\times \Big[w_1(Q_1,Q_2,\tau)\, F_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -(Q_1+Q_2)^2)\, F_{\pi^0\gamma^*\gamma^*}(-Q_2^2, 0)$$
$$+ w_2(Q_1,Q_2,\tau)\, F_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)\, F_{\pi^0\gamma^*\gamma^*}(-(Q_1+Q_2)^2, 0) \Big]$$

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- $Q_i^2 = -q_i^2$ and $\tau = \cos\theta$
- integrations run over the lengths $Q_i = |Q_i|$ and the angle θ
- $w_1(Q_1, Q_2, \tau)$ & $w_2(Q_1, Q_2, \tau)$: weight functions
- $F_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)$: space-like form factor

Pion-pole contribution to a_μ

Weight functions $w_1(Q_1, Q_2, \tau)$ and $w_2(Q_1, Q_2, \tau)$: Nyffeler, 2016

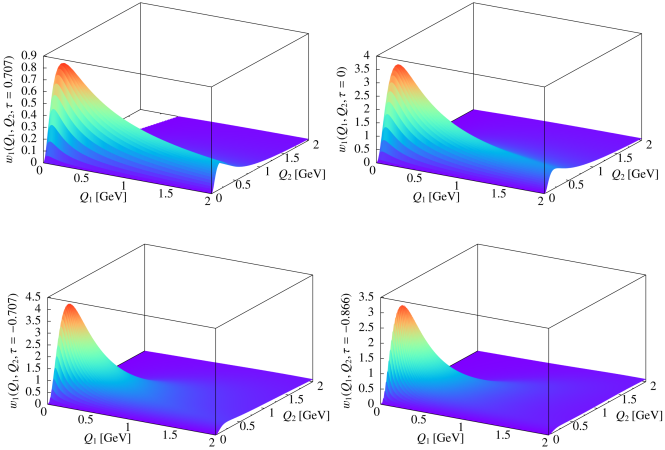
$$w_1(Q_1, Q_2, \tau) = \left(-\frac{2\pi}{3}\right) \sqrt{1-\tau^2}\, \frac{Q_1^3 Q_2^3}{Q_2^2 + m_\pi^2}\, T_1(Q_1, Q_2, \tau),$$

$$w_2(Q_1, Q_2, \tau) = \left(-\frac{2\pi}{3}\right) \sqrt{1-\tau^2}\, \frac{Q_1^3 Q_2^3}{(Q_1 + Q_2)^2 + m_\pi^2}\, T_2(Q_1, Q_2, \tau)$$

- $w_{1/2}(Q_1, Q_2, \tau)$ are dimensionless
- $w_2(Q_1, Q_2, \tau)$ symmetric under $Q_1 \leftrightarrow Q_2$
- $w_{1/2}(Q_1, Q_2, \tau) \to 0$ for $Q_i \to 0$ and $\tau \to \pm 1$
- $w_1(Q_1, Q_2, \tau) \sim 1/Q_1$ for $Q_1 \to \infty$, $w_1(Q_1, Q_2, \tau) \sim 1/Q_2^2$ for $Q_2 \to \infty$
- $w_2(Q_1, Q_2, \tau) \sim 1/Q_i^3$ for $Q_i \to \infty$

Pion-pole contribution to a_μ

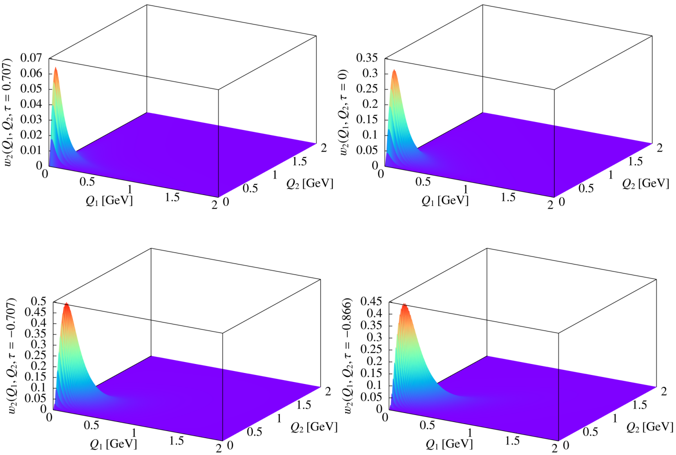
The weight function $w_1(Q_1, Q_2, \tau)$ as a function of Q_1 and Q_2:



- Maximum peaks in the range $(0.1 - 0.2)\,\mathrm{GeV}$
- Becomes negative in some momenta range for small θ values

Pion-pole contribution to a_μ

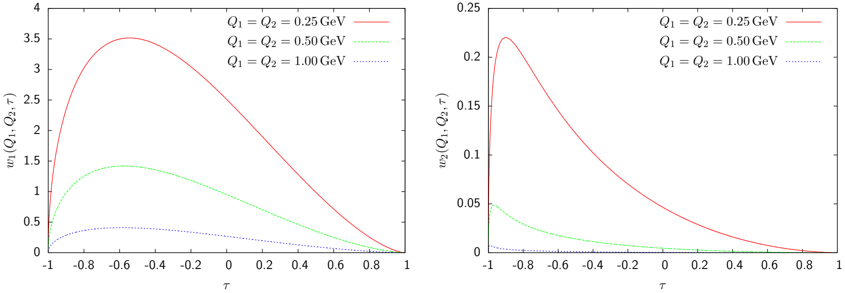
The weight function $w_2(Q_1, Q_2, \tau)$ as a function of Q_1 and Q_2:



- Always positive, an order of magnitude smaller than $w_1(Q_1, Q_2, \tau)$
- Maximum peak positions in the range $Q_1 = Q_2 \leq 0.2\,\mathrm{GeV}$

Pion-pole contribution to a_μ

The plots for $w_1(Q_1, Q_2, \tau)$ and $w_2(Q_1, Q_2, \tau)$ as functions of $\tau = \cos\theta$:



- Achieve their largest values in the range $\tau < 0$
- Become globally smaller for $Q_1 = Q_2 > 0.25$ GeV

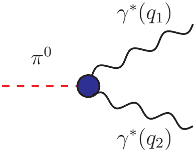
Pion-pole contribution to a_μ

- In short summary, $w_{1/2}(Q_1, Q_2, \tau)$ are concentrated in $Q_i \leq$ 0.5 GeV.
- Dominant part of the pion-pole contribution arises from the low-energy region.
- Exactly the region where the pion transition form factor is precisely & model-independently determined using dispersion relations.

Definition and dispersion relations

- Defined by the matrix element of two electromagnetic currents $j_\mu(x)$

$$i\int \mathrm{d}^4x\, e^{iq_1\cdot x}\,\langle 0|T\{j_\mu(x)j_\nu(0)\}|\pi^0(q_1+q_2)\rangle$$
$$= -\epsilon_{\mu\nu\rho\sigma}\, q_1^\rho q_2^\sigma F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2)$$



- Normalization fixed by the Adler–Bell–Jackiw anomaly:

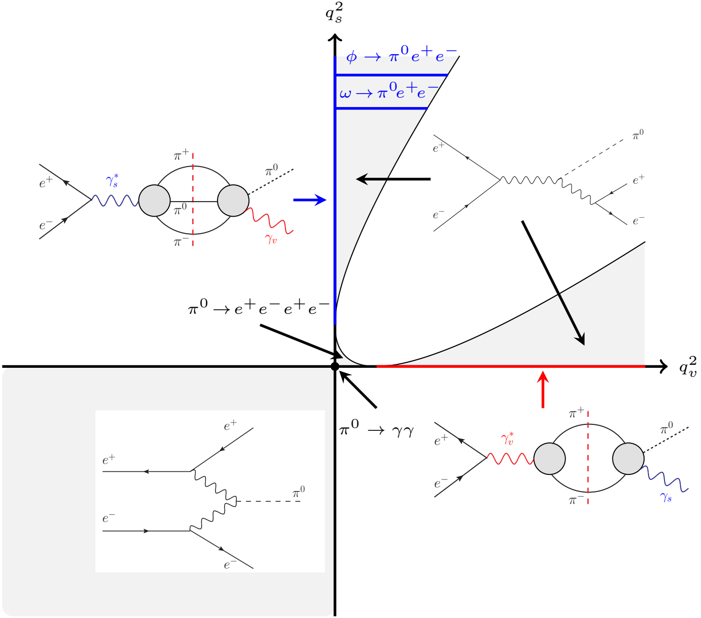
$$F_{\pi^0\gamma^*\gamma^*}(0,0) = \frac{1}{4\pi^2 F_\pi} \equiv F_{\pi\gamma\gamma}$$

$F_\pi = 92.28(9)\,\mathrm{MeV}$: pion decay constant C. Patrignani et al., 2016

- Bose symmetry and isospin decomposition:

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = F_{\pi^0\gamma^*\gamma^*}(q_2^2,q_1^2) = F_{vs}(q_1^2,q_2^2) + F_{vs}(q_2^2,q_1^2)$$

Definition and dispersion relations

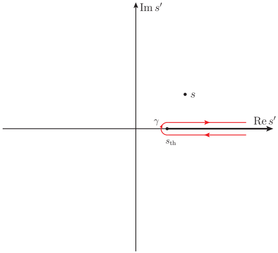


Definition and dispersion relations

Dispersion relations combining analyticity, unitarity & crossing:

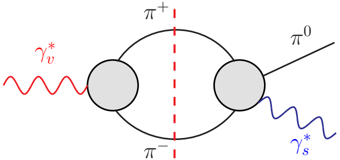
- Microcausality $\Rightarrow$ analyticity
- Unitarity $\Rightarrow 2\,\mathrm{Im}\,M_{fi} = \sum_n \int \mathrm{d}\Pi_n\, M^*_{fn}\, M_{in}$
- Particle-antiparticle transformation $\Rightarrow$ crossing

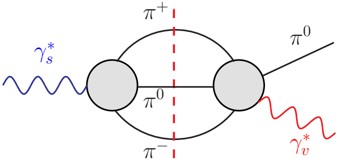
$$M(s) = \frac{1}{2\pi i}\int_{s_{\mathrm{th}}}^{\infty} \frac{\mathrm{disc}\,M(s')}{s'-s}\,\mathrm{d}s' = \frac{1}{\pi}\int_{s_{\mathrm{th}}}^{\infty} \frac{\mathrm{Im}\,M(s')}{s'-s}\,\mathrm{d}s'$$



Definition and dispersion relations

Dispersive reconstruction from the lowest-lying hadronic intermediate states:





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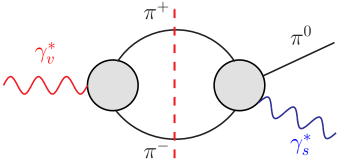
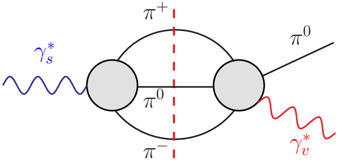


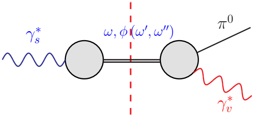


Isovector photon: 2 pions

- $\gamma_v^* \to \pi^+\pi^- \to \gamma_s^*\pi^0$
- $\mathrm{disc} \propto$ pion vector form factor $\times\ \gamma_s^* \to 3\pi$ amplitude

Isoscalar photon: 3 pions

- $\gamma_s^* \to \pi^+\pi^-\pi^0 \to \gamma_v^*\pi^0$
- Dominated by resonances $\omega,\ \phi,\ \omega',\ \&\,\omega''$



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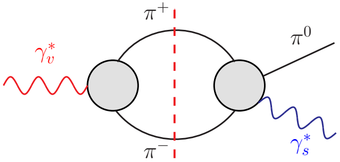
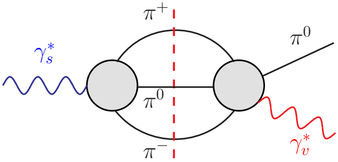


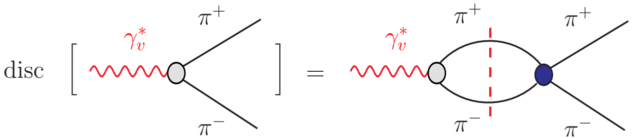


Building blocks of the dispersive treatment:

- Pion vector form factor $F_\pi^V(s)$
- Partial wave amplitude $f_1(s, q^2)$ for the $\gamma_s^*(q) \to \pi^+\pi^-\pi^0$ reaction

Definition and dispersion relations

Pion vector form factor $F_\pi^V(s)$:

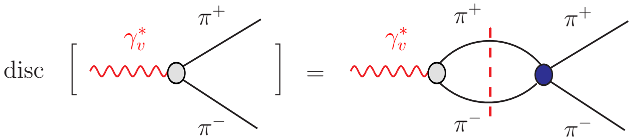


disc $F_\pi^V(s) = 2i\,\mathrm{Im}\,F_\pi^V(s) = 2iF_\pi^V(s)\sin\delta_1^1(s)\,e^{-i\delta_1^1(s)}\theta(s-4M_\pi^2)$

Watson's final-state theorem: phase of $F_\pi^V(s)$ is given by $\delta_1^1(s)$ Watson, 1954

Definition and dispersion relations

Pion vector form factor $F_\pi^V(s)$:



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Watson's final-state theorem: phase of $F_\pi^V(s)$ is given by $\delta_1^1(s)$ Watson, 1954

Solution:

$$F_\pi^V(s) = P(s)\Omega(s), \quad \Omega(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^\infty \mathrm{d}s'\,\frac{\delta_1^1(s')}{s'(s'-s)}\right\}$$

- $\Omega(s)$ is the Omnès function Omnès, 1958
- $P(s)$ polynomial, $P(0)=1$ from charge conservation
- $\pi\pi$ P-wave phase shift $\delta_1^1(s)$ from Roy equations

Definition and dispersion relations

The $\gamma_s^*(q) \to \pi^+\pi^-\pi^0$ decay amplitude $\mathcal{M}(s,t,u;q^2)$:

$$\mathcal{M}(s,t,u;q^2) = i\epsilon_{\mu\nu\rho\sigma} n^\mu p_+^\nu p_-^\rho p_0^\sigma \mathcal{F}(s,t,u;q^2)$$

Decompose into single-variable functions:

$$\mathcal{F}(s,t,u;q^2) = \mathcal{F}(s,q^2) + \mathcal{F}(t,q^2) + \mathcal{F}(u,q^2)$$

Normalization from the Wess–Zumino–Witten anomaly:

$$\mathcal{F}(0,0,0;0) = \frac{1}{4\pi^2 F_\pi^3} \equiv F_{3\pi}$$

Discontinuity equation:

$$\text{disc}\,\mathcal{F}(s,q^2) = 2i(\mathcal{F}(s,q^2) + \hat{\mathcal{F}}(s,q^2))\theta(s-4M_\pi^2)\sin\delta_1^1(s)\,e^{-i\delta_1^1(s)}$$

- $\mathcal{F}(s,q^2)$: right-hand cut
- $\hat{\mathcal{F}}(s,q^2)$: left-hand cut; angular averages of $\mathcal{F}(t,q^2)$ & $\mathcal{F}(u,q^2)$

Definition and dispersion relations

A once-subtracted dispersive solution to the discontinuity equation:

Hoferichter et al., 2014

$$\mathcal{F}(s,q^2) = a(q^2)\Omega(s)\Big\{1 + \frac{s}{\pi}\int_{4M_\pi^2}^{\infty} \mathrm{d}s' \, \frac{\hat{\mathcal{F}}(s',q^2)\sin\delta_1^1(s')}{s'(s'-s)|\Omega(s')|}\Big\}$$

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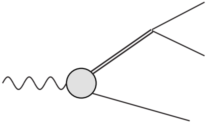
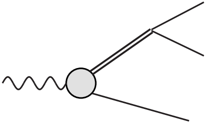
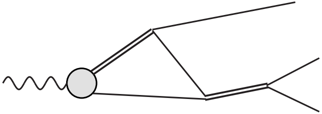
$$\mathcal{F}(s,q^2) = a(q^2)\Omega(s)\left\{1 + \frac{s}{\pi}\int_{4M_\pi^2}^{\infty} \mathrm{d}s' \frac{\hat{\mathcal{F}}(s',q^2)\sin\delta_1^1(s')}{s'(s'-s)|\Omega(s')|}\right\}$$

$\hat{\mathcal{F}}(s,q^2)$ absent:

$\mathcal{F}(s,q^2) =$ 

$\hat{\mathcal{F}}(s,q^2)$ present:

$\mathcal{F}(s,q^2) =$  $+$  $+\cdots$

- Incorporated crossed-channel interactions

Definition and dispersion relations

$a(q^2)$ fit to different $e^+e^- \to 3\pi$ cross-section data with parameterization:

$$a(q^2) = \frac{F_{3\pi}}{3} + \frac{q^2}{\pi} \int_{s_{\text{thr}}}^{\infty} \mathrm{d}s' \frac{\operatorname{Im} \mathcal{A}(s')}{s'(s'-q^2)} + C_n(q^2)$$

$$\mathcal{A}(q^2) = \sum_V \frac{c_V}{M_V^2 - q^2 - i\sqrt{q^2}\,\Gamma_V(q^2)}, \qquad V = \omega, \phi, \omega', \omega''$$

$$C_n(q^2) = \sum_{i=1}^{n} c_i \big(z(q^2)^i - z(0)^i\big), \qquad z(q^2) = \frac{\sqrt{s_{\text{inel}} - s_1} - \sqrt{s_{\text{inel}} - q^2}}{\sqrt{s_{\text{inel}} - s_1} + \sqrt{s_{\text{inel}} - q^2}}$$

Definition and dispersion relations

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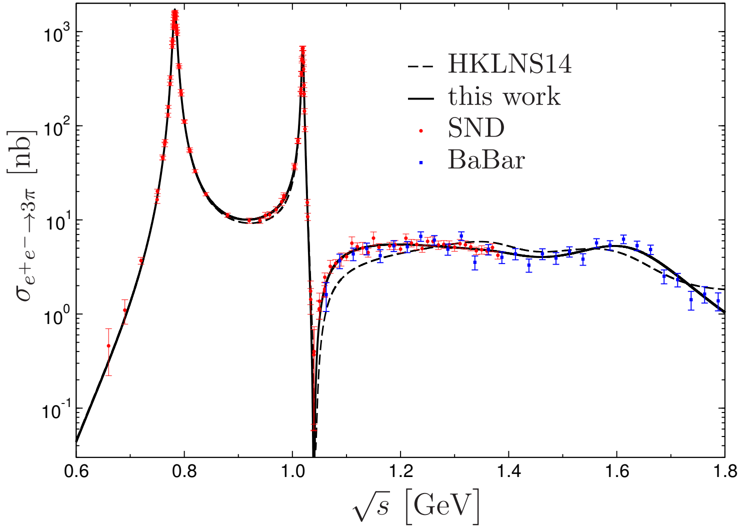
$$C_n(q^2) = \sum_{i=1}^{n} c_i \big(z(q^2)^i - z(0)^i\big), \qquad z(q^2) = \frac{\sqrt{s_{\rm inel} - s_1} - \sqrt{s_{\rm inel} - q^2}}{\sqrt{s_{\rm inel} - s_1} + \sqrt{s_{\rm inel} - q^2}}$$

- S-wave cusp eliminated
- Asymptotic behavior of $C_n(q^2)$ controlled
- Exact implementation of $\gamma_s^*(q) \to 3\pi$ anomaly:

$$\frac{F_{3\pi}}{3} = \frac{1}{\pi}\int_{s_{\rm thr}}^{\infty} \mathrm{d}s' \frac{\mathrm{Im}\,a(s')}{s'}$$

Definition and dispersion relations

6 (7) parameters c_ω, c_ϕ, $c_{\omega'}$, $c_{\omega''}$, c_1, c_2 & (c_3) fit to $e^+e^- \to 3\pi$ data:



- Substantially improved above the ϕ peak

Double-spectral representation

$F_{vs}(q_1^2, q_2^2)$ fulfills a dispersion relation: Hoferichter et al., 2014

$$F_{vs}(q_1^2, q_2^2) = \frac{1}{12\pi^2} \int_{4M_\pi^2}^{\infty} \mathrm{d}x \frac{q_\pi^3(x)\big(F_\pi^V(x)\big)^* f_1(x, q_2^2)}{x^{1/2}(x - q_1^2)},$$

$q_\pi(s) = \sqrt{s/4 - M_\pi^2}$, $f_1(s, q^2) = \mathcal{F}(s, q^2) + \hat{\mathcal{F}}(s, q^2)$: $\gamma_s^*(q) \to 3\pi$ P-wave

Go to doubly-space-like kinematics by writing another dispersion relation:

$$F_{vs}(-Q_1^2, q_2^2) = \frac{1}{\pi} \int_{s_{\text{thr}}}^{s_{\text{is}}} \mathrm{d}y \frac{\operatorname{Im} F_{vs}(-Q_1^2, y)}{y - q_2^2}$$

Double-spectral representation of the form factor:

$$F^{\text{disp}}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2) = \frac{1}{\pi^2} \int_{4M_\pi^2}^{s_{\text{iv}}} \mathrm{d}x \int_{s_{\text{thr}}}^{s_{\text{is}}} \frac{\rho^{\text{disp}}(x, y)\,\mathrm{d}y}{(x + Q_1^2)(y + Q_2^2)},$$

$$\rho^{\text{disp}}(x, y) = \frac{q_\pi^3(x)}{12\pi\sqrt{x}} \operatorname{Im}\Big[\big(F_\pi^V(x)\big)^* f_1(x, y)\Big] + [x \leftrightarrow y]$$

Double-spectral representation

Effective pole term:

$F^{\text{disp}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$ fulfills the chiral anomaly $F_{\pi\gamma\gamma}$ by around 90%

⇒ Introduce an effective pole term

$$F^{\text{eff}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{g_{\text{eff}}}{4\pi^2 F_\pi} \frac{M_{\text{eff}}^4}{(M_{\text{eff}}^2 - q_1^2)(M_{\text{eff}}^2 - q_2^2)}$$

g_{eff} fixed by fulfilling the chiral anomaly

$g_{\text{eff}} \sim 10\%$, ⇒ small

M_{eff} fit to singly-virtual data excluding BaBar above $5\,\text{GeV}^2$ Gronberg et al., 1998, Aubert et al., 2009, Uehara et al., 2012

$M_{\text{eff}} \sim 1.5$–$2\,\text{GeV}$, ⇒ reasonable

Matching to the asymptotic behavior

Asymptotically, $F_{\pi^0\gamma^*\gamma^*}$ should fulfill Brodsky, Lepage, 1979-1981

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = -\frac{2F_\pi}{3}\int_0^1 \mathrm{d}u \frac{\phi_\pi(u)}{uq_1^2 + (1-u)q_2^2} + \mathcal{O}\left(\frac{1}{q_i^4}\right)$$

Pion distribution amplitude:

$$\phi_\pi(u,\mu) = 6u(1-u)\left[1 + \sum_{n=1}^{\infty} a_{2n}(\mu) C_{2n}^{3/2}(2u-1)\right]$$

Change the expression into a form:

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = -\frac{4F_\pi}{3}\frac{f(\omega)}{q_1^2+q_2^2} + \mathcal{O}\left(q_i^{-4}\right),$$

with

$$\omega = \frac{q_1^2 - q_2^2}{q_1^2 + q_2^2}, \quad f(\omega) = \int_0^1 \mathrm{d}u \frac{\phi_\pi(u)}{u(1-\omega) + (1-u)(1+\omega)}$$

Matching to the asymptotic behavior

Brodsky–Lepage (BL) limit: Brodsky, Lepage, 1979-1981, G. Eichmann et al., 2017

$$\lim_{Q^2\to\infty} F_{\pi^0\gamma^*\gamma^*}(-Q^2,0) = \frac{2\,F_\pi}{Q^2}$$

Operator product expansion (OPE): Nesterenko, Radyushkin, 1983, Novikov et al., 1984, Manohar, 1990

$$\lim_{Q^2\to\infty} F_{\pi^0\gamma^*\gamma^*}(-Q^2,-Q^2) = \frac{2\,F_\pi}{3\,Q^2}$$

Some form factor models:

$$\lim_{Q^2\to\infty} F^{\text{VMD}}_{\pi^0\gamma^*\gamma^*}(-Q^2,0)\sim 1/Q^2, \qquad \lim_{Q^2\to\infty} F^{\text{VMD}}_{\pi^0\gamma^*\gamma^*}(-Q^2,-Q^2)\sim 1/Q^4$$

$$\lim_{Q^2\to\infty} F^{\text{LMD}}_{\pi^0\gamma^*\gamma^*}(-Q^2,0)\sim \mathcal{C}, \qquad \lim_{Q^2\to\infty} F^{\text{LMD}}_{\pi^0\gamma^*\gamma^*}(-Q^2,-Q^2)\sim 1/Q^2$$

$$\lim_{Q^2\to\infty} F^{\text{LMD+V}}_{\pi^0\gamma^*\gamma^*}(-Q^2,0)\sim 1/Q^2, \qquad \lim_{Q^2\to\infty} F^{\text{LMD+V}}_{\pi^0\gamma^*\gamma^*}(-Q^2,-Q^2)\sim 1/Q^2,$$

$$\lim_{Q_1^2\to\infty} F^{\text{LMD+V}}_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_2^2\neq 0)\sim \mathcal{C}$$

Matching to the asymptotic behavior

Change the asymptotic form to a dispersion relation by a simple change of variables $u \to x/(x-q_2^2)$ for space-like virtuality q_2^2: Khodjamirian, 1999

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{1}{\pi}\int_0^\infty \mathrm{d}x \frac{\mathrm{Im}\, F_{\pi^0\gamma^*\gamma^*}(x, q_2^2)}{x - q_1^2},$$

with

$$\mathrm{Im}\, F_{\pi^0\gamma^*\gamma^*}(x, q_2^2) = \frac{2\pi F_\pi}{3(x-q_2^2)}\phi_\pi\Big(\frac{x}{x-q_2^2}\Big)$$

We promote the asymptotic form into a double-spectral representation:

$$F^{\mathrm{asym}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{1}{\pi^2}\int_0^\infty\int_0^\infty \mathrm{d}x\mathrm{d}y \frac{\rho^{\mathrm{asym}}(x,y)}{(x-q_1^2)(y-q_2^2)},$$

$$\rho^{\mathrm{asym}}(x,y) = -2\pi^2 F_\pi x y \delta''(x-y)$$

Matching to the asymptotic behavior

Decomposition of the pion-transition form factor:

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = \frac{1}{\pi^2}\int_0^{s_\mathrm{m}} \mathrm{d}x \int_0^{s_\mathrm{m}} \mathrm{d}y \frac{\rho^{\mathrm{disp}}(x,y)}{(x-q_1^2)(y-q_2^2)} + \frac{1}{\pi^2}\int_{s_\mathrm{m}}^{\infty} \mathrm{d}x \int_{s_\mathrm{m}}^{\infty} \mathrm{d}y \frac{\rho^{\mathrm{asym}}(x,y)}{(x-q_1^2)(y-q_2^2)}$$
$$+ \frac{1}{\pi^2}\int_0^{s_\mathrm{m}} \mathrm{d}x \int_{s_\mathrm{m}}^{\infty} \mathrm{d}y \frac{\rho(x,y)}{(x-q_1^2)(y-q_2^2)} + \frac{1}{\pi^2}\int_{s_\mathrm{m}}^{\infty} \mathrm{d}x \int_0^{s_\mathrm{m}} \mathrm{d}y \frac{\rho(x,y)}{(x-q_1^2)(y-q_2^2)}$$

- s_m: continuum threshold
- Dispersive $\rho^{\mathrm{disp}}(x,y)$ for low energies
- Asymptotic $\rho^{\mathrm{asym}}(x,y)$ for asymptotic region
- $\rho(x,y)$ not known rigorously, $\rho^{\mathrm{asym}}(x,y)$ applied in mixed regions vanishes

$\Rightarrow$ All constraints can be fulfilled discarding mixed regions

Matching to the asymptotic behavior

This defines the asymptotic contribution

$$F^{\mathsf{asym}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = 2F_\pi \int_{s_\mathrm{m}}^{\infty} \mathrm{d}x \frac{q_1^2 q_2^2}{(x - q_1^2)^2 (x - q_2^2)^2}$$

- Does not contribute for the singly-virtual kinematics
- Restores the asympotics for singly/doubly-virtual kinematics

The final representation:

$$\boxed{F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = F^{\mathsf{disp}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) + F^{\mathsf{eff}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) + F^{\mathsf{asym}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)}$$

- Reconstructed from all lowest-lying singularities
- Fulfills the asymptotic constraints at $\mathcal{O}\left(1/Q^2\right)$
- Full freedom to account for the tension of BaBar/Belle space-like data

Numerical results

Uncertainty estimates:

- The uncertainty in $F_{\pi\gamma\gamma}$ at 1.4% from PrimEx — Larin et al., 2011
  - Varying the coupling g_{eff}
- Dispersive uncertainties estimated by
  - Varying the cutoffs between 1.8 and $2.5\,\text{GeV}$
  - Different $\pi\pi$ phase shifts — Caprini et al., 2012, García-Martín et al., 2011, Schneider et al., 2012
  - Different representations of $F_\pi^V(s)$
  - Different conformal polynomial fits to $e^+e^- \to 3\pi$

Numerical results

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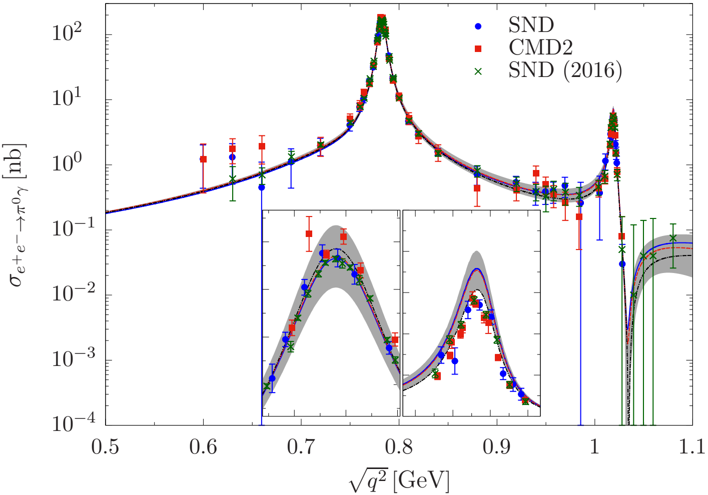
  - Different representations of $F_\pi^V(s)$
  - Different conformal polynomial fits to $e^+e^- \to 3\pi$
- BL limit uncertainty by ${}^{+20}_{-10}\%$ — Aubert et al., 2009, Uehara et al., 2012
  - Varying the mass parameter M_{eff}
  - Completely covers 3σ band
- Asymptotic part $s_{\text{m}} = 1.7(3)\,\text{GeV}^2$ — Khodjamirian, 1999, Agaev et al., 2011, Mikhailov et al., 2016
  - Expected from light-cone sum rules

Numerical results

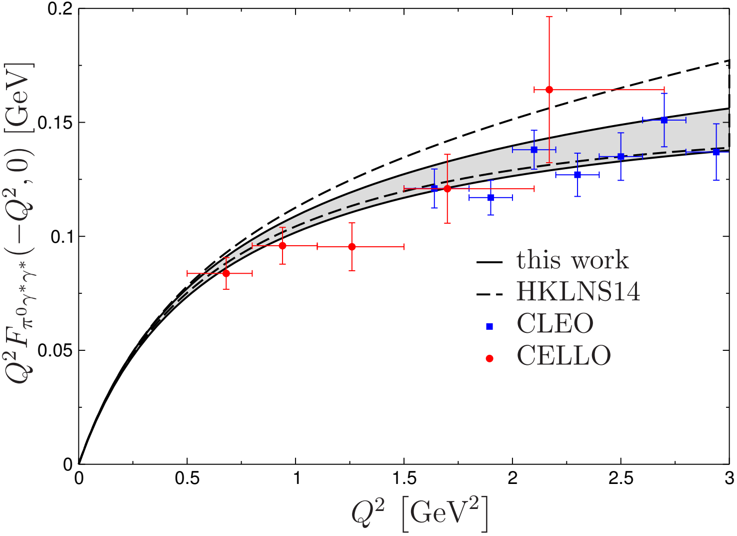
Our predictions of the time-like form factor in $e^+e^- \to \pi^0\gamma$:

- Entirely based on the dispersive framework
- Input quantities $F_{\pi\gamma\gamma}$, $F_{3\pi}$, $\delta_1^1(s)$ and $e^+e^- \to 3\pi$ data



Numerical results

Singly-virtual space-like transition form factor in $Q^2 F_{\pi^0\gamma^*\gamma^*}(-Q^2, 0)$:



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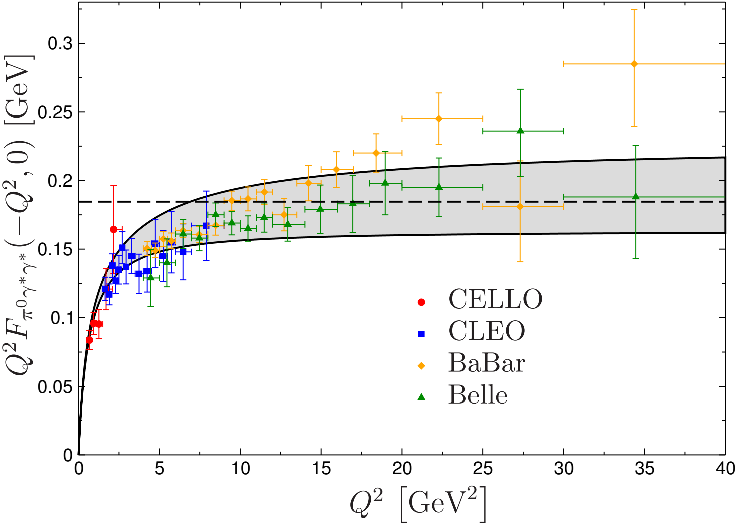
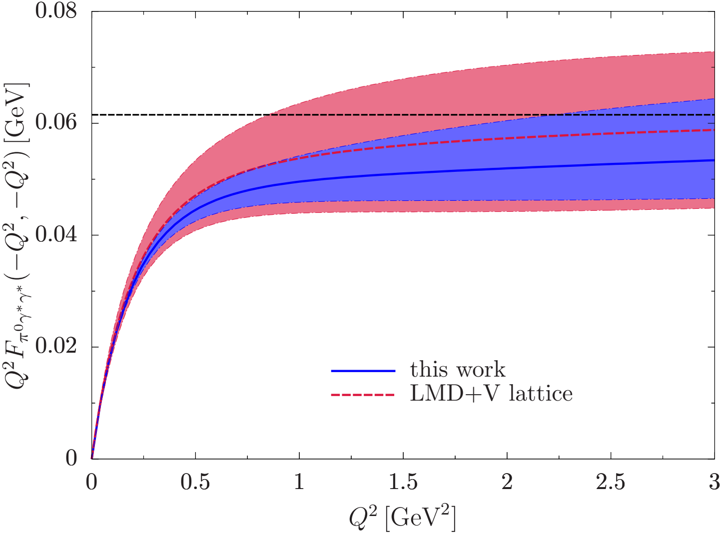


Numerical results

Diagonal form factor in $Q^2 F_{\pi^0\gamma^*\gamma^*}(-Q^2, -Q^2)$ in comparison to LMD+V fit to lattice: Gérardin et al., 2016



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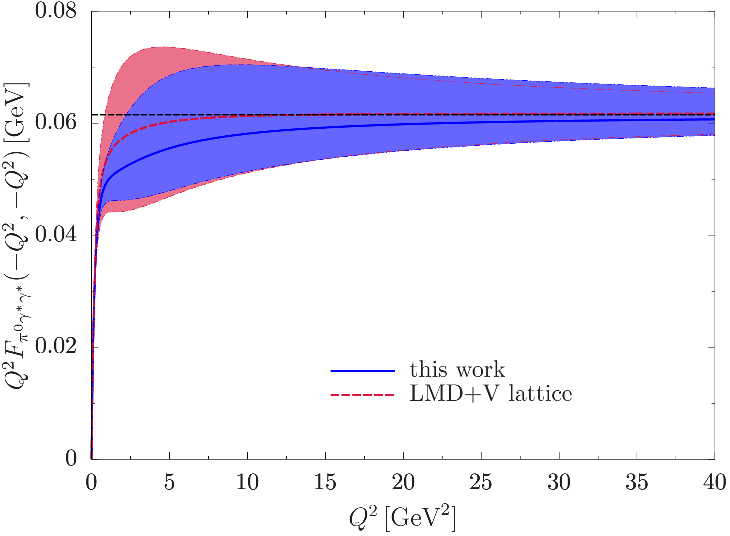
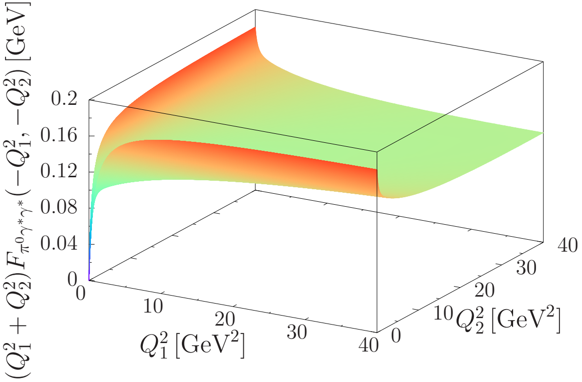


Numerical results

$(Q_1^2 + Q_2^2) F_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)$ as a function of Q_1^2 and Q_2^2:

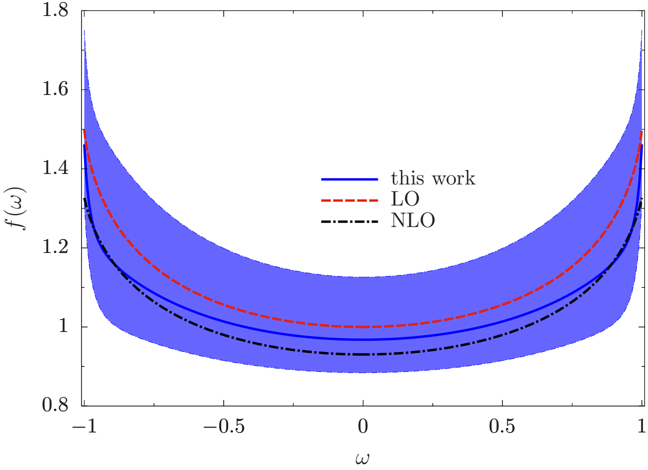


- $1/Q_i^2$ behavior in the entire domain of space-like virtualities

  $\Rightarrow$ Hard to obtain in resonance models

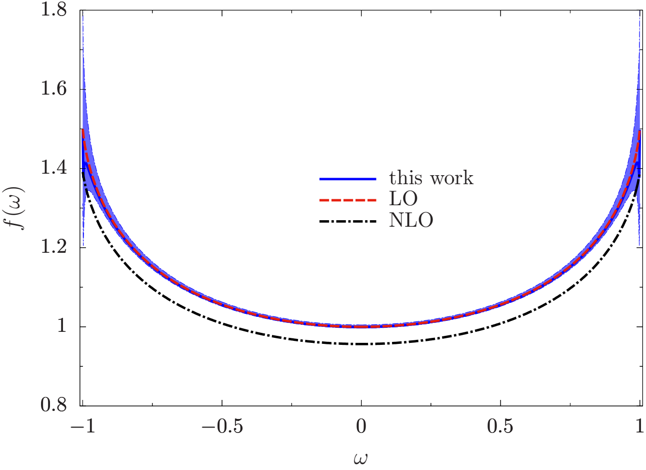
Numerical results

$f(\omega)$ versus ω calculated at $Q_1^2 + Q_2^2 = 35\,\mathrm{GeV}^2$:



Numerical results

$f(\omega)$ versus ω calculated at $Q_1^2 + Q_2^2 = 1.6 \times 10^3\,\mathrm{GeV}^2$:



Numerical results

Pion-pole contribution to a_μ from the final representation:

$$\begin{aligned} a_\mu^{\pi^0\text{-pole}} &= 62.6(1.7)_{F_{\pi\gamma\gamma}}(1.1)_{\text{disp}}\binom{2.2}{1.4}_{\text{BL}}(0.5)_{\text{asym}} \times 10^{-11} \\ &= 62.6^{+3.0}_{-2.5} \times 10^{-11} \end{aligned}$$

- First complete data-driven determination
- Fully controlled uncertainty estimates

The slope parameter:

$$\begin{aligned} a_\pi &= \frac{M_{\pi^0}^2}{F_{\pi\gamma\gamma}} \frac{\partial}{\partial q^2} F_{\pi^0\gamma^*\gamma^*}(q^2, 0)\Big|_{q^2=0} \\ &= 31.5(2)_{F_{\pi\gamma\gamma}}(8)_{\text{disp}}(3)_{\text{BL}} \times 10^{-3} = 31.5(9) \times 10^{-3} \end{aligned}$$

- In comparison to $a_\pi = 30.7(6) \times 10^{-3}$ (HKLNS14) Hoferichter et al., 2014

  $\Rightarrow$ Larger value expected from matching

Conclusions and outlook

- Dispersive reconstruction of the pion transition form factor
  - Incorporated all the lowest-lying singularities
  - Matched to perturbative QCD
- Data-driven determination of $a_\mu^{\pi^0\text{-pole}}$ with carefully estimated improvable uncertainties
  - Preliminary updated 0.85% $F_{\pi\gamma\gamma}$ uncertainty from PrimEx-II
  - Dispersive inputs may be consolidated with COMPASS, BESIII
  - BL limit may be clarified from BELLE II
  - Doubly virtual form factor comparing to lattice QCD, DSE
- Applications to η and η' transition form factors
- $\pi^0 \to e^+e^-$, HVP 3π channels, $\cdots$

Much obliged for your attention!

"$g-2$ is not an experiment: it is a way of life."

John Adams (CERN Director General 1971 - 1980)

Back up

$$\hat{T}_1(q_1,q_2;p) = -\frac{16}{3}\left((q_1\cdot q_2)^2 - q_1^2 q_2^2\right)m_\mu^2 - \frac{16}{3}q_1^2(p\cdot q_2)^2 + p\cdot q_1\left(\frac{16}{3}p\cdot q_2 q_1\cdot q_2 - \frac{8}{3}q_2^2 q_1\cdot q_2\right)$$
$$+ p\cdot q_2\left(8q_1^2q_2^2 - \frac{16}{3}(q_1\cdot q_2)^2\right),$$
$$\hat{T}_2(q_1,q_2;p) = -\frac{8}{3}\left((q_1\cdot q_2)^2 - q_1^2q_2^2\right)m_\mu^2 - \frac{8}{3}q_2^2(p\cdot q_1)^2 - \frac{8}{3}q_1^2(p\cdot q_2)^2 - \frac{4}{3}q_1^2 p\cdot q_2\left(q_2^2 + q_1\cdot q_2\right)$$
$$+ p\cdot q_1\left(\frac{4}{3}\left(q_1^2 + q_1\cdot q_2\right)q_2^2 + \frac{16}{3}p\cdot q_2 q_1\cdot q_2\right),$$
$$T_1(Q_1,Q_2;\tau) = \frac{Q_1\left(\sigma_1^E - 1\right)\left(Q_1\tau\left(\sigma_1^E+1\right) + 4Q_2\left(\tau^2-1\right)\right) - 4\tau m_\mu^2}{Q_1Q_2Q_3^2m_\mu^2} + X\frac{8\left(\tau^2-1\right)\left(2m_\mu^2 - Q_2^2\right)}{Q_3^2m_\mu^2},$$
$$T_2(Q_1,Q_2;\tau) = \frac{Q_1^2\tau\left(\sigma_1^E-1\right)\left(\sigma_1^E+5\right) + Q_2^2\tau\left(\sigma_2^E-1\right)\left(\sigma_2^E+5\right) + 4Q_1Q_2\left(\sigma_1^E+\sigma_2^E-2\right) - 8\tau m_\mu^2}{2Q_1Q_2Q_3^2m_\mu^2}$$
$$+ X\left(\frac{8\left(\tau^2-1\right)}{Q_3^2} - \frac{4}{m_\mu^2}\right),$$

with

$$X = \frac{1}{Q_1Q_2x}\operatorname{atan}\left(\frac{zx}{1-z\tau}\right), \quad x = \sqrt{1-\tau^2},$$
$$z = \frac{Q_1Q_2}{4m_\mu^2}(1-\sigma_1^E)(1-\sigma_2^E), \quad \sigma_i^E = \sqrt{1+\frac{4m_\mu^2}{Q_i^2}},$$
$$Q_3^2 = Q_1^2 + 2Q_1Q_2\tau + Q_2^2.$$

Back up

The VMD model:

$$F^{\mathrm{VMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{M_\rho^2 M_\omega^2}{8\pi^2 F_\pi}\left[\frac{1}{(q_1^2 - M_\rho^2)(q_2^2 - M_\omega^2)} + (q_1^2 \leftrightarrow q_2^2)\right]$$

The LMD model:

$$F^{\mathrm{LMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{F_\pi}{3}\,\frac{c_V - (q_1^2 + q_2^2)}{(q_1^2 - M_\rho^2)(q_2^2 - M_\rho^2)}$$

where

$$c_V = \frac{3M_\rho^4}{4\pi^2 F_\pi^2}$$

The LMD+V model:

$$F^{\mathrm{LMD+V}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{F_\pi}{3}\,\frac{h_7 - q_1^2\, q_2^2\,(q_1^2 + q_2^2) - h_2\, q_1^2\, q_2^2 - h_5\,(q_1^2 + q_2^2)}{(q_1^2 - M_\rho^2)(q_2^2 - M_\rho^2)(q_1^2 - M_{\rho'}^2)(q_2^2 - M_{\rho'}^2)}$$

where

$$h_2 = -4(M_\rho^2 + M_{\rho'}^2) + (16/9)\,\delta^2 = -10.63\ \text{GeV}^2,$$

$$h_5 = (6.93 \pm 0.26)\ \text{GeV}^4, \quad h_7 = \frac{3M_\rho^4 M_{\rho'}^4}{4\pi^2 F_\pi^2}$$

Back up

NLO pQCD result:

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = -\frac{2F_\pi}{3}\int_0^1 \mathrm{d}u \frac{\phi_\pi(u)}{uq_1^2+(1-u)q_2^2}\left(1+\frac{C_F\alpha_s(\mu_s^2)}{2\pi}f(u,-q_1^2,-q_2^2,-\mu^2)\right),$$

$$\begin{aligned} f(u,q_1^2,q_2^2,\mu^2) &= -\frac{9}{2}+\frac{L_{12}(L_{12}-2)}{2}\left(1-\frac{q_1^2q_2^2}{(q_1^2-q_2^2)^2u(1-u)}\right)+\frac{3}{2}L_{12} \\ &\quad -\frac{q_1^2}{2(q_1^2-q_2^2)}\left(1-\frac{q_2^2}{(q_1^2-q_2^2)(1-u)}\right)L_1(L_1-2)+\frac{q_2^2}{2(q_1^2-q_2^2)u}(L_{12}-L_2) \\ &\quad +\frac{q_2^2}{2(q_1^2-q_2^2)}\left(1+\frac{q_1^2}{(q_1^2-q_2^2)u}\right)L_2(L_2-2)-\frac{q_1^2}{2(q_1^2-q_2^2)(1-u)}(L_{12}-L_1), \end{aligned}$$

$$L_i = \log\frac{q_i^2}{\mu^2}, \qquad L_{12} = \log\frac{uq_1^2+(1-u)q_2^2}{\mu^2}, \qquad C_F = \frac{N_c^2-1}{2N_c} = \frac{4}{3}.$$

$$F_{\pi^0\gamma^*\gamma^*}(-Q^2,0) = \frac{2F_\pi}{Q^2}\left(1-\frac{5}{3}\frac{\alpha_s(-Q^2)}{\pi}\right),$$

$$F_{\pi^0\gamma^*\gamma^*}(-Q^2,-Q^2) = \frac{2F_\pi}{3Q^2}\left(1-\frac{\alpha_s(-2Q^2)}{\pi}\right).$$