Inhomogeneous chiral condensates

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Michael Buballa

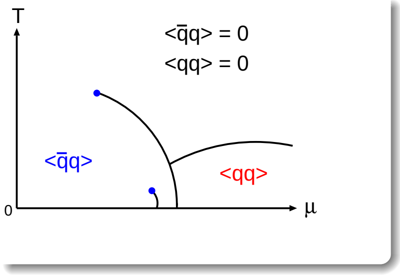
Theoriezentrum, Institut für Kernphysik, TU Darmstadt

Lunch Club Seminar, JLU Gießen, December 5, 2018

CRC-TR 211
Strong-interaction matter under extreme conditions

Motivation

- QCD phase diagram (standard picture):



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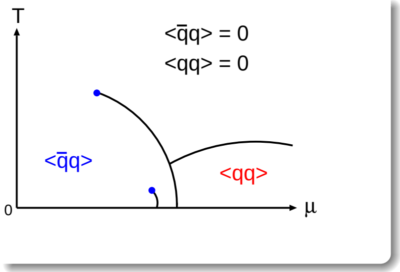

- assumption: $\langle \bar{q}q \rangle$, $\langle qq \rangle$ constant in space

Motivation

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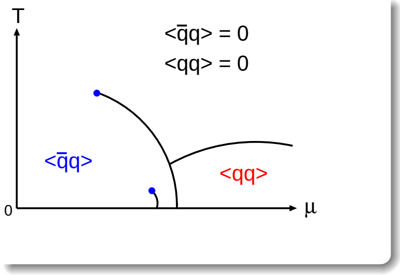
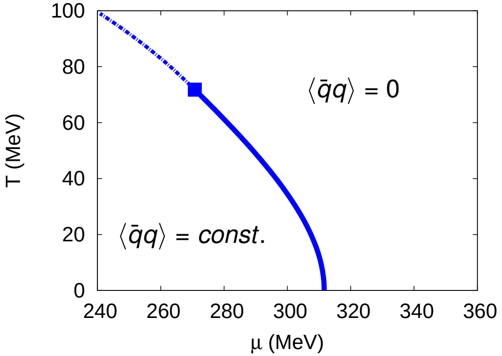
- QCD phase diagram (standard picture):



- assumption: $\langle \bar{q}q \rangle$, $\langle qq \rangle$ constant in space
- How about non-uniform phases ?

NJL-model studies

homogeneous phases only



[D. Nickel, PRD (2009)]

NJL-model studies

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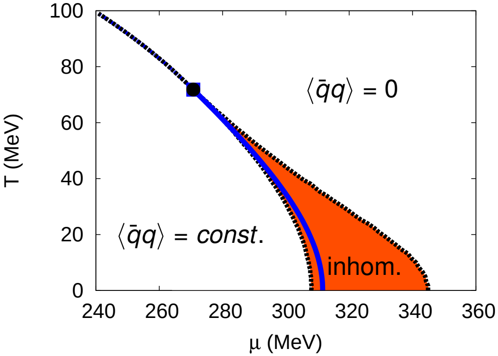
including inhomogeneous phase



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NJL-model studies

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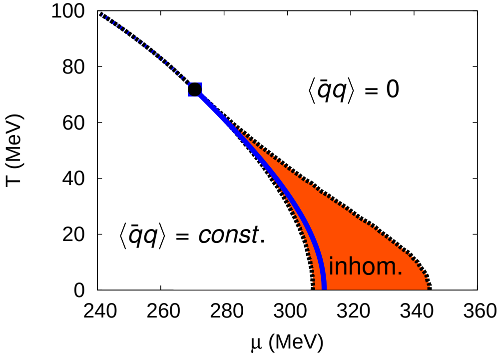

[D. Nickel, PRD (2009)]

- 1st-order phase boundary completely covered by the inhomogeneous phase!
- Critical point → Lifshitz point
  [D. Nickel, PRL (2009)]

NJL-model studies

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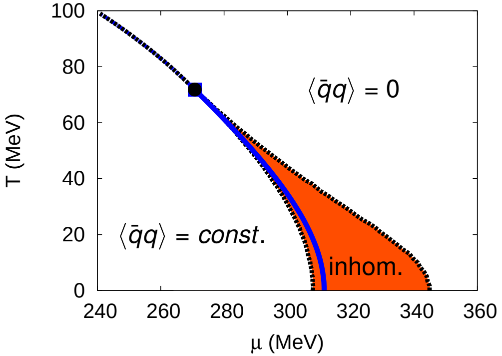
including inhomogeneous phase



[D. Nickel, PRD (2009)]

- 1st-order phase boundary completely covered by the inhomogeneous phase!
- Critical point → Lifshitz point
  [D. Nickel, PRL (2009)]
- Inhomogeneous phase rather robust under model extensions and variations:
  - vector interactions
  - Polyakov-loop dynamics
  - including strange quarks
  - isospin imbalance
  - magnetic fields

  [MB, S. Carignano, PPNP (2015)]

Questions addressed in this talk:

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- What is the effect of nonzero bare quark masses?

  [MB, S. Carignano, arxiv:1809.10066 [hep-ph]]

- What is the influence of strange quarks?

- based on:
  - MB, S. Carignano, arxiv:1809.10066 [hep-ph]
  - MB, S. Carignano, submitted to PoS (proceedings QCHS 2018)

Nonzero bare quark masses

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Nonzero bare quark masses

- What is the effect of going away from the chiral limit?

Nonzero bare quark masses

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- Andersen, Kneschke, PRD (2018):

  No inhomogeneous phase in the 2-flavor quark-meson model for $m_\pi > 37.1$ MeV

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$m_{u,d}$ = 0, 5 MeV, 10 MeV

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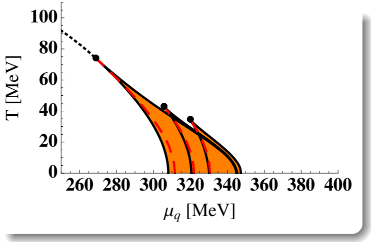
  Inhomogeneous phase in 2-flavor NJL gets smaller but still reaches the CEP



Nonzero bare quark masses

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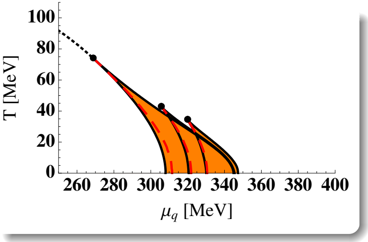

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- Can we investigate this more systematically?

NJL Model

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- Lagrangian:

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi + G\left[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\vec{\tau}\psi)^2\right]$$

NJL Model

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- bosonize: $\sigma(x) = \bar{\psi}(x)\psi(x), \quad \vec{\pi}(x) = \bar{\psi}(x)i\gamma_5\vec{\tau}\psi(x)$

$$\Rightarrow \quad \mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m + 2G_S(\sigma + i\gamma_5\vec{\tau}\cdot\vec{\pi})\right)\psi - G\left(\sigma^2 + \vec{\pi}^{\,2}\right)$$

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- mean-field approximation:

$$\sigma(x) \to \langle\sigma(x)\rangle \equiv \phi_S(\vec x), \quad \pi_a(x) \to \langle\pi_a(x)\rangle \equiv \phi_P(\vec x)\,\delta_{a3}$$

  - $\phi_S(\vec x)$, $\phi_P(\vec x)$ time independent classical fields
  - retain space dependence !

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  - $\phi_S(\vec{x})$, $\phi_P(\vec{x})$ time independent classical fields
  - retain space dependence !

- mean-field Lagrangian:

$$\mathcal{L}_{MF} = \bar{\psi}(x)\left[i\partial\!\!\!/ - m + 2G\left(\phi_S(\vec{x}) + i\gamma_5\tau_3\phi_P(\vec{x})\right)\right]\psi(x) - G\left[\phi_S^2(\vec{x}) + \phi_P^2(\vec{x})\right]$$

Mean-field thermodynamic potential

- mean-field thermodynamic potential:

$$\Omega_{MF}(T,\mu) = -\frac{T}{V}\ln\int\mathcal{D}\bar{\psi}\mathcal{D}\psi\,\exp\left(\int_{x\in[0,\frac{1}{T}]\times V}(\mathcal{L}_{MF}+\mu\bar{\psi}\gamma^0\psi)\right)$$

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- $\mathcal{L}_{MF}$ bilinear in ψ and $\bar\psi$ $\Rightarrow$ quark fields can be integrated out:

$$\Omega_{MF}(T,\mu) = -\frac{T}{V}\mathbf{Tr}\log\left(\frac{S^{-1}}{T}\right) + G\,\frac{1}{V}\int d^3x\;\left(\phi_S^2(\vec x)+\phi_P^2(\vec x)\right)$$

  - inverse dressed propagator: $\mathcal{S}^{-1}(x) = i\partial\!\!\!/ + \mu\gamma^0 - m + 2G_S\left(\phi_S(\vec x) + i\gamma_5\tau_3\phi_P(\vec x)\right)$
  - **Tr**: functional trace over Euclidean $V_4 = [0,\frac{1}{T}]\times V$, Dirac, color, and flavor

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$\Rightarrow \quad \Omega_{MF} = \Omega_{MF}[\phi_S(\vec{x}),\phi_P(\vec{x})]$ minimization extremly difficult !

Ginzburg-Landau analysis

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- Simplifications:
  - chiral limit $m = 0$ (will be relaxed later)
  - $\phi_P = 0$ (to simplify the notation, can be included straightforwardly)

$\rightarrow$ order parameter $M(\vec{x}) = -2G\phi_S(\vec{x})$ ("constituent quark mass")

$\rightarrow$ $\Omega_{MF} = \Omega_{MF}[M]$

Ginzburg-Landau analysis

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  $\rightarrow$ $\Omega_{MF} = \Omega_{MF}[M]$

- Assumptions: M, $|\nabla M|$ small (holds near the LP)

  $\rightarrow$ expansion of the thermodynamic potential.

$$\Omega[M] = \Omega[0] + \frac{1}{V}\int_V d^3x \left\{ \alpha_2 M^2(\vec{x}) + \alpha_{4,a} M^4(\vec{x}) + \alpha_{4,b} |\vec{\nabla} M(\vec{x})|^2 + \ldots \right\}$$

  - $\alpha_n = \alpha_n(T, \mu)$: GL coefficients
  - chiral symmetry: only even powers allowed
  - stability: higher-order coeffs. positive

Tricritical and Lifshitz point

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- GL expansion: $\Omega[M] = \Omega[0] + \frac{1}{V} \int_V d^3x \left\{ \alpha_2 M^2 + \alpha_{4,a} M^4 + \alpha_{4,b} |\vec{\nabla} M|^2 + ... \right\}$

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- case 1: $\alpha_{4,b} > 0 \Rightarrow$ gradients disfavored $\Rightarrow$ homogeneous

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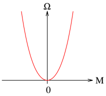
  case 1.1: $\alpha_{4,a} > 0$

  - $\alpha_2 > 0 \Rightarrow$ restored phase



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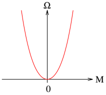
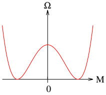
- $\alpha_2 < 0 \;\Rightarrow$ hom. broken phase





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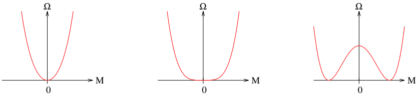
  - 2nd-order p.t. at $\alpha_2 = 0$



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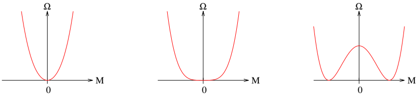
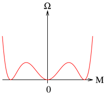


<u>case 1.2:</u> $\alpha_{4,a} < 0$

- 1st-order phase trans. at $\alpha_2 > 0$



Tricritical and Lifshitz point

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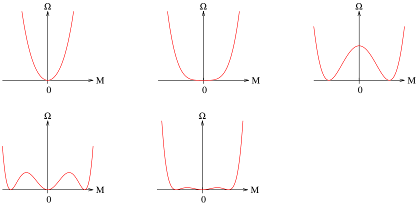

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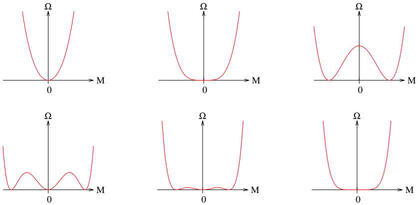

$\Rightarrow$ tricritical point (TCP): $\alpha_2 = \alpha_{4,a} = 0$

Tricritical and Lifshitz point

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  case 1.1: $\alpha_{4,a} > 0$

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- case 2: $\alpha_{4,b} < 0$
  - inhomogeneous phase possible

Tricritical and Lifshitz point

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  $\Rightarrow$ tricritical point (TCP): $\alpha_2 = \alpha_{4,a} = 0$

- case 2: $\alpha_{4,b} < 0$
  - inhomogeneous phase possible
  - 2nd-order phase boundary inhom. - restored: $\alpha_{4,b} < 0, \ \alpha_2 > 0$
    finite wavelength, amplitude $\to 0$

Tricritical and Lifshitz point

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  <u>case 1.2:</u> $\alpha_{4,a} < 0$

  - 1st-order phase trans. at $\alpha_2 > 0$

- <u>case 2:</u> $\alpha_{4,b} < 0$

  - inhomogeneous phase possible

  Lifshitz point (LP): $\alpha_2 = \alpha_{4,b} = 0$

  - 2nd-order phase boundary inhom. - restored: $\alpha_{4,b} < 0,\ \alpha_2 > 0$
    finite wavelength, amplitude $\to 0$

Away from the chiral limit

- $m \neq 0$: no chirally restored solution $M = 0$

  $\rightarrow$ expand about a priory unknown constant mass M_0:

$$\Omega[M] = \Omega[M_0] + \frac{1}{V} \int d^3x \, \left(\alpha_1 \delta M + \alpha_2 \delta M^2 + \alpha_3 \delta M^3 + \alpha_{4,a} \delta M^4 + \alpha_{4,b} (\nabla \delta M)^2 + \ldots \right)$$

  - small parameters: $\delta M(\vec{x}) \equiv M(\vec{x}) - M_0, \quad |\nabla \delta M(\vec{x})|$
  - GL coefficients: $\alpha_j = \alpha_j(T, \mu, M_0)$
  - odd powers allowed
  - require M_0 = extremum of Ω at given T and μ

    $\Rightarrow \quad \alpha_1(T, \mu, M_0) = 0 \quad \rightarrow \quad M_0 = M_0(T, \mu)$ (= homogeneous gap equation)

CEP and pseudo Lifshitz point

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- GL expansion:

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- case 1: $\alpha_{4,b} > 0 \;\Rightarrow$ homogeneous

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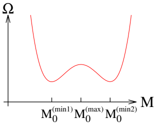

  - no restored phase, but 1st-order ph. trans. between different minima possible

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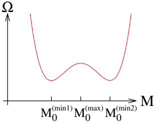

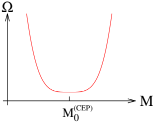

  - no restored phase, but 1st-order ph. trans. between different minima possible
  - 2 minima + 1 maximum $\rightarrow$ 1 minimum

$\Rightarrow$ critical endpoint (CEP): $\alpha_2 = \alpha_3 = 0$

CEP and pseudo Lifshitz point

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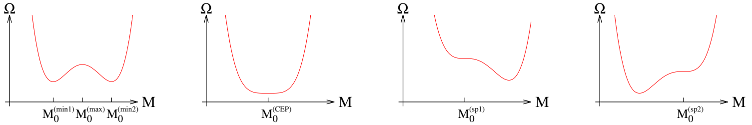

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  - spinodals: left: $\alpha_2 = 0, \alpha_3 < 0$, right: $\alpha_2 = 0, \alpha_3 > 0$,

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    $\Rightarrow$ as in the chiral limit: $\alpha_{4,b} < 0,\ \alpha_2 > 0$

    $\to$ pseudo Lifshitz point (PLP): $\alpha_2 = \alpha_{4,b} = 0$

Summarizing: GL analysis of critical and Lifshitz points

- chiral limit ($m = 0$):
  - expansion about $M = 0$
  - TCP: $\alpha_2 = \alpha_{4,a} = 0$
  - LP: $\alpha_2 = \alpha_{4,b} = 0$
- away from the chiral limit ($m \neq 0$):
  - expansion about $M_0(T, \mu)$ solving $\alpha_1(T, \mu, M_0) = 0$
  - CEP: $\alpha_2 = \alpha_3 = 0$
  - PLP: $\alpha_2 = \alpha_{4,b} = 0$

Determination of the GL coefficients

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- NJL mean-field thermodynamic potential:

$$\Omega_{MF}(T,\mu) = -\frac{T}{V}\mathbf{Tr}\log\left(\frac{S^{-1}}{T}\right) + G\,\frac{1}{V}\int d^3x\;\left(\phi_S^2(\vec{x}) + \phi_P^2(\vec{x})\right)$$

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  - $S_0^{-1}(x) = i\partial\!\!\!/ + \mu\gamma^0 - M_0$ inverse propagator of a free fermion with mass M_0

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- expand logarithm:

$$\log(S_0^{-1} - \delta M) = \log(S_0^{-1}) + \log(1 - S_0\delta M) = \log(S_0^{-1}) - \sum_{n=1}^{\infty}\frac{1}{n}(S_0\delta M)^n$$

Determination of the GL coefficients

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- Thermodynamic potential: $\Omega_{MF} = \sum_{n=0}^{\infty} \Omega^{(n)}$

$\Omega^{(n)}$: contribution of order $(\delta M)^n$:

$$\Omega^{(0)} = -\frac{T}{V}\,\mathbf{Tr}\,\log S_0^{-1} + \frac{1}{V}\int_V d^3x\,\frac{(M_0 - m)^2}{4G}$$

$$\Omega^{(1)} = \frac{T}{V}\,\mathbf{Tr}\,(S_0\delta M) + \frac{M_0 - m}{2G}\,\frac{1}{V}\int_V d^3x\,\delta M(\vec{x})\,,$$

$$\Omega^{(2)} = \frac{1}{2}\frac{T}{V}\,\mathbf{Tr}\,(S_0\delta M)^2 + \frac{1}{4G}\,\frac{1}{V}\int_V d^3x\,\delta M^2(\vec{x})\,,$$

$$\Omega^{(n)} = \frac{1}{n}\frac{T}{V}\,\mathbf{Tr}\,(S_0\delta M)^n \quad \text{for } n \geq 3.$$

Determination of the GL coefficients

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- functional trace:

$$\mathbf{Tr}\,(S_0\delta M)^n = 2N_c \int \prod_{i=1}^{n} d^4x_i\, \mathrm{tr_D}\left[S_0(x_n, x_1)\delta M(\vec{x}_1) S_0(x_1, x_2)\delta M(\vec{x}_2) \ldots S_0(x_{n-1}, x_n)\delta M(\vec{x}_n)\right]$$

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- gradient expansion: $\delta M(\vec{x}_i) = \delta M(\vec{x}_1) + \nabla M(\vec{x}_1)\cdot(\vec{x}_i - \vec{x}_1) + \ldots$

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- final steps:
  - Insert momentum-space rep. of the free propagators S_0 and turn out all but one d^4x_i integrals.
  - Compare results with GL expansion of Ω_{MF} to read off the GL coefficients.

GL coefficients: results

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- Resulting coefficients:

$$\alpha_1 = \frac{M_0 - m}{2G} + M_0 F_1 , \qquad \alpha_2 = \frac{1}{4G} + \frac{1}{2} F_1 + M_0^2 F_2 , \qquad \alpha_3 = M_0 \left(F_2 + \frac{4}{3} M_0^2 F_3 \right) ,$$

$$\alpha_{4,a} = \frac{1}{4} F_2 + 2 M_0^2 F_3 + 2 M_0^4 F_4 , \qquad \alpha_{4,b} = \frac{1}{4} F_2 + \frac{1}{3} M_0^2 F_3$$

- $F_n = 8 N_c \int \frac{d^3p}{(2\pi)^3} \, T \sum_j \frac{1}{[(i\omega_j + \mu)^2 - \vec{p}^{\,2} - M_0^2]^n} , \qquad \omega_j = (2j+1)\pi T$

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- chiral limit:
  - $m = 0 \;\Rightarrow\; M_0 = 0$ solves gap equation $\alpha_1 = 0$
  - $M_0 = 0 \;\Rightarrow\; \alpha_3 = 0$ (no odd powers)
  - $M_0 = 0 \;\Rightarrow\; \alpha_{4,a} = \alpha_{4,b} \;\Rightarrow\;$ TCP = LP [Nickel, PRL (2009)]

GL coefficients: results

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- towards the chiral limit:

  - $M_0 \to 0 \;\Rightarrow\; \alpha_3, \alpha_{4ba}, \alpha_{4,b} \propto F_2 \;\Rightarrow\;$ CEP $\to$ TCP = LP

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TECHNISCHE UNIVERSITÄT DARMSTADT

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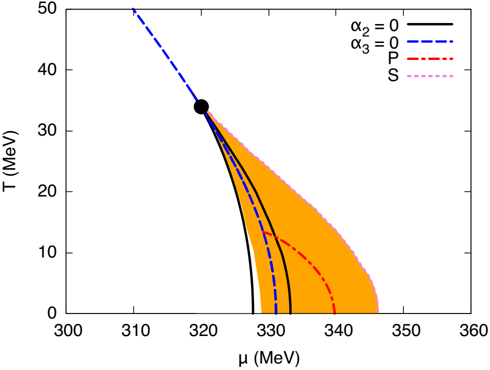
  The CEP coincides with the PLP!

Results:

- phase diagram for m = 10 MeV:



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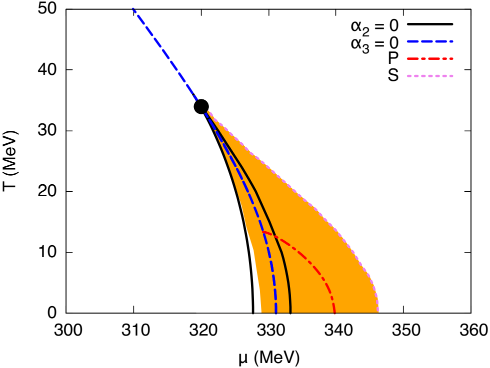

- dominant instability in the scalar channel

Results:

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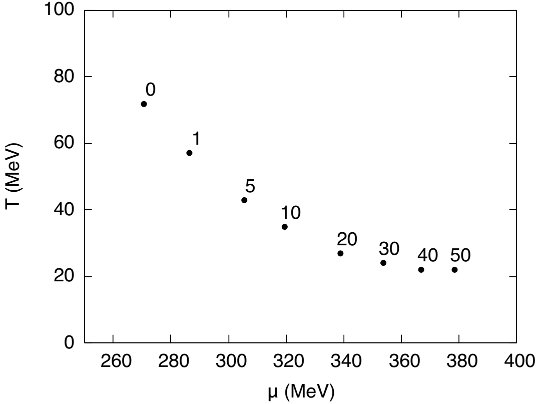
- position of the CEP=PLP for different *m*:



m/MeV	m_π/MeV
0.	0.
1.	43.
5.	96.
10.	135.
20.	191.
30.	235.
40.	271.
50.	303.

Including strange quarks

Motivation

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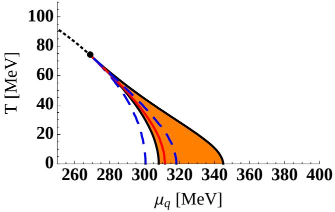
- 2-flavor NJL: TCP $\rightarrow$ LP, CEP $\rightarrow$ PLP



[D. Nickel, PRD (2009)]

Motivation

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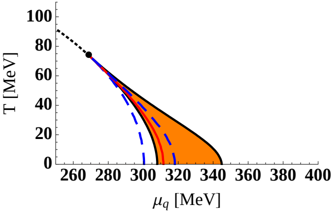
- Is this also true in QCD?



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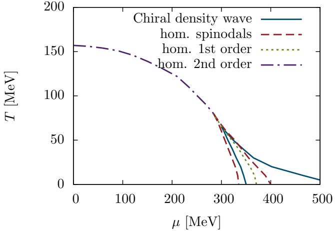

[D. Müller et al. PLB (2013)]

Motivation

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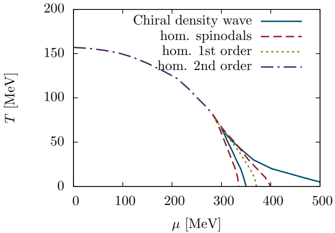

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  - CEP reaches T-axis
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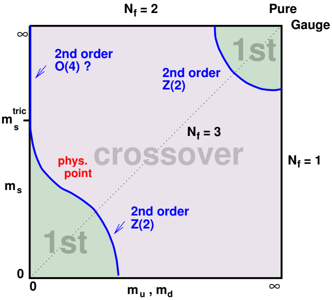
  - chance to study the inhomogeneous phase on the lattice!



[from de Forcrand et al., POSLAT 2007]

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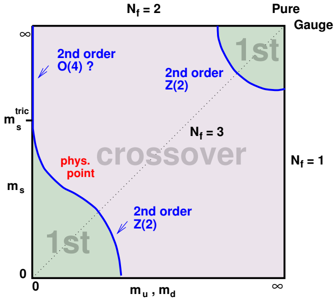
- Here: Ginzburg-Landau study for 3-flavor NJL



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3-flavor NJL model

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- Lagrangian: $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - \hat{m})\psi + \mathcal{L}_4 + \mathcal{L}_6$
  - fields and bare masses: $\psi = (u, d, s)^T, \quad \hat{m} = \mathrm{diag}_f(0, 0, m_s)$
  - 4-point interaction: $\mathcal{L}_4 = G \sum_{a=0}^{8} \left[(\bar{\psi}\tau_a\psi)^2 + (\bar{\psi}i\gamma_5\tau_a\psi)^2 \right]$
  - 6-point ('t Hooft) interaction: $\mathcal{L}_6 = -K \left[\det_f \bar{\psi}(1 + \gamma_5)\psi + \det_f \bar{\psi}(1 - \gamma_5)\psi \right]$

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  - fields and bare masses: $\psi = (u, d, s)^T$, $\hat{m} = \mathrm{diag}_f(0, 0, m_s)$
  - 4-point interaction: $\mathcal{L}_4 = G \sum_{a=0}^{8} \left[(\bar{\psi}\tau_a\psi)^2 + (\bar{\psi}i\gamma_5\tau_a\psi)^2\right]$
  - 6-point ('t Hooft) interaction: $\mathcal{L}_6 = -K\left[\det_f \bar{\psi}(1+\gamma_5)\psi + \det_f \bar{\psi}(1-\gamma_5)\psi\right]$
- Mean fields:
  - light sector: $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle \equiv \frac{S}{2}$, $\langle \bar{u}i\gamma_5 u \rangle = -\langle \bar{d}i\gamma_5 d \rangle \equiv \frac{P}{2}$

    $(\Rightarrow\ \langle \bar{\psi}_\ell \psi_\ell \rangle \equiv \langle \bar{u}u \rangle + \langle \bar{d}d \rangle = S,\quad \langle \bar{\psi}_\ell i\gamma_5\tau_3\psi_\ell \rangle \equiv \langle \bar{u}i\gamma_5 u \rangle - \langle \bar{d}i\gamma_5 d \rangle = P)$
  - strange sector: $\langle \bar{s}s \rangle \equiv S_s$, $\langle \bar{s}i\gamma_5 s \rangle = 0$
  - no flavor-nondiagonal mean fields
  - allow for inhomogeneities: $S = S(\vec{x})$, $P = P(\vec{x})$, $S_s = S_s(\vec{x})$

Mean-field Thermodynamic Potential

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- $\Omega_{MF}(T,\mu) = -\frac{T}{V}\text{Tr}\,\log\left(i\partial\!\!\!/ + \mu\gamma^0 - \hat{M}\right) + \frac{1}{V}\int d^3x\,\mathcal{V}(\vec{x})$
  - dressed "masses": $\hat{M}_{u,d}(\vec{x}) = -(2G - KS_s(\vec{x}))\big(S(\vec{x}) \pm i\gamma_5 P(\vec{x})\big)$

    $\hat{M}_s(\vec{x}) = m_s - 4GS_s(\vec{x}) + \frac{1}{2}K\left(S^2(\vec{x}) + P^2(\vec{x})\right)$
  - "potential field": $\mathcal{V}(\vec{x}) = G\big(S^2(\vec{x}) + P^2(\vec{x}) + 2S_s(\vec{x})\big) - KS_s(\vec{x})\big(S^2(\vec{x}) + P^2(\vec{x})\big)$

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- $K = 0$: light and strange sectors decouple!

  $\hat{M}_{u,d} = -2G\big(S \pm i\gamma_5 P\big), \quad \hat{M}_s(\vec{x}) = m_s - 4GS_s; \qquad \mathcal{V} = G\big(S^2 + P^2\big) + 2GS_s$

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- Chiral density wave ansatz for the light sector:

  $S(\vec{x}) = \phi_0\cos(\vec{q}\cdot\vec{x}), \quad P(\vec{x}) = \phi_0\sin(\vec{q}\cdot\vec{x}), \quad S_s = \phi_s = const.$

  $\Rightarrow \quad \hat{M}_{u,d} = \Delta\, e^{\pm i\gamma_5\vec{q}\cdot\vec{x}}, \quad \Delta \equiv -(2G - K\phi_s)\phi_0,$

  $M_s = const., \quad \mathcal{V} = const.$

  consistent with the literature [Moreira et al., PRD (2014)]

Ginzburg-Landau expansion

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- Difficulty at $m_s \neq 0$: No $SU(3)_L \times SU(3)_R$ restored solution
- $m_u = m_d = 0$

  $\Rightarrow$ Expand about two-flavor restored solution $S = P = 0$:

$$\Omega_{MF}[S, P, S_s] = \Omega_{MF}[0, 0, S_s^{(0)}] + \frac{1}{V} \int d^3x\, \Omega_{GL}[S(\vec{x}), P(\vec{x}), X(\vec{x})]$$

  - strange condensate: $S_s(\vec{x}) = S_S^{(0)} + X(\vec{x})$
  - $S_S^{(0)}$: homogeneous solution of the gap equation for $S = P = 0$ at given T and μ
  - Expand Ω_{GL} in S, P and X, and their gradients.

Ginzburg-Landau potential

- Define: $\Delta_\ell = -2G(S + iP), \quad \Delta_s = -4GX$

  $[\Delta_i] = \text{(mass)} \quad \rightarrow \quad$ counting scheme: $\mathcal{O}(\vec{\nabla}) = \mathcal{O}(\Delta_i)$

Ginzburg-Landau potential

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- Resulting structure:

$$\Omega_{GL} = a_2|\Delta_\ell|^2 + a_{4,a}|\Delta_\ell|^4 + a_{4,b}|\vec{\nabla}\Delta_\ell|^2$$
$$+ b_1\Delta_s + b_2\Delta_s^2 + b_3\Delta_s^3 + b_{4,a}\Delta_s^4 + b_{4,b}(\vec{\nabla}\Delta_s)^2$$
$$+ c_3|\Delta_\ell|^2\Delta_s + c_4|\vec{\nabla}\Delta_\ell|^2(\vec{\nabla}\Delta_s)^2 \qquad + \mathcal{O}(\Delta_i^5)$$

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- Stationarity condition: $\frac{\partial\Omega_{GL}}{\partial\Delta_s}\Big|_{\Delta_\ell=\Delta_s=0} = 0 \quad \Leftrightarrow \quad b_1 = 0$

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$$\Rightarrow \quad M_s^{(0)} = m_s - 16N_c\, G\, T\sum_n \int \frac{d^3p}{(2\pi)^3}\frac{M_s^{(0)}}{(i\omega_n+\mu)^2 - \vec{p}^{\,2} - M_s^{(0)\,2}}$$

(= gap equation for $M_s^{(0)} \equiv \hat{M}_s|_{S=P=X=0} = m_s - 4GS_S^{(0)}$)

Eliminating the strange condensate

TECHNISCHE UNIVERSITÄT DARMSTADT

- Extremizing Ω_{MF} w.r.t. $\Delta_s(\vec{x})$

  $\rightarrow$ Euler-Lagrange equation $\frac{\partial \Omega_{GL}}{\partial \Delta_s} - \partial_i \frac{\partial \Omega_{GL}}{\partial \partial_i \Delta_s} = 0$

  $\Leftrightarrow \quad \Delta_s = -\frac{c_3}{2b_2} |\Delta_\ell|^2 + \mathcal{O}(|\Delta_\ell|^4)$

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CP and LP don't coincide anymore!

Discussion

- Relevant GL coefficients (no guarantee yet!):

$$a_2 = \frac{1}{4G}(1+2\delta) + (1+\delta)^2\, 4N_c \frac{1}{V_4}\sum \frac{1}{p^2} + \frac{K}{2G^2} N_c \frac{1}{V_4}\sum \frac{M_s^{(0)}}{p^2 - M_s^{(0)\,2}}$$

$$a_{4,a} = (1+\delta)^4\, 2N_c \frac{1}{V_4}\sum \frac{1}{p^4} + \frac{K^2}{32G^4} N_c \frac{1}{V_4}\sum \frac{p^2 + M_s^{(0)\,2}}{\left[p^2 - M_s^{(0)\,2}\right]^2}$$

$$a_{4,b} = (1+\delta)^2\, 2N_c \frac{1}{V_4}\sum \frac{1}{p^4}$$

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  - $K = 0 \;\Rightarrow\; \delta = 0 \;\Rightarrow\;$ CP=LP

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- Interesting limits:
  - $K = 0 \;\Rightarrow\; \delta = 0 \;\Rightarrow\;$ CP=LP
  - $m_s \to 0 \;\Rightarrow\; M_s^{(0)}, S_s^{(0)}, \delta \to 0 \;\Rightarrow\;$ LP $\to$ LP(K=0) $\neq$ CP

Discussion

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- Numerical survey of the general case still to be done.

Conclusions

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- Ginzburg-Landau analysis of the effect of bare quark masses and strange quarks the inhomogeneous chiral phase in the NJL model
- nonzero $m_{u,d}$:
  - PLP coincides with CEP
  - dominant instability towards inhomogeneities in the scalar channel
  - numerical result: inhomogeneous phase survives large (higher than physical) quark masses
- strange quarks:
  - CP and LP no longer agree as a consequence of the axial anomaly
  - detailed numerical study to be done
- studies in the QM model underway
- QCD?