The critical endpoint of QCD in a finite volume

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Based on:
JB, Isserstedt, Fischer, Schaefer (in preparation)

ITP Lunch Club, JLU Gießen
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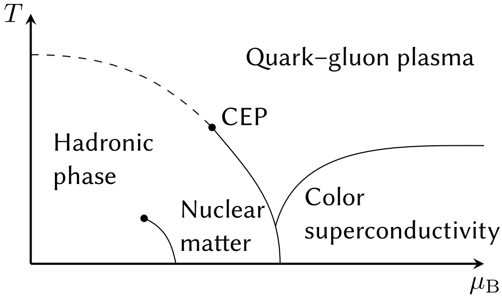
Outline

Why Finite Volume?



- Goal of many experiments is to locate CEP in QCD phase diagram
- “Fireball” of heavy-ion collisions has finite spatial extent
- Impact of volume effects on CEP is important for comparison between experiment and theory
- Lattice QCD is by construction formulated in a finite volume

Dyson–Schwinger Equations

Master DSE

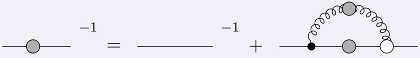
$$0 = \int \mathcal{D}\varphi \, \frac{\delta}{\delta\varphi} \exp(-\mathcal{S}[\varphi] + \langle \varphi, J \rangle) = \left\langle -\frac{\delta \mathcal{S}}{\delta\varphi} + J \right\rangle$$

- Exact equations of motion of Euclidean n-point functions
- Non-perturbative, functional approach
- Obtained by taking appropriate number of functional derivatives of master DSE (and setting $J = 0$)
- Wide range of applications
  - Phase diagram, thermodynamics, spectral functions
  - Together with BSEs: Hadron physics (spectroscopy, decays, ...)
  - ...

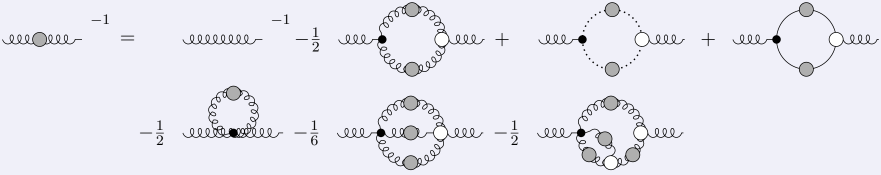
Reviews: Fischer, PPNP 105 (2019) 1
Eichmann, Sanchis-Alepuz, Williams, Alkofer, Fischer, PPNP 91 (2016) 1

DSEs of QCD Propagators

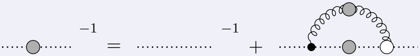
Quark propagator



Gluon propagator

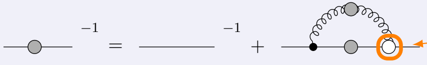


Ghost propagator



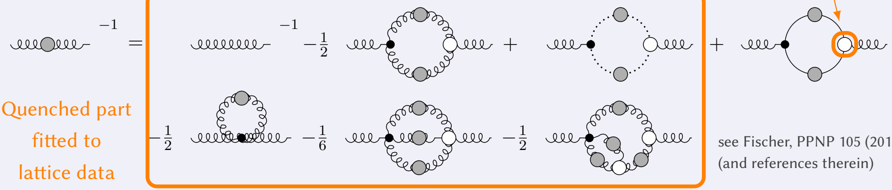
Truncation Scheme

Quark propagator



Quark–gluon vertex ansatz

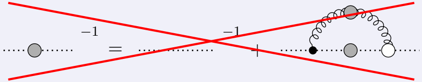
Gluon propagator



Quenched part fitted to lattice data

see Fischer, PPNP 105 (2019) 1 (and references therein)

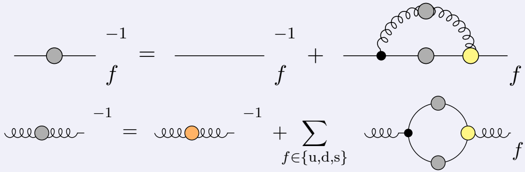
Ghost propagator



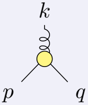
Not needed

Truncated Set of DSEs

Truncated DSEs



Quark–gluon vertex ansatz



$$\Gamma_\mu^f(k,p,q) = \gamma_\mu \Gamma(k,p,q) \left(\delta_{\mu 4} \Gamma_4^{f,\mathrm{BC}}(p,q) + (1-\delta_{\mu 4}) \Gamma_s^{f,\mathrm{BC}}(p,q) \right)$$

Quenched gluon propagator

$D_{\mu\nu}^{\mathrm{que}}(k) = D_{\mu\nu}^{\mathrm{que}}(k;T)$ (Temperature-dependent fit to lattice data)

reference for lattice data: Maas, Pawlowski, von Smekal, Spielmann, PRD 85 (2012) 034037

Review: Matsubara Formalism

- Finite temperature ($T = \beta^{-1}$) $\to$ bounded imaginary time ($\tau := \mathrm{i}t$) integration

$$\int_{-\infty}^{\infty} \mathrm{d}\tau \, L \;\to\; \int_{0}^{\beta} \mathrm{d}\tau \, L$$

- Spin–statistics theorem dictates boundary conditions (BC):

$$\begin{aligned} \varphi(\tau + \beta) &= +\varphi(\tau) \quad \text{for bosons,} \\ \varphi(\tau + \beta) &= -\varphi(\tau) \quad \text{for fermions} \end{aligned}$$

- Together: bounded integral + (anti-)periodicity $\to$ discrete Fourier transform

$\Rightarrow$ Only discrete energies possible

Inclusion of Temperature and Chemical Potential

(Temporal) Matsubara frequencies

$$\omega_n = \begin{cases} 2n\pi T & \text{for bosons}\,, \\ (2n+1)\pi T & \text{for fermions}\,, \end{cases} \qquad n \in \mathbb{Z}$$

- At finite T, energy integral becomes sum over Matsuabara frequencies

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}q_0}{2\pi} K(q_0) \to T \sum_{n=-\infty}^{\infty} K(\omega_n)$$

- Chemical potential corresponds to imaginary shift of energy

$$\omega_n \to \tilde{\omega}_n := \omega_n + \mathrm{i}\mu$$

Quark and Gluon Dressing Functions

- Solve DSEs in terms of scalar dressing functions

- Quark: according to Dirac structure

$$S_f^{-1}(\tilde{\omega}_n, \boldsymbol{p}) = \mathrm{i}\tilde{\omega}_n \gamma_4 C_f(\tilde{\omega}_n, \boldsymbol{p}) + \mathrm{i}\boldsymbol{p} \cdot \boldsymbol{\gamma} A_f(\tilde{\omega}_n, \boldsymbol{p}) + B_f(\tilde{\omega}_n, \boldsymbol{p})$$

- Gluon: transversal (T) and longitudinal (L) w.r.t. heat bath

$$D_{\nu\sigma}(k) = \frac{Z^{\mathrm{T}}(k)}{k^2} P^{\mathrm{T}}_{\nu\sigma}(k) + \frac{Z^{\mathrm{L}}(k)}{k^2} P^{\mathrm{L}}_{\nu\sigma}(k)$$

QCD Phase Diagram

Order parameter: quark condensate

$$\langle\overline{\psi}\psi\rangle_f \sim \sum_{\omega_n} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \, \mathrm{Tr}\big[S_f(\omega_n + \mathrm{i}\mu, \boldsymbol{q})\big]$$

- $\langle\overline{\psi}\psi\rangle_f$ is divergent for $m_f > 0$. Therefore, define regularized condensate:

$$\Delta_{\mathrm{us}} := \langle\overline{\psi}\psi\rangle_{\mathrm{u}} - \frac{Z_{\mathrm{u}}^{\mathrm{m}}}{Z_{\mathrm{s}}^{\mathrm{m}}} \frac{m_{\mathrm{u}}}{m_{\mathrm{s}}} \langle\overline{\psi}\psi\rangle_{\mathrm{s}}$$

Pseudocritical temperature:

$$T_{\mathrm{c}} := \arg\max_{T} \left| \frac{\partial \Delta_{\mathrm{us}}}{\partial T} \right|$$

Finite Volume Framework I

- Feasible shape as ansatz: cube with edge length L

$$\int_{\mathbb{R}^3} \mathrm{d}^3x\, \mathcal{L} \to \int_{[0,L]^3} \mathrm{d}^3x\, \mathcal{L}$$

- For quarks, free to choose between periodic or antiperiodic BC:

$$\psi(\boldsymbol{x} + L\boldsymbol{e}_i) = +\psi(\boldsymbol{x}) \quad \text{for periodic BC (PBC),}$$
$$\psi(\boldsymbol{x} + L\boldsymbol{e}_i) = -\psi(\boldsymbol{x}) \quad \text{for antiperiodic BC (ABC)}$$

- Again, only discrete values possible in momentum space

- *Side note: (anti-)periodicity in both space and time corresponds geometrically to a hyper**torus***

Finite Volume Framework II

Spatial Matsubara modes

$$\omega_n^L = \begin{cases} 2n\pi/L & \text{for PBC}\,, \\ (2n+1)\pi/L & \text{for ABC}\,, \end{cases} \qquad n \in \mathbb{Z}$$

- Momentum integrals become sums

$$\int \frac{\mathrm{d}^3 q}{(2\pi)^3}\, K(\boldsymbol{q}) \to \frac{1}{L^3} \sum_{\boldsymbol{n}\in\mathbb{Z}^3} K(\boldsymbol{q_n})\,,$$

where $\boldsymbol{q_n} := \sum_{i=1}^{3} \omega_{n_i}^L \boldsymbol{e}_i$ are allowed momentum vectors

- Analogy: finite temperature $\leftrightarrow$ finite volume (where $L \mathrel{\hat{=}} \beta$)

Illustration of Momentum Summation

- Create cube of points
- Sort points with respect to radius
- Only consider complete shells

$\rightarrow$ Leads to $\mathrm{O}(3)$-symmetric cutoff

$\rightarrow$ Reduces hypercubic artifacts

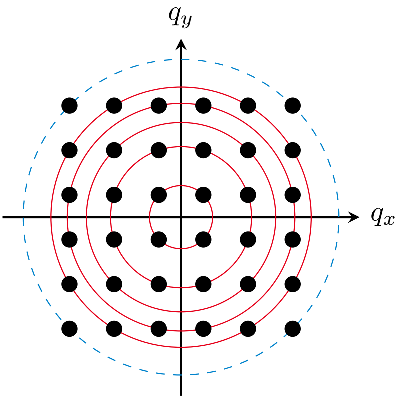


Figure: 2D ABC momentum grid

Naïve Approach: Pure Torus

Pure torus momentum integration

$$\int \frac{\mathrm{d}^3 q}{(2\pi)^3}\, K(\boldsymbol{q}) \to \frac{1}{L^3} \sum_{\boldsymbol{n}\in\mathbb{Z}^3} K(\boldsymbol{q_n})\,,$$

Downside:

- Need to save all vectors on torus up to UV cutoff

$\rightarrow$ Cutoff $\Lambda > 10\,\mathrm{GeV}$ unfeasible (finite-size effects)

$\rightarrow$ Need to renormalize in (possibly) not sufficiently perturbative regime

Renormalization

- Two renormalization constants to fix: Z_f^2 and Z_f^{m}
- Usual renormalization condition:

$$A_f(\nu) \overset{!}{=} 1 \quad \text{and} \quad B_f(\nu) \overset{!}{=} m_f$$

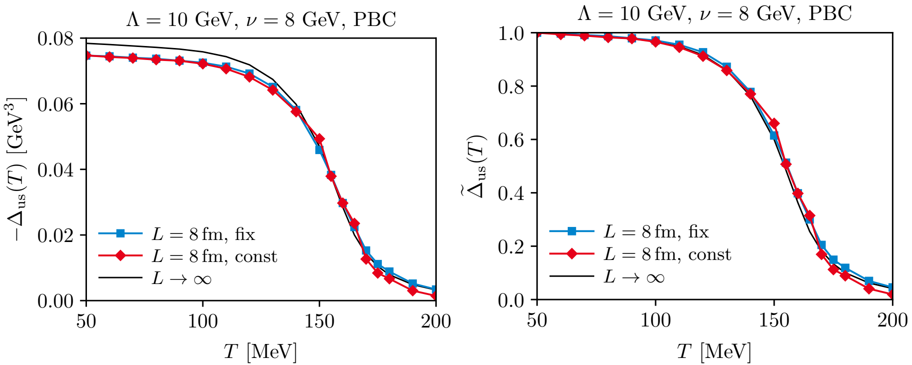
at $T = 0$ with renormalization point $\nu = 80\,\mathrm{GeV}$

- Condition for pure torus:

$$A_f(\omega_0, \nu)\Big|_{L=8\,\mathrm{fm}} \overset{!}{=} A_f(\omega_0, \nu)\Big|_{L\to\infty}$$
$$B_f(\omega_0, \nu)\Big|_{L=8\,\mathrm{fm}} \overset{!}{=} B_f(\omega_0, \nu)\Big|_{L\to\infty}$$

at $T = 50\,\mathrm{MeV}$ with renormalization point $\nu = 8\,\mathrm{GeV}$

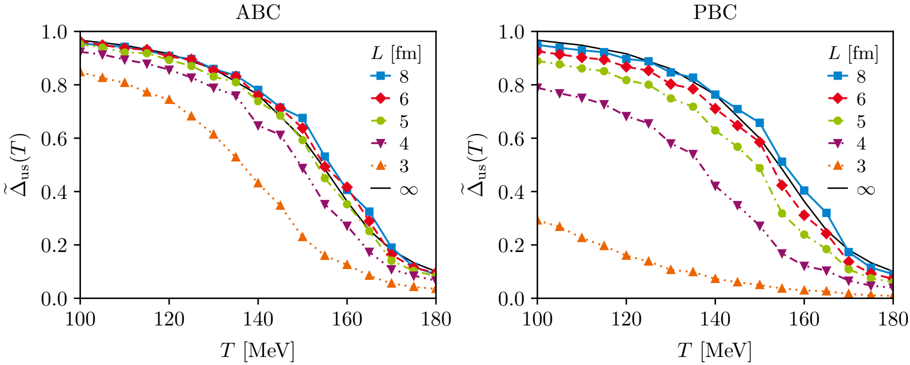
Pure Torus Results: (Re-)Normalization



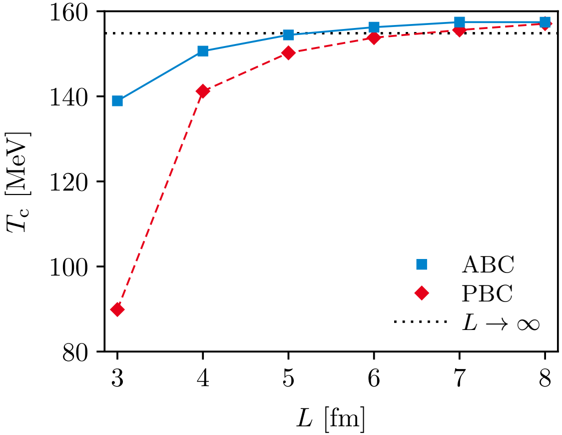
Normalized condensate

$$\widetilde{\Delta}^{L}_{\mathrm{us}}(T) := \frac{\Delta^{L}_{\mathrm{us}}(T)}{\Delta^{8\,\mathrm{fm}}_{\mathrm{us}}(50\,\mathrm{MeV})}\,, \quad \widetilde{\Delta}^{\infty}_{\mathrm{us}}(T) := \frac{\Delta^{L}_{\mathrm{us}}(T)}{\Delta^{\infty}_{\mathrm{us}}(50\,\mathrm{MeV})}$$

Pure Torus Results: Quark Condensate at $\mu = 0$

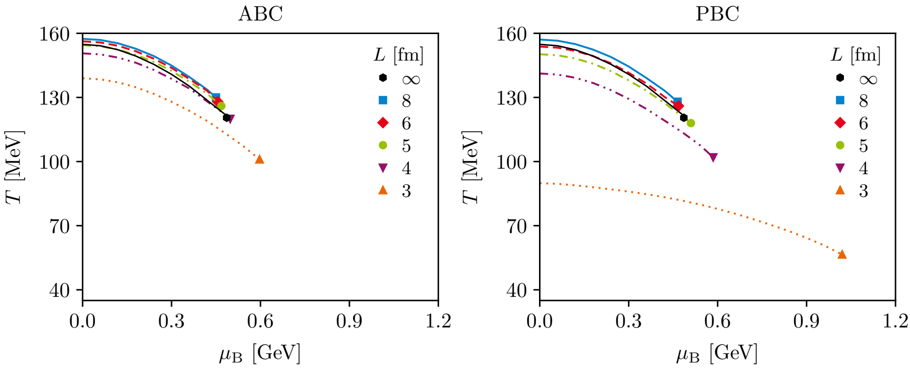


Pure Torus Results: Pseudocritical Temperature at $\mu = 0$



JB, Isserstedt, Fischer, Schaefer (in preparation)

Pure Torus Results: QCD Phase Diagram



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Advanced Approach: Continuum-improved Torus

Continuum-improved torus momentum integration

$$\int \frac{\mathrm{d}^3 q}{(2\pi)^3} \, K(\boldsymbol{q}) \;\rightarrow\; \frac{1}{L^3} \sum_{\boldsymbol{n}\in\mathbb{Z}^3}^{|\boldsymbol{q}_{\boldsymbol{n}}|<\Lambda_{\mathrm{vol}}} K(\boldsymbol{q}_{\boldsymbol{n}}) + \int_{|\boldsymbol{q}|>\Lambda_{\mathrm{vol}}} \frac{\mathrm{d}^3 q}{(2\pi)^3} K(\boldsymbol{q})$$

- Λ_{vol} determined by number of points in torus (dependent on L)
- Chosen such that increase does not alter results

Advantages:

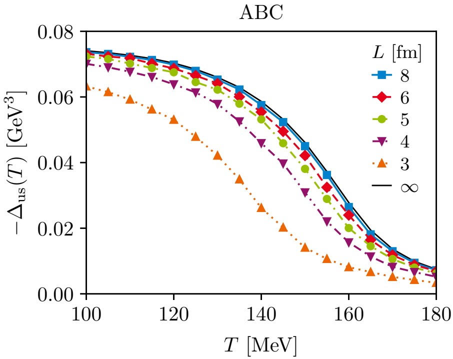
- No more restrictions on UV cutoff

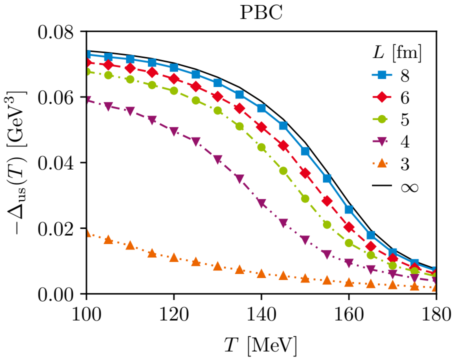
$\rightarrow$ No finite-size effects

$\rightarrow$ Possible to renormalize deep in UV (at same point as infinite volume)

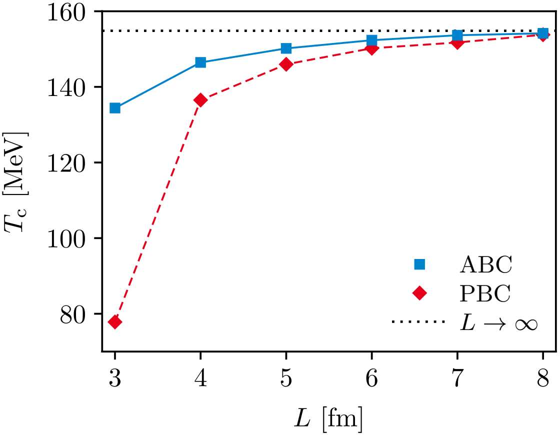
$\rightarrow$ Together: should be much more consistent with infinte-volume results

Improved Torus Results: Quark Condensate at $\mu = 0$





Improved Torus Results: Pseudocritical Temperature



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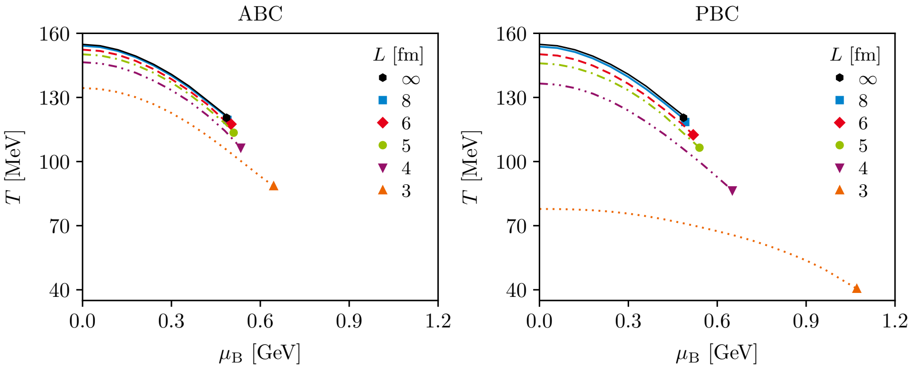
Improved Torus Results: QCD Phase Diagram



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Conclusion and Outlook

Conclusion:

- Studied finite volume effects on QCD phase diagram for different boundary conditions using DSEs beyond rainbow–ladder
- Strong volume effects for $L \leq 4\,\mathrm{fm}$ especially for PBC
- Investigated two different approaches: naïve pure torus ansatz and advanced continuum-improved torus
- Advanced approach does not suffer from finite-size effects and yields qualitatively the same results with consistent infinite-volume limit

Outlook:

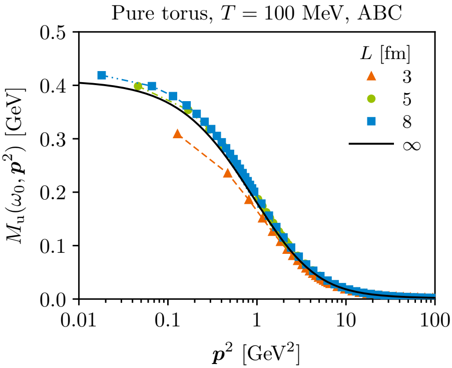
- Quark number fluctuations in finite volume
- Spherical volume (MIT bag model)
- Spectral functions

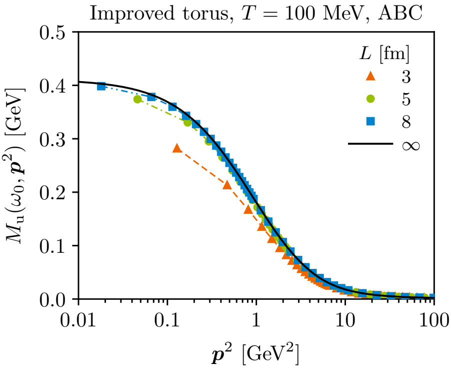
Backup slides

Dressing Functions

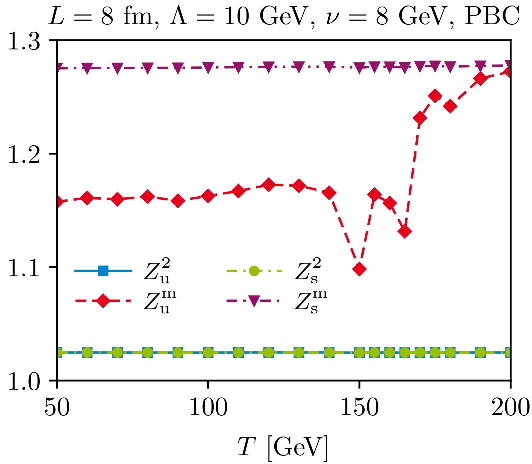
Renormalization point invariant mass function

$$M_f(\omega_n, \boldsymbol{p}) := \frac{B_f(\omega_n, \boldsymbol{p})}{A_f(\omega_n, \boldsymbol{p})}$$





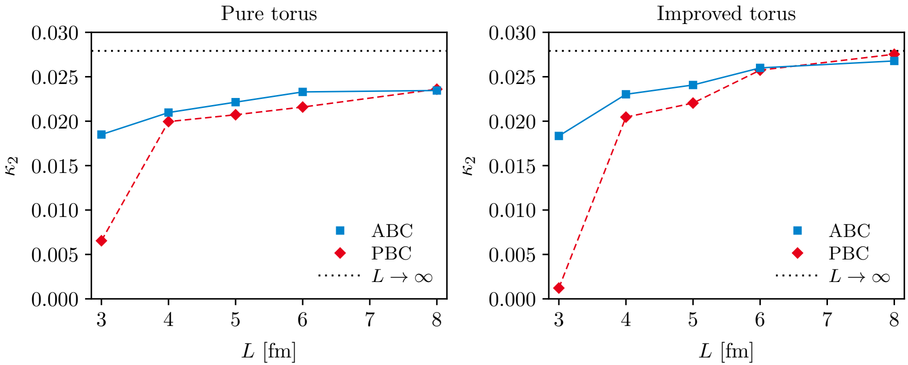
Pure Torus Renormalization Constants



Curvature of the Crossover Transition Line

Parametrization of crossover line (with curvature κ_2)

$$\frac{T_c(\mu_B)}{T_c} = 1 - \kappa_2 \left(\frac{\mu_B}{T_c}\right)^2 - \kappa_4 \left(\frac{\mu_B}{T_c}\right)^4 + \dots$$



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