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Simulating the leading order strong coupling expansion of $SU(2)$ lattice gauge theory in a dual representation

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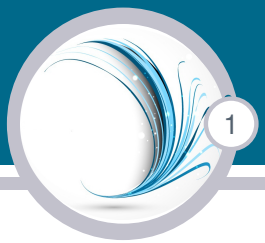
Content



Introduction and motivation

- Sign problems or complex action problems are a common challenge in Lattice Field Theory.
- Possible sources:
  - Finite density
  - Finite chemical potential
  - Topological term
- One approach to solve these problems: *dualisation method*
  - Exactly rewrite the partition sum in terms of new *dual* variables.
  - New degrees of freedom are:
    - World lines of conserved flux for matter fields
    - World sheets for gauge fields
- Dual representations well studied for Abelian theories, more difficult for non-abelian theories.

Introduction and motivation

- This thesis considers dual representations for pure $SU(2)$ lattice gauge theory. Considering this is useful because:
  - Simplest non-Abelian theory
  - Model building theory
- Initially does not have a sign problem.
- The sign problem is introduced in the dualisation procedure.
- Goals of the thesis:
  - Develop a method to solve the "artificial" sign problem.
  - Check validity of the method by numerical simulation.

Lattice regularization of $SU(N)$ gauge theory

Consider the action of a $SU(N)$ gauge theory, given by

$$S[\bar{\psi},\psi,A] = \underbrace{\frac{1}{2g^2}\int d^4x \ \mathrm{Tr}[F_{\mu\nu}(x)F_{\mu\nu}(x)]}_{S_G[A]} + \tag{1}$$

$$\underbrace{\sum_{f=1}^{N_f}\int d^4x\ \bar{\psi}^{(f)}(x)[m^{(f)} + \not{D}(x)]\psi^{(f)}(x)}_{S_F[\bar{\psi},\psi,A]} \quad , \tag{2}$$

with

- $F_{\mu\nu} = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + i\,[A_\mu(x), A_\nu(x)]$,
- $D_\mu(x) = \partial_\mu + igA_\mu(x)$.

Lattice regularization of $SU(N)$ gauge theory

Let $\Omega(x)$ denote a $SU(N)$ gauge transformation. The transformation properties of the fields are:

- $\psi'(x) = \Omega(x)\psi(x), \quad \bar{\psi}'(x) = \bar{\psi}(x)\Omega(x)^\dagger,$
- $D'_\mu(x) = \Omega(x)D_\mu(x)\Omega(x)^\dagger$
- $A'_\mu(x) = \Omega(x)A_\mu(x)\Omega(x)^\dagger - \frac{i}{g}\Omega(x)(\partial_\mu\Omega(x)^\dagger).$

Lattice discretization:
Make the following replacements and identifications:

- $x \to n \in \Lambda_4,$
- $\partial_\mu\psi(x) \to \frac{\psi(n+\hat{\mu})-\psi(n-\hat{\mu})}{2a},$
- $\psi'(n) = \Omega(n)\psi(n),$
- $\Omega(x) \to \Omega(n) \in SU(N)$
- $\Omega(n)^\dagger\Omega(n) = \mathbb{I}$

Lattice regularization of $SU(N)$ gauge theory

$$S_F[\bar{\psi}, \psi, A] = \sum_{f=1}^{N_f} \int d^4x \, \bar{\psi}^{(f)}(n)[m^{(f)} + \not{D}(n)]\psi^{(f)}(n) \quad , \tag{3}$$

Under these transformations the massterm of the fermionic action is invariant.

However, the term with the covariant derivative is not, which can be seen in the following:

$$\bar{\psi}'(n)\gamma_\mu\psi'(n+\hat{\mu}) = \bar{\psi}(n)\underbrace{\Omega(n)^\dagger\Omega(n+\hat{\mu})}_{\neq\mathbb{I}}\gamma_\mu\psi(n+\hat{\mu}) \quad , \tag{4}$$

To fix this, we introduce quantities called gauge links $U_\mu(n)$ with which Eq. (4) becomes:

$$\bar{\psi}(n)U'_\mu(n)\psi(n+\hat{\mu}) \to \bar{\psi}(n)\Omega(n)^\dagger U'_\mu(n)\Omega(n+\hat{\mu})\psi(n+\hat{\mu}) \quad . \tag{5}$$

These gauge links obey the transformation properties:

$$U'_\mu(n) = \Omega(n)U_\mu(n)\Omega(n+\hat{\mu})^\dagger \quad . \tag{6}$$

Lattice regularization of $SU(N)$ gauge theory

Figure: Graphical representation of the gauge link variables.

The gauge links live on the links of the lattice and have the graphical representation shown in figure above.

Lattice regularization of $SU(N)$ gauge theory

For the gauge action

$$S_G[A] = \frac{1}{2g^2} \int d^4x \ \mathrm{Tr}[F_{\mu\nu}(x) F_{\mu\nu}(x)] \ , \tag{7}$$

we have to build a gauge invariant quantity using the gauge links $U_\mu(n)$. The simplest gauge invariant quantity is the trace over the product of gauge links arranged along squares of the lattice. This product of gauge links is called a plaquette and is given as:

$$p_{\mu\nu}(n) = U_\mu(n) U_\nu(n+\hat{\mu}) U_\mu(n+\hat{\nu})^\dagger U_\nu(n)^\dagger \ . \tag{8}$$

Using the plaquettes and using $\beta = \frac{2N}{g^2}$ one identifies the gauge action for a $SU(N)$ theory to be:

$$S_G[U] = \frac{\beta}{N} \sum_{n \in \Lambda_4} \sum_{\mu<\nu} \mathcal{R}e(\mathrm{Tr}[\mathbb{I} - p_{\mu\nu}(n)]) \ . \tag{9}$$

Lattice regularization of $SU(N)$ gauge theory

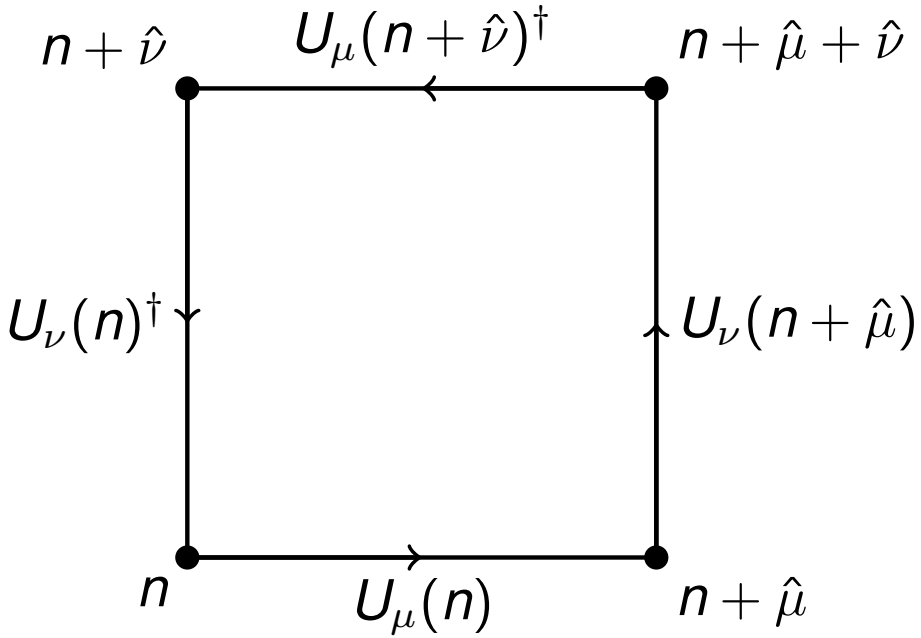


Figure: Graphical illustration of a plaquette on the lattice.

The plaquettes have a very nice geometric representation which is shown in the figure above. Figure taken from [3].

Dualisation of $SU(2)$ lattice gauge theory

Starting point is the partition sum

$$Z = \int D[U]\, e^{-S_G[U]} \quad , \tag{10}$$

with Wilson action

$$S_G[U] = -\frac{\beta}{2} \sum_{x,\mu<\nu} \mathrm{Tr}[U_{x,\mu} U_{x+\hat{\mu},\nu} U^{\dagger}_{x+\hat{\nu},\mu} U^{\dagger}_{x,\nu}] \quad , \tag{11}$$

where

$$U_{x,\mu} \in SU(2). \tag{12}$$

The integral measure is given by:

$$\int D[U] = \prod_{x,\mu} \int dU_{x,\mu}, \tag{13}$$

with $dU_{x,\mu}$ denoting the Haar measure.

Dualisation of $SU(2)$ lattice gauge theory

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Choose following parametrisation of $SU(2)$ matrix:

$$U_{x,\mu} = \begin{bmatrix} \cos(\theta_{x,\mu}) e^{i\alpha_{x,\mu}} & \sin(\theta_{x,\mu}) e^{i\beta_{x,\mu}} \\ -\sin(\theta_{x,\mu}) e^{-i\beta_{x,\mu}} & \cos(\theta_{x,\mu}) e^{-i\alpha_{x,\mu}} \end{bmatrix} \quad , \tag{12}$$

with $\theta_{x,\mu} \in [0, \frac{\pi}{2}]$ and $\alpha_{x,\mu}, \beta_{x,\mu} \in [0, 2\pi]$. The Haar measure reads:

$$dU_{x,\mu} = 2\, d\theta_{x,\mu} \cos(\theta_{x,\mu}) \sin(\theta_{x,\mu}) \frac{d\alpha_{x,\mu}}{2\pi} \frac{d\beta_{x,\mu}}{2\pi} \quad . \tag{13}$$

Dualisation of $SU(2)$ lattice gauge theory

First step towards the dualisation, as given in [1], is to rewrite the trace and matrix multiplication explicitly as:

$$S_G[U] = -\frac{\beta}{2} \sum_{x,\mu<\nu} \sum_{a,b,c,d=0}^{1} U^{ab}_{x,\mu} U^{bc}_{x+\hat{\mu},\nu} U^{dc\,*}_{x+\hat{\nu},\mu} U^{ad\,*}_{x,\nu} \quad , \tag{14}$$

where the $U^{ab}_{x,\mu}$ are now complex numbers.
When expanding the Boltzmann factors in a power series, the partition sum becomes

$$Z = \int D[U] \prod_{x,\mu<\nu} \prod_{a,b,c,d} \sum_{p^{abcd}_{x,\mu\nu}=0}^{\infty} \frac{\left(\frac{\beta}{2}\right)^{p^{abcd}_{x,\mu\nu}}}{p^{abcd}_{x,\mu\nu}!} \Big(\underbrace{U^{ab}_{x,\mu} U^{bc}_{x+\hat{\mu},\nu} U^{dc\,*}_{x+\hat{\nu},\mu} U^{ad\,*}_{x,\nu}}_{ACC} \Big)^{p^{abcd}_{x,\mu\nu}} \quad , \tag{15}$$

where the expansion indices $p^{abcd}_{x,\mu\nu} \in \mathbb{N}_0$ will be identified as our dual variables. The shorthand ACC stands for Abelian Colour Cycles.

Dualisation of $SU(2)$ lattice gauge theory

Abelian Color Cycles have a nice geometrical representation:

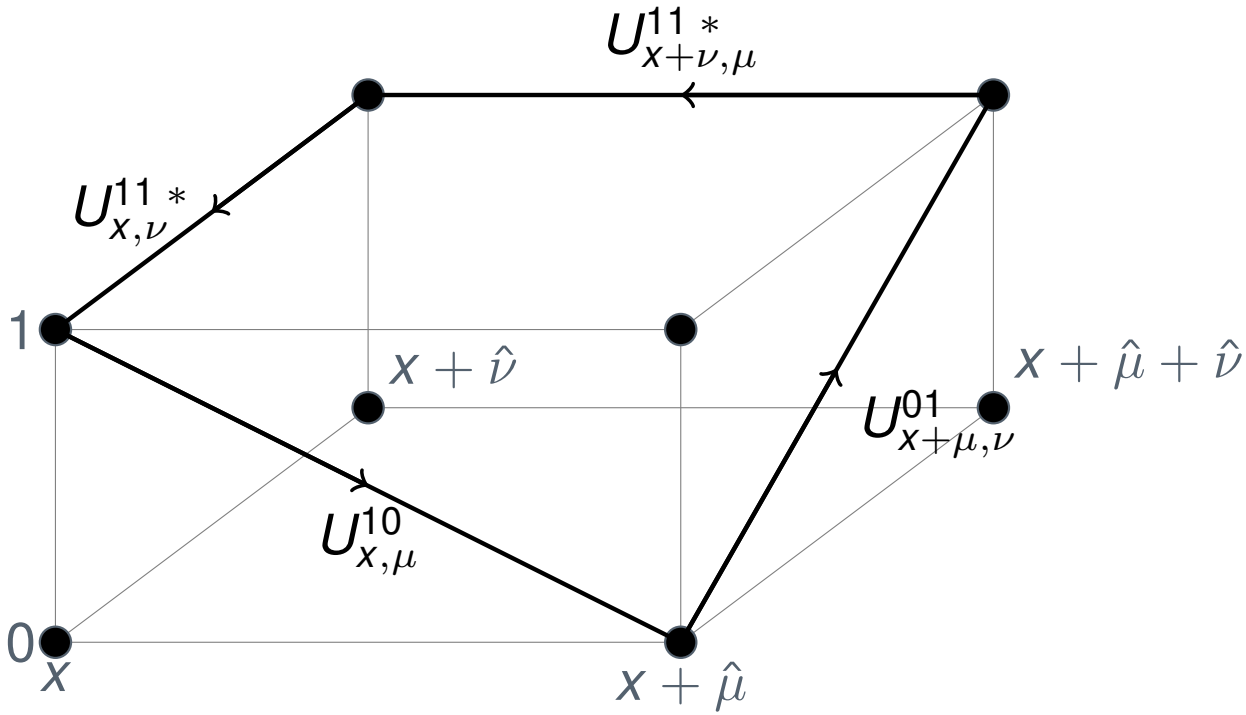


Figure: Graphical illustration of an example ACC here defined by the color index combination 1011. Figure adapted from [1].

Dualisation of $SU(2)$ lattice gauge theory

Inserting now the parametrisation we chose and rearranging the terms we arrive at:

$$
\begin{aligned}
Z = & \sum_{\{p\}} W_\beta[p] \prod_{x,\mu} (-1)^{\mathcal{J}^{01}_{x,\mu}} \\
& \times 2 \int_0^{\frac{\pi}{2}} d\theta_{x,\mu} (\cos(\theta_{x,\mu}))^{1+\mathcal{S}^{00}_{x,\mu}+\mathcal{S}^{11}_{x,\mu}} (\sin(\theta_{x,\mu}))^{1+\mathcal{S}^{01}_{x,\mu}+\mathcal{S}^{10}_{x,\mu}} \\
& \times \int_0^{2\pi} \frac{d\alpha_{x,\mu}}{2\pi} \, e^{i\alpha_{x,\mu}[\mathcal{J}^{00}_{x,\mu}-\mathcal{J}^{11}_{x,\mu}]} \int_0^{2\pi} \frac{d\beta_{x,\mu}}{2\pi} \, e^{i\beta_{x,\mu}[\mathcal{J}^{01}_{x,\mu}-\mathcal{J}^{10}_{x,\mu}]} \, , \qquad (16)
\end{aligned}
$$

where the $\mathcal{J}$ and $\mathcal{S}$ here are flux variables and are given as follows:

$$
\mathcal{J}^{ab}_{x,\mu} = \sum_{\nu:\mu<\nu} [p^{abss}_{x,\mu\nu} - p^{ssba}_{x-\hat{\nu},\mu\nu}] - \sum_{\rho:\rho<\mu} [p^{assb}_{x,\rho\mu} - p^{sabs}_{x-\hat{\rho},\rho\mu}] \qquad (17)
$$

$$
\mathcal{S}^{ab}_{x,\mu} = \sum_{\nu:\mu<\nu} [p^{abss}_{x,\mu\nu} + p^{ssba}_{x-\hat{\nu},\mu\nu}] + \sum_{\rho:\rho<\mu} [p^{assb}_{x,\rho\mu} + p^{sabs}_{x-\hat{\rho},\rho\mu}] \qquad (18)
$$

Dualisation of $SU(2)$ lattice gauge theory

By performing the integrals in Equation (16) we arrive at the final dualised version of the partition sum, which reads:

$$Z = \sum_{\{p\}} W_\beta[p]\; W_H[p]\; (-1)^{\sum_{x,\mu} \mathcal{J}^{01}_{x,\mu}} \prod_{x,\mu} \delta(\mathcal{J}^{00}_{x,\mu} - \mathcal{J}^{11}_{x,\mu})\; \delta(\mathcal{J}^{01}_{x,\mu} - \mathcal{J}^{10}_{x,\mu}) \quad .$$

Here $W_\beta[p]$ and $W_H[p]$ are weight factors given as:

$$W_\beta[p] = \left[\prod_{x,\mu<\nu} \prod_{a,b,c,d} \frac{\left(\frac{\beta}{2}\right)^{p^{abcd}_{x,\mu\nu}}}{p^{abcd}_{x,\mu\nu}!} \right] \quad , \tag{19}$$

and

$$W_H[p] = \prod_{x,\mu} \frac{\left(\frac{\mathcal{S}^{00}_{x,\mu}+\mathcal{S}^{11}_{x,\mu}}{2}\right)!\left(\frac{\mathcal{S}^{01}_{x,\mu}+\mathcal{S}^{10}_{x,\mu}}{2}\right)!}{\left(\frac{\mathcal{S}^{00}_{x,\mu}+\mathcal{S}^{11}_{x,\mu}+\mathcal{S}^{01}_{x,\mu}+\mathcal{S}^{10}_{x,\mu}}{2}+1\right)!} \quad . \tag{20}$$

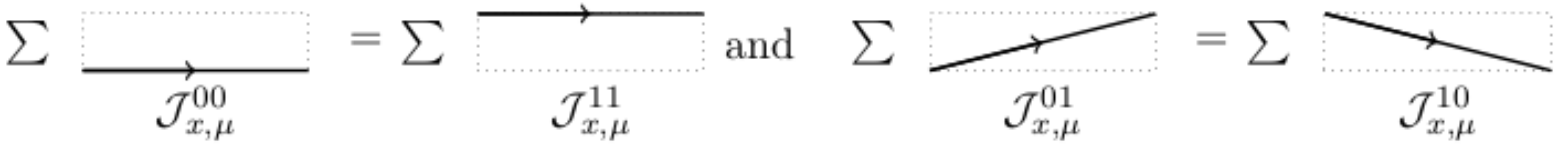


Figure: Graphical illustration of the constraints of the $\mathcal{J}^{ab}_{x,\mu}$-fluxes. Figure adapted from [1].

Dualisation of $SU(2)$ lattice gauge theory
Observables

Some interesting simple observables are:

- The plaquette expectation value:

$$\langle U_p \rangle \equiv \frac{1}{6V} \frac{\partial \ln(Z)}{\partial\left(\frac{\beta}{2}\right)} = \frac{1}{6V} \left\langle \sum_{x,\mu<\nu} \mathrm{Tr}\left[U_{x,\mu}\, U_{x+\hat{\mu},\nu}\, U^{\dagger}_{x+\hat{\nu},\mu}\, U^{\dagger}_{x,\nu}\right] \right\rangle \tag{21}$$

- The plaquette susceptibility:

$$\chi_p = \frac{\partial \langle U_p \rangle}{\partial\left(\frac{\beta}{2}\right)} = \frac{1}{6V} \frac{\partial^2 \ln(Z)}{\partial\left(\frac{\beta}{2}\right)^2} \tag{22}$$

Approaches for simulation

Here we will shortly introduce the approaches we used:

a) Factoring out the center group of $SU(2)$ in search for new constraints,

b) Introduce a new index to try and resum the different contributions to obtain, positive weights

c) Truncate the dual theory and relate it to a Qubit representation.

Approaches for simulation
a) Factoring out the center group of $SU(2)$

The main idea was to change the representation of the gauge links in the following way

$$U^{ab}_{x,\mu} \to \sigma_{x,\mu} U^{ab}_{x,\mu} \quad , \tag{23}$$

where the $\sigma_{x,\mu}$ are from the center group of $SU(2)$, i.e, $\mathbb{Z}_2$. The dualisation is analogous, but we can solve the $\mathbb{Z}_2$ part in closed form to obtain a new constraint of the form:

$$\prod_{x,\mu} E \left(\sum_{\nu} \sum_{a,b,c,d} \left[p^{abcd}_{x,\mu\nu} - p^{abcd}_{x-\hat{\nu},\mu\nu} \right] \right) \quad . \tag{24}$$

However, it can be shown that this constraint corresponds to the $\mathcal{J}$-flux constraint and is thus not new.

Approaches for simulation
b) Index parametrisation

Idea here is to start from the following partition function

$$Z = \sum_{\{p\}} W_\beta[p] \int D[U] \underbrace{\prod_{x,\mu<\nu} \left(\mathrm{Tr}\left[U_{x,\mu} U_{x+\hat{\mu},\nu} U^\dagger_{x+\hat{\nu},\mu} U^\dagger_{x,\nu} \right] \right)^{p_{x,\mu\nu}}}_{(*)}, \quad (25)$$

Write $(*)$ as

$$(*) = \sum_{\{a,b,c,d\}} \prod_{x,\mu<\nu} \prod_{j=1}^{p_{x,\mu\nu}} U^{a^j_{x,\mu\nu} b^j_{x,\mu\nu}}_{x,\mu} U^{b^j_{x,\mu\nu} c^j_{x,\mu\nu}}_{x+\hat{\mu},\nu} U^{*\ d^j_{x,\mu\nu} c^j_{x,\mu\nu}}_{x+\hat{\nu},\mu} U^{*\ a^j_{x,\mu\nu} d^j_{x,\mu\nu}}_{x,\nu}, \quad (26)$$

with

$$\sum_{\{a,b,c,d\}} = \prod_{x,\mu<\nu} \prod_{j=1}^{p_{x,\mu\nu}} \sum_{a^j_{x,\mu\nu}} \sum_{b^j_{x,\mu\nu}} \sum_{c^j_{x,\mu\nu}} \sum_{d^j_{x,\mu\nu}}, \quad (27)$$

and introducing an index j to account for the exponentiation.

Approaches for simulation
b) Index parametrisation

Using this new index *j* and using a clever representation of the matrix elements we carry out the dualisation procedure. Analysing the terms in the exponent, we can identify a certain combination of them as conserved color fluxes. We then rewrite the whole partition sum in terms of these color flux variables.

Although this approach seemed very interesting and promising, we decided to abandon it. The reason for this was, that this approach did not solve the problem comprehensively. For a simulation a more conventional way was more appealing.

Approaches for simulation
c) Truncation of the dual theory

The idea for this approach is to truncate the dualised theory such that it matches the terms expected for a *Qubit regularisation* [2].

For this we introduce new dual variables:

$$V^{abcd}_{x,\mu\nu} \equiv p^{abcd}_{x,\mu\nu} - p^{\bar{a}\bar{b}\bar{c}\bar{d}}_{x,\mu\nu} \tag{28}$$

$$D^{abcd}_{x,\mu\nu} \equiv p^{abcd}_{x,\mu\nu} \cdot p^{\bar{a}\bar{b}\bar{c}\bar{d}}_{x,\mu\nu}, \tag{29}$$

where the letters with a bar denote the anti-color index.

We restrict these new variables to the following values:

$$V^{abcd}_{x,\mu\nu} \in \{0, \pm 1\} \tag{30}$$

$$D^{abcd}_{x,\mu\nu} \in \{0, 1\}\ , \tag{31}$$

thus truncating the theory. I.e., the dual variables $p^{abcd}_{x,\mu\nu}$ can only take the values 0 or 1.

Approaches for simulation
c) Truncation of the dual theory

The variables of the previous slide can be related to the ACC as follows

$p^{abcd}_{x,\mu\nu}$ / $p^{\bar{a}\bar{b}\bar{c}\bar{d}}_{x,\mu\nu}$	0	1
0	$V^{abcd}_{x,\mu\nu} = 0$ $D^{abcd}_{x,\mu\nu} = 0$	$V^{abcd}_{x,\mu\nu} = +1$ $D^{abcd}_{x,\mu\nu} = 0$
1	$V^{abcd}_{x,\mu\nu} = -1$ $D^{abcd}_{x,\mu\nu} = 0$	$V^{abcd}_{x,\mu\nu} = 0$ $D^{abcd}_{x,\mu\nu} = 1$

Approaches for simulation
c) Truncation of the dual theory

With the introduction of a *plaquette state*: $|D^{abcd}_{x,\mu\nu}, V^{abcd}_{x,\mu\nu}\rangle$ we are able to complete the analogy to the qubit representation as we obtain a singlet-triplet representation:

$$\begin{aligned} &\text{plaquette singlet:} \quad |1,0\rangle \\ &\text{plaquette triplet:} \quad \begin{cases} |0,+1\rangle \\ |0,0\rangle \\ |0,-1\rangle \end{cases} \end{aligned} \tag{32}$$

Furthermore this representation simplifies the weight factors of the original theory quite a lot. This truncated version of the theory is the one we used for the simulation.

Simulation and results

To check the validity of our approach we simulate the theory in the conventional representation and the truncated dual theory and compared the results. All the simulations are carried out via Monte Carlo methods, in particular the Metropolis algorithm.

First I will summarise shortly why we need Monte Carlo methods and how they work. After this I will show the structure of our program and which which updates we can perform.

Simulation and results

Vacuum expectation values for observables are obtained via:

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int D[\phi]\, e^{-S[\phi]}\, \mathcal{O}[\phi] \quad . \tag{33}$$

This integral cannot be solved exactly $\rightarrow$ use Monte Carlo to approximate it by the average of the observable evaluated on N sample field configurations $\tilde{\phi}$. These are generated randomly according to a probability distribution $P[\phi]$.

Simulation and results

Short overview of a Metropolis algorithm:

- Generate proposal field configuration $\tilde{\phi}$ near the current one
- Calculate the Metropolis ratio: $\rho = \frac{P[\tilde{\phi}]}{P[\phi]}$.
- Generate random number r, uniformly distributed in the interval [0,1].
- Accept new configuration if $r < \rho$.

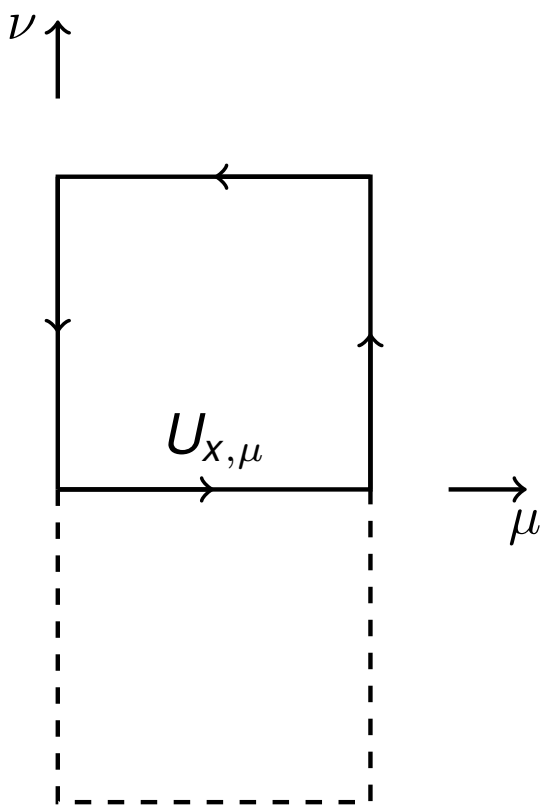
Updating the configurations using the above steps is called a *sweep*.

Simulation and results

The structure of our programs is as follows:

- Choose initial parameters, such as
  - Number of equilibrating steps
  - Number of skip steps (for autocorrelation)
  - Number of measurements
  - Values of the inverse gauge coupling for the simulation
- Update the configurations
  - Update the single links in the conventional representation
  - Update the configurations via sheet, double and cube updates
- Compute the observable
- Perform error analysis

Simulations and results
Graphical illustration of the possible updates.



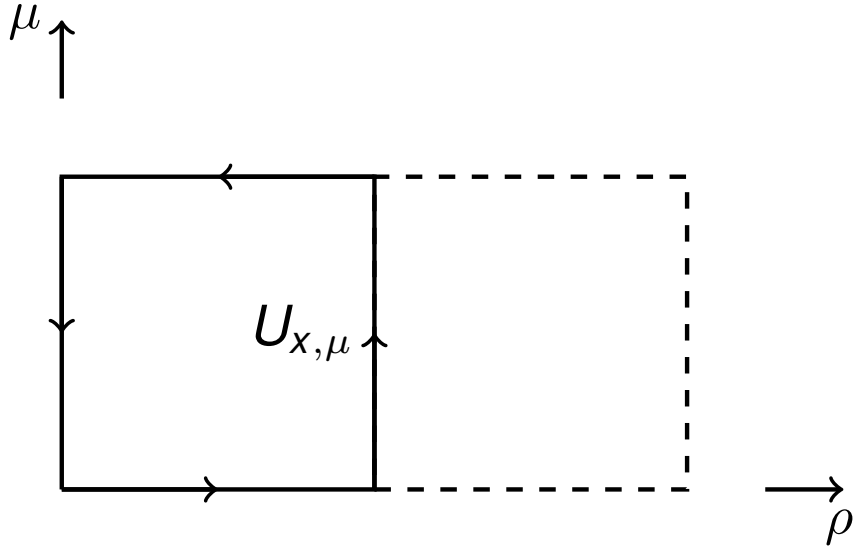


Figure: Illustration of the different cases of two neighbouring plaquettes which contain the gauge link $U_{x,\mu}$ we want to update.

Simulations and results
Graphical illustration of the possible updates.

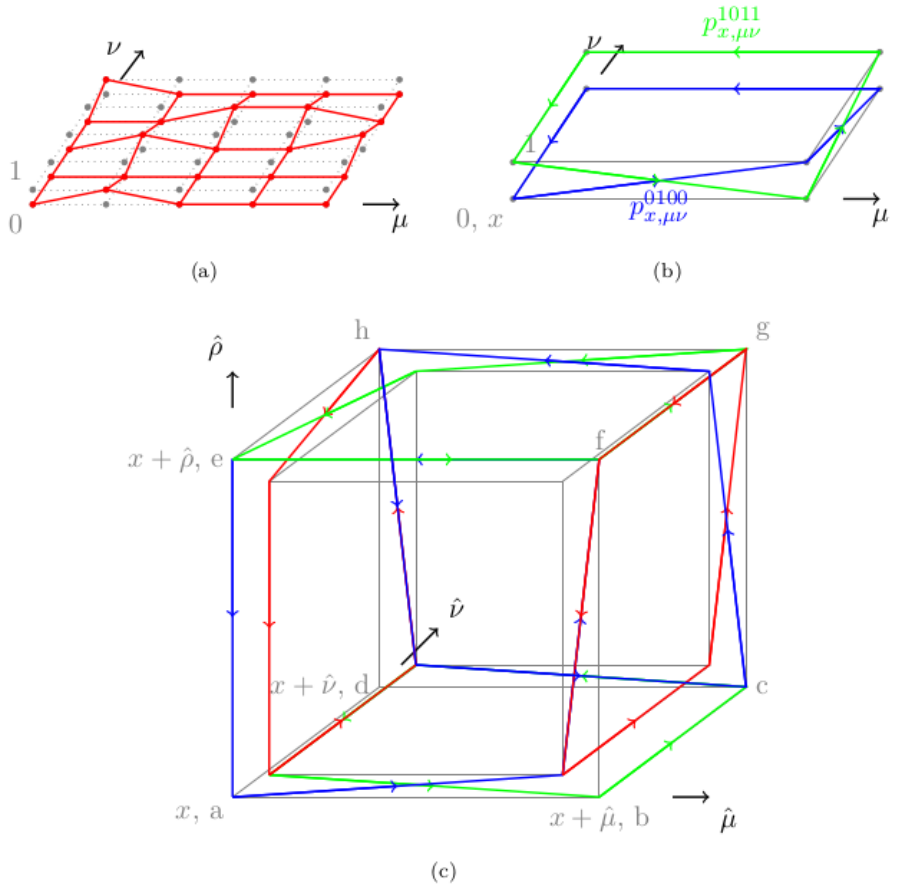


Figure: Graphical illustration of the update schemes for the dual simulation.

Simulation and results

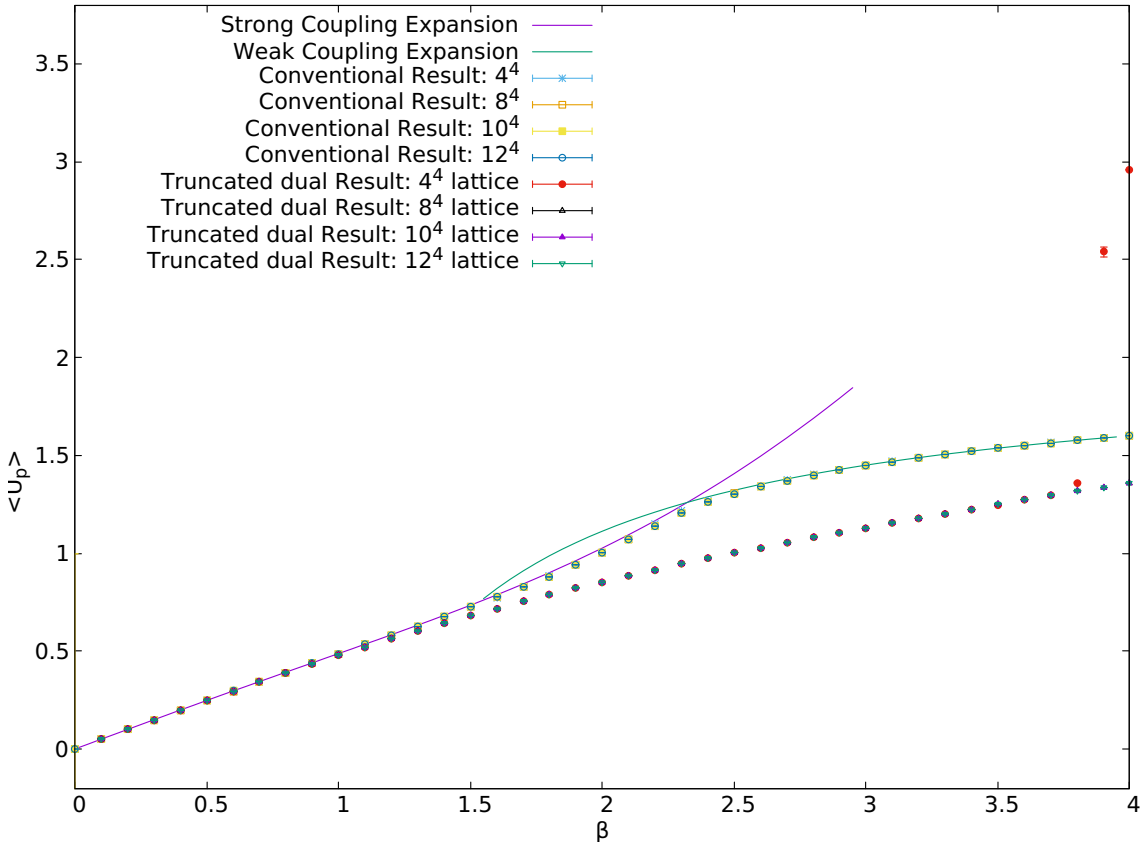


Figure: Comparison of the plaquette expectation value for the conventional and the dual truncated simulation.

Simulation and results

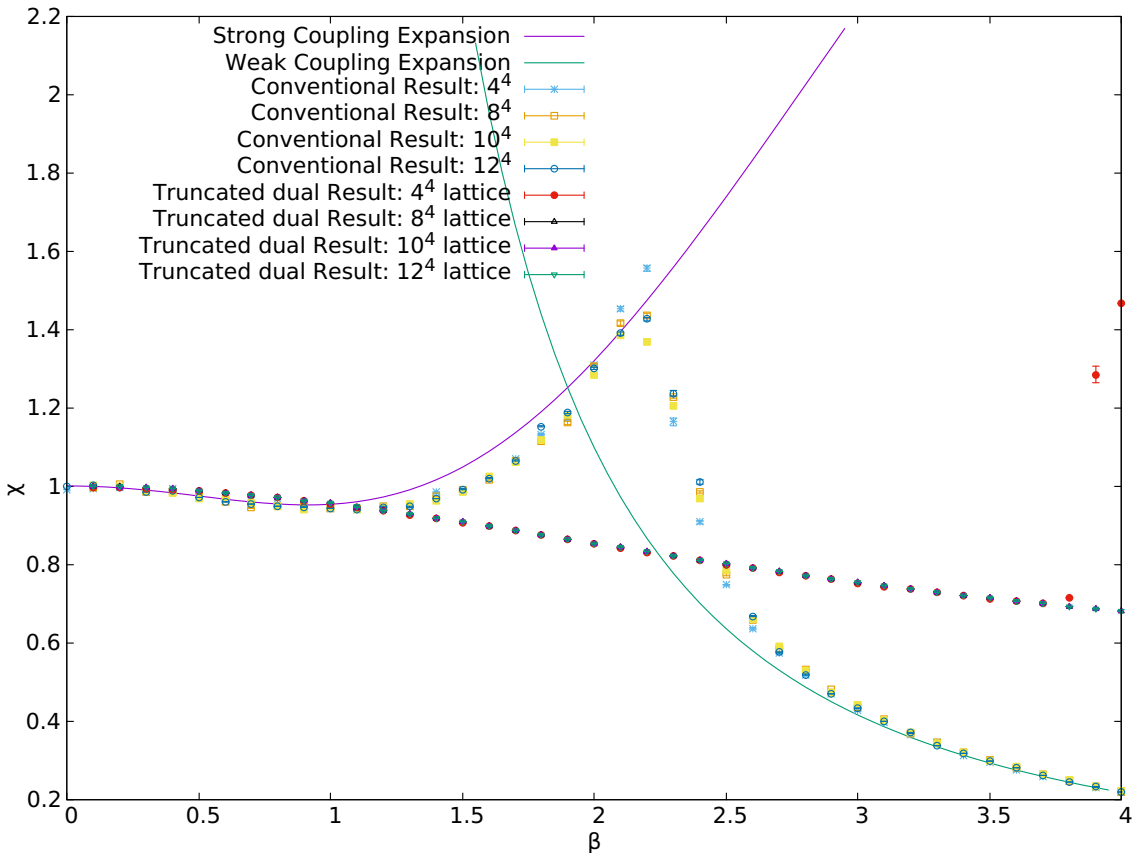


Figure: Comparison of the susceptibility for conventional and dual truncated simulation.

Summary

Summary:

- In our work we tried to overcome a sign problem introduced by the dualisation of $SU(2)$ lattice gauge theory.
- We explored three simulation approaches, of which the truncation approach was the most promising.
- We simulated the theory to check the validity of the approach.
- Using the Dual truncated approach:
  - Reproduced the results in the strong coupling region
  - No relative signs appear in the simulation. Thus the sign problem has been overcome in the truncation approach.

Outlook

Outlook:

- Include higher orders in the truncation. This should be possible since negative terms appear only at $\mathcal{O}(\beta^4)$.
- It would be very interesting to see, whether we can apply this truncation approach to $SU(3)$, as this is the next higher Lie group and the gauge group of QCD.
- Find new resummation techniques.

[1] C. Gattringer, C. Marchis. *Abelian color cycles: a new approach to strong coupling expansion and dual representations for non-abelian lattice gauge theory.* Nuclear Physics B, 916:627–646, March 2016.

[2] H. Singh, S. Chandrasekharan. *Qubit regularization of the O(3) sigma model.* Phys. Rev. D, 100:054505, Sep 2019. doi: 10.1103/PhysRevD.100.054505.

[3] C. Gattringer, C.B. Lang. *Quantum Chromodynamics on the Lattice: An Introduc- tory Presentation*. Lecture Notes in Physics ISSN 0075-8450. Springer Heidelberg Dordrecht London New York, 2009. ISBN 978-3-642-01849-7.