Inhomogeneous phases in NJL-type models: An echo from $1+1$ dimensions

Laurin Pannullo, Goethe University Frankfurt

Lunch Club Seminar, JLU, 12.07.2023

in collaboration with Marc Winstel and Marc Wagner.

Based on
[L. Pannullo, (2023), arXiv: 2306.16290],
[L. Pannullo, M. Winstel, (2023), arXiv: 2305.09444],
[L. Pannullo, M. Wagner, M. Winstel, *PoS.* **LATTICE2022** (2023)].

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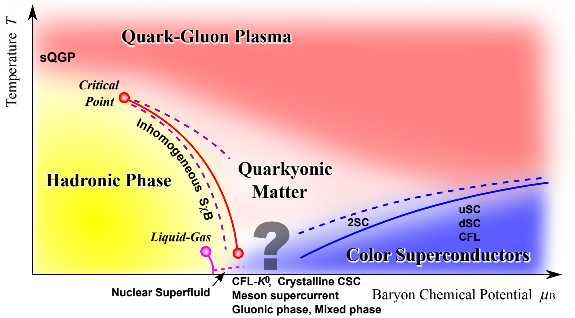
GOETHE UNIVERSITÄT FRANKFURT AM MAIN

CRC–TR 211
Strong-interaction matter under extreme conditions

Outline

- Motivation - QCD, inhomogeneous phases and the Gross-Neveu model
- The dimensionality puzzle of inhomogeneous phases in the Gross-Neveu model
- Part I: General Lorentz-(pseudo)scalar four-fermi model in $2+1$-dimensions
- Part II: The Gross-Neveu model in non-integer dimensions

Motivation (I) – QCD phase diagram



[K. Fukushima, T. Hatsuda, *Reports on Prog. Phys.* **74** (2011)]

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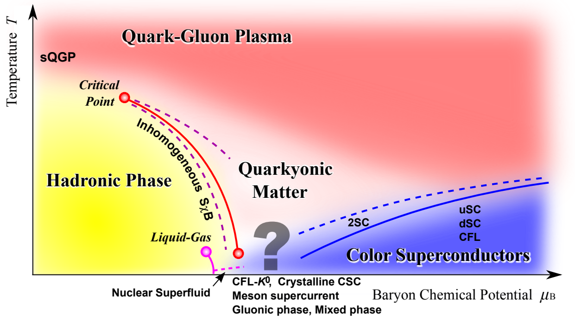

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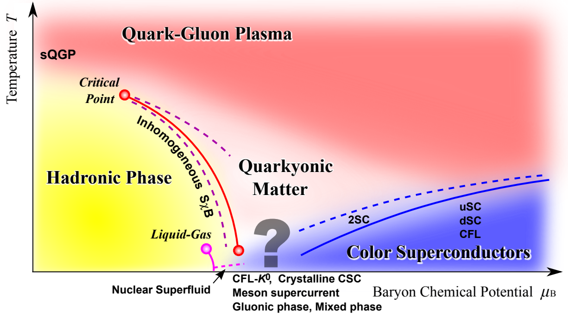

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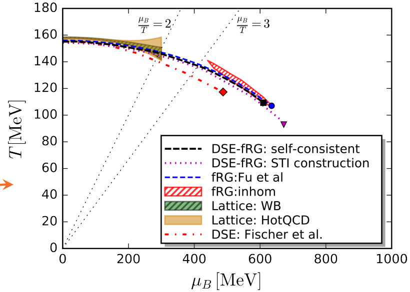

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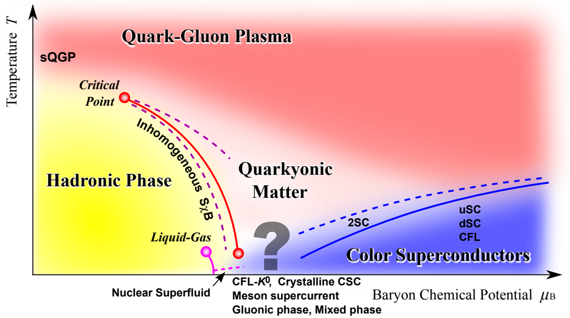

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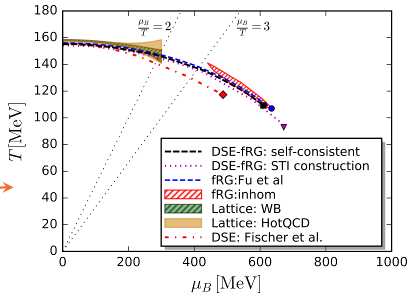

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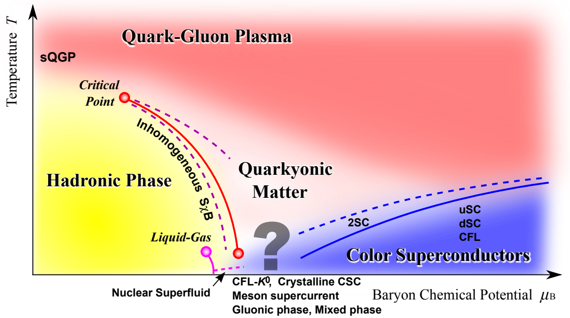

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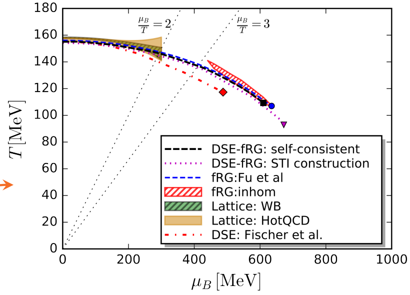

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  - Maybe chiral inhomogeneous phases?

Motivation (II) – What is a chiral inhomogeneous phase?

- Possible chiral phases
  - $\langle\bar{\psi}\psi\rangle(x) = \text{const.} = 0:$ Symmetric phase (SP)
  - $\langle\bar{\psi}\psi\rangle(x) = \text{const.} \neq 0:$ Homogeneously broken phase (HBP)

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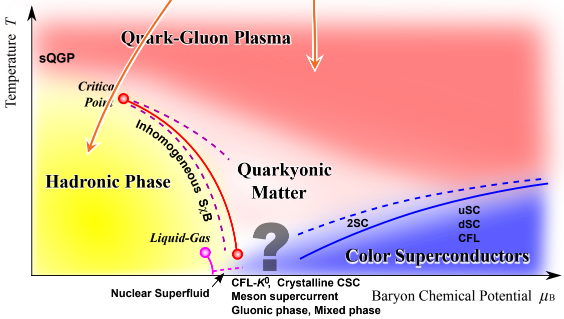

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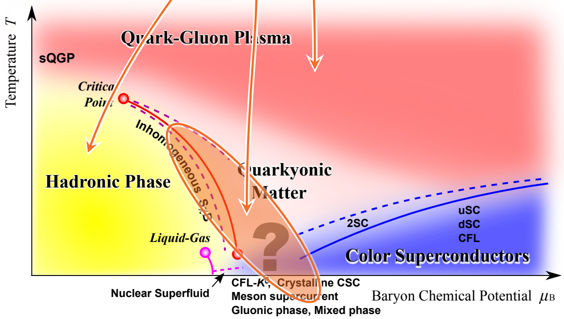

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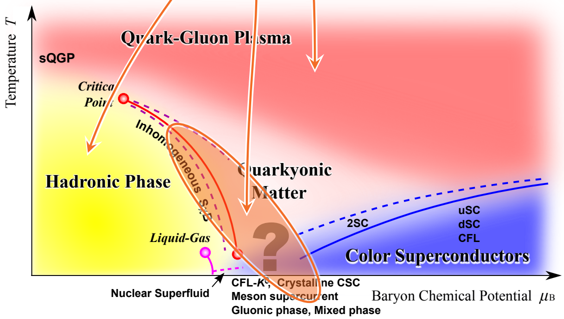
- IP breaks chiral symmetry and translational invariance **(!)**
- Well known in condensed matter, exotic in high energy physics



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Motivation (III) – Where to find a chiral inhomogeneous phase?

Gross-Neveu (GN) model in $1+1$ dimensions

$$S[\bar{\psi}, \psi] = \int \mathrm{d}^2 x \left[\bar{\psi}(\not{\partial} + \gamma_0 \mu)\psi - \frac{\lambda}{2N} \left(\bar{\psi}\psi\right)^2\right]$$

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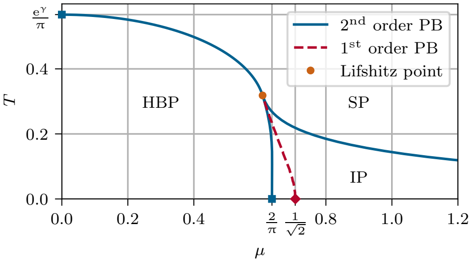

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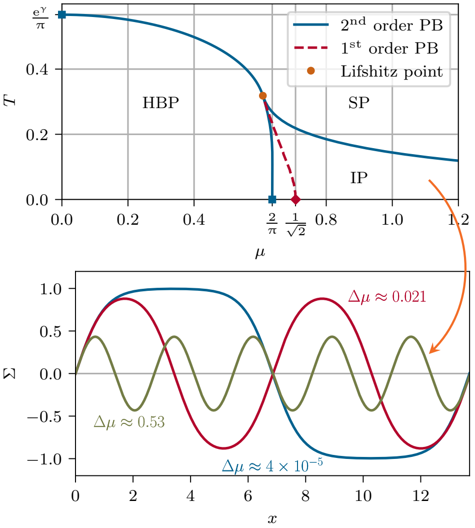
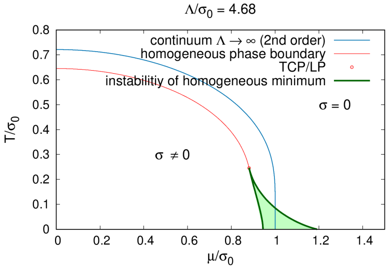


[M. Thies, K. Urlichs, *Phys. Rev. D.* **67** (2003)]

Motivation (IV) – Inhomogeneous phases in higher dimensions

GN model in $2+1$ dimensions



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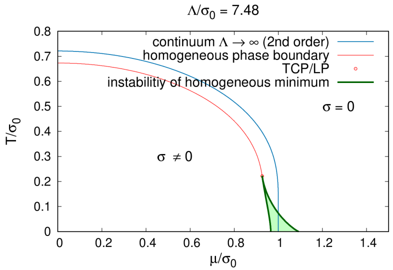

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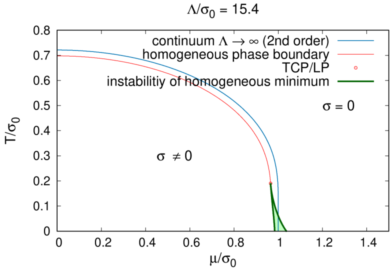

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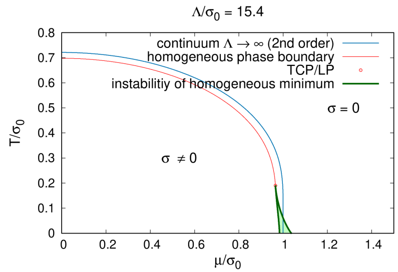

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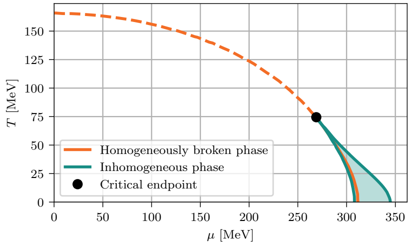

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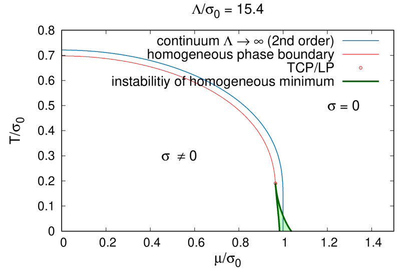

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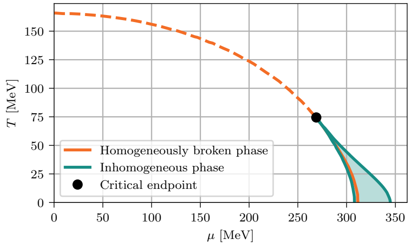

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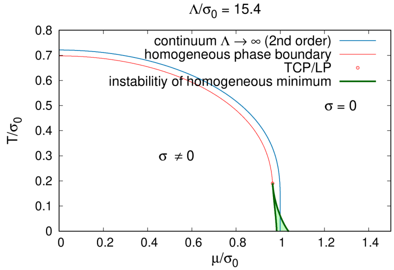

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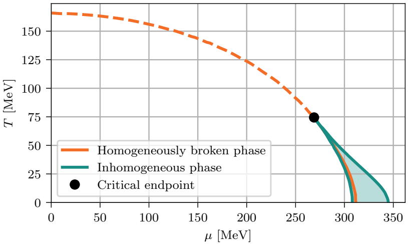

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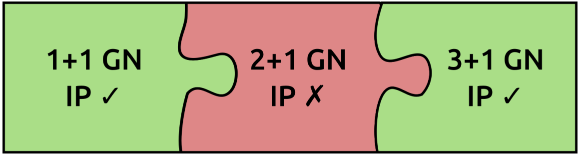

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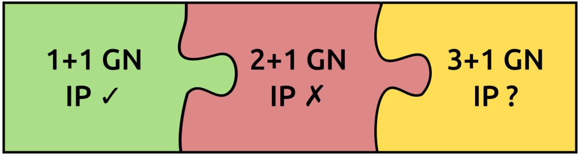

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  1. the GN model is too simple for IPs in $2+1$-dimensions
     $\Rightarrow$ consider a more general model in $2+1$-dimensions (Part I)
  2. $3+1$-dimensional results are "regulator artifacts" and the $3+1$ puzzle piece is also "red"
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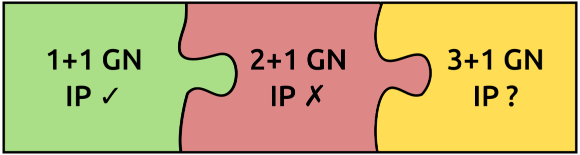

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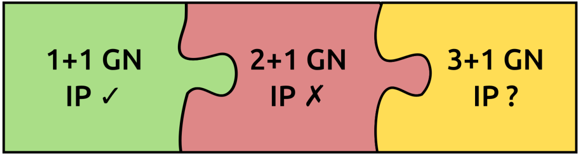


Part I: General model in $2+1$-dimensions

[L. Pannullo, M. Winstel, (2023), arXiv: `2305.09444`]

General model in $2 + 1$-dimensions

- Consider model with 16 Lorentz-(pseudo)scalar interaction channels

$$\mathcal{L} = \bar{\psi}\left(\not{\partial} + \gamma_0 \mu\right)\psi - \sum_{j=1}^{16} \frac{\lambda_j}{2N}\left(\bar{\psi}\, c_j\, \psi\right)^2, \quad c_j = \left(1, \mathrm{i}\gamma_4, \mathrm{i}\gamma_5, \gamma_{45}, \vec{\tau}, \mathrm{i}\vec{\tau}\gamma_4, \mathrm{i}\vec{\tau}\gamma_5, \vec{\tau}\gamma_{45}\right)_j$$

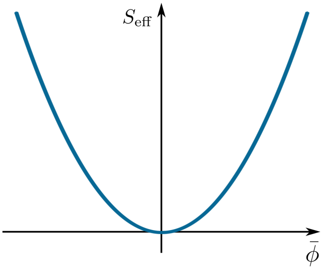
- $U(4N)$ symmetry for $\lambda_j = \lambda \Rightarrow$ homogeneous phase diagram identical to GN model
- Hubbard-Stratonovich, large-N limit similar to GN model
- search for IPs with stability analysis

Stability analysis (I) – General idea

- Homogeneous fields

$$\phi(x) = \bar{\phi}$$

- Minimum of S_{eff} is easy to obtain.



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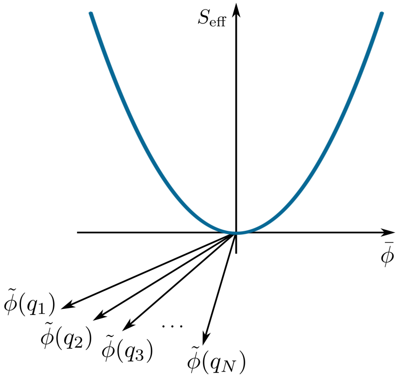
- In general fields have full space dependence

$$\phi(x) = \bar{\phi} + \phi_s(x) = \bar{\phi} + \sum\!\!\!\!\!\!\!\int_j \tilde{\phi}_s(q_j)\, \mathrm{e}^{\mathrm{i}xq_j}$$



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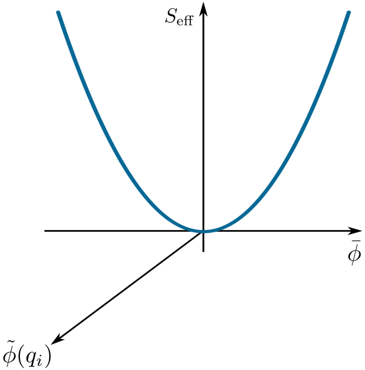

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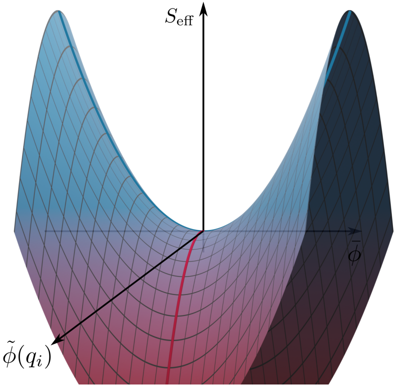
$$\phi(x) = \bar{\phi} + \phi_s(x)$$
$$= \bar{\phi} + \sum\!\!\!\!\!\!\int_j \tilde{\phi}_s(q_j)\, \mathrm{e}^{\mathrm{i}xq_j}$$

- Former homogeneous minimum might only be saddle point
- Full dependence of S_{eff} on $\phi(x)$ extremely difficult or impossible

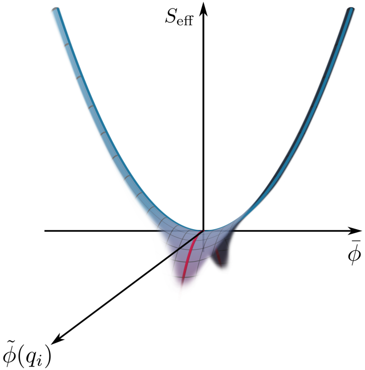


Stability analysis (I) – General idea

- Consider only inhomogeneous perturbations

$$\phi(x) = \bar{\phi} + \delta\phi_s(x)$$
$$= \bar{\phi} + \sum\!\!\!\!\!\!\int_j \delta\tilde{\phi}_s(q_j)\, \mathrm{e}^{\mathrm{i}xq_j}$$

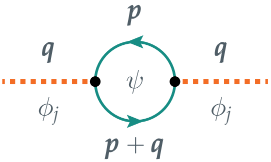
- Investigate curvature at homogeneous minimum



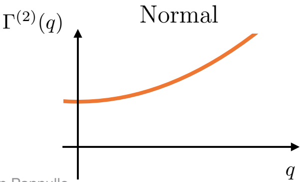
Stability analysis (II) – Two-point function

- Curvature of the action in direction $\delta\tilde{\phi}_j(q)$ is given by the bosonic two-point function $\Gamma_j^{(2)}(\boldsymbol{q})$
- Simple quantity in the mean-field approximation

$$\Gamma_j^{(2)}(\boldsymbol{q}) = \frac{1}{\lambda} +$$



$$= \frac{1}{\lambda} + \int \frac{\mathrm{d}^D p}{(2\pi)^D}\, \mathrm{Tr}\left[S(p)\, c_j S(p+\boldsymbol{q}) c_j\right]$$



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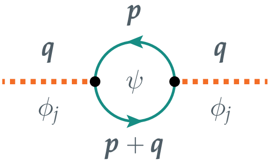

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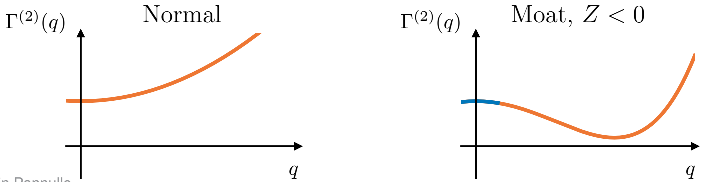
- negative Z indicates moat regime



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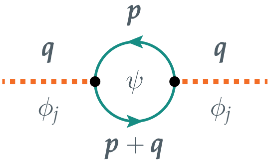

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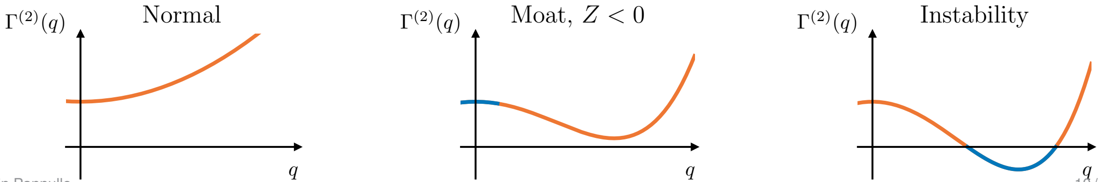
- negative values indicate instability for mode q



Results of the stability analysis (I)

- One finds for all channels two-point functions of the form

$$\Gamma_j^{(2)}(\boldsymbol{q}) = \underbrace{\frac{1}{\lambda} - l_1(M^2)}_{\text{independent of } \boldsymbol{q}} + L_{2,j}(q^2, M^2),$$

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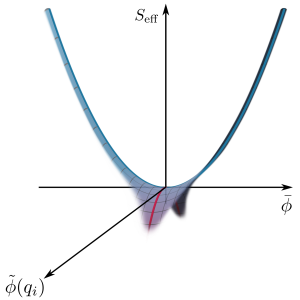

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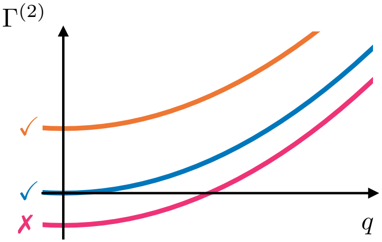

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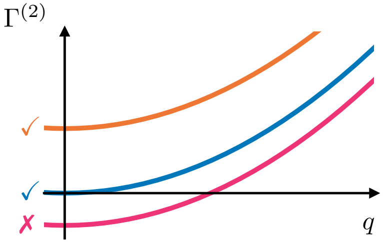

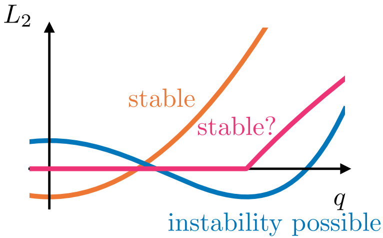

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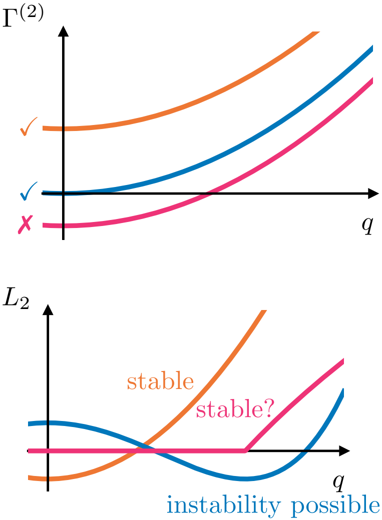
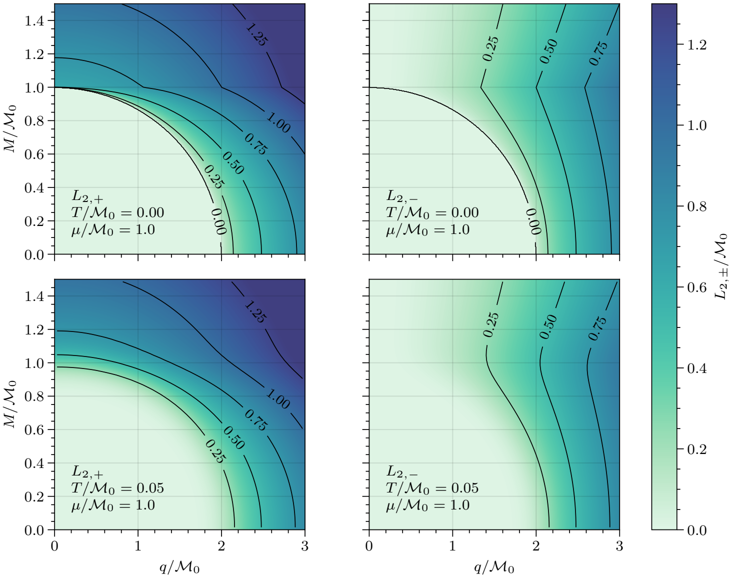
- offset depends on details of model
- if q-dependent part $L_{2,j}$ is monotonically rising function, no instability possible and no moat regime
- For all j either $L_{2,j} = L_{2,+}$ or $L_{2,j} = L_{2,-}$



Results of the stability analysis (II)

- consider all possible expansion points M
- $L_{2,\pm}$ monotonically rising function in q for all M, μ, T
- no instability towards an IP for any combination of interaction channels, e.g., GN, NJL etc.



Extensions

Yukawa-type extensions

- Yukawa-type extensions of four-fermi models

$$S_Y[\vec{\phi}] = \underbrace{\frac{S_{\text{eff}}[\vec{\phi}]}{N}}_{\text{four-fermi part}} + \int \mathrm{d}^3x \left[\frac{1}{2h^2} \left(\partial_\nu \vec{\phi}(x) \right)^2 + \sum_{n>1} \kappa_n \Big(\sum_j \phi_j^2(x) \Big)^n \right], \tag{1}$$

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Other chemical potentials:

- we also considered some other chemical potentials such as chiral and isospin
- in this general framework the two-point function cannot be diagonalized for arbitrary set of interaction channels
  $\Rightarrow$ have to neglect some interaction channels, which are physically relevant, e.g., charged pions
- for remaining subsetsno instability there

Conclusion and outlook of Part I

Conclusion:

- Absence of inhomogeneous phase in $2+1$ dimensions not restricted to GN model
- Lorentz-(pseudo)scalar interaction channels show no instability when subjected to quark chemical potential
- also no evidence of a moat regime

Outlook:

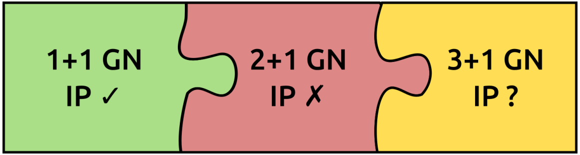
- Other interaction channels: Vector interactions promising – investigated by **Marc Winstel**

Part II: Gross-Neveu model in non-integer spatial dimensions $1 \leqslant d < 3$

[L. Pannullo, (2023), arXiv: `2306.16290`]

The dimensionality puzzle of IPs in the GN model

- result in $2+1$ dimensions not due to limited symmetry group



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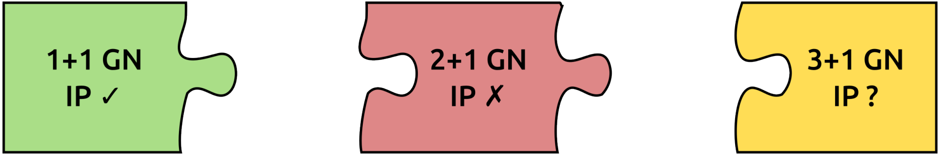

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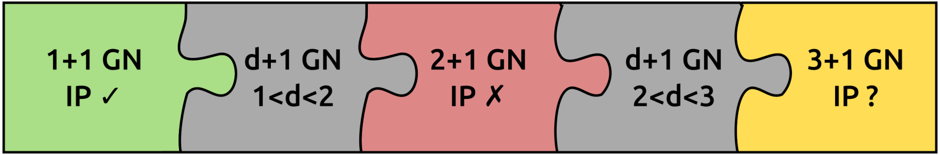
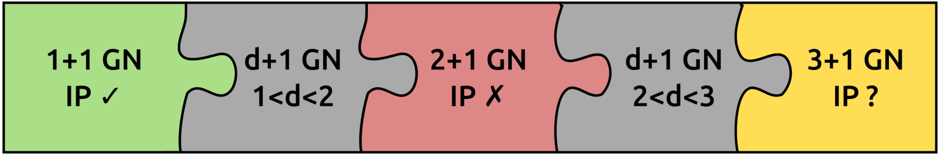


The dimensionality puzzle of IPs in the GN model

- result in $2+1$ dimensions not due to limited symmetry group
- maybe pieces are missing?
- consider renormalized GN model in non-integer spatial dimensions $1 \leqslant d < 3$
- perform stability analysis to understand how the IP vanishes between $d=1$ and $d=2$
- approach $d=3$ from below to see if an instability begins to develop again



Do's and Dont's in non-integer dimensions

First, let's think about what we are allowed to do in non-integer dimensions:

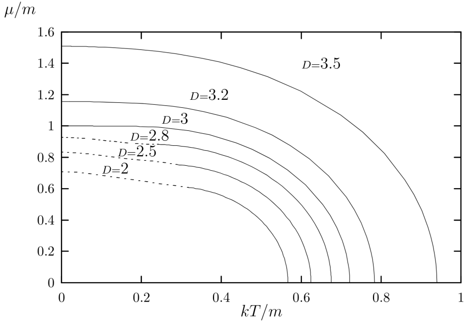
[G. 't Hooft, M. Veltman, *Nucl. Phys. B.* **44** (1972)]

- $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu,\nu}$: ✓
- $\{\gamma_\mu, \gamma_5\} = 0$: ✗
  $\Rightarrow$ avoid everything that depends on γ_5 (such as the NJL)
- Chiral symmetry : ✗
- Chirality : ?
- Angles between vectors: ?
- interpret a lot into results at a specific non-integer d: ✗
  $\Rightarrow$ it is more about trends between integer dimensions

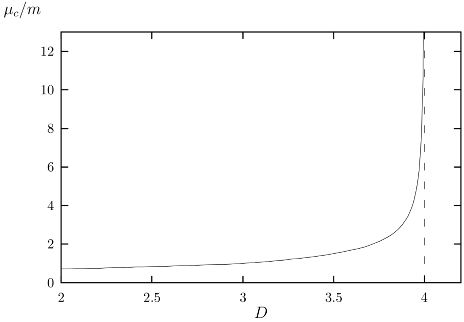
Homogeneous phase diagram of the GN model in $1 \leqslant d < 3$

- Model renormalizable in $1 \leqslant d < 3$ just as in integer dimensions (via Gap equation)
- First: Restrict to homogeneous σ, investigate dependence of phase diagram on spacetime dimensions $D = d + 1$ [T. Inagaki *et al.*, *Int. J. Mod. Phys. A.* **10** (1995)]

Phase boundaries (dashed: 1st order line)



Critical μ at $T = 0$



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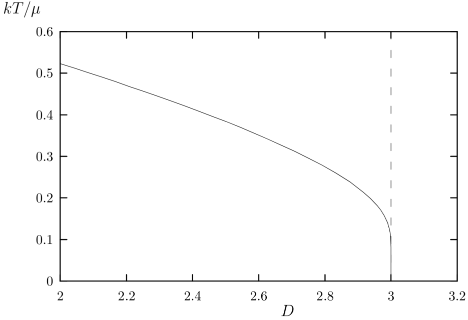
Position of critical point



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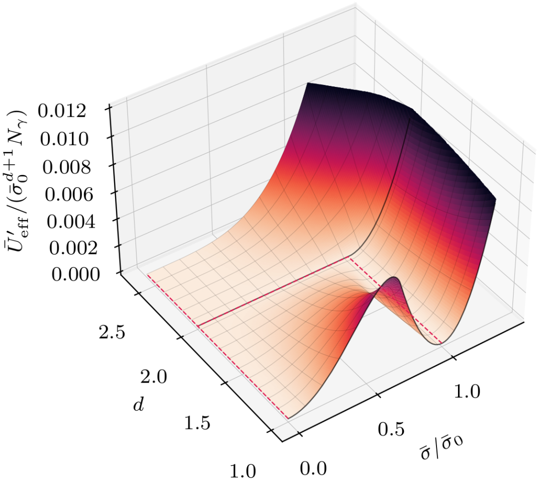
Homogeneous effective potential at $T = 0$

- Homogeneous effective potential at $T = 0$ and at $\mu = \mu_c$
- Critical point meets $T = 0$ at $d = 2$
  $\Rightarrow \bar{U}_{\text{eff}}$ is completely flat



Stability analysis – non-integer d calculation

- Two-point function can again be decomposed into

$$\Gamma^{(2)}(\bar{\sigma}^2, \mu, q^2, d) = \frac{1}{\lambda} - N_\gamma l_1\left(\bar{\sigma}^2, \mu, d\right) + L_2(\bar{\sigma}^2, \mu, \boldsymbol{q}, d)$$

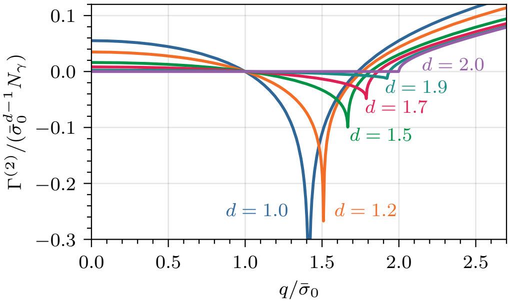
- Momentum dependent part at $T = 0$

$$L_2(\bar{\sigma}^2, \mu, \boldsymbol{q}, d) = \tfrac{1}{2}\left(q^2 + 4\bar{\sigma}^2\right) N_\gamma \int_{-\infty}^{\infty} \tfrac{\mathrm{d}p_0}{(2\pi)} \int \tfrac{\mathrm{d}^d p}{(2\pi)^d} \tfrac{1}{((p_0 - \mathrm{i}\mu)^2 + \bar{\sigma}^2 + (\boldsymbol{p}+\boldsymbol{q})^2)((p_0 - \mathrm{i}\mu)^2 + \bar{\sigma}^2 + \boldsymbol{p}^2)}$$

- Angle dependence can be removed with Feynman trick
  - resulting integral cannot be solved in closed form for arbitrary d
  - but can easily be integrated numerically
- Bosonic wave-function renormalization can be given in a completely closed form

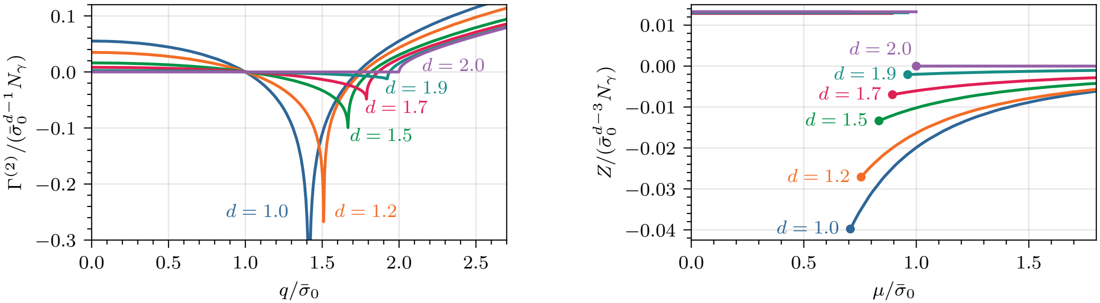
Results – $1 \leqslant d \leqslant 2$ (I)

- $\Gamma^{(2)}$ at $T = 0$ and at homogeneous critical chemical potential μ_c shows instability for $d < 2$
- Diverges only for $d = 1$, but has cusp for $d > 1$
- Instability gradually weakens until it vanishes together with critical point at $d = 2$
- At $d = 2$
  - Two-point function exhibits flat region similar to $\bar{U}_{\text{eff}}$
  - Urlichs found also a degeneracy of homogeneous configurations and inhomogeneous configurations at $T = 0$ and $\mu = \mu_c$ [K. Urlichs, (2007)]
  - Maybe not only the homogeneous effective potential is flat, but also the inhomogeneous potential



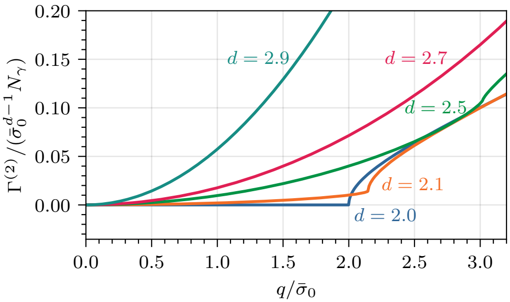
Results – $1 \leqslant d \leqslant 2$ (II)

- Wave-function renormalization Z indicates moat regime as expected.



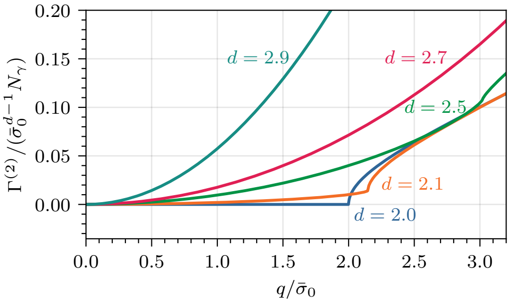
Results – $2 \leqslant d < 3$ (I)

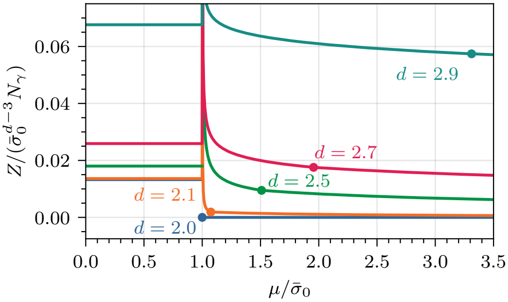
- $\Gamma^{(2)}$ at homogeneous critical chemical potential μ_c approaches parabolic shape for increasing d
- In conflict with $3+1$ results; difference is that the present results are renormalized
  $\Rightarrow$ IPs in $3+1$-dimensional GN and NJL are solely generated by the necessary regulator
- same effect observed in renormalizable $2+1$ GN [M. Buballa *et al.*, *Phys. Rev. D.* **103** (2021)]



Results – $2 \leqslant d < 3$ (II)

- Wave-function renormalization Z shows no moat regime
- Z diverges at $\mu/\bar{\sigma}_0 = 1$
  - kink in $\Gamma^{(2)}$ is located at $q^2/4 = \mu^2 - \bar{\sigma}^2$
  - for $d > 2$, one can find $\mu^2 = \bar{\sigma}^2 \Rightarrow Z$ diverges within broken phase



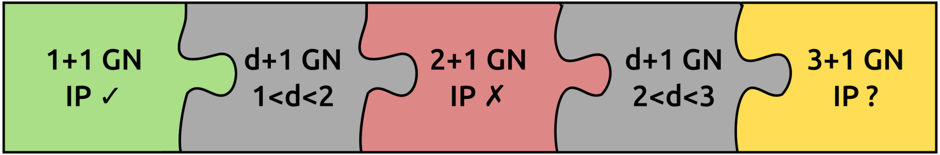


Alternate interpretation

- in $3+1$-dimensional GN/NJL model one has to keep the regulator at finite value
- Some use dimensional regularization
  - There one introduces a mass scale M_0 since the regulator d is dimensionless
  - One then tunes λ and $d(< 3)$ via IR observables (e.g. $\langle\bar{\psi}\psi\rangle_{\mathrm{vac.}}$ and f_π)
- $\Rightarrow$ we really just calculated the regulator dependence of the $3+1$-dimensional Gross-Neveu model for dim. reg.
- strong regularization then just procures low dimensional phase diagram

Conclusion of Part II

- Solely increasing the number of dimensions causes the IP to vanish at $d = 2$
  - here the critical point meets $T = 0$
  - Maybe flat (inhomogeneous) effective action at $d = 2$ at $T = 0$ and $\mu = \mu_c$?
- No instability for $2 < d < 3 \Rightarrow$ IP in $d = 3$ generated by regulator
- alternate view of d-dimensional study: study of regulator dependence of dim. reg.



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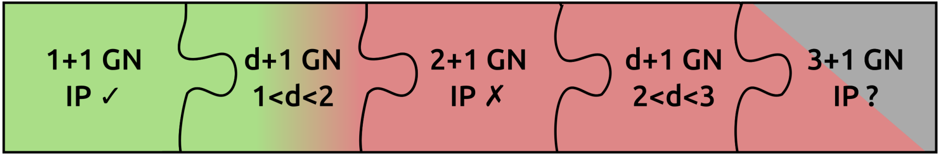

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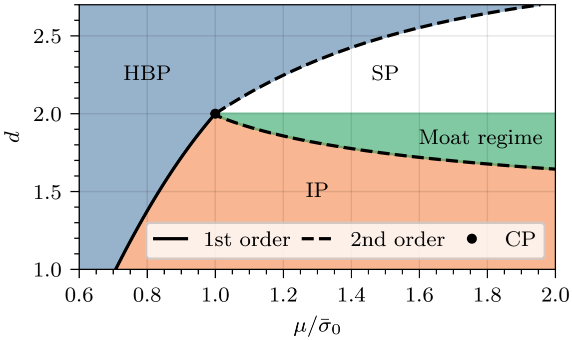

Outlook of Part II

- calculations for finite T
- calculation for finite regulator
  $\Rightarrow$ connect to $3+1$-dimensional results
- consider 1-dimensional ansatz functions embedded in d-dimensional space