The Anderson Transition in Lattice QCD

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Based on:
R. Kehr, D. Smith, L. von Smekal, arXiv:2304.13617

Outline

1. Motivation
2. QCD on the lattice
3. Numerical methods and lattice setup
4. Results and analysis
5. Conclusion and outlook

Outline

Motivation

- Principal transitions within the QCD phase diagram:
  Chirally broken → Chirally restored & Confined → Deconfined phase

Open question

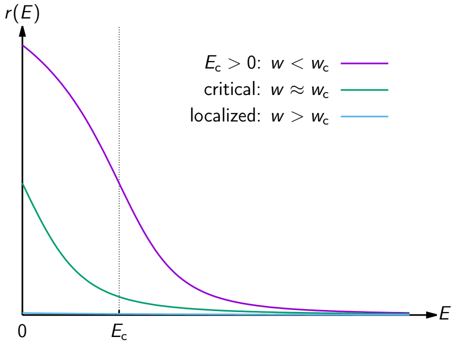
Is chiral symmetry breaking related to confinement and if so, how?

- QCD Anderson transition seems to be related to both phenomena
  ⇒ Might provide the answer to this question
- In this work: Focus on relation to chiral symmetry breaking

- Term *Anderson transition* originates from condensed matter physics
  [P. W. Anderson, 1958] [F. Evers, A. D. Mirlin, arXiv:0707.4378]
  - Describes metal-insulator transition in disordered solids
  - In metal phase low-lying eigenmodes of Hamiltonian are delocalized
    ⇒ Conductivity
  - Above critical disorder all eigenmodes localized ⇒ No conductivity

Anderson transition

- Delocalized modes separated from localized modes by energy threshold E_c (*mobility edge*)
- Above critical disorder strength w_c all modes are localized



- Similar transition in QCD [M. Giordano, T. G. Kovács, arXiv:2104.14388]
  - Hamilton operator $\rightarrow$ Dirac operator
  - Disorder strength $\rightarrow$ Temperature
  - Low-lying modes are localized
  - Higher ones delocalized
  - Below T_0 all modes are delocalized (no mobility edge)

Relation to . . .

. . . (de)confinement

- Eigenmodes tend to localize in the sinks of Polyakov loop [L. Holicki, E.-M. Ilgenfritz, L. von Smekal, arXiv:1810.01130]
- In quenched QCD the vanishing of mobility edge coincides with the deconfining phase transition [T. G. Kovács, R. Á. Vig, arXiv:1706.03562]

. . . chiral symmetry restoration/breaking

- Previous work suggests $T_0 = T_{\mathrm{pc}}$, where T_{pc} is the pseudocritical temperature of the chiral crossover (nonvanishing quark mass)
- No Goldstone bosons in the chiral limit, if near-zero modes are localized [M. Giordano, arXiv:2206.11109]
  $\Rightarrow T_0 \geq T_{\mathrm{c}}$ (T_{c}: temperature of the chiral phase transition)
- Near-zero modes produce chiral condensate (Banks-Casher relation) [T. Banks, A. Casher, 1980]

Outline

QCD in the continuum

- QCD expectation value:

$$\langle O \rangle = \frac{1}{\mathcal{Z}} \int_{\mathcal{C}} \mathcal{D}[\psi, \overline{\psi}, A] \, O[\psi, \overline{\psi}, A] \, \mathrm{e}^{-S[\psi, \overline{\psi}, A]}$$

- Partition function:

$$\mathcal{Z} = \int_{\mathcal{C}} \mathcal{D}[\psi, \overline{\psi}, A] \, \mathrm{e}^{-S[\psi, \overline{\psi}, A]}$$

- Action: $S\left[\psi, \overline{\psi}, A\right] = S_{\mathrm{F}}\left[\psi, \overline{\psi}, A\right] + S_{\mathrm{G}}[A]$

- Gauge action for finite temperature T (Euclidean):

$$S_{\mathrm{G}}[A] = \frac{1}{2g^2} \int_0^{\frac{1}{T}} \mathrm{d}t \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} \; \mathrm{tr}(F_{\mu\nu}(x) F_{\mu\nu}(x))$$

QCD in the continuum

- Fermion action:

$$S_{\mathrm{F}}\left[\psi, \bar{\psi}, A\right] = \int_0^{\frac{1}{T}} \mathrm{d}t \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}\; \bar{\psi}(x)\, D_{\mathrm{c}}(x)\, \psi(x)$$

Dirac operator

$$D_{\mathrm{c}}(x) = \gamma_\mu(\partial_\mu + \mathrm{i}\, A_\mu(x)) + m$$

- Anticommutes with γ_5 for $m = 0$:

$$\left\{ D_{\mathrm{c}}\big|_{m=0}\,,\; \gamma_5 \right\} = 0$$

- Massless action invariant under chiral rotations:

$$\begin{aligned} \psi &\mapsto \exp\left(\mathrm{i}\alpha\, \gamma_5\right) \psi \\ \bar{\psi} &\mapsto \bar{\psi}\, \exp\left(\mathrm{i}\alpha\, \gamma_5\right) \end{aligned}$$

Fermion action on the lattice

- Want to keep symmetries of continuum action if possible
- Discretize four-dimensional spacetime (with lattice spacing a)
- Central difference quotient for the partial derivative

- Naive discretization of the fermion action:

$$S_{\text{naive}}\left[\psi,\overline{\psi},U\right] = a^4 \sum_{n\in\Lambda} \overline{\psi}(n)\left(m\psi(n) + \gamma_\mu \frac{U_\mu(n)\psi(n+e_\mu) - U_{-\mu}(n)\psi(n-e_\mu)}{2a}\right)$$

Group valued link variables

$$U_\mu(n) = \exp(\mathrm{i}\, aA_\mu(n))$$

- Negative direction:

$$U_{-\mu}(n) := U_\mu(n-e_\mu)^\dagger$$

Doubling problem

- Read off lattice Dirac operator:

$$S_{\text{naive}}\left[\psi, \overline{\psi}, U\right] = a^4 \sum_{n,l \in \Lambda} \overline{\psi}(n)\, D_{\text{naive}}(n,l)\, \psi(l)$$

- Consider free naive Dirac operator ($U_\mu \equiv \mathbb{1}$)
- Take Fourier transform and invert

Free naive fermion propagator

$$\widetilde{D}^0_{\text{naive}}(p)^{-1}\Big|_{m=0} = \frac{-\mathrm{i}a\,\gamma_\mu \sin(p_\mu a)}{\sum_{\nu=1}^4 \sin^2(p_\nu a)} \quad \xrightarrow{a\to 0} \quad \frac{-\mathrm{i}\gamma_\mu p_\mu}{p^2}$$

- Correct continuum limit but $2^4 = 16$ poles:

$$p = a^{-1}(p_1, p_2, p_3, p_4) \quad \text{with} \quad p_\mu \in \{0, \pi\}$$

Wilson fermions

- Wilson Dirac operator (momentum space):

$$\widetilde{D}^0_{\mathrm{W}}(p) = \widetilde{D}^0_{\mathrm{naive}}(p) \underbrace{+ \frac{1}{a} \sum_{\mu=1}^{4} (1 - \cos(p_\mu a))}_{\text{Wilson term}}$$

- Removes fermion doublers
- Vanishes for physical pole $p = (0, 0, 0, 0)$

- Wilson operator in position space ($\gamma_{-\mu} := -\gamma_\mu$):

$$D_{\mathrm{W}}(n, l) = \left(m + \frac{4}{a}\right) \delta_{n,l} - \frac{1}{2a} \sum_{\mu=\pm 1}^{\pm 4} (1 - \gamma_\mu)\, U_\mu(n)\, \delta_{n+e_\mu, l}\, \sigma_\mu(n_4)$$

Chiral symmetry on the lattice

- From now m=0 $\Rightarrow$ Wilson operator still breaks chiral symmetry:

$$\{D_{\mathrm{W}}, \gamma_5\} \neq 0$$

- Nielsen-Ninomiya theorem:
  Chiral symmetric and doubler free lattice action does not exist!

- Ginsparg-Wilson equation:

$$\{D, \gamma_5\} = aD\gamma_5 D \quad \Leftrightarrow \quad D\,\gamma_5\left(1-\frac{a}{2}D\right) + \left(1-\frac{a}{2}D\right)\gamma_5\,D = 0$$

- Modify chiral rotations:

$$\begin{aligned} \psi \quad &\mapsto \quad \exp\left(\mathrm{i}\alpha\,\gamma_5\left(1-\frac{a}{2}D\right)\right)\psi \\ \bar{\psi} \quad &\mapsto \quad \bar{\psi}\,\exp\left(\mathrm{i}\alpha\left(1-\frac{a}{2}D\right)\gamma_5\right) \end{aligned}$$

Ginsparg-Wilson fermions

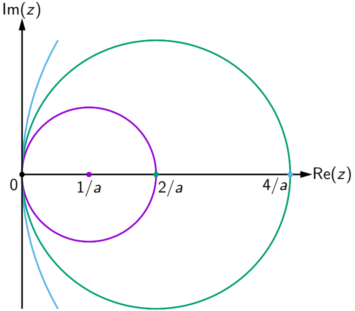
- Consider γ_5-hermitian Dirac operator:

$$\gamma_5 D \gamma_5 = D^\dagger$$

$\Rightarrow$ Eigenvalues are either real or come in complex conjugate pairs

- Ginsparg-Wilson equation:
  v_λ eigenmode with eigenvalue $\lambda \quad \Rightarrow \quad v_{\lambda^*} = \gamma_5 v_\lambda$

- γ_5-hermiticity and Ginsparg-Wilson equation:
  $\Rightarrow$ Eigenvalues lie on circle in complex plane with center at $1/a$ and radius $1/a$ (Ginsparg-Wilson circle)



Overlap fermions

Overlap operator

$$D_{\text{ov}} = \frac{1}{\tilde{a}} (1 + \operatorname{sgn} K)$$

- γ_5-hermitian kernel operator K:

$$\gamma_5 K \gamma_5 = K^\dagger \quad \Rightarrow \quad \operatorname{sgn} K = \frac{K}{\sqrt{K^\dagger K}} \quad \text{is well defined}$$

- In this work:

$$K = aD_{\text{W}} - (1 + s)$$

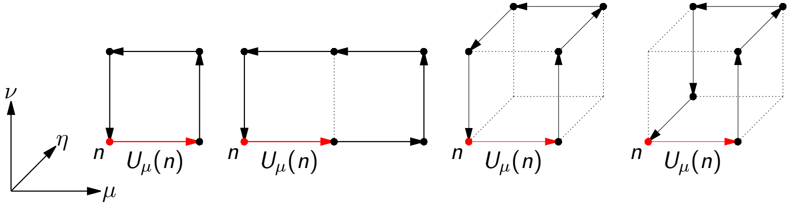
- Let $\tilde{a} = a/(1 + s)$:

$$\Rightarrow \quad D_{\text{ov}} \xrightarrow{a \to 0} D_{\text{c}}$$

- Not strictly local due to inverse square root
- Still exponential decay:

$$\|D_{\text{ov}}(n, l)\| \leq C \exp(-c \|n - l\|)$$

Lattice gauge action



- Plaquette:

$$P_{\mu\nu}(n) = U_\mu(n)\, U_\nu(n+e_\mu)\, U_\mu(n+e_\nu)^\dagger\, U_\nu(n)^\dagger$$

- Wilson gauge action:

$$S_{\mathrm{W}}[U] = \frac{2}{g^2} \sum_{n\in\Lambda} \sum_{\mu<\nu} \mathrm{Re}\,\mathrm{tr}(\mathbb{1} - P_{\mu\nu}(n))$$

Expectation value on the lattice

- Rearrange and integrate out fermions:

$$\langle O \rangle = \frac{1}{\mathcal{Z}} \int_{\mathcal{C}_\mathsf{G}} \mathcal{D}[U]\, \mathrm{e}^{-S_\mathsf{G}[U]}\, \mathcal{Z}_\mathsf{F}[U]\, \langle O \rangle_\mathsf{F}[U]$$

- Measure for gauge field:

$$\mathcal{D}[U] = \prod_{n \in \Lambda} \prod_{\mu=1}^{4} \mathrm{d}\,U_\mu(n)$$

- Fermion determinant:

$$\mathcal{Z}_\mathsf{F}[U] = -a^4 \det(D)$$

- Generate set of gauge field configurations $U_1, U_2, \ldots U_k$:

$$\mathrm{d}P[U] = \frac{1}{\mathcal{Z}} \mathcal{D}[U]\, \mathrm{e}^{-S_\mathsf{G}[U]}\, \mathcal{Z}_\mathsf{F}[U]$$

- Approximate path integral by Monte Carlo simulation:

$$\langle O \rangle \approx \frac{1}{k} \sum_{i=1}^{k} \langle O \rangle_\mathsf{F}[U_i]$$

Outline

Implementation of the overlap operator

- High-dimensional matrix:

$$d = \underbrace{4}_{\text{Dirac}} \cdot \underbrace{3}_{\text{color}} \cdot \underbrace{\prod_{\mu=1}^{4} N_\mu}_{\text{lattice sites}}$$

- $N_s = 24,\ N_t = 4$: $d \approx 660\,\mathrm{k}$ $\Rightarrow d^2 \approx 440$ billion complex entries $\gtrapprox 3.2\,\mathrm{TB}$ storage (single precision)
- $N_s = 48,\ N_t = 20$: $\gtrapprox 5\,\mathrm{PB}$
- No chance of storing the whole operator!

- Implement matrix-vector product $D_{\mathrm{ov}} v$ instead
- Rational approximation for sign function:

$$\operatorname{sgn} K = \frac{1}{\sqrt{(\gamma_5 K)^2}} \approx \alpha_0 + \sum_{j=1}^{k} \frac{\alpha_j}{(\gamma_5 K)^2 + \beta_j}$$

- Solve system of equations $\forall\, j$:

$$\left((\gamma_5 K)^2 + \beta_j\right) w_j = v$$

- Conjugate gradient method

Krylov subspace methods

- Due to high dimension entire diagonalization not feasible!
- Compute small part of spectrum using Krylov subspace methods

- p-dimensional Krylov subspace to matrix A and start vector x_1:

$$\mathcal{K}_p(A, x_1) = \text{span}\{x_1, Ax_1, A^2x_1, \dots A^{p-1}x_1\}$$

- Let $u_1, u_2, \dots u_p$ be an orthonormal basis of the Krylov subspace
- Projection onto subspace:

$$H = U^\dagger AU$$

- Compute eigenvalues θ_i and eigenvectors y_i of H
- Approximate eigenpairs of A:

$$\lambda_i \approx \theta_i \qquad \text{(Ritz values)}$$
$$v_i \approx Uy_i \qquad \text{(Ritz vectors)}$$

- Best approximations in $\mathcal{K}_p(A, x_1)$

Krylov-Schur method (SLEPc library)

Arnoldi method

- Successively extend Krylov subspace with $\omega := Au_j$
- Compute matrix elements $H_{ij} = u_i^\dagger \omega$
- Orthogonalize ω to $u_1, u_2, \dots u_j$
- Compute $H_{j+1,j} = \|\omega\|_2$ and set $u_{j+1} = \omega / \|\omega\|_2$

- H takes upper Hessenberg form of dimension p
- Arnoldi decomposition:

$$AU = UH + (Au_p)e_p^\dagger$$

- Krylov decomposition:

$$AU = UB + u_{p+1} b_{p+1}^\dagger$$

- Krylov-Schur: B has 1×1 or 2×2 blocks on the diagonal

Repeat

- Build Krylov decomposition using Arnoldi method
- Transform into Krylov-Schur decomposition
- Reorder blocks according to desired eigenvalues
- Truncate Krylov subspace

Mixed action setup

- Compute low-lying eigenmodes of overlap operator with Wilson kernel:

$$D_{\mathrm{ov}} = \frac{1+s}{a}\left(1 + \operatorname{sgn} K\right), \quad \text{where} \quad K = aD_{\mathrm{W}} - (1+s)$$

- Configurations from *twisted mass at finite temperature* collaboration [F. Burger, E.-M. Ilgenfritz, M. P. Lombardo, A. Trunin, arXiv:1805.06001]
    - Twisted mass Wilson fermions at maximal twist, Iwasaki gauge action
    - $N_{\mathrm{f}} = 2+1+1$: two degenerate light & physical strange, charm quarks
    - Pseudocritical temperature T_{pc} from disconnected chiral susceptibility
    - Lattice spacing a from nucleon mass [C. Alexandrou et al., arXiv:1406.4310]

- Mixed action setup well studied [K. Cichy et al., arXiv:1211.1605]
    - Adopting lattice spacing a and matching quark mass (respectively m_π) $\Rightarrow$ Consistent continuum limit for f_π, m_{N} and m_Δ
    - Locality: $s = 0.4$ optimal value for $N_{\mathrm{f}} = 2$ and Symanzik gauge action

Gauge configurations

Set of ensembles	N_s	N_t	T / MeV	T/T_{pc}	# conf.	$\frac{\text{modes}}{\text{conf.}}$
A370 $a = 0.0936(13)$ fm $m_\pi = 364(15)$ MeV $T_{pc} = 185(8)$ MeV	24	4	527(7)	2.85(13)	200	200
		5	422(6)	2.28(10)	200	160
		6	351(5)	1.90(9)	200	135
		7	301(4)	1.63(7)	150	115
		8	264(4)	1.42(6)	200	100
		9	234(3)	1.27(6)	200	90
		10	211(3)	1.14(5)	250	80
		11	192(3)	1.04(5)	200	75
		12	176(2)	0.95(4)	200	70
	32	13	162(2)	0.88(4)	50	150
		14	151(2)	0.81(4)	50	140
D370 $a = 0.0646(7)$ fm $m_\pi = 369(15)$ MeV $T_{pc} = 185(4)$ MeV	32	3	1018(11)	5.50(13)	120	400
		6	509(6)	2.75(7)	120	200
		14	218(2)	1.18(3)	160	85
		16	191(2)	1.03(2)	160	75
	40	18	170(2)	0.92(2)	40	150
	48	20	153(2)	0.83(2)	3	200
D210 $a = 0.0646(7)$ fm $m_\pi = 213(9)$ MeV $T_{pc} = 158(5)$ MeV	48	4	764(8)	4.83(16)	10	1000
		6	509(6)	3.22(11)	10	700
		8	382(4)	2.42(8)	10	500
		10	305(3)	1.93(6)	10	400
		12	255(3)	1.61(5)	10	350
		14	218(2)	1.38(5)	10	300
		16	191(2)	1.21(4)	10	250
		18	170(2)	1.07(4)	10	225
		20	153(2)	0.97(3)	-	-
		24	127(1)	0.81(3)	-	-

- N_s: Number of lattice sites in each space direction
- Volume $= L^3$:

$$L = aN_s$$

- N_t: Number of lattice sites in temporal direction
- Temperature:

$$T = \frac{1}{aN_t}$$

Outline

Stereographic projection

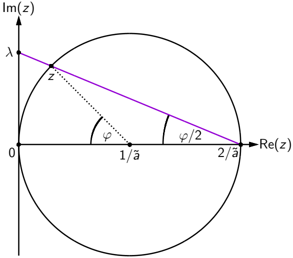
- Mimic continuum limit by projecting eigenvalues on imaginary axis
- Require $f(z) = z + \mathcal{O}(\tilde{a}z^2)$ and $f(2/\tilde{a}) = \infty$ [W. Bietenholz, K. Jansen, S. Shcheredin, arXiv:0306022]:

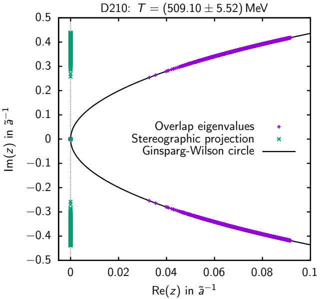
$$f(z) = \frac{z}{1 - \frac{\tilde{a}}{2}z}$$

(Möbius transformation)

- Translates to stereographic projection:

$$\lambda := f(\lambda) = \frac{\mathrm{i}\,\mathrm{Im}\,\lambda}{1 - \frac{\tilde{a}}{2}\,\mathrm{Re}\,\lambda} = \mathrm{i}\,\frac{2}{\tilde{a}}\tan\frac{\varphi}{2}$$

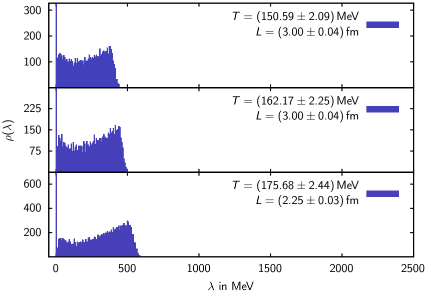




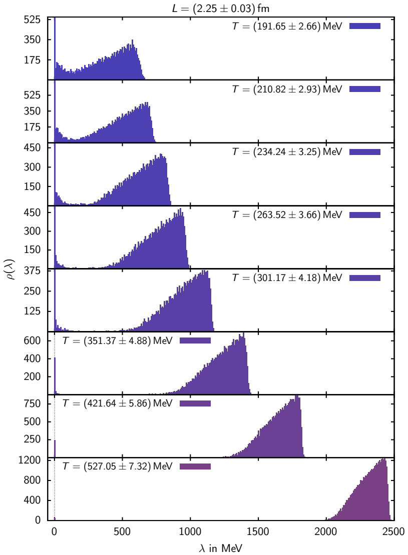
Distribution A370: $a = 0.0936(13)\,\mathrm{fm}$, $m_\pi = 364(15)\,\mathrm{MeV}$

- Banks-Casher relation:

$$\langle\overline{\psi}\psi\rangle = -\pi \lim_{\lambda\to 0}\lim_{m\to 0}\lim_{V\to\infty}\rho(\lambda)$$

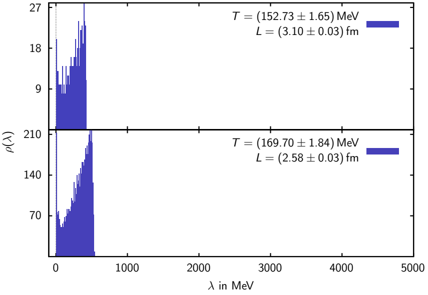


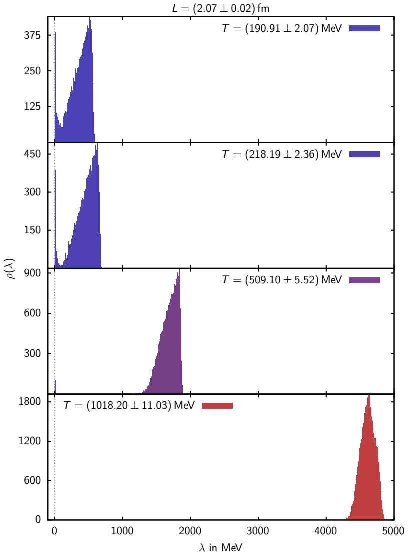
- $T_{\mathrm{pc}} = 185(8)\,\mathrm{MeV}$



Distribution D370: $a = 0.0646(7)\,\text{fm}$, $m_\pi = 369(15)\,\text{MeV}$

- Roughly same m_π, T_{pc} and volume but smaller lattice spacing a





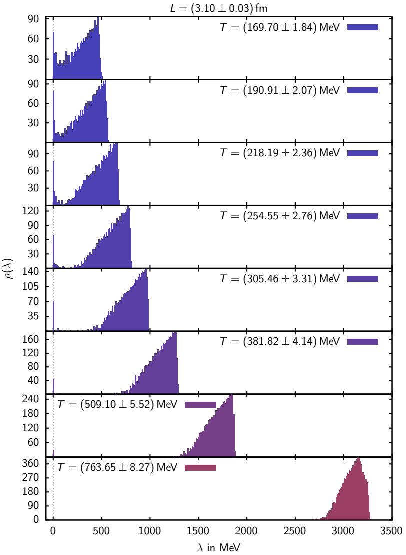
- $T_{\text{pc}} = 185(4)\,\text{MeV}$

Distribution D210: $a = 0.0646(7)\,\mathrm{fm}$, $m_\pi = 213(9)\,\mathrm{MeV}$

- Keep a but reduce m_π, T_{pc} and increase volume
- However smallest $m_\pi L \approx 3.3$ (at least > 3, optimally > 4)

- $$T_{pc} = 158(5)\,\mathrm{MeV}$$

- Near-zero modes do not instantly vanish when increasing T above T_{pc} (chiral crossover)
- For increasing temperature an increasingly wider gap emerges (Banks-Casher gap)



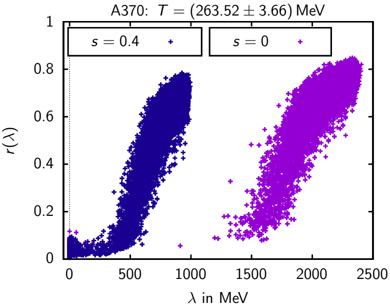
Localization measure

- Need measure of localization to reveal QCD Anderson transition
- Relative eigenmode volume:

$$r(\lambda) = \frac{P_2^{-1}(\lambda)}{|\Lambda|} \in [1/|\Lambda|, 1]$$

with *inverse participation ratio*
$P_2(\lambda) = \sum_{i \in \Lambda} (v_\lambda(i)^\dagger v_\lambda(i))^2$

  - Maximally localized on n_0:
    $v_\lambda(n)^\dagger v_\lambda(n) = \delta_{n,n_0}$
    $\Rightarrow r(\lambda) = 1/|\Lambda|$
  - Maximally delocalized:
    $v_\lambda(n)^\dagger v_\lambda(n) = 1/|\Lambda|$
    $\Rightarrow r(\lambda) = 1$



- Transition clearly visible
- Need criterion to quantify position $\hat{=}$ mobility edge
- Employ inflection point (IP)

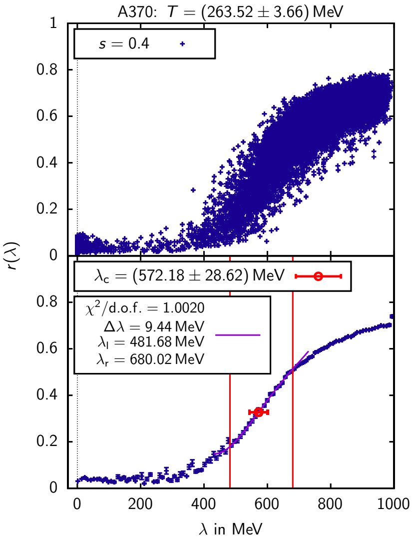
Mobility edge

Strategy for determining the IP

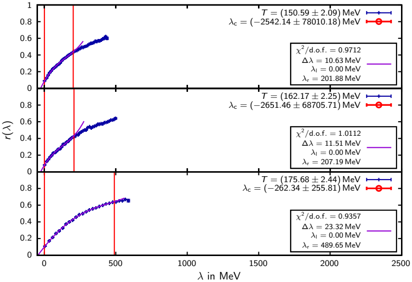
- Average over small bins in λ ($\approx$ expectation value of $r(\lambda)$)
- Obtain data with standard error
- Vary binsize and fit window
- Find fits with $\chi^2/\mathrm{d.o.f.} \approx 1$
- Read off inflection point

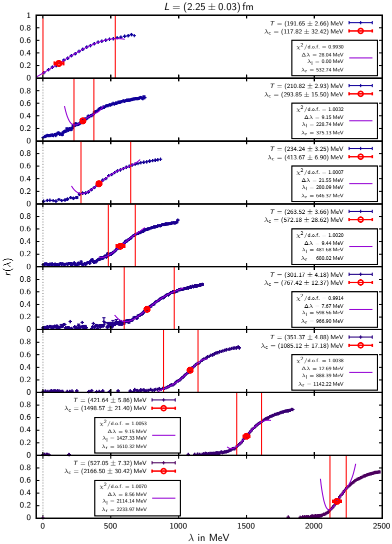
Fit Taylor polynomial (at IP)

$$r(\lambda) = r_\mathrm{c} + b(\lambda - \lambda_\mathrm{c}) + 0(\lambda - \lambda_\mathrm{c})^2 + c(\lambda - \lambda_\mathrm{c})^3 + d(\lambda - \lambda_\mathrm{c})^4$$



A370: $a = 0.0936(13)\,\mathrm{fm}$, $m_\pi = 364(15)\,\mathrm{MeV}$

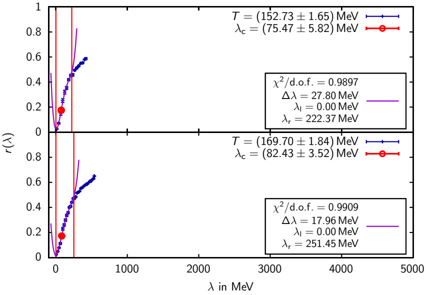




- Mobility edge vanishes below
  $T_{\mathrm{pc}} = 185(8)\,\mathrm{MeV}$
- Consistent with earlier work
  [M. Giordano et al., arXiv:1410.8392]
  [L. Holicki, E.-M. Ilgenfritz, L. von Smekal, arXiv:1810.01130]

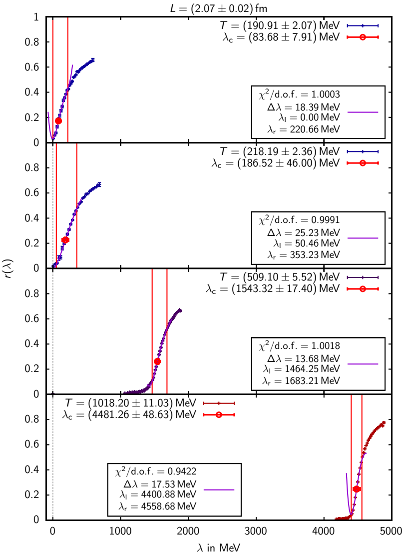
D370: $a = 0.0646(7)$ fm, $m_\pi = 369(15)$ MeV

- Smaller lattice spacing a



- Mobility edge does not vanish below

$$T_{\mathrm{pc}} = 185(4)\,\mathrm{MeV}$$



D210: $a = 0.0646(7)$ fm, $m_\pi = 213(9)$ MeV

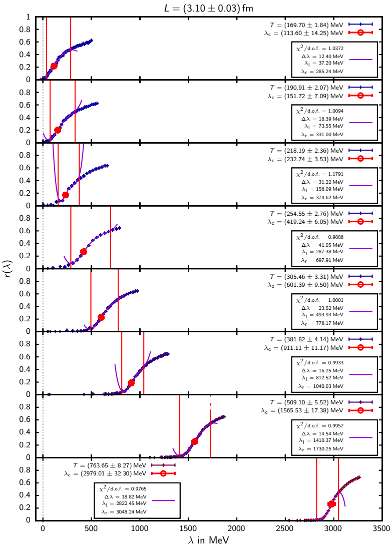
- Move closer towards physical limit: reduce m_π, T_{pc}, increase volume

- Find mobility edge for all evaluated ensembles, but all temperatures are above

$$T_{pc} = 158(5)\,\text{MeV}$$

Key question

Where does the mobility edge vanish in the physical limit?



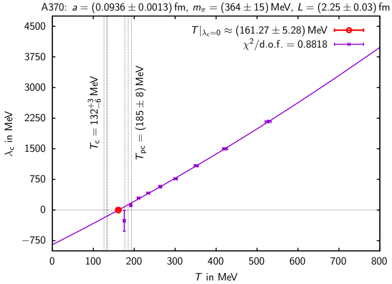
Temperature extrapolation of the mobility edge

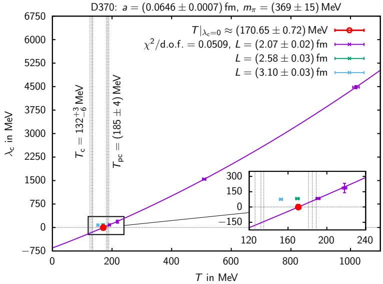
Fit Taylor polynomial (at the zero)

$$\lambda_c(T) = b(T - T_0) + c(T - T_0)^2$$

- A370 consistent with linear dependence and vanishing at T_{pc} as seen in earlier work
  [L. Holicki, E.-M. Ilgenfritz,
  L. von Smekal, arXiv:1810.01130]
- D370 shows slight curvature and no vanishing at T_{pc}, supported by data points to larger volumes

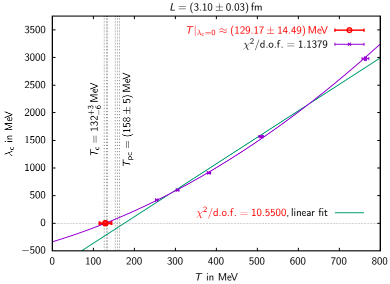
- In [M. Giordano et al., arXiv:1410.8392] actually slight curvature as well



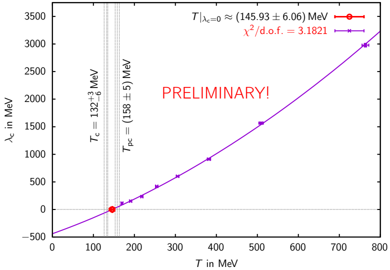


Extrapolation D210: $a = 0.0646(7)$ fm, $m_\pi = 213(9)$ MeV

- Linear fit breaks down
- There is definitely a curvature
- Zero coincides with T_c $(m \to 0)$
  [H.-T. Ding et al., arXiv:1903.04801]
- Data still far away from T_{pc} & T_c



- Include lower temperatures
- T_0 gets slightly shifted upwards but still below T_{pc}
- Bad χ^2/d.o.f. and e.g. fit for $T = 218(2)$ MeV not convincing
  $\Rightarrow$ Better statistics and/or analysis needed

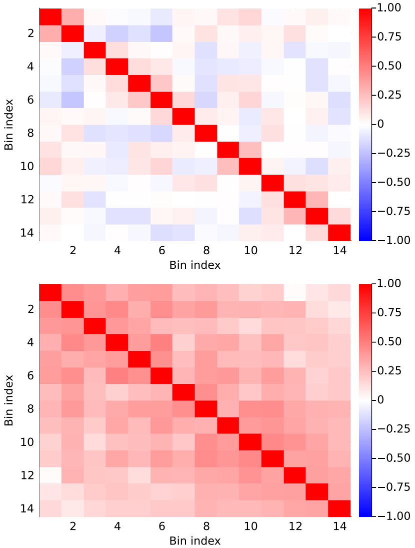


Correlation analysis

- Criterion $\chi^2/\text{d.o.f.} \approx 1$ requires statistically independent errors
- Same configurations for each bin $\Rightarrow$ Bins might be correlated
- Quantify by computing Pearson correlation coefficient of bin X and Y:

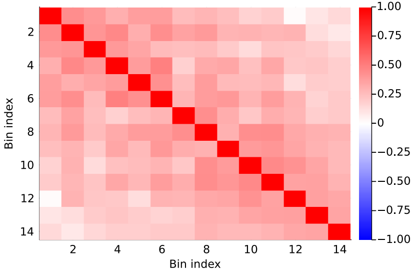
$$\rho_{X,Y} = \frac{\mathrm{cov}(X,Y)}{\sigma_X \sigma_Y}$$

- Correlation of λ (top) and $r(\lambda)$ (bottom) for $N_t = 4$ of A370
- Correlations for λ negligible but not for $r(\lambda)$



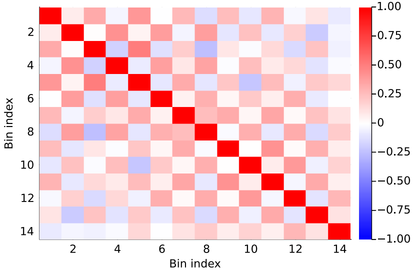
Reduce correlation

- Correlation especially strong for neighboring bins
  $\Rightarrow$ Pick different configurations: Even index for bin i, odd for bin $i+1$ (alternate picking)
- Correlation for relative volume strongly reduced (bottom)



- Repeat analysis with new method
- Small difference (a for alternating)

Data set	T_0 / MeV	$T_{0,a}$ / MeV
A370	161(5)	158(6)
D370	171(1)	168(2)
D210	129(14)	137(9)



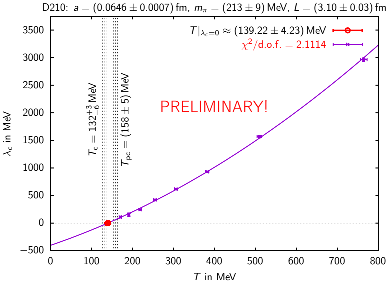
Scaling (second order phase transition)?

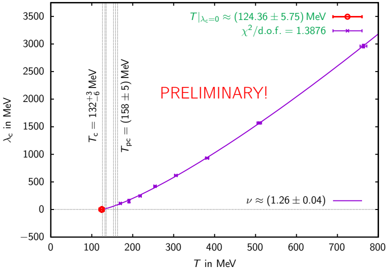
- Repeat quadratic fit for D210
- T_0 gets shifted downwards again
- Better χ^2/d.o.f. but still not convincing

- Try scaling fit:

$$\lambda_c(T) = b(T - T_0)^\nu$$

- T_0 moves downwards back to T_c
- Acceptable χ^2/d.o.f., however just 5 configurations per bin
- $\nu \approx 1.437$ for unitary Anderson model [L. Ujfalusi, I. Varga, arXiv:1501.02147]





Outline

Conclusion and outlook

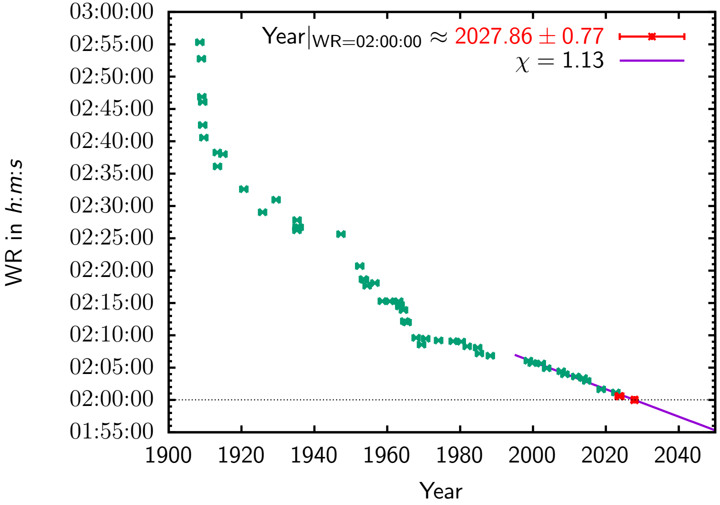
- Mobility edge λ_c depends nonlinear on the temperature (scaling?)
- Evidence for vanishing of λ_c at T_c and especially not at T_{pc}
- Data on lower temperatures and closer to physical limit necessary

  *$\Rightarrow$ Reduce computational costs
  - Improve program (if possible)
  - Apply UV-smoothing to lattice gauge configurations

- Employ other definitions of localization [A. Alexandru, I. Horváth, arXiv:2103.05607] $\Rightarrow$ Find infrared mobility edge *(...)
- Annihilation of both mobility edges as alternative scenario
- Both scenarios combined possible as well
- Determine mobility edge more precisely by finite-size analysis *(...)

- Study correlation of eigenmodes with Polyakov loop

Extra: (Official) Marathon world record extrapolation



$\Rightarrow$ Magical two hour mark will be broken in 2028(1)!

Thank you!