# Examination Paper 

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"Bound States of Quarks and Antiquarks in the Bethe-Salpeter Formalism"

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"Gebundene Zustände von Quarks und Antiquarks im Bethe-Salpeter-Formalismus"

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#### Abstract

In this thesis we will determine the physical properties of quarks, the $\pi$-meson (pion) and the $\rho$-meson based on the corresponding quark Dyson-Schwinger equation and the meson Bethe-Salpeter equation for bound states of quarks and antiquarks. We investigate the dynamical mass generation due to the spontaneously broken chiral symmetry in quantum chromodynamics (QCD) and the effects on the masses of hadronic bound states. We will also calculate the leptonic decay constants of the $\pi$ - and the $\rho$-meson and the coupling strength of the strong decay $\rho \rightarrow \pi \pi$.


## Kurzzusammenfassung

In dieser Arbeit werden auf Grundlage der Dyson-Schwinger-Gleichung für Quarks und der Bethe-Salpeter-Gleichungen für das $\pi$-Meson (Pion) und das $\rho$-Meson die physikalischen Eigenschaften von Quarks und gebundenen Zuständen von Quarks und Antiquarks untersucht. Dabei untersuchen wir die dynamische Massengenerierung durch die spontan gebrochene, chirale Symmetrie in der Quantenchromodynamik (QCD) und deren Auswirkungen auf die Massen hadronischer Bindungen. Ebenso werden die leptonischen Zerfallskonstanten des $\pi$ - und des $\rho$-Mesons berechnet, sowie die Kopplungsstärke des hadronischen Zerfalls $\rho \rightarrow \pi \pi$.

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## 1 Introduction

> "I learned very early the difference between knowing the name of something and knowing something."

\author{

- Richard Feynman
}

For centuries, scientists all over the world try to encode the phenomena of our world such that we become able to understand it. Studying the mechanics that surrounds us, Sir Isaac Newton found a mathematic model to describe the kinematics of heavy objects of our daily life properly. However, the difficult physical problems surmount the visible and go beyond classical mechanics. The fundamental theories of the heaviest and fastest objects and the lightest objects are the general relativity (A. Einstein: 1916) and the quantum mechanics (E. Schrödinger: 1926, W. Heisenberg: 1927, P. Dirac: 1928, ...). The problem of combining these theories is that the equations of motion in quantum mechanics, e.g. the Schrödinger equation, does not exist in a covariant form, in which the relativity is formulated. P. Dirac solved this problem by formulating a covariant equation of motion for spin $1 / 2$ particles, the Dirac equation. Dirac realized that the solutions of this equation are four dimensional spinors, which distinguish between spin orientation as well as positive and negative energy solutions. We identify the latter as the solutions of antiparticles - so, the existence of antiparticles had been determined by Dirac's equation for the first time.

The next breakthrough was the development of quantum field theories, which are relativistic field theories which satisfy quantization properties. These theories combine several principles and are the actual theories which characterize the behavior of particles in small distances. The occurring expressions are fairly complicated in all their glory but a graphical tool, the Feynman diagrams, leads to comprehension of what actually happens in small dimensions. Since the formulation of the general relativity, we know that mass and energy are equivalent due to the relation

$$
E=m c^{2}
$$

In this thesis, we will see, by using the algebra of the associated quantum field theory, that mass can be generated dynamically. From the standard model we know about several particles like leptons, quarks and hadrons. The current ("bare") quark masses are wellknown, but by experiments we observe that about $99 \%$ of the hadronic mass has to be generated dynamically. A simple example is the proton, which consists of two u- and one d-quark. Its mass is $928 \mathrm{MeV}^{*}$ and, since $m_{\mathrm{u}}=2.3 \mathrm{MeV}$ and $m_{\mathrm{d}}=4.8 \mathrm{MeV}$ [1], a simple addition of masses is not expedient anymore. For this reason, a basic concept like the mass seems to become much tougher by dipping deeper into the matter. This thesis takes a closer look at the mass generation of the $\rho$ - and the $\pi$-meson. The determination of baryons as bound states differs and can be done within a Faddeev-approach, which is not part of this thesis, but can be consulted e.g. in [2]. The meson mass spectrum of light

[^0]pseudoscalar and vector mesons [3] shows that the masses of light mesons are, similar to the baryon case, much higher than the sum of the "bare" quarks too. Furthermore, the vector mesons are much heavier than their pseudoscalar equivalents and can be interpreted as the angular-momentum-excited meson states of the pseudoscalar ones. To give an impression, the experimentally observed mass of a charged pion $\pi^{ \pm}$is given by 139.6 MeV , whereas the mass of the corresponding vector particle with $J=1$, the $\rho$-meson, is experimentally given by 775.3 MeV [1]. A phenomenological formula for the masses of all light mesons is given by
\[

$$
\begin{equation*}
M_{\overline{\mathrm{q} q}}=m_{\mathrm{q}}+m_{\overline{\mathrm{q}}}+\Delta M_{\mathrm{ss}}, \tag{1.1}
\end{equation*}
$$

\]

in which $\Delta M_{\mathrm{ss}}$ is the mass shift induced by the spin-spin interaction [3]. In this thesis we will treat this mass generation of light pseudoscalar and vector mesons mathematically in Bethe-Salpeter formalism. This formalism is based on the inner algebra of quantum field theory and leads to bound states of quarks and antiquarks by satisfying specific conditions. The required essentials are given by a detailed analysis of the quark DysonSchwinger equation, a self-consistent equation, which determines the propagation of the quark and the antiquark including all possible interactions, absorptions and emissions. Based on conservation laws in the standard model, we know how mesons could degrade and one can investigate these decays in experiments. According to the decay modes of the charged pions $\pi^{ \pm}$, the (by far) most propable leptonic decay of e.g. the $\pi^{+}$is [1]

$$
\pi^{+} \rightarrow \mu^{+} \nu_{\mu}
$$

The equivalent, dominant leptonic decay of the $\rho$ is given by

$$
\rho \rightarrow \mu^{+} \mu^{-} .
$$

These decays are determined by their decay constants and the quantum field theoretical determination is very complex, so it needs detailed analyses. We will treat these decay constants in course of this thesis by translating the corresponding Feynman diagrams into integral equations. But in case of the $\rho$ meson, the most probable decay is by far not


Figure 1: The hadronic decay $\rho \rightarrow \pi \pi$ with an unknown vertex.
leptonic, but rather hadronic. That means, the $\rho$ degrades into two other hadrons, which are in this case pions. In terms of this thesis we will treat the hadronic $\rho^{0} \rightarrow \pi^{+} \pi^{-}$decay. In Feynman diagrams, this decay looks like Fig. 1.
The goal is to know exactly the vertex structure, which is located behind the orange surface. Then, based on the Feynman-rules, we can directly formulate the expressions we need to do our calculations by looking at the diagram. But to do so, we have to know a lot about the ingredients of this process.

At first this thesis will give an introduction into basic physical principles which have to be understood to have the necessary theoretical background. To be more precise, starting by basic principles of field theories like the QCD, we will clarify the subject of (dynamical chiral) symmetry breaking and the underlying algebraic structures in this chapter. Then, we will introduce the quark Dyson-Schwinger equation as an equation of motion of the corresponding quark propagator. To determine bound states of one quark and one antiquark, we introduce the Bethe-Salpeter equation and express the interaction between the quark and the antiquark after applying a truncation scheme, the Rainbow-Ladder truncation, in a way in which the calculations of several properties simplify. Then we will outline the results of the calculations done in terms of this thesis, accompanied by the explicit mathematical approach. These calculations involve physical quantities like the effective quark masses, the $\pi$ - and $\rho$-meson masses and the leptonic decay constants. In the end we will calculate the hadronic coupling constant of the hadronic decay $\rho \rightarrow \pi \pi$ introduced in the previous paragraph. We will also discuss how far the chosen interaction is able to reproduce suitable approximations of the experimental values of the physical quantities, which are calculated. Finally, we will reflect the results of the calculations and give an outlook of what advantages the insights could bring for further research activities.
The appendix includes important conventions and relations, which were used by doing the calculations. Furthermore, important derivations can be seen in.
Due to the phenomena mentioned above, the dynamical mass generation and the decays, this thesis takes a closer look at the properties of particles and tries to reproduce the experimental values by using a suitable model.

While conducting research in terms of this thesis, the introducing quote of Richard Feynman described my mood fairly often, because it mirrors exactly my thoughts by taking a look behind these scenes for the first time.

## 2 Physical Principles

### 2.1 The Lagrange Formalism in Field Theories

In classical mechanics, the equations of motion are interesting because they exactly determine the physical system consisting of point masses. There are several ways to get to these equations, one of them is the Lagrange formalism. The so called Lagrange function (or equivalently the Lagrangian) is the difference between the kinetic energy $T$ and the potential $V$ :

$$
\begin{equation*}
L=T-V \tag{2.1}
\end{equation*}
$$

We get the equations of motion by applying the Lagrangian to the Euler-Lagrange formalism*:

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t} \frac{\partial L}{\partial\left(\partial_{t} q_{i}\right)}-\frac{\partial L}{\partial q_{i}}=0 \tag{2.2}
\end{equation*}
$$

To describe particles and interactions in quantum field theories like the QED and the QCD, we transfer this formalism and determine the system using the Lagrange density $\mathcal{L}\left(\phi, \partial_{t} \phi, t\right)$ (that depends on the fields $\phi$, their time derivatives and time), or equivalently, the action $S$ that is given by

$$
\begin{equation*}
S=\int \mathrm{d}^{4} x \mathcal{L} \tag{2.3}
\end{equation*}
$$

Supposing the action does not vary under infinitesimal small symmetry transformations ${ }^{\dagger}$ $\left(0=\mathrm{d} S=\frac{\partial S}{\partial \phi} \mathrm{~d} \phi\right)$ we get the equations of motion (in analogy to the Euler-Lagrange formalism in classical mechanics) by using

$$
\begin{equation*}
\frac{\partial \mathcal{L}}{\partial \psi}-\partial_{\mu} \frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \psi\right)}=0 . \tag{2.4}
\end{equation*}
$$

Based on the Dirac equation (see A.1) ${ }^{\ddagger}$ we can formulate the Lagrangian for a free spin $1 / 2^{\S}$ particle as

$$
\begin{equation*}
\mathcal{L}=\bar{\psi}\left(i \gamma_{\mu} p^{\mu}-m\right) \psi \tag{2.5}
\end{equation*}
$$

By plugging this into the Lagrange formalism we get back to the Dirac equations as the equations of motion of four dimensional spinors $\psi$ and, respectively, their Dirac adjoint spinors $\bar{\psi}=\psi^{\dagger} \gamma^{0}$ :

$$
\begin{align*}
& \left(i \gamma_{\mu} p^{\mu}-m\right) \psi=0  \tag{2.6}\\
& \bar{\psi}\left(i \gamma_{\mu} p^{\mu}-m\right)=0 \tag{2.7}
\end{align*}
$$

[^1]Symmetries. It is fairly interesting to take a look at the symmetries of the Lagrangian. The physics in a wave function is given by its absolute value, hence, the Lagrangian has to stay invariant under (complex) phase shifts which correspond to the unitary group $U(1)$ that includes all global phase shifts $e^{i \alpha}$. With the definitions of a spinor and its Dirac adjoint we can quickly see that the transformed Lagrangian stays invariant, because of $\psi^{\prime}=e^{i \alpha} \psi$ and $\bar{\psi}^{\prime}=e^{-i \alpha} \bar{\psi}$.
The motivation by developing field theories like QED and QCD was i.a. to convert global gauge invariances into local gauge invariances, which means, that the Lagrangian has to stay the same under selected space dependent rotations. In QED that means the Lagrangian remains invariant under local transformations of the unitary group $U(1)$, which now includes all local phase shifts $u=e^{i q \alpha(x)}$. In QCD we get additional degrees of freedom due to the color charge that corresponds to $S U(3)_{C}$.* The field theories now demand an invariant Lagrangian under rotations in $U(1)$, respectively $S U(3)_{C}$.
It follows that new massless ${ }^{\dagger}$ vector fields occur, which we identify as the spin-1 exchange boson of the electromagnetic interaction, the photon (QED), and the one of the strong interaction, the gluon (QCD).
As a result, the QCD Lagrangian, which is particularly relevant in this thesis, looks different, includes more terms than the "free" Lagrangian (Eq. 2.5) and contains both, the Dirac particle $\psi$ and a resulting field strength tensor $\mathcal{G}^{\mu \nu}$ : [4]

$$
\begin{align*}
\mathcal{L} & =\bar{\psi}\left(-\gamma_{\mu} D^{\mu}-m\right) \psi+\frac{1}{4} \mathcal{G}_{\mu \nu}^{a} \mathcal{G}_{a}^{\mu \nu}  \tag{2.8}\\
& =\bar{\psi}\left(-\gamma_{\mu} \partial^{\mu}-m\right) \psi-g_{0} \bar{\psi}\left(\gamma_{\mu} T^{a} A_{a}^{\mu}\right) \psi+\frac{1}{4} \mathcal{G}_{\mu \nu}^{a} \mathcal{G}_{a}^{\mu \nu} \tag{2.9}
\end{align*}
$$

Here $D^{\mu}$ is the so called covariant derivative,

$$
\begin{equation*}
D^{\mu}=\partial^{\mu}+i g_{0} A^{\mu}, \quad A^{\mu}=T^{a} A_{a}^{\mu} \tag{2.10}
\end{equation*}
$$

which is brought in by demanding the local gauge invariance. $\gamma_{\mu}$ are the Dirac matrices and $T_{a}$ are the generators of the $S U(3)_{C} \cdot \mathcal{G}$ is defined by

$$
\begin{equation*}
\mathcal{G}_{\mu \nu}^{a}=\partial_{\mu} A_{\nu}^{a}-\partial_{\nu} A_{\mu}^{a}+g_{0} f^{a b c} A^{b \mu} A^{c \nu}, \tag{2.11}
\end{equation*}
$$

in which the fields $A^{i \mu}$ are the eight gluon fields and $f_{a b c}$ are the structure constants of the $S U(3)$-algebra.
The first term describes the kinetic energy and the mass of the Dirac particle, the second term is the interaction between the Dirac particle and the eight gluon fields and lastly, the third term describes the energy density of the eight gluon fields. In contrast to the QED Lagrangian, the non-Abelian $S U(3)_{C}$ implies a commutator $\left[T_{a}, T_{b}\right.$ ] that is non-vanishing, thus, higher terms proportional to $A^{3}$ and $A^{4}$ occur. We interpret these terms as the gluon

[^2]self interactions and conclude that gluons have to carry a color charge because otherwise such a self interaction is inexplainable.*
These self interactions lead us to the many ways a quark as a Dirac particle can propagate. We will analyze these combinations by facing the corresponding Dyson-Schwinger equations (DSEs).

### 2.2 Symmetry Breaking

### 2.2.1 Principles

As broached in the previous chapter, symmetries are interesting properties of the Lagrangian. In the cause of this thesis, we will see that symmetry breaking is the reason why the "effective" quark masses differ from the "bare" quark masses.

Example. Breaking symmetries is a concept that is fairly interesting because it describes several processes in physics. A demonstrative example for this is the behavior of a ferromagnet: its net magnetization obliterates at any temperature larger than its Curie Temperature $T_{c}$ and it behaves like a paramagnet. If the system cools down and reaches a temperature $T \lesssim T_{c}$, spontaneous magnetization occurs. This defines a new ground state of the system. The question is now whether we can exactly describe the magnetization shortly before $T_{c}$ is undercut. ${ }^{\dagger}$ The answer is "No!", because now there are infinitely many, energy degenerated ground states that can be realized. To sum it up by definition:

A symmetry $U^{\alpha}$ is called spontaneously broken, if $\mathcal{L}$ is (and equivalently the equations of motion are) invariant under $U^{\alpha}$, whereas the ground state is not.
It follows that the ground state has less symmetries than the Lagrangian itself. In the magnetization example we can describe the symmetries with $S O(N)$ groups - the rotational groups in spatial coordinates. Before the temperature went to $T<T_{c}$ the system is perfectly $S O(3)$-symmetric because we can rotate the system in any direction without changing the micro state. For $T \lesssim T_{c}$ we cannot do so. Here, we can simply rotate around $\mathbf{M}^{\ddagger}$ as the rotational axis to guarantee the invariance. That reduces the symmetry by one dimension and leaves us with a $S O(2)$-symmetry.
For the sake of completeness, we provide two more definitions:
A symmetry $U^{\alpha}$ is called exact, if $\mathcal{L}$ and the ground state is invariant under $U^{\alpha}$. A symmetry $U^{\alpha}$ is called explicitly broken, if neither $\mathcal{L}$, nor the ground state is invariant under $U^{\alpha}$.

[^3]
### 2.2.2 Chiral Symmetry Breaking

Besides spin, helicity and some other abstract properties we can attribute to a particle, an additional property is added, the chirality. We define two operators,

$$
\begin{equation*}
P_{L}=\frac{1}{2}\left(\mathbb{1}-\gamma^{5}\right) \quad \text { and } \quad P_{R}=\frac{1}{2}\left(\mathbb{1}+\gamma^{5}\right) \tag{2.12}
\end{equation*}
$$

in which $\gamma^{5}$ is the fifth Dirac matrix.
These operators act on the four dimensional spinors, satisfy the completeness relation

$$
\begin{equation*}
P_{L}+P_{R}=\mathbb{1} \tag{2.13}
\end{equation*}
$$

and define new chiral spinors:

$$
\begin{align*}
\psi_{+} & :=P_{L} \psi
\end{align*} \quad \psi_{-}:=P_{R} \psi ~ 子 ~\left(\bar{\psi}_{+}:=\bar{\psi} P_{R} \quad \bar{\psi}_{-}:=\bar{\psi} P_{L}\right.
$$

We call the spinor $\psi_{-}$right-handed and the spinor $\psi_{+}$left-handed. This denotation arises from the particles in the relativistic limit $(\beta \rightarrow 1)$, in which helicity and spin are equivalent. In QCD of massless fermions the Dirac equation is given by the so called Weyl-equation,

$$
\begin{equation*}
\not D \psi=i \gamma^{\mu} p_{\mu} \psi=0 \tag{2.15}
\end{equation*}
$$

We call the approximation $m \rightarrow 0$ the chiral limit. For light quarks such as the $u-$, d- or s-quark, this approximation is fairly justified because their masses are, in specific scales, negligibly small. At this point, it is advantageous to transform the matrices and the spinors into their Weyl representation, in which the Weyl equation is given by

$$
\left(\begin{array}{rr} 
& i \sigma^{\mu} p_{\mu}  \tag{2.16}\\
-i \bar{\sigma}^{\mu} p_{\mu} &
\end{array}\right)\binom{\psi_{+}}{\psi_{-}}=0
$$

where $\sigma^{\mu}=(\mathbb{1}, \boldsymbol{\sigma})$ and $\bar{\sigma}^{\mu}=(\mathbb{1},-\boldsymbol{\sigma})$.
The reason why the Dirac spinor has this form in Weyl representation is because the spinor rotation transformation and the boost transformation are both block diagonal. Hence, this Dirac spinor representation is reducible. [5] To verify that both entries are indeed given by the two chiral spinors $\psi_{ \pm}$we defined in (2.14), we take a look at $\gamma^{5}$, which becomes diagonal in Weyl representation,

$$
\gamma_{\mathrm{Weyl}}^{5}=\left(\begin{array}{ll}
\mathbb{1} &  \tag{2.17}\\
& -\mathbb{1}
\end{array}\right) .
$$

The eigenvalues of $P_{L}$ and $P_{R}$ are $\pm 1$ and correspond to the eigenvectors $(1,0)^{T}$ and $(0,1)^{T}$
so that the two spinors decouple into two seperate equations

$$
\begin{align*}
& i \sigma^{\mu} \partial_{\mu} \psi_{+}=0  \tag{2.18}\\
& i \bar{\sigma}^{\mu} \partial_{\mu} \psi_{-}=0
\end{align*}
$$

Furthermore, this decoupling emerges as a separation in the Lagrangian*:

$$
\begin{equation*}
\mathcal{L}=\bar{\psi}_{+} \not D \psi_{+}+\bar{\psi}_{-} \not D \psi_{-} \tag{2.19}
\end{equation*}
$$

As mentioned earlier, there are three relevant flavours that can be justified by the chiral limit: the u-, the d- and the s-quark. Thus, because of the flavour-independence of $I D$, there has to be a complete $U(3)$-symmetry in flavour space for both spinors. We can partition ${ }^{\dagger}$ this symmetry into a direct product:

$$
\begin{equation*}
U(3)_{L} \otimes U(3)_{R}=S U(3)_{L} \otimes S U(3)_{R} \otimes U(1)_{V} \otimes U(1)_{A} \tag{2.20}
\end{equation*}
$$

The invariance of $\mathcal{L}$ under the $S U(3)$ s is called "chiral symmetry". $U(1)_{V}$ is called the vector symmetry and $U(1)_{A}$ the axial symmetry. Attached to the decoupling, the $S U(3) \mathrm{s}$ should rotate both chiral states seperately and independent with

$$
\begin{align*}
& \psi_{-} \rightarrow \exp \left(i \frac{\lambda^{a}}{2} \theta_{a}^{-}\right) \psi_{-} \\
& \psi_{+} \rightarrow \exp \left(i \frac{\lambda^{a}}{2} \theta_{a}^{+}\right) \psi_{+} \tag{2.21}
\end{align*}
$$

We identify $\lambda^{a}(a=1 \ldots 8)$ with the eight Gell-Mann matrices that act on the flavour spinors $\psi_{ \pm}=\left(u_{ \pm}, d_{ \pm}, s_{ \pm}\right)$.
For any of the $2 \times 8=16$ generators of the symmetry the Noether's theorem implies one conserved quantity due to the corresponding conserved Noether currents.


Figure 2: The occurring Goldstone bosons due to spontaneous symmetry breaking, illustrated using the examples of $N_{f}=2$ and $N_{f}=3$.

[^4]The Goldstone theorem. Depending on the spontaneously broken chiral symmetry, the Goldstone theorem predicts, that whenever a symmetry is broken spontaneously, massless bosons (the so called Goldstone bosons) occur. Integrating the (indeed small) mass of light quarks into the massless Lagrangian yields that the Goldstone bosons become massive and, in terms of this thesis, appear as pions*. The number of Goldstone bosons is defined by the number of generators / degrees of freedom which get lost due to symmetry breaking. The prominent examples of QCD, the spontaneous breaking of $S U\left(N_{f}\right) \times S U\left(N_{f}\right)$ symmetries with $N_{f}=2,3$, are illustrated in Fig. 2.

### 2.3 Dyson Schwinger Equations

Dyson Schwinger equations (DSEs) are an infinite set of coupled integral equations and represent the equations of motions of the Green's functions in quantum field theory. They are used in several quantum field theories to determine the propagation of particles like the electron (in QED), the quark (in QCD) and others. This becomes clear when we realize that the $n$-point Green's functions that solve the DSEs are nothing more than the propagators of our particles. The "bare" propagators finally have to get "dressed" with the aid of dressing functions to represent a possible solution of the respective DSE. Based on Feynman diagrams, it is, in field theories, possible to express the DSEs as Feynman diagrams and assign every expression in the DSE to an element of the diagram. In this thesis we thematize the quark DSE which will be introduced later on.

A quark can propagate in several ways, thus, when we describe a general propagation, we have to consider every possibility we can realise by using Feynman diagrams. Due to the additional gluon self interaction (see Eq. 2.11) there are various of such possibilities in QCD. We can illustrate them by Feynman diagrams (see Fig. 3).

To summarize the three prominent elements in the Feynman diagrams, the emissions, the absorptions and the interactions, we can introduce a new quantity, the quark self energy $\Sigma(p)$, also referred to as the "one particle irreducible" [6]. This quantity includes all possible propagations of a quark, in which one gluon is emitted at a point $\mu$ and absorbed by the quark at a point $\nu$. It is shown as the iterative part in the second row in Fig. 3.
We distinguish between the fully dressed quark propagator $S(p)$ and the bare quark propagator $S_{0}(p)$ and can formulate the "graphical equation" from Fig 3 by the self-consistency series ${ }^{\dagger}$

$$
\begin{equation*}
S(p)=S_{0}(p)+S_{0}(p) \Sigma(p) S(p) \tag{2.22}
\end{equation*}
$$

By writing the occuring sum out "in full" and reformulating it as a geometric series, (see A.2) we can express the inverse quark propagator $S^{-1}$ by the following equation, also known as gap equation:

$$
\begin{equation*}
S^{-1}(p)=S_{0}^{-1}-\Sigma(p) \tag{2.23}
\end{equation*}
$$

[^5]

Figure 3: The self-consistency problem of the full quark propagator. Straight lines correspond to propagating quarks, spring lines correspond to propagating gluons. A blob is equivalent to the summation of all possible insertions and is expressed mathematically by Green's functions. The blue blobs are fully dressed quark and gluon propagators, the red blob is the quark gluon vertex.

The bare quark propagator is a two point Green's function and, up to the color and flavour structure, is given by the fermion propagator

$$
\begin{equation*}
S_{0}(p)=\frac{-i \not p+m}{p^{2}+m^{2}} \tag{2.24}
\end{equation*}
$$

in which $m$ is the fermion mass* and $p^{2}$ is the absolute value of the four momentum. The inverse propagator $S_{0}^{-1}(p)$ is given by

$$
\begin{equation*}
S_{0}(p)^{-1}=i \not p+m . \tag{2.25}
\end{equation*}
$$

To describe the full propagator, we introduce the dressing functions $A\left(p^{2}\right)$ and $B\left(p^{2}\right)$ so that the inverse bare propagator transforms into the inverse full propagator in the following way [7]:

$$
\begin{equation*}
S_{0}^{-1}=i \not p+m \xrightarrow{\text { dressing }} i \not p A\left(p^{2}\right)+B\left(p^{2}\right)=S^{-1} . \tag{2.26}
\end{equation*}
$$

In this form, the propagator $S$ can satisfy the gap equation.
For the full solution of the gap equation, we insert the full expression for the self energy [8]:

$$
\begin{equation*}
S^{-1}(p)=i \not p+m+\frac{4 g^{2}}{3} \int \frac{\mathrm{~d}^{4} q}{(2 \pi)^{4}} \gamma_{\mu} S(q) \Gamma_{\nu}(q, p) D^{\mu \nu}(p-q) \tag{2.27}
\end{equation*}
$$

Thereby we identify $D^{\mu \nu}$ as the full gluon propagator and $\Gamma_{\nu}$ as the quark gluon vertex. $g$ is a constant coupling strength and the prefactor of $4 / 3$ stems from the color trace. ${ }^{\dagger}$ The expression for $\Sigma$ is a consequence of the Feynman rules - the integral, for example, results from the indeterminate momentum in the loop. The indices $\mu$ and $\nu$ are Dirac indices and represent where the gluon is emitted and absorbed. (see Fig. 4 and A.2)

[^6]

Figure 4: The gap equation in Feynman diagrams.

### 2.4 The Bethe-Salpeter Equation

To determine a bound state of quarks, in the easiest case mesons, a more complicated formalism has to be consulted, which includes the solutions of the quark Dyson-Schwinger equations (DSEs) and satisfies the Ward-Takahashi identities (see chapter 2.5). Furthermore, the ingredients have to contain all possible interactions between the constituent quarks. The formalism that is particular relevant in this thesis is the Bethe-Salpeter formalism, which lets us calculate specific properties of mesons, e.g. pions, and diquarks. The idea is to take the exact QCD equation for scattering processes, in which the interaction can be determined by a 4 -point Green's function, the full propagator $G$. This propagator can be expressed by a bare propagator $G_{0}$ as a product of the two fully dressed fermion propagators, and the scattering $T$-Matrix, which describes the interactions between the quarks [7], [9]:

$$
\begin{equation*}
G=G_{0}+G_{0} T G_{0} \tag{2.28}
\end{equation*}
$$

$T$ can be expressed with a Dyson equation,

$$
\begin{equation*}
T=K+K G_{0} T, \tag{2.29}
\end{equation*}
$$

where $K$ is the scattering kernel. We choose the ansatz

$$
\begin{equation*}
T \sim \frac{\Gamma \bar{\Gamma}}{p^{2}+m^{2}} \tag{2.30}
\end{equation*}
$$

with $\Gamma$ as an amplitude and $\bar{\Gamma}$ the corresponding, charge conjugated amplitude. $T$ diverges for "on-shell" particles which satisfy the energy mass relation $\left(p^{2}=-m^{2}\right)$. By this, Eq. (2.29) simplifies* to

$$
\begin{equation*}
\Gamma=K G_{0} \Gamma \tag{2.31}
\end{equation*}
$$

Eq. (2.31) is called the Bethe-Salpeter equation (BSE). The solutions $\Gamma$ of this equation are called Bethe-Salpeter amplitudes (BSAs) and have to satisfy the corresponding BSE as an eigenvalue equation to the eigenvalue 1 . Thus, every discrete eigenstate $\Gamma_{0}$, which satisfies $K G_{0} \Gamma_{0}=1 \cdot \Gamma_{0}$ represents a potential physical state. The BSE is illustrated

[^7]

Figure 5: The Bethe-Salpeter equation (2.31) in Feynman diagrams.
graphically in Fig. 5. It contains every relevant element that has to be considered: The two quark propagators denote the quark and the antiquark in the meson individually, the BSA denotes the bound state between the quarks, and the scattering kernel $K$ includes all the interactions between the constituents.

As an abstract quantity that should determine physical quantities, the BSA has to satisfy a canonical normalization condition. Taking the derivative of $G$ yields

$$
\begin{equation*}
\bar{\Gamma}\left[\frac{\mathrm{d} G_{0}}{\mathrm{~d} P^{2}}+G_{0} \frac{\mathrm{~d} K}{\mathrm{~d} P^{2}} G_{0}\right] \Gamma=-1 \tag{2.32}
\end{equation*}
$$

In case the scattering kernel does not depend on $P$, the second term vanishes and the task is to differentiate the product of the two point Green's functions, $G_{0}=G_{(2)} G_{(2)}$, in a way that the normalization condition finally reads

$$
\begin{equation*}
\bar{\Gamma}\left[\frac{\mathrm{d} G_{(2)} G_{(2)}}{\mathrm{d} P^{2}}\right] \Gamma=-1 \tag{2.33}
\end{equation*}
$$

For mesons, the Bethe-Salpeter amplitude (BSA) as an integral equation* is given by [10]:

$$
\begin{equation*}
\left[\Gamma^{j}(p, P)\right]_{t u}=\int_{q}^{\Lambda} K_{t u}^{r s}(q, p ; P)\left[\chi^{j}(q ; P)\right]_{s r} \tag{2.34}
\end{equation*}
$$

In this equation, $q$ is the relative and $P$ the absolute momentum of the treated meson. The explicit definition of the integral notation is given by the definition (A.13). The indices $j, r, s, t$ and $u$ are color, flavour and Dirac indices and $\chi^{j}(q, P)$ is defined as the BetheSalpeter wave function, which is given by

$$
\begin{equation*}
\chi^{j}(q, P)=S\left(q_{+}\right) \Gamma^{j}(q, P) S\left(q_{-}\right) \tag{2.35}
\end{equation*}
$$

The quantities $q_{+}=q+\eta P$ and $q_{-}=q+(\eta-1) P$, with the parameter $\eta \in[0,1]$, are the momenta of the constituents.

[^8]The general solution is given by [2], [11]

$$
\begin{equation*}
\Gamma^{\mu}(p, P)=-\frac{4 g^{2}}{3} \int_{q}^{\Lambda} D^{\mu \nu}(p-q) \gamma_{\mu} S\left(q_{+}\right) \Gamma^{\mu}(q, P) S\left(q_{-}\right) \Gamma_{\nu}^{\mathrm{qg}}(p, q) \tag{2.36}
\end{equation*}
$$

which is quite similar to the quark self energy (see Eq. (2.27)). Here, we gave the quarkgluon vertex an additional index ' qg ' to distinguish it from the Bethe-Salpeter amplitude $\Gamma^{\mu}$.
The meson BSA can also be expressed as a linear combination of its basis elements, which are the underlying Dirac structures. One distinguishes between different quantum numbers $J^{P}$ so that [12]

$$
\Gamma^{(\mu)}= \begin{cases}\sum_{i} F_{i}(p, P) T_{i}^{(\mu)}(p, P) & : J^{P}=0^{+}, 1^{-}  \tag{2.37}\\ \sum_{i} \gamma_{5} F_{i}(p, P) T_{i}^{(\mu)}(p, P) & : \quad J^{P}=0^{-}, 1^{+}\end{cases}
$$

The quantities $T_{i}^{(\mu)} \left\lvert\, \begin{aligned} & 1 \leq i \leq 4: J=0 \\ & 1 \leq i \leq 8: J=1\end{aligned}\right.$ are the corresponding twelve linear independent Dirac structures and can be taken from [12]. An index $\mu$ denotes the vectorial character, hence, it has to be considered, iff $J=1$. The quantities $F_{i}$ are, convention-dependent, odd or even functions in the arguments.
Pseudoscalar (pion) BSA. The BSA of pseudoscalar mesons such as the pion can be decomposed into four of these structures* and is given by its general form ${ }^{\dagger}$ :

$$
\begin{equation*}
\Gamma_{0^{-}}(p, P)=\gamma_{5}\left[F_{1}(p, P)-i \not P F_{2}(p, P)-i \not p(p \cdot P) F_{3}(p, P)-[\not P, \not p] F_{4}(p, P)\right] \tag{2.38}
\end{equation*}
$$

A combination of Eqs. (2.36) and (2.38) leads to equations that can be solved algebraically, indeed with the aid of a special interaction.
In order to determine observable quantities we have to translate the canonical normalization condition of Eq. (2.33) into an equation that includes the quantities we are working with. This can be done with the aid of the Feynman rules. The indeterminate loop momentum $q$ implies an integral, the fermion loop brings in a factor of -1 and, working out the implicitness of this equation, we have to take the trace over color, flavour and Dirac indices and order the propagators in the following way:

$$
\begin{equation*}
1=\left.\frac{\mathrm{d}}{\mathrm{~d} P^{2}} \operatorname{tr} \int_{q}^{\Lambda} \bar{\Gamma}(q, K) S\left(q_{+}\right) \Gamma(q, K) S\left(q_{-}\right)\right|_{P^{2}=-m_{\pi}^{2}} \tag{2.39}
\end{equation*}
$$

in which $\bar{\Gamma}(q, P)=\mathcal{C} \Gamma^{T}(-q,-P) \mathcal{C}^{T}$ is the charge conjugated BSA. Fig. 6 shows the normalization scheme as a Feynman diagram. On the mass shell, the charge conjugated, pseudoscalar BSA is given by (cf. A.2):

$$
\begin{equation*}
\bar{\Gamma}(p, P)=\Gamma(p,-P) \tag{2.40}
\end{equation*}
$$

[^9]

Figure 6: The normalization scheme as a Feynman diagram. $\Gamma$ is the BSA and $\bar{\Gamma}$ is the charge conjugated BSA. Arrows denote fully dressed quark propagators.

By using this, we can calculate the leptonic decay constant, which is defined as an ingredient of the on-shell residue of the axial current transition matrix element:

$$
\begin{equation*}
\langle 0| j_{5}^{\mu}|\pi\rangle=-i P^{\mu} f_{\pi} \cdot \mathrm{e}^{-i x \cdot P} \tag{2.41}
\end{equation*}
$$

The pseudoscalar current transition matrix element defines another quantity, $r_{\pi}$, which cannot be interpreted as an observable quantity: [13]

$$
\begin{equation*}
\langle 0| j_{5}|\pi\rangle=-i r_{\pi} \cdot \mathrm{e}^{i x \cdot P} \tag{2.42}
\end{equation*}
$$

Translating these into a representation in momentum space yields: [10]

$$
\begin{equation*}
\delta^{i j} f_{\pi} P_{\mu}=Z_{2} \int_{q}^{\Lambda} \operatorname{tr}\left[\frac{\sigma^{i}}{2} \gamma_{5} \gamma_{\mu} S\left(q_{+}\right) \Gamma_{\pi}^{j}(q, P) S\left(q_{-}\right)\right] \tag{2.43}
\end{equation*}
$$

and

$$
\begin{equation*}
i \delta^{i j} r_{\pi}=Z_{4} \int_{q}^{\Lambda} \operatorname{tr}\left[\frac{\sigma^{i}}{2} \gamma_{5} S\left(q_{+}\right) \Gamma_{\pi}^{j}(q, P) S\left(q_{-}\right)\right] \tag{2.44}
\end{equation*}
$$

in which $Z_{2}$ and $Z_{4}=Z_{2} Z_{m}$ are renormalization constants. The corresponding Feynman diagram to the leptonic decay is given by Fig. 7. We see that the BSA couples to the axial vector vertex $\Gamma_{5 \mu}=\gamma_{5} \gamma_{\mu}$.
The quantities $f_{\pi}$ and $r_{\pi}$ are related by the following equation:

$$
\begin{equation*}
f_{\pi} m_{\pi}^{2}=2 m r_{\pi} \tag{2.45}
\end{equation*}
$$

Furthermore, the decay constant is an ingredient of the Gell-Mann-Oakes-Renner (GMOR) relation, which predicts [13]

$$
\begin{equation*}
f_{\pi}^{2} m_{\pi}^{2}=-2 m_{c}\langle\overline{\mathrm{q}} \mathrm{q}\rangle / N_{f} \tag{2.46}
\end{equation*}
$$

in which the decay constant, the quark condensate* and the pion mass are related.

[^10]

Figure 7: The leptonic decay e.g. of the pion as a Feynman diagram. The BSA couples to the axial vector vertex $\Gamma_{5 \mu}=\gamma_{5} \gamma_{\mu}$. The momenta $q_{ \pm}$are defined by $q_{+}=q+\eta P$ and $q_{-}=q+(\eta-1) P$. Arrows denote fully dressed quark propagators.

Vector ( $\rho$ ) BSA. With the aid of the Dirac composition for vector particles, we can also determine the corresponding BSA as a sum:

$$
\begin{equation*}
\Gamma_{1^{-}}^{\mu}(p, P)=\sum_{i=1}^{8} F_{i}(p, P) T_{i}^{\mu}(p ; P) \tag{2.47}
\end{equation*}
$$

The canonical normalization condition of the vector BSA reads, similar to the pseudoscalar case,

$$
\begin{equation*}
1=\left.\frac{1}{3} \frac{\mathrm{~d}}{\mathrm{~d} P^{2}} \operatorname{tr} \int_{q}^{\Lambda} \mathrm{T}_{\mu \nu} \bar{\Gamma}^{\nu}(q,-K) S\left(q_{+}\right) \Gamma^{\mu}(q, K) S\left(q_{-}\right)\right|_{P^{2}, K^{2}=-m_{\rho}^{2}} . \tag{2.48}
\end{equation*}
$$

The prefactor of $1 / 3$ stems from averaging over the three different polarizations and $T_{\mu \nu}$ is the transverse projector which will be elucidated in the context of Eq. (2.57), the actual, "truncated" $\rho$-BSA.
The leptonic decay constant is determined by

$$
\begin{equation*}
f_{\rho} m_{\rho}=\frac{Z_{2} N_{c}}{\sqrt{3}} \int_{q}^{\Lambda} \operatorname{tr}\left[\gamma_{\mu} S\left(q_{+}\right) \Gamma^{\mu}(q, P) S\left(q_{-}\right)\right] . \tag{2.49}
\end{equation*}
$$

Here, the color- and the flavour-trace has been already taken, thus, $\operatorname{tr}$ denotes only the Dirac trace.

### 2.5 Truncation Schemes

Solving problems like the dynamical mass generation of quarks and mesons is quite challenging and, especially the latter, is still a subject of latest research. As often the case, truncation schemes can be applied to simplify the calculations. In this thesis, we will reduce the propagation possibilities to a set of terms that is easy to handle. However, by reducing problems in this way, one has to apply truncation schemes that satisfy some important identities, the Ward-Takahashi identities (WTIs). The relevant identity in this thesis is the axial-vector WTI (AVWTI), which ensures e.g. that the effects of chiral sym-
metry breaking remain preserved. It provides a connection between the quantities that have been applied to solve the quark DSE and the quantities that have to be applied to solve the meson BSE. The explicit form of the AVWTI is given by

$$
\begin{equation*}
\gamma^{5} \Sigma\left(q_{-}\right)+\Sigma\left(q_{+}\right) \gamma^{5}=-\int K(p, q ; P)\left(\gamma^{5} S\left(q_{-}\right)+S\left(q_{+}\right) \gamma^{5}\right) \tag{2.50}
\end{equation*}
$$

One can see that the scattering kernel as an ingredient of the BSE, and the self energy of the quark DSE, underlie a definite relation. In any truncation scheme we have to consider this, otherwise problems may arise. We will see that in chapter 3.2 when we implement a little bug into the equations, which violates the WTIs.

### 2.5.1 The Rainbow Truncation and the Contact Interaction

The scheme that will be applied to solve the quark DSE is the "Rainbow truncation", in which the quark is merely allowed to emit a gluon and absorb it equivalently. While doing so, the gluon has to propagate free and without any interactions or reactions except for self interactions. When a gluon is emitted, other gluons can be emitted too, but shall not interact with each other. The name, Rainbow truncation, is easy to comprehend due to the "allowed" Feynman-diagrams, which look like rainbows from a higher order on (see Fig. 8). While the Rainbow Truncation only modifies the quark-gluon vertex, the contact interaction brings additional simplifications and demands a simple interaction scheme, in which the (actually dynamical) coupling strength $\alpha\left(k^{2}\right)$ is set to a constant value.

## Effects of the Rainbow Truncation on the Quark DSE

The expressions for the full gluon propagator $D^{\mu \nu}$ and the quark-gluon vertex $\Gamma_{\nu}$ obey their own DSEs and appear complicated in all their glory. Here, the Rainbow truncation and the contact interaction lead to simplifications.
In Rainbow-truncation, we set [2], [14]

$$
\begin{equation*}
\Gamma_{\nu}(q, p)=\gamma_{\nu} \tag{2.51}
\end{equation*}
$$

which has its base in the twelve independent tensor structures. (see A.2) The full gluon propagator $D^{\mu \nu}$ in Landau gauge is strictly given by [8]:

$$
\begin{equation*}
g^{2} D^{\mu \nu}(k)=\left[\delta^{\mu \nu}-\frac{k^{\mu} k^{\nu}}{k^{2}}\right] \cdot \frac{4 \pi \alpha\left(k^{2}\right)}{k^{2}} \tag{2.52}
\end{equation*}
$$

Furthermore, in contact interaction, we demand the invariant charge $\alpha\left(k^{2}\right)$ to be constant, so $\alpha\left(k^{2}\right) \equiv \alpha$ [6], and modify the interaction so that

$$
\begin{equation*}
g^{2} D^{\mu \nu}(k)=\frac{\delta^{\mu \nu}}{m_{G}^{2}} \tag{2.53}
\end{equation*}
$$

which is the basic property of the contact interaction model.
$m_{G}$ is the "gluon mass scale", which is generated dynamically in QCD and quantifies the coupling strength. The dependence of the quark mass on this quantity will be researched in chapter 3.1.2. Eq. (2.53) simplifies the problem and it preserves the symmetries of our system. However, it entails some contradictory attributes compared to today's QCD, which we will see in chapter 3 .


Figure 8: The quark propagator in Rainbow truncation.

### 2.5.2 The (Rainbow-)Ladder Truncation

The description of mesons by the Bethe-Salpeter formalism can be truncated by modifying the operator $K . K$ is the term that manages the interaction, e.g. the exchange of gluons between the two constituent quarks in the meson. The simplest interaction is a 1 -gluon exchange among the two quarks. To apply such an exchange is called the "Ladder truncation" because, similar to the Rainbow truncation, the interaction has the appearance of a ladder when drawn as a Feynman diagram. This is shown in Fig. 9.
We can combine the two truncation schemes with the "Rainbow-Ladder truncation". Here, the scattering kernel admits a gluon-exchange with the properties set in the Rainbow truncation. As mentioned in the beginning of this chapter, there is a specific connection between the quantities in the DSE and the BSE given by the AVWTI. One conclusion is that the applied approximations for the gluon propagator and the quark-gluon vertex have to be inherited. This touches the quantities 2.51 and 2.53 and the effects of this will be discussed in the following paragraph.


Figure 9: The Bethe-Salpeter equation in Rainbow-Ladder truncation.

## Effects of the Rainbow-Ladder Truncation on the Meson BSE

Similar to the quark DSE, the integral BSE, Eq. (2.36), includes the terms $\Gamma_{\nu}$ and $D^{\mu \nu}$. As discussed in chapter 2.5.2, we have to apply the implemented approximations of the Rainbow truncation and the contact interaction in the way that $\Gamma_{\nu}=\gamma_{\nu}$ and $D^{\mu \nu}=\delta^{\mu \nu} / m_{G}^{2}$. Plugging this into Eq. (2.36) delivers the correlation

$$
\begin{equation*}
\Gamma(P)=-\frac{4}{3 m_{G}^{2}} \int_{q}^{\Lambda} \gamma^{\nu} \chi(q, P) \gamma_{\nu} . \tag{2.54}
\end{equation*}
$$

One can see that we forced $D^{\mu \nu}=$ const., which yields an independence of $\Gamma$ on the relative momentum $k$, for which we can easily set $k=0$. By this, Eq. (2.38) simplifies in so far as the latter two summands vanish and we can reformulate the pion BSA as ([14] [2], A.2)

$$
\begin{equation*}
\Gamma_{0^{-}}=\gamma_{5}\left[i E_{0^{-}}+\frac{\gamma \cdot P}{M} F_{0^{-}}\right] . \tag{2.55}
\end{equation*}
$$

Factors like the imaginary unit $i$ and $1 / M$ are inserted as optional factors and get compensated in the partial amplitudes. Hence, depending on conventions, the values for the amplitudes can differ source-by-source but they represent all the same physics.
Finally, combining Eqs. (2.54) and (2.55) leads us to an eigenvalue equation with the eigenvalue 1,

$$
\binom{E_{0^{-}}}{F_{0^{-}}}=\left(\begin{array}{ll}
\mathcal{K}_{E E} & \mathcal{K}_{E F}  \tag{2.56}\\
\mathcal{K}_{F E} & \mathcal{K}_{F F}
\end{array}\right)\binom{E_{0^{-}}}{F_{0^{-}}},
$$

which can be solved algebraically.* As solutions of the eigenvalue equation, the amplitudes $E_{0^{-}}$and $F_{0^{-}}$depend on the corresponding input quantities. These input quantities are the current quark masses of the two constituents, which are together equivalent to one corresponding pion mass. Therefore, the amplitudes are often stated as $E_{0^{-}}(P)$ and $F_{0^{-}}(P)$


Figure 10: The bound state pseudoscalar pion BSA as a vertex in a Feynman diagram ( $\Gamma_{\pi} \equiv \Gamma_{0^{-}}$). The incoming quark and the incoming antiquark carry the momenta $q_{ \pm}$, with $q_{+}-q_{-}=$ $P$. Based on the momentum conservation at any vertex the absolute pion momentum is given by $P$. The grey helix lines denote the gluon emissions and absorptions which are included in the RLT. Arrows denote fully dressed quark propagators.

[^11]with $P=\left(i m_{0^{-}}, \mathbf{0}\right) \in \mathbb{C} \times \mathbb{R}^{3}$.
The $\rho$-BSA Dirac decomposition, considering the properties already mentioned, yields the following expression [11]:
\[

$$
\begin{equation*}
\Gamma_{\rho}^{\mu}(p, P)=\gamma_{\mathrm{T}}^{\mu} E_{\rho}(P)+\frac{\sigma^{\mu \nu} P_{\nu}}{M} F_{\rho}(P) \tag{2.57}
\end{equation*}
$$

\]

The subscript T denotes the transversalized gamma matrix.* In Rainbow-Ladder truncation $F_{\rho}=0$ holds based on the chosen interaction (2.53). Hence, the complete $\rho$-BSA can be determined by its leading partial amplitude, $E_{\rho}$.

[^12]
## 3 Calculations

### 3.1 Solving the Gap Equation in Rainbow Truncation

### 3.1.1 Mathematical Approach

With the assumptions of the Rainbow approximation, the gap equation (2.27) becomes much simpler (see A.2):

$$
\begin{equation*}
S^{-1}(p)=i \not p+m+\frac{4}{3 m_{G}^{2}} \int_{q}^{\Lambda} \gamma_{\mu} S(q) \gamma^{\mu} \tag{3.1}
\end{equation*}
$$

By substituting $S^{-1}$ with the ansatz from Eq. (2.26), we obtain

$$
\begin{equation*}
i \not p A\left(p^{2}\right)+B\left(p^{2}\right)=i \not p+m+\frac{4}{3 m_{G}^{2}} \int_{q}^{\Lambda} \gamma_{\mu}\left[\frac{-i q A\left(q^{2}\right)+B\left(q^{2}\right)}{q^{2} A^{2}\left(q^{2}\right)+B^{2}\left(q^{2}\right)}\right] \gamma^{\mu} \tag{3.2}
\end{equation*}
$$

Multiplying (3.2) with $-i \not p$ from the left side yields

$$
\begin{equation*}
p^{2} A\left(p^{2}\right)-i \not p B\left(p^{2}\right)=p^{2}-i \not p m-\frac{4}{3 m_{G}^{2}} \int_{q}^{\Lambda} \gamma_{\mu}\left[\frac{-\not p q A\left(q^{2}\right)-i \not p B\left(q^{2}\right)}{q^{2} A^{2}\left(q^{2}\right)+B^{2}\left(q^{2}\right)}\right] \gamma^{\mu} \tag{3.3}
\end{equation*}
$$

To determine the explicit form of $A\left(p^{2}\right)$ and $B\left(p^{2}\right)$, we use the tracelessness of the four gamma matrices, and thus the tracelessness of the Feynman-slashed momentum $\not \varnothing=i \gamma^{\mu} \partial_{\mu}$. We take the trace of Eqs. (3.2) and (3.3), and we integrate the expressions in hyperspherical coordinates*. The results are:

$$
\begin{equation*}
A\left(p^{2}\right)=1 \quad ; \quad B\left(p^{2}\right)=m+\frac{1}{3 m_{G}^{2} \pi^{2}} \int_{0}^{\Lambda^{2}} \mathrm{~d} q^{2} \frac{q^{2} B\left(q^{2}\right)}{q^{2} A^{2}\left(q^{2}\right)+B^{2}\left(q^{2}\right)} \tag{3.4}
\end{equation*}
$$

Recalling the dressing in (2.26) it makes more sense to interpret $B\left(p^{2}\right) / A\left(p^{2}\right)$ as the effective quark mass. So, starting from this point of the thesis, we redefine the quantities $B\left(p^{2}\right) \equiv$ $M\left(p^{2}\right)$ and $m \equiv m_{c}$ due to the fact that $A\left(p^{2}\right)=1$. By combining the expressions in (3.4) we get the self-consistent integral equation for the dynamically generated quark mass ${ }^{\dagger}$ :

$$
\begin{equation*}
M\left(p^{2}\right)=m_{c}+\frac{1}{3 m_{G}^{2} \pi^{2}} \cdot \int_{0}^{\Lambda^{2}} \mathrm{~d} q^{2} \frac{q^{2} M}{q^{2}+M^{2}} \tag{3.5}
\end{equation*}
$$

This integral is a case in point for the divergences of momentum integrals in quantum field theories. ${ }^{\ddagger}$ The insertion of a hard ultraviolet cutoff parameter ensures the validity of the calculations done in terms of this thesis. In this calculation we set $\Lambda=0.873 \mathrm{GeV}$ and $m_{G}=0.132 \mathrm{GeV}$. [15]
To solve this integral equation, we have to employ numerical calculation methods. In this

[^13]thesis, the solution of (3.5) will be found iteratively and the integral will be solved with the Gauß-Legendre integration method, which is discussed in chapter A.3.
In a first calculation, the relation $M\left(m_{c}\right)$ will be analyzed. In a second calculation, we research the effects of a varying gluon mass scale $m_{G}$ on $M$ for fix values of $m_{c}$.
A quantity that will be interesting too, is the chiral quark condensate $\langle\bar{q} q\rangle$. It redefines the QCD vacuum and permits that it is not empty, but rather full of particles. Mathematically, the quark condensate is determined by
\[

$$
\begin{equation*}
-\langle\overline{\mathrm{q}} \mathrm{q}\rangle=\mathcal{N} \cdot \operatorname{tr}_{D} \int_{q}^{\Lambda} S_{\mathrm{chiral}}(q), \tag{3.6}
\end{equation*}
$$

\]

in which $\mathcal{N}=Z_{2} Z_{m} N_{c}$ is a normalization constant, $S_{\text {chiral }}$ is the fully dressed quark propagator in the chiral limit and $\operatorname{tr}_{D}$ is the trace over Dirac indices. [7]
Plugging in Eqs. (2.26), (3.4) and (3.5) and integrating in hyperspherical coordinates delivers*

$$
\begin{equation*}
-\langle\overline{\mathrm{q}} \mathrm{q}\rangle=\left.\frac{\mathcal{N}}{4 \pi^{2}} \int_{0}^{\Lambda^{2}} \mathrm{~d} q^{2} \frac{q^{2} M}{q^{2}+M^{2}}\right|_{m_{c}=0} \tag{3.7}
\end{equation*}
$$

If the chiral symmetry is spontaneously broken, this expression will be non-vanishing ${ }^{\dagger}$, and as a consequence of this (spontaneous) symmetry breaking, the Goldstone theorem predicts bosons to occur in the QCD vacuum.

### 3.1.2 Results

To solve (3.5), the effective mass had been evaluated at 1000 points in the interval $[0,10] \ni$ $m_{c}{ }^{\ddagger}$ with a tolerance range ${ }^{\S}$ of $\varepsilon=10^{-7} \mathrm{GeV}$. The plot of $M$ is shown in Fig. 11.
One sees that the effective quark masses are much larger than the current quark masses (of light quarks). The plot of the ratio $M\left(m_{c}\right) / m_{c}$ is also shown in Fig. 11 and supports the findings. For the lightest quarks, the $u$ and the $d$ quark, the effective quark mass is about 100 to 1000 times of the current quark mass. That correlates with the assertion in the introduction claiming that approximately $99 \%$ of the hadronic mass is generated dynamically.
We did not consider heavier quarks like the $\mathrm{s}, \mathrm{c}, \mathrm{b}$, t quarks yet, so we have to keep in mind that the hadronic mass is not only constituted of light quarks, but in fact, the greater the quark mass is, the lower is the relative effect of this dynamic generation.
It is fairly interesting to see that even in the chiral limit the (current-)massless quarks get an effective mass of round about 355 MeV . The results, finally, lead us to the question of why the effective quark mass is so much higher than the current quark mass. To research that, one can vary the quantity that regulates the interaction strength.

[^14]

Figure 11: Left panel: The relation $M\left(m_{c}\right)$ for current quark masses $m_{c} \in[0,10] \mathrm{MeV}$. Right panel: The ratio $M\left(m_{c}\right) / m_{c}$ in a logarithmic scale.


Figure 12: The relation between $M$ and $m_{G}^{-2}$ with fix values for $m_{c}$.
Left panel: Overview for $m_{G}^{-2} \in[0,70] \mathrm{GeV}^{-2}$. Right panel: Overview for a small coupling, $m_{G}^{-2} \rightarrow 0$, or equivalently $m_{G} \rightarrow \infty$. As one would expect for a vanishing coupling, the effective quark mass is equal to the current quark mass mass.

As mentioned earlier, the gluon mass scale $m_{G}$ can cause this interaction strength and influences the solution of Eq. (3.5). In previous calculations this quantity got a fixed value ( $m_{G}=0.132 \mathrm{GeV}$ ) which corresponds to the experimental results for the masses and decay constants, which we will calculate in the following chapters. At this point, it is interesting to investigate how this factor influences with the solution, therefore, it makes sense to vary the prefactor $m_{G}^{-2}$, which occurs in Eq. (3.5). The result of an equivalent procedure as in the previous calculation, just with a various gluon mass scale $m_{G}$ and fix values for $m_{c}$, delivers the dependence that is plotted in Fig. 12. The right plot is cut out of the left plot for a large gluon mass scale.
One can readily see that for $m_{G}^{-2} \rightarrow 0$, or equivalently $m_{G} \rightarrow \infty$, the effective quark mass is equal to the current quark mass. For $m_{G}^{-2}>0$, we can distinguish two cases. Going to the chiral limit, the mass generation does not start until a specific value of $m_{G}^{-2}=38.4 \mathrm{GeV}^{-2}$ that corresponds to a value $m_{G} \approx 161.896 \mathrm{MeV}=: m_{G}^{\text {crit }}$, so we can summarize for the
chiral limit:

$$
\begin{cases}m_{G} \lesssim m_{G}^{\text {crit }} & : \text { mass is generated dynamically. }  \tag{3.8}\\ m_{G} \gtrsim m_{G}^{\text {crit }} & : M=m_{c}=0\end{cases}
$$

Going away from the chiral limit, we observe a dynamically generated mass even for values $m_{G} \lesssim m_{G}^{\text {crit }}$, but we can see that for small values of $m_{G}^{-2}$ the additional mass due to the dynamical mass generation is only a small fraction of the sum of the quark masses. This fraction rises for higher current quark masses as we can see in Fig. 12. For a gluon mass scale $m_{G} \gtrsim m_{G}^{\text {crit }}$ the generated mass constitutes the (by far) greatest part of the effective mass.


Figure 13: The quark condensate $-\langle\overline{\mathrm{q}} \mathrm{q}\rangle$ as a function of $m_{G}^{-2}$ up to the normalization factor $\mathcal{N}$ in the chiral limit ( $m_{c}=0$ ).

To interpret what happens at this critical value of $m_{G}$, we determine the quark condensate $\left\langle\overline{\mathrm{q}}\right.$ 〉 as a function of $m_{G}^{-2}$. The corresponding plot* is shown in Fig. 13.
Similar to Fig. 12, the quark condensate is equal to zero for $m_{G} \gtrsim m_{G}^{\text {crit }}$, but rises instantaneously starting from this critical value, which implies that the QCD vacuum changes dramatically. A condesate occurs, and thermodynamically this represents a phase transition. Regarding the discussions in chapter 2.2 .2 , these sections are called "chiral symmetric" $\left(m_{G} \gtrsim m_{G}^{\text {crit }}\right)$ and "chiral broken" $\left(m_{G} \lesssim m_{G}^{\text {crit }}\right)$. One conclusion of that is that the initial conditions of our universe set that the hadron masses are like they are. In an alternative universe, in which the coefficient $m_{G}$ would look different, the hadron masses consisting of light quarks would differ too, and in a hypothetic universe in which $m_{G}>m_{G}^{\text {crit }}$, the chiral symmetry would not be broken spontaneously and the hadron masses would be almost consistent with the sum of their current quark masses. As mentioned before, $m_{G}$ had been set to a constant value for the calculations of the real effective quark masses, which corresponds with the experiment, as well as with the fact that $m_{G}^{\exp }<m_{G}^{\text {crit }}$ verifies the theory implying that chiral symmetry is actually broken spontaneously.
On the basis of this knowledge, it becomes obvious why the occurring Goldstone bosons

[^15]have to be pions. As a consequence of the Goldstone theorem of massless fermions the Goldstone bosons have to be massless too. The fact that the light quarks are not massless* can be brought in as a perturbation. Indeed, this perturbation is slight, therefore it seems reasonable that the effect on the bosons is small too. Thus, recalling the introduction, the only logical boson that can be built out of the quark condensate is the (by far) lightest hadronic boson, the (pseudoscalar) pion, which is composed of u- and d-quarks.

[^16]
### 3.2 Solving the Pseudoscalar Bethe-Salpeter Equation in Rainbow-Ladder Truncation

### 3.2.1 Mathematical Approach

As mentioned in chapters 2.4 and 2.5.2, the Bethe-Salpeter equation is an eigenvalue equation to the eigenvalue 1 , which can be expressed with the aid of the $\mathcal{K}$-matrix (Eq. 2.56 ) and it can be solved algebraically in the truncation used in this work. As in chapter 3.1.1, the solution will be derivated numerically. The four matrix elements $\mathcal{K}_{i j}$ (cf. A.2),

$$
\begin{align*}
& \mathcal{K}_{E E}=-4 \cdot \mathcal{N} \iint_{q, z}^{\Lambda} \frac{\left(q_{+} \cdot q_{-}\right)+M^{2}}{\left(q_{+}^{2}+M^{2}\right)\left(q_{-}^{2}+M^{2}\right)}  \tag{3.9}\\
& \mathcal{K}_{E F}=4 m_{0^{-}}^{2} \cdot \mathcal{N} \iint_{q, z}^{\Lambda} \frac{1}{\left(q_{+}^{2}+M^{2}\right)\left(q_{-}^{2}+M^{2}\right)}  \tag{3.10}\\
& \mathcal{K}_{F E}=-2 M^{2} \cdot \mathcal{N} \iint_{q, z}^{\Lambda} \frac{1}{\left(q_{+}^{2}+M^{2}\right)\left(q_{-}^{2}+M^{2}\right)}  \tag{3.11}\\
& \mathcal{K}_{F F}=2 \cdot \mathcal{N} \iint_{q, z}^{\Lambda} \frac{M^{2}-m_{0^{-}}^{-2} \cdot\left(2\left(q_{+} \cdot P\right)\left(q_{-} \cdot P\right)+m_{0^{-}}^{2}\left(q_{+} \cdot q_{-}\right)\right)}{\left(q_{+}^{2}+M^{2}\right)\left(q_{-}^{2}+M^{2}\right)} \tag{3.12}
\end{align*}
$$

are integral equations and can also be solved with the Gauss-Legendre method.
Here, the quantities $q_{ \pm}$are the momenta the constituents carry. Fig. $14^{*}$ shows the BSE as a Feynman diagram including all momenta which are involved. We see, based on momentum conservation on any vertex, that $q_{+}-q_{-}=P$ with $P=\left(i m_{\pi}, \mathbf{0}\right)$. The quantity $q$ is an indeterminate loop momentum, which cancels out at the vertex, thus, this is the momentum we have to integrate over. Since the momentum routing parameter $\eta$ only specifies the amount of the pion momentum the constituents carry, it doesn't touch the total momentum, and hence, should not influence the physics.

We demand real on-shell particles which satisfy the relativistic energy-mass relation $P^{2}=-m_{\pi}^{2}$ by definition. ${ }^{\dagger}$ In this form, the solution space is two dimensional in the $M-m_{\pi}$-surface. $M$ as the "full" quark mass can also be expressed with the aid of the cur-


Figure 14: The meson BSE including all momenta.

[^17]

Figure 15: The collapse of the solution space of the homogeneous pion BSE due to the eigenvalue condition. Left panel: The yellow surface denotes the solution space, the orange line denotes the solutions that satisfy the eigenvalue condition (3.13). Right panel: The solution space collapsed into an one-dimensional space. The red crosses denote values of $m_{c}$, which will be used in the calculation to determine the $m_{\pi}-m_{c}$-relation.
rent quark mass $m_{c}$, in a way that the solution space blends into the $m_{c}-m_{\pi}$-surface. An additional condition which is imposed by Eq. (2.56) is the eigenvalue condition,

$$
\begin{equation*}
\chi\left(m_{\pi}, m_{c}\right)=\operatorname{det}(\mathcal{K}-\mathbb{1}) \stackrel{!}{=} 0, \tag{3.13}
\end{equation*}
$$

which demands the characteristic polynomial to vanish, and ensures that the solution space collapses further into a one dimensional space that is isomorphic to the $m_{c}$-ray. Schematically, this is illustrated in Fig. 15. For this, a simple root-finding algorithm, the bisection method, is applied. Consequently, the plot of $m_{\pi}\left(m_{c}\right)$ will be analyzed. For the values of $M$, the constituent mass, we have to consult the results from chapter 3.1.2. The result will be checked with the aid of some numerical methods which lead to the eigenvalues of the matrix.
Despite the fact that the momentum routing parameter $\eta$ should not influence the physics, we don't fix it, but rather vary it during the calculations.
As mentioned in chapter 2.4, physical observables can only be determined by the BSA when it's normalized. Thus, we have to use the derived matrix elements of $\mathcal{K}$ to make the amplitudes $E_{(\pi)}$ and $F_{(\pi)}$ satisfy Eq. (2.39). This is possible because every eigenvalue has its own appropriate eigenspace. Because $\mathcal{K} \in \mathbb{R}^{2 \times 2}$ and only one of the eigenvalues is equal to 1 , the eigenspace is a one dimensional subspace of the $\mathbb{R}^{2}$. The eigenvalue condition lets the eigenspace collapse into a well-defined 2-tuple that represents the normalized BSA. Evaluating Eq. (2.39) leads to the following equation that has to be solved:

$$
\begin{align*}
1= & \frac{3}{P \pi^{3}} \frac{\mathrm{~d}}{\mathrm{~d} P} \iint_{q, z}^{\Lambda}\left\{-M^{2}-\left(q_{+} \cdot q_{-}\right) \cdot E^{2}-2(P \cdot K) \cdot E F\right. \\
& \left.+\left[K^{2}+M^{-2} \cdot\left(2\left(q_{+} \cdot K\right)\left(q_{-} \cdot K\right)-K^{2}\left(q_{+} \cdot q_{-}\right)\right)\right] \cdot F^{2}\right\} \\
& \times\left[\left(q_{+}^{2}+M^{2}\right)\left(q_{-}^{2}+M^{2}\right)\right]^{-1} \tag{3.14}
\end{align*}
$$

For the derivation and a solution technique, A. 2 can be consulted.
With a normalized BSA we can calculate the leptonic decay constant $f_{\pi}$. Bringing Eqs.
(2.43) and (2.44) into a computable form*, we obtain

$$
\begin{equation*}
f_{\pi}=\frac{3}{\pi^{3}} \iint_{q, z}^{\Lambda} \frac{M E-\left(M+M^{-1} P^{-2}\left(2\left(q_{+} \cdot P\right)\left(q_{-} \cdot P\right)-P^{2}\left(q_{+} \cdot q_{-}\right)\right)\right) F}{\left(q_{+}^{2}+M^{2}\right)\left(q_{-}^{2}+M^{2}\right)} \tag{3.15}
\end{equation*}
$$

and, for $r_{\pi}$,

$$
\begin{equation*}
r_{\pi}=\frac{3}{\pi^{3}} \iint_{q, z}^{\Lambda} \frac{\left(M^{2}+\left(q_{+} \cdot q_{-}\right)\right)+P^{2} F}{\left(q_{+}^{2}+M^{2}\right)\left(q_{-}^{2}+M^{2}\right)} \tag{3.16}
\end{equation*}
$$

One can verify the relations (2.45) and (2.46) in the following way: We define the functions

$$
\begin{equation*}
H\left(m_{c}\right)=\left|f_{\pi} m_{\pi}^{2}-2 m_{c} r_{\pi}\right| \quad \text { and } \quad G\left(m_{c}\right)=\left|f_{\pi}^{2} m_{\pi}^{2}+2 m_{c}\langle\overline{\mathrm{q}} \mathrm{q}\rangle / N_{f}\right| \tag{3.17}
\end{equation*}
$$

and, w.l.o.g., the left-hand side of both equations:

$$
\begin{equation*}
H_{1}\left(m_{c}\right)=\left|f_{\pi} m_{\pi}^{2}\right| \quad \text { and } \quad G_{1}=\left|f_{\pi}^{2} m_{\pi}^{2}\right| \tag{3.18}
\end{equation*}
$$

Assuming that (2.45) and (2.46) hold, it follows directly that $H=0$ and $G=0$ for any $m_{c}$. Since $H_{1}$ is one summand of $H$ and $G_{1}$ is one summand of $G$, both of them have to be cancelled out by their counterpart, the other summand. To give an example: $H=0$, if $H_{1}=\left|f_{\pi} m_{\pi}^{2}\right| \stackrel{!}{=}\left|2 m_{c} r_{\pi}\right|$. Thus, if the dimension of $H$ is much smaller than the dimension of $H_{1}$ (analogous for $G$ and $G_{1}$ ), we can conclude that the associated relations are verified in a good approximation.

### 3.2.2 Results

The pion mass as a function of the current mass, is shown graphically in Fig. 16 in both momentum routings. We can see that in the chiral limit the pions are, as the bosons the Goldstone theorem predicts, namely massless and define the QCD vacuum for light quarks. Apparently, a rising fermion mass larger than 0 covaries with a quickly gaining pion mass, so a very small quark current mass lasts for a pion, which is more massive than its "bare" constituents. An interesting fact is that in this model the momentum routing parameter influences the pion mass. In more general models, one can show that the parameter cannot influence the physics in reality. Interestingly, in the asymmetric case, for a quark mass $m_{c}$ of round about 7.8 MeV , we obtain the experimental value of a (charged) pion, 140 MeV . This does not accord with the well known quark masses of the up- and down-quarks $m_{\mathrm{u}}$ and $m_{\mathrm{d}}$ but needs to be understood as an effect of the chosen coupling in Eq. (2.53). The input quantities have to be chosen so that the experimental values can be reconstructed. Theories that dip deeper in the actual QCD get much closer to the experimentally observed masses and decay constants. [16]
As mentioned in chapter 2.5 , it is interesting what happens when we implement a little bug into our equations so that the WTIs (here: the AVWTI) are not satisfied anymore. We do this by transforming $\Gamma_{\mu}^{\mathrm{qg}}=\gamma_{\mu} \rightarrow 0.9 \gamma_{\mu}$. The result is plotted in Fig. 17 and shows

[^18]

Figure 16: The pion mass $m_{\pi}$ as a function of the "bare" mass of the single (anti-)quark. Black graph: The pion mass in a symmetric momentum routing: $\eta=1$.
Red graph: The pion mass in an asymmetric momentum routing: $\eta=0.5$.
Blue graph: The difference between the pion masses for different momentum routings. Despite the fact that the momentum routing parameter must not influence the physics, it does in this approximation.
us that a correct modeling of the chiral limit is not given anymore. The pions would be massive for a current quark mass $m_{c}=0$ and thus cannot be the Goldstone bosons of the QCD of light quarks anymore. But as seen by analyzing the quark DSE, chiral symmetry is definitely broken, which has to imply massless bosons according to the Goldstone theorem, and since we consider only two flavors, the pion is the single logical candidate for being this boson. That verifies, that the AVWTI ensures that the effects of chiral symmetry breaking remain conserved.
For different momentum routings we also obtain different decay constants $f_{\pi}$. These decay constants are shown in Fig. 18 and are plotted against the current quark mass $m_{c}$. We see that from 0 to 10 MeV the decay just rises by round about $0.63 \%$, which shows a weak dependence of $f_{\pi}$ on $m_{c}$.
Relations (2.45) and (2.46) let us interpret the calculated values regarding their validity, when we assume that both relations hold for the correct physical quantities. By comparing the functions $H$ and $H_{1}$, and respectively $G$ and $G_{1}$ (Fig. 19), it turns out that only the asymmetric momentum routing satisfies the relations in good approximation. Thus, the asymmetric routing seems to be a better choice when reproducing physical quantities which underlie the concept of dynamical chiral symmetry breaking. The fact that the "dimension difference" decreases for quark masses $\gtrsim 10 \mathrm{MeV}$ is not that surprising against the backdrop of the range of validity of this model, because it is just able to describe light quarks.
The amplitude is canonically normalized for $E=3.421$ and $F=0.497$, hence, $E$ is the leading amplitude. Therefore, it is not that surprising that a BSA which is solely


Figure 17: The pion mass $m_{\pi}$ as a function of the current quark mass with breaching the WTIs due to a weak bug implemented within the quark-gluon vertex. Asymmetric momentum routing is implemented.
determined by $E$ leads to significant approximations of the physical quantities too. To give an example: In the asymmetric case, the pion mass for an unmodified current quark mass, $m_{c}=7.8 \mathrm{MeV}$, would be 123.1 MeV , which corresponds to an error of only $12 \%$ when compared to the physical value of $m_{\pi}$.


Figure 18: The decay constant and its weak dependence on the current quark mass. The different momentum routings bring in an absolute difference of 0.0095 MeV or, equivalently, $9 \%$.


Figure 19: Plots of the functions $H, H_{1}, G$ and $G_{1}$ for different momentum routings in a doublelogarithmic scale. L.h.s.: The results for $\eta=1$. ("asymmetric case") R.h.s.: The results for $\eta=0.5$. ("symmetric case")
Comparable results were plotted in the same scale. One can see that only in the asymmetric case both functions differ by more than two dimensions, thus, in this case the relations (2.45) and (2.46) also hold.

### 3.3 Solving the Vector Bethe-Salpeter Equation in the Rainbow-Ladder Truncation

### 3.3.1 Mathematical approach

The contact interaction (2.53) yields that the leading amplitude determines the bound state by itself, so the vectorial BSA is determined by

$$
\begin{equation*}
\Gamma_{\rho}^{\mu}(p ; P)=\gamma_{\mathrm{T}}^{\mu} E_{\rho}(P) \tag{3.19}
\end{equation*}
$$

and it satisfies the homogeneous BSE for bound states,

$$
\begin{equation*}
\Gamma_{\rho}^{\mu}=-\frac{4}{3 m_{G}^{2}} \int_{q}^{\Lambda} \gamma_{\nu} \chi_{\rho}^{\mu} \gamma^{\nu} \tag{3.20}
\end{equation*}
$$

Multiplying $\gamma_{\mu}$ by the left side yields

$$
\begin{equation*}
f\left(m_{\rho}, m_{c}, \Lambda, m_{G}\right)=-3+\frac{1}{3 m_{G}^{2} \pi^{3}} \iint_{q, z}^{\Lambda} \frac{6 M^{2}+2\left(q_{+} \cdot q_{-}\right)+4\left(q_{+} \cdot \hat{P}\right)\left(q_{-} \cdot \hat{P}\right)}{\left(q_{+}^{2}+M^{2}\right)\left(q_{-}^{2}+M^{2}\right)} \stackrel{!}{=} 0 \tag{3.21}
\end{equation*}
$$

To analyze the ability of the Rainbow-Ladder-truncated theory to determine the bound state of the $\rho$-meson in a good approximation, we distinguish again between the two momentum routings and check if the asymmetric one, which determines the phenomena of $\mathrm{D} \chi \mathrm{SB}$, and/or the symmetric one yields good values for the $\rho$-meson mass. For that we plot $f\left(m_{\rho}\right)$ with constant values of $m_{c}, \Lambda$ and $m_{G}$ and look for a root that provides a bound state.
The normalization condition (2.48) reads, computed with the expression (3.19),

$$
\begin{equation*}
1=\frac{1}{P \pi^{3}} \frac{\mathrm{~d}}{\mathrm{~d} P} \iint_{q, z}^{\Lambda} \frac{3 M^{2}+\left(q_{+} \cdot q_{-}\right)+2\left(q_{+} \cdot \hat{P}\right)\left(q_{-} \cdot \hat{P}\right)}{\left(q_{+}^{2}+M^{2}\right)\left(q_{-}^{2}+M^{2}\right)} \cdot E^{2} \tag{3.22}
\end{equation*}
$$

Once the BSA is normalized, the decay constant can be calculated by the evaluated form of (2.49), which is given by

$$
\begin{equation*}
f_{\rho}=\frac{Z_{2}}{m_{\rho} \pi^{3}} \iint_{q, z}^{\Lambda} \frac{3 M^{2}+\left(q_{+} \cdot q_{-}\right)+2\left(q_{+} \cdot \hat{P}\right)\left(q_{-} \cdot \hat{P}\right)}{\left(q_{+}^{2}+M^{2}\right)\left(q_{-}^{2}+M^{2}\right)} \cdot E \tag{3.23}
\end{equation*}
$$

### 3.3.2 Results and Further Approach

Choosing the coupling strength $m_{G}$ and the cutoff $\Lambda$ like it's done in previous calculations and set $m_{c}$ to a value that yields a suitable pion mass $(\simeq 140 \mathrm{MeV})$ leads to the plots for $f$ shown in Fig. 20. One can see that only the symmetric routing yields a bound state for the $\rho$-meson, whereas the asymmetric routing does not produce a bound state. Interestingly, the bound state in the symmetric routing provides a good approximation of the actual value of the $\rho$-meson, whereas the asymmetric routing does not. Hence, the model in this form cannot determine the effects of $\mathrm{D} \chi \mathrm{SB}$ and the $\rho$-meson mass at once. We note that


Figure 20: Left panel: $f\left(m_{\rho}\right)$ for $\eta=1$ and $m_{c}=7.8 \mathrm{MeV}$. The function does not have a root in the probed range of $[0,1200] \mathrm{MeV}$, thus a bound state does not exist.
Right panel: $f\left(m_{\rho}\right)$ for $\eta=0.5$ and $m_{c}=6.6 \mathrm{MeV}$. The function has a root at $m_{\rho} \approx$ 734 MeV , which accords in a good approximation with the actual value ( 775.26 MeV ).
the actual pion decay constant of 93 MeV [5] is not reached in both cases, but it is rather a rough approximation. (see Fig. 18)

One idea to customize the model is to vary the input parameters $m_{c}, m_{G}$ and $\Lambda$, which had been chosen before, to rectify the output values $m_{\rho}, m_{\pi}$ and $f_{\pi}$. However, the asymmetric case, which reproduces the chiral limit correctly, leads to some problems. After varying the input parameters systematically, it turns out that the decay constant and the $\rho$-meson mass cannot be brought to their actual values at once. This has to do with the circumstance that the decay constant is strongly dependent on the coupling strength here and merely weakly dependent on the other input parameters, whereas the $\rho$-meson mass is strongly dependent on the coupling strength and the cutoff parameter. It follows that both values only covary in a positive way, which means that a raise of one quantity results in a raise of the other quantity. To give an example of what this could mean: By ensuring ideal values for $m_{\pi}$ and $f_{\pi}$, a lower bound for the $\rho$-meson mass is given by 1894 MeV .
To proceed, we take a closer look at the symmetric case $(\eta=0.5)$ and hazard the consequences that the chiral limit and the $\rho$-meson properties cannot be reproduced at once. Although we get a good approximation for $m_{\rho}$ and $m_{\pi}$ using the input parameters $m_{c}$, $m_{G}$ and $\Lambda$ as in other cases before, we vary these parameters systematically like we did searching for bound states in the asymmetric case to fix the errors occurring in the calculated quantities, e.g. $f_{\pi}$, as good as we can. A possible choice that reproduces excellent approximations is given by Tab. 1.

| input triple | output triple |
| :---: | :---: |
| $m_{c}=7.9 \mathrm{MeV}$ | $m_{\pi}=139.9 \mathrm{MeV}$ |
| $m_{G}=0.094 \mathrm{GeV}$ | $f_{\pi}=93.0 \mathrm{MeV}$ |
| $\Lambda=0.709 \mathrm{GeV}$ | $m_{\rho}=773.4 \mathrm{MeV}$ |

Table 1: The input and output triples in the calculations of the $\rho$-meson mass for a symmetric momentum routing $(\eta=0.5)$. The output triple provides an excellent approximation of the actual values, which are $m_{\pi}=139.57 \mathrm{MeV}, f_{\pi}=93 \mathrm{MeV}$ and $m_{\rho}=775.25 \mathrm{MeV}$.

With these values we can try to normalize the vector-BSA with Eq. (3.22). By doing this, it turns out that the function $E^{-2}\left(m_{\rho}\right)$ is negative definite for meson masses of $m_{\rho} \lesssim 0.873 \mathrm{GeV}$ and it oscillates heavily for $\rho$-masses above this value. Therefore, the BSA is either imaginary or not stable under small mass variations. Going further and choosing the asymmetric momentum routing yields a similar result. Here, the function oscillates heavily until about 800 MeV , it is stable up to 950 MeV and then falls instantaneously into the area of negative values, which would yield again that $E_{\rho} \notin \mathbb{R}$. These functions are imaged in the upper panel of Fig. 21. Based on this, we can conclude that by choosing the contact interaction, the "hard cutoff model" is not able to yield a normalized vector-BSA in the sense of Eq. (3.23).
It is quite interesting that other research groups obtain bound states which come much closer to the actual value of $m_{\rho}$ by using the same truncation scheme. An example is the one that is used in sources [2] and [11], in which the bound state for the $\rho$-meson appears at $m_{\rho}=928 \mathrm{MeV}$ for the asymmetric momentum routing. This difference arises from the varying regularization schemes. Based on the divergence of the integral for the quark mass, we introduced the hard cutoff parameter $\Lambda$, whereas in [2] the authors implemented exponential functions which let the integrand attenuate, so that the integral converges without touching the bounds of integration:

$$
\begin{equation*}
\frac{1}{s+M^{2}} \xrightarrow{\text { regularization }} \frac{\mathrm{e}^{-\left(s+M^{2}\right) \tau_{\mathrm{uv}}^{2}}-\mathrm{e}^{-\left(s+M^{2}\right) \tau_{\mathrm{ir}}^{2}}}{s+M^{2}} \tag{3.24}
\end{equation*}
$$

Implementing this regularization into Eq. (3.21) instead of the hard cutoff indeed changes the $m_{\rho}$-dependence of $f$ dramatically and yields a bound state at $m_{\rho}=900.4 \mathrm{MeV}$. (see Fig. 22)
Based on the rough accordance with the results of the authors of [2], we can implement the new regularization into our normalization condition (3.22). It turns out that in both routings we get a smooth and slowly varying function, but neither in the asymmetric nor in the symmetric case the BSA becomes real because $E^{-2}\left(m_{\rho}\right)<0$ for all $m_{\rho}$ in the attractive range of 0.5 GeV up to 1.0 GeV . These functions appear in the middle panel of Fig. 21. Surprisingly, the momentum routing has almost no influence on the normalization condition in this regularization. This contrasts with every calculation done so far and represents a special property of this regularization, because it seems that it resolves the contradiction that the relative momentum influences the physics.


Figure 21: The different behaviour of the function $1 / E^{2}$ after implementing different regularization schemes. Upper left: hard cutoff, asymmetric routing $(\eta=1)$ and standard input parameters. Upper right: hard cutoff, symmetric routing ( $\eta=0.5$ ) and optimized input parameters. Middle left: implemented exponential regularization, asymmetric routing, standard input parameters. Middle right: implemented exponential regularization, asymmetric routing, standard input parameters. Bottom: regularization scheme and input setting by [2], resp. [11]. Only the bottom behaviour is attractive due to its slow variation, which yields stable values.


Figure 22: The function $f\left(m_{\rho}\right)$ dependent on the rho quark mass. The function has a root at $m_{\rho}=900.4 \mathrm{MeV}$ and indicates a bound state for this mass.

Finally we can try to reproduce the normalization used in [2] and [11] in the exact way the authors did. The corresponding condition reads*

$$
\begin{equation*}
\frac{1}{E_{\rho}^{2}}=\frac{3}{\pi^{2}} \int_{0}^{1} \mathrm{~d} \alpha \alpha(1-\alpha) \int_{0}^{\infty} \mathrm{d} q^{2} q^{2} \frac{\mathrm{~d}}{\mathrm{~d} P^{2}} P^{2} \frac{\mathrm{~d}}{\mathrm{~d} \omega} \frac{\mathrm{e}^{-\left(q^{2}+\omega\right) \tau_{u v}^{2}}-\mathrm{e}^{-\left(q^{2}+\omega\right) \tau_{i r}^{2}}}{q^{2}+\omega} . \tag{3.25}
\end{equation*}
$$

Evaluating this yields $E_{\rho}=1.648 \cdot i \in \mathbb{C}$, which in RL-truncation determines the full vector-BSA. Unlike we have done before, we do not label this solution as a nonphysical result because it accords, up to the factor of $i$, with the results in the associated sources in good approximation. If the imaginary factor stems from a missing minus in the expression given in [2], our amplitude would be fine. Implementing the exact input parameters used in these sources indeed accords, up to the same factor of $i$, exact with the results of the paper too. Thus, we assume carefully that there is a typographical error. For the sake of completeness, the plot of $E_{\rho}^{-2}\left(m_{\rho}\right)$ in the sense of (3.25) is imaged in the bottom panel of Fig. 21.
Since only the regularization scheme applied in [2] and [11] provides suitable quantities, we are forced to continue using it instead of the 'hard cutoff'. As we have done it while dealing with the pion, the calculation of the leptonic decay constant follows. Here, we can contrast the corresponding expression for $f_{\rho}$ in Eq. (3.23) with the published one in [11], which reads

$$
\begin{equation*}
f_{\rho}=-\left.\frac{9 E_{\rho}}{2 m_{\rho}} \cdot K_{\gamma}\left(P^{2}\right)\right|_{P^{2}=-m_{\rho}^{2}} \tag{3.26}
\end{equation*}
$$

The explicit form of $K_{\gamma}$ can be consulted in A.2, especially in the derivation of (3.25). Taking a closer look at the ingredients of this definition - or being more precise, their

[^19]units - exhibits that it cannot be correct.* Hence, we have to look at the results we get by using (3.23). Again, we distinguish between the hard cutoff and the regularization with the aid of exponential functions and we find that, similar to the calculations before, both regularizations yield different values. With the hard cutoff we obtain $f_{\rho}=0.0448 \mathrm{GeV}$, whereas the exponential regularization yields $f_{\rho}=0.0162 \mathrm{GeV}$. Compared to the value obtained in $[11]^{\dagger}$ these results are too small. These discrepancies will be analyzed in further research, but, for time reasons, not in terms of this thesis.

[^20]
### 3.4 Calculation of the Hadronic Coupling Constant of the $\rho \rightarrow \pi \pi$ Decay

### 3.4.1 Mathematical Approach

The hadronic decay $\rho \rightarrow \pi \pi$ as one of the simplest decays we can treat in QCD has already been mentioned in chapter 1 . We are looking for a diagram which can determine this decay in a simple way at most. Such a diagram is given by a fermion (quark) triangle diagram like the one that is shown in Fig. 23. This triangle includes a fermion loop of quarks and determines the decay in a highly plain way. Because of the preparations in this thesis by treating all phenomena in RL-truncation, this triangle includes several possibilities of the quarks to emit and to absorb gluons. Even the simple 1-gluon-exchange between two quarks is included by the Ladder truncation.
The incoming $\rho$-momentum is given by $Q$; the outgoing pion momenta are given by $p_{1}$ and $p_{2}$ with $p_{1}+p_{2}=Q$ due to momentum conservation at every vertex. Furthermore, we define an additional momentum $P=1 / 2\left(p_{1}-p_{2}\right)$ so that the three quark momenta can be written as

$$
\begin{equation*}
q=k+P / 2 \quad ; \quad q_{+}=k-P / 2+Q / 2 \quad ; \quad q_{-}=k-P / 2-Q / 2 \tag{3.27}
\end{equation*}
$$

$k$ is the indeterminate loop momentum, over which we have to integrate. Based on the Feynman rules, we can write this vertex as [16]

$$
\begin{equation*}
\Lambda_{\mu}(Q, P)=\operatorname{tr}_{\mathrm{sc}} \int_{k}^{\Lambda} S(q) \Gamma\left(q, q_{+}\right) S\left(q_{+}\right) \Gamma_{\mu}\left(q_{+}, q_{-}\right) S\left(q_{-}\right) \Gamma\left(q_{-}, q\right) \tag{3.28}
\end{equation*}
$$

In this expression, we only implement the leading pion BSA $E_{\pi}$. As shown in chapter 3.2.2, this quantity provides good approximations even without $F_{\pi}$. We treat the $\rho^{0} \rightarrow \pi^{+} \pi^{-}$ decay and the corresponding state of the uncharged $\rho^{0}$ is given by $|\rho\rangle=(|u \bar{u}\rangle-|d \bar{d}\rangle) / \sqrt{2}$.


Figure 23: The hadronic decay $\rho \rightarrow \pi \pi$ as a Feynman diagram. Thin lines denote quark propagators, grey helix lines denote possible gluon emissions/absorptions included in Rainbow-Ladder truncation.

Similarly we have to write the vertex in an equivalent form*:

$$
\begin{equation*}
\Lambda_{\mu}^{\rho^{0}}(Q, P)=\frac{\Lambda_{\mu}(Q, P)-\Lambda_{\mu}(Q,-P)}{\sqrt{2}} \tag{3.29}
\end{equation*}
$$

The coupling constant $g$ of the strong decay is defined by [16]

$$
\begin{equation*}
\Lambda_{\mu}^{\rho^{0}}=2 P_{\mu}^{\mathrm{T}} \cdot g_{\rho \rightarrow \pi \pi} \tag{3.30}
\end{equation*}
$$

A clever choice regarding the momenta is important, because it simplifies the integrals so that we can integrate, in this case, over one angle trivially. We set our coordinate system in a way in which the $\rho$-meson is in rest, hence $Q=\left(i m_{\rho}, \mathbf{0}\right)$, which is nothing else than the center of mass frame. To solve the integrals, we partition the momenta equally to both pions after the decay, so the pions have equal energies $p_{j}^{0}$ and subtended three-momenta in the $x_{1}$-direction. So, we choose

$$
\begin{equation*}
p_{1}=\left(i m_{\rho} / 2, a, 0,0\right) \quad \text { and } \quad p_{2}=\left(i m_{\rho} / 2,-a, 0,0\right) \tag{3.31}
\end{equation*}
$$

which yields

$$
\begin{equation*}
P=(0, a, 0,0) \quad \text { and } \quad a=\sqrt{m_{\rho}^{2} / 4-m_{\pi}^{2}} \tag{3.32}
\end{equation*}
$$

because $p_{j}^{2}=-m_{\pi}^{2}$ for any pion momentum.
We get an explicit expression for the hadronic coupling constant $g$ by applying the projector $\Pi=P_{\mathrm{T}}^{\mu} /\left(2 P_{\mathrm{T}}^{2}\right)$ on both sides of Eq. (3.30). From $P \cdot Q=0$ it follows immediately that $P_{\mathrm{T}}^{\mu}=P^{\mu}$. This yields the following expression for the coupling constant:

$$
\begin{equation*}
g_{\rho \rightarrow \pi \pi}=\frac{\Lambda(Q, P) \cdot P-\Lambda(Q,-P) \cdot P}{\sqrt{8} \cdot P^{2}} \tag{3.33}
\end{equation*}
$$

Explicitly, the expressions $\Lambda(Q, \pm P) \cdot P$ read $^{\dagger}$

$$
\begin{align*}
\Lambda(Q, \pm P) \cdot P=\frac{i}{2 \pi^{3}} \cdot N_{\mathrm{c}} \iiint_{k, z, y}^{\Lambda} & {\left[M^{2}\left[(q \cdot K)-\left(q_{+} \cdot K\right)-\left(q_{-} \cdot K\right)\right]-\left(q \cdot q_{+}\right)\left(q_{-} \cdot K\right)\right.} \\
& \left.-\left(q \cdot q_{-}\right)\left(q_{+} \cdot K\right)+\left(q_{+} \cdot q_{-}\right)(q \cdot K)\right] \cdot E_{\pi}^{2} \cdot E_{\rho} \\
& \times\left.\left[\left(q^{2}+M^{2}\right)\left(q_{+}^{2}+M^{2}\right)\left(q_{-}^{2}+M^{2}\right)\right]^{-1}\right|_{K_{\mu}=(0, a, 0,0)} ^{P_{\mu} \rightarrow \pm P_{\mu}} \tag{3.34}
\end{align*}
$$

[^21]
### 3.4.2 Results

Doing the calculations mentioned in the previous chapter yields a hadronic coupling constant $g=9.65$. $^{*}$ One can compare that to the results which can be found e.g. in [16]. Here, the corresponding value is given by $g=8.8$ for $(\mathrm{v}, \mathrm{p})=(1,1)$. This notation, introduced in [16], denotes how much pseudoscalar or vector BSA-amplitudes we consider. In this case we only regard the leading amplitudes, namely $E_{\pi}$ and $E_{\rho}$, so the comparison with the literature value of the $(\mathrm{v}, \mathrm{p})=(1,1)$ case is reasonable. ${ }^{\dagger}$ The relative deviation between the calculated quantity in terms of this thesis and the literature value is $9.7 \%$, which can be caused by several differences regarding the different approaches between this work and [16]. Firstly, the interaction chosen in this thesis is the contact interaction (2.53), which differs from the one chosen in the reference. A vital point is that the authors did not set the coupling constant $\alpha\left(k^{2}\right)$ constant like we did in this thesis. Therefore, they work with a effective coupling that is dependent on the momentum $k$. We will discuss that in chapter 4. Secondly, the authors didn't put the exact cutoff value $\Lambda$ in their writing. Hence, even this value is unknown for us. However, the approach described in the previous chapter leads to a good approximation of the value calculated in the reference. The experimental value is given by $g=6.02$ [16]. So, the relative deviation of the results of this thesis from this experimental value is given by $60.3 \%$. In view of the fact that we chose our truncation so that it contains only the simplest forms of the quark-gluon vertex and the gluon propagator, this deviation seems not to be critical, but rather shows us that this truncation is able to reproduce suitable results, even at that level.

[^22]
## 4 Concluding Remarks

In this thesis we saw that based on the simplest interaction we can choose to determine properties of quarks and light mesons, one can reproduce appropriate physical values like masses and decay constants. However, at some points, e.g. the calculation of the leptonic decay constant of the $\rho$-meson, one could realize that this model gets at its limits and (apparently) plain changes like a different regularization scheme can cause critical discrepancies (see Fig. 21). Furthermore, conditions like the segmentation of the quark momenta inside a meson, which clearly cannot influence the physical properties, do so anyway.
While introducing truncation schemes and the contact interaction we did a momentous assumption when we demanded the "running coupling" $\alpha\left(k^{2}\right)$ to be a constant $\alpha$. Maybe it would be more senseful to apply the Rainbow-Ladder truncation without applying the contact interaction by setting $g^{2} D_{\mu \nu}=$ const.
This would bring in an effective coupling $\mathcal{G}\left(k^{2}\right)$ so that

$$
\begin{equation*}
Z_{1} g^{2} D_{\mu \nu}(k) \Gamma_{\nu}^{i}(q, P) \rightarrow \mathcal{G}\left(k^{2}\right) D_{\mu \nu}^{\mathrm{free}} \gamma_{\nu} \frac{\lambda^{i}}{2} \tag{4.1}
\end{equation*}
$$

This approach is equal to that in [16] and entails that neither the two latter summands of Eq. (2.38), nor the second summand of Eq. (2.57) vanish, and hence, we can include higher ordered terms. An ansatz for the $\mathcal{G}$ looks quite complicated and is given by

$$
\begin{equation*}
\frac{\mathcal{G}\left(k^{2}\right)}{k^{2}}=\frac{4 \pi^{2} D k^{2}}{\omega^{6}} \mathrm{e}^{-k^{2} / \omega^{2}}+\frac{4 \pi^{2} \gamma_{m} \mathcal{F}\left(k^{2}\right)}{\frac{1}{2} \ln \left[\tau+\left(1+k^{2} / \Lambda_{\mathrm{QCD}}^{2}\right)^{2}\right]} \tag{4.2}
\end{equation*}
$$

in which $\gamma_{m}=12 / 33-2 N_{f}$ and $\mathcal{F}=\left(1-\exp \left(-s / 4 m_{t}^{2}\right) / s\right.$. Furthermore, $m_{t}=0.5 \mathrm{GeV}$, $\tau=\mathrm{e}^{2}-1, N_{f}=4$ and $\Lambda_{\mathrm{QCD}}=0.234 \mathrm{GeV}$. The parameters $\omega$ and $D$ are fit parameters to guarantee a suitable description of the physical observables, and hence, play a similar role as $\Lambda$ and $m_{G}$ in the contact interaction model.
Ref. [16] gives an impression why choosing a non-constant effective coupling is probably

| quantity | this thesis | model exact [16] | experiment |
| :---: | :---: | :---: | :---: |
| $m_{\mathrm{u} / \mathrm{d}}$ | 7.8 | 5.5 | $5-10$ |
| $m_{\pi}$ | 139.8 | 138 | 138.5 |
| $f_{\pi}$ | $161^{*}$ | 131 | 131 |
| $m_{\rho}$ | 900 | 735 | 770 |
| $f_{\rho}$ | - | 207 | 216 |
| $g_{\rho \rightarrow \pi \pi}$ | 9.65 | 5.14 | 6.02 |

Table 2: The comparison between the calculations in this thesis, the calculations in RL-truncation assuming a non-constant effective coupling $\mathcal{G}\left(k^{2}\right)$ like it's done in [16] and the experimental values, also taken from [16]. We see that the effective coupling yields better approximations of the experimental values.
a more suitable one then that one used in this thesis is. The computed values together with these out of this thesis and the experimental values are shown explicitly in Tab. 2. Comparing the "model exact" values computed with an effective coupling $\mathcal{G}\left(k^{2}\right)$ with those obtained in the course of this thesis and the experimental values shows that the calculations in RL-truncation without using the contact interaction yields better approximations of the experimental values than the contact interaction model does. Even the $\rho$-meson leptonic decay constant can be computed, which yields a good approximation, whereas the contact interaction model doesn't give senseful values.
We conclude that the description of mesons like the pion and the $\rho$-meson require a type of model, which goes beyond the model used in this thesis. To dip deeper into the properties of the mesons, one has to consult these models and look for possibilities to customize them, so that experimental values could be determined more precise.

## A Appendix

## A. 1 Euclidean Conventions and Relations

In this thesis Euclidean conventions are applied, which means that

$$
\begin{equation*}
a \cdot b=\sum_{i=0}^{3} a_{i} b_{i}=a_{i} b^{i}=a^{i} b_{i}=\delta_{i j} a^{i} b^{j} \tag{A.1}
\end{equation*}
$$

Associated with the transition from Minkowski to Euclidean metric the momentum four vector had been Wick rotated such that

$$
\begin{equation*}
(E, \boldsymbol{p}) \rightarrow(i E, \boldsymbol{p})=: p^{\mu} \tag{A.2}
\end{equation*}
$$

Thus, a four vector $p \in \mathbb{C} \times \mathbb{R}^{3}$ is spacelike, iff $p^{2}>0$.

## A.1. 1 Gamma Matrices

The matrices $\gamma_{\mu}(\mu=0 \ldots 5, \mu \neq 4)$ are the gamma or Dirac matrices and can be expressed via the Pauli matrices $\sigma_{\mu}$. A possible Euclidean representation, in which the underlying Dirac (Minkowski) representation had been Wick rotated in the way mentioned before, is the following:

$$
\gamma_{0}=\left(\begin{array}{cc}
\mathbb{1} &  \tag{A.3}\\
& -\mathbb{1}
\end{array}\right) \quad ; \quad \gamma_{\mu}=\left(\begin{array}{cc} 
& i \sigma_{\mu} \\
-i \sigma_{\mu} &
\end{array}\right) \quad ; \quad \gamma_{5}=\left(\begin{array}{ll} 
& \mathbb{1} \\
\mathbb{1} &
\end{array}\right)
$$

with

$$
\mathbb{1} \equiv \mathbb{1}_{2 \times 2}
$$

In Euclidean metric, the gamma matrices are hermitian,

$$
\begin{equation*}
\left(\gamma_{\mu}\right)^{\dagger}=\gamma_{\mu} \tag{A.4}
\end{equation*}
$$

and satisfy the Clifford algebra,

$$
\begin{equation*}
\left\{\gamma_{\mu}, \gamma_{\nu}\right\}=2 \delta_{\mu \nu} \tag{A.5}
\end{equation*}
$$

In this form the Dirac equation for a positive-energy on-shell spinor reads

$$
\begin{equation*}
\left(i \gamma_{\mu} p^{\mu}-m\right) \psi=0 \tag{A.6}
\end{equation*}
$$

## $\gamma$-Relations

The following relations for Dirac matrices can be fairly useful:

- $\gamma_{5}=\left(\gamma_{5}\right)^{-1}$
- $\left\{\gamma_{5}, \gamma_{\mu}\right\}=0$
- $\left\{\gamma_{\mu}, \gamma_{\nu}\right\}=2 \delta_{\mu \nu} \cdot \mathbb{1}$
- $\gamma_{\mu} \gamma^{\mu}=4 \cdot \mathbb{1}$
- $\gamma_{\mu} \gamma_{\nu} \gamma^{\mu}=-2 \gamma_{\nu}$
- $\gamma_{\mu} \gamma_{\nu} \gamma_{\rho} \gamma^{\mu}=4 \delta_{\nu \rho} \cdot \mathbb{1}$
- $\gamma_{\mu} \gamma_{\nu} \gamma_{\rho} \gamma_{\sigma} \gamma^{\mu}=-2 \gamma_{\sigma} \gamma_{\rho} \gamma_{\nu}$


## Trace rules

When taking the traces of Dirac matrices one is glad to draw on trace rules. With $\operatorname{tr}\left(\gamma_{\mu} \gamma^{\nu} \ldots\right)=: \operatorname{tr}_{\gamma}\left({ }_{\mu}{ }^{\nu} \ldots\right)$

- $\operatorname{tr}_{\gamma}(\mu)=0$
- $\operatorname{tr}_{\gamma}\left(\mu^{\mu}\right)=16$
- $\operatorname{tr}_{\gamma}\left({ }_{\mu \nu}\right)=4 \delta_{\mu \nu}$
- $\operatorname{tr}_{\gamma}\left({ }_{\mu \nu \rho \sigma}\right)=4\left(\delta_{\mu \nu} \delta_{\rho \sigma}-\delta_{\mu \rho} \delta_{\nu \sigma}+\delta_{\mu \sigma} \delta_{\nu \rho}\right)$
- $\operatorname{tr}_{\gamma}(\underbrace{\alpha \ldots \omega}_{\text {odd } \#})=0$


## Feynman slash notation

A component-wise sum of the Dirac matrix vector and a four vector can be expressed shorter with the so called Feynman slash,

$$
\begin{equation*}
\gamma_{\mu} A^{\mu}=: \mathbb{A} . \tag{A.7}
\end{equation*}
$$

Comfortably, based on the Clifford algebra anticommutator relation $\left\{\gamma_{\mu}, \gamma_{\nu}\right\}=2 \delta_{\mu \nu}$, this notation lets the scalar product invariant:

$$
\begin{equation*}
A B=A \cdot B . \tag{A.8}
\end{equation*}
$$

## A.1.2 Integration

In the four dimensional spacetime, the occurring integrals are solved in hyperspherical coordinates, the 4 -dimensional version of the well known spherical coordinates.
Here the four components are given by

$$
\begin{align*}
& x_{0}=r \cos (\psi) \\
& x_{1}=r \sin (\psi) \cos (\theta) \\
& x_{2}=r \sin (\psi) \sin (\theta) \cos (\phi) \\
& x_{3}=r \sin (\psi) \sin (\theta) \sin (\phi) \tag{A.9}
\end{align*}
$$

Consequently, the corresponding Hessenberg Jacobian to the coordinate transformation $\Phi$ reads

$$
\mathcal{J}_{\Phi}=\left(\begin{array}{cccc}
\cos (\psi) & -r \sin (\phi) & &  \tag{A.10}\\
\sin (\psi) \cos (\theta) & r \cos (\psi) \cos (\theta) & -r \sin (\psi) \sin (\theta) & \\
\sin (\psi) \sin (\theta) \cos (\phi) & r \cos (\psi) \sin (\theta) \cos (\phi) & r \sin (\psi) \cos (\theta) \cos (\phi) & -r \sin (\psi) \sin (\theta) \sin (\phi) \\
\sin (\psi) \sin (\theta) \sin (\phi) & r \cos (\psi) \sin (\theta) \sin (\phi) & r \sin (\psi) \cos (\theta) \sin (\phi) & r \sin (\psi) \sin (\theta) \cos (\phi)
\end{array}\right)
$$

and some trigonometric theorems yield the functional determinant

$$
\begin{equation*}
\operatorname{det} \mathcal{J}_{\Phi}=r^{3} \sin ^{2}(\psi) \sin (\theta) \tag{A.11}
\end{equation*}
$$

with

$$
\begin{align*}
V_{\text {hypersphere }}(R) & =\int_{0}^{R} \mathrm{~d} r r^{3} \int_{0}^{\pi} \mathrm{d} \psi \sin ^{2}(\psi) \int_{0}^{\pi} \mathrm{d} \theta \sin (\theta) \int_{0}^{2 \pi} \mathrm{~d} \phi \\
& =\frac{1}{2} \cdot \int_{0}^{R^{2}} \mathrm{~d} r^{2} r^{2} \int_{-1}^{1} \mathrm{~d} z \sqrt{1-z^{2}} \int_{-1}^{1} \mathrm{~d} y \int_{0}^{2 \pi} \mathrm{~d} \phi . \tag{A.12}
\end{align*}
$$

In several calculations we shorten occurring integral expressions like

$$
\begin{gather*}
\left.\int \frac{\mathrm{d}^{4} q}{(2 \pi)^{4}}\right|_{q \leq \Lambda} \rightarrow \int_{q}^{\Lambda},  \tag{A.13}\\
\int_{0}^{\Lambda} \mathrm{d} q q^{3} \int_{-1}^{1} \mathrm{~d} z \sqrt{1-z^{2}} \rightarrow \iint_{q, z}^{\Lambda} . \tag{A.14}
\end{gather*}
$$

and

$$
\begin{equation*}
\int_{0}^{\Lambda} \mathrm{d} q q^{3} \int_{-1}^{1} \mathrm{~d} z \sqrt{1-z^{2}} \int_{-1}^{1} \mathrm{~d} y \rightarrow \iiint_{q, z, y}^{\Lambda} \tag{A.15}
\end{equation*}
$$

## A.1.3 Flavour- and Color-Space Conventions

In this thesis we use flavour- and color factors which correspond to a pion decay constant of 93 MeV . Other conventions in which this value (and equivalent ones) differ by factors of $\sqrt{2}$ are also familiar. (e.g. in [7])
The Bethe-Salpeter amplitude space composition can be expressed as

$$
\begin{equation*}
\Gamma \sim \operatorname{Dirac} \otimes \delta_{A B} \otimes \delta_{\mathrm{ab}}^{\mathrm{e}} . \tag{A.16}
\end{equation*}
$$

$A, B$ are color indices, $\mathrm{e} \in\{+,-, 0\}$ and $\mathrm{a}, \mathrm{b}$ are flavour-indices and set the pion charge. The axial-vector vertex, to which the meson BSAs couple regarding leptonic decays, can be composed in in an analogous way:

$$
\begin{equation*}
\Gamma_{5 \mu} \sim \gamma_{5} \gamma_{\mu} \sim \operatorname{Dirac} \otimes \delta_{A B} \otimes \delta_{\mathrm{ab}}^{e} \tag{A.17}
\end{equation*}
$$

## A. 2 Associated Derivations

## Relation (2.23)

Starting by

$$
\begin{equation*}
S(p)=S_{0}(p)+S_{0}(p) \Sigma(p) S(p) \tag{A.18}
\end{equation*}
$$

we expand this into a (geometric) series

$$
\begin{align*}
S(p) & =S_{0}(p)+S_{0}(p) \Sigma(p) S_{0}(p)+S_{0}(p) \Sigma(p) S_{0}(p) \Sigma(p) S_{0}(p)+\ldots \\
& =S_{0}(p) \cdot\left[1+\Sigma(p) S_{0}(p)+\Sigma(p) S_{0}(p) \Sigma(p) S_{0}(p)+\ldots\right] \\
& =S_{0}(p) \underbrace{\sum_{j=0}^{\infty}\left(\Sigma(p) S_{0}(p)\right)^{j}} \\
& =S_{0}(p) \cdot \frac{1}{1-\Sigma(p) S_{0}(p)} \tag{A.19}
\end{align*}
$$

By multiplying with $S^{-1}$ from the left and with $\left(1-\Sigma(p) S_{0}(p)\right) \cdot S_{0}^{-1}$ from the right we get the wanted relation:

$$
\begin{equation*}
S_{0}^{-1}(p)-\Sigma(p)=S^{-1}(p) \tag{A.20}
\end{equation*}
$$

## Relation (2.31)

Working with the definitions

$$
\tilde{T}:=T G_{0} \quad \text { and } \quad \tilde{K}:=K G_{0},
$$

we derive:

$$
\begin{align*}
\tilde{T} & =T G_{0}  \tag{A.21}\\
& =\left(K+K G_{0} T\right) G_{0} \\
& =\left(K+K G_{0} K+K G_{0} K G_{0} K+\ldots\right) G_{0} \\
& =\tilde{K}+\tilde{K} \tilde{K}+\tilde{K} \tilde{K} \tilde{K}+\ldots \\
& =\tilde{K}(1+\tilde{T}) \tag{A.22}
\end{align*}
$$

Plugging in the ansatz for $T$ :

$$
\begin{align*}
\mathcal{N} \frac{\Gamma \bar{\Gamma}}{P^{2}+M^{2}} G_{0} & =K G_{0}\left(1+\mathcal{N} \frac{\Gamma \bar{\Gamma}}{P^{2}+M^{2}} G_{0}\right) \\
& =\left(K+K G_{0} \mathcal{N} \frac{\Gamma \bar{\Gamma}}{P^{2}+M^{2}}\right) G_{0} \\
& =\frac{1}{P^{2}+M^{2}}(K \underbrace{\left(P^{2}+M^{2}\right)}_{\underbrace{\text { on-shell }} 0}+K G_{0} \mathcal{N} \Gamma \bar{\Gamma}) \tag{A.23}
\end{align*}
$$

It follows:

$$
\begin{equation*}
\Gamma \bar{\Gamma}=K G_{0} \Gamma \bar{\Gamma} \Rightarrow \Gamma=K G_{0} \Gamma . \tag{A.24}
\end{equation*}
$$

## Relation (2.40)

We start with the definition of $\bar{\Gamma}$ and use the properties of the gamma matrices mentioned in chapter A.1:

$$
\begin{align*}
\bar{\Gamma}(q, P) & =\mathcal{C} \Gamma^{T}(-q,-P) \mathcal{C}^{T}=\gamma_{0} \gamma_{2} \gamma_{5}\left(i E+\gamma_{\mu}^{T} P^{\mu} M^{-1} F\right) \gamma_{2}^{T} \gamma_{0}^{T} \\
& =\gamma_{5} i E+\gamma_{0} \gamma_{2} \gamma_{5} \gamma_{0} \gamma_{2} \gamma_{0} P^{0} M^{-1} F=\gamma_{5}\left(i E-\gamma_{0} \gamma_{2} \gamma_{2} \gamma_{0} \gamma_{\mu} P^{\mu} M^{-1} F\right) \\
& =\gamma_{5}\left(i E-\gamma_{\mu} P^{\mu} M^{-1} F\right)=\Gamma(q,-P) . \tag{A.25}
\end{align*}
$$

## Relation (2.51)

The quark gluon vertex can be expressed with the aid of three independent four vectors and four types of scalars, which gives twelve independent tensor structures [18] such that

$$
\begin{equation*}
\Gamma_{\nu} \in\left\{\gamma_{\mu}, p_{\mu}, q_{\mu}\right\} \otimes\{\mathbb{1}, \not p, \not q,[\not p, q]\}, \tag{A.26}
\end{equation*}
$$

thus, the quark-gluon vertex is given by a sum

$$
\begin{equation*}
\Gamma_{\nu}=\sum_{i=1}^{12} \tau_{\nu}^{i} T^{i}(p, q) \tag{A.27}
\end{equation*}
$$

In RL-truncation we consider only the first term, so the vertex is determined by the product of $\gamma^{\mu} \otimes \mathbb{1}$. By demanding $\alpha\left(k^{2}\right) \equiv \alpha$, the prefactor $T_{1}$ becomes constant and can be set 1 . We obtain

$$
\begin{equation*}
\Gamma_{\nu}=\gamma_{\nu} \tag{A.28}
\end{equation*}
$$

## Relation (2.56)

To derive this expression, we will at first introduce some notations which shorten the terms:

$$
a:=-\frac{4}{3 m_{G}^{2}} \quad ; \quad b^{-1}:=\left(q_{-}^{2}+M^{2}\right)\left(q_{+}^{2}+M^{2}\right)
$$

With that we can combine Eqs. (2.54) and (2.55), by equaling them and plug in the Dirac decomposition into the integrand:

$$
\begin{equation*}
a \int_{q}^{\Lambda} b \gamma_{\nu}\left(-i \gamma_{\alpha} q_{+}^{\alpha}+M\right) \gamma_{5}\left(i E+\gamma_{\beta} \hat{P}^{\beta} F\right)\left(-i \gamma_{\epsilon} q_{-}^{\epsilon}+M\right) \gamma^{\nu}=\gamma_{5}\left(i E+\gamma_{\beta} \hat{P}^{\beta} F\right) \tag{A.29}
\end{equation*}
$$

To deal with the gamma matrices we write this expression out "in full":

$$
\begin{align*}
a \int_{q}^{\Lambda} b & \left\{\left[-\left(\gamma_{\nu} \gamma_{\alpha} \gamma_{5} \gamma_{\epsilon} \gamma^{\nu}\right) \cdot i q_{+}^{\alpha} q_{-}^{\epsilon}+\left(\gamma_{\nu} \gamma_{\alpha} \gamma_{5} \gamma^{\nu}\right) q_{+}^{\alpha}+\left(\gamma_{\nu} \gamma_{5} \gamma_{\epsilon} \gamma^{\nu}\right) M q_{-}^{\epsilon}+\left(\gamma_{\nu} \gamma_{5} \gamma^{\nu}\right) i M^{2}\right] \cdot E\right. \\
& +\left[\left(\gamma_{\nu} \gamma_{\alpha} \gamma_{5} \gamma_{\beta} \gamma_{\epsilon} \gamma^{\nu}\right) q_{+}^{\alpha} P^{\beta} M^{-1}-\left(\gamma_{\nu} \gamma_{\alpha} \gamma_{5} \gamma_{\beta} \gamma^{\nu}\right) i q_{+}^{\alpha} P^{\beta}-\left(\gamma_{\nu} \gamma_{5} \gamma_{\beta} \gamma_{\epsilon} \gamma^{\nu}\right) i P^{\beta} q_{-}^{\epsilon}\right. \\
& \left.\left.+\left(\gamma_{\nu} \gamma_{5} \gamma_{\beta} \gamma^{\nu}\right) M P^{\beta}\right] \cdot F\right\}=\gamma_{5}\left(i E+\gamma_{\beta} \hat{P}^{\beta}\right) F \tag{A.30}
\end{align*}
$$

Obviously we can isolate $E$ by multiplying $\gamma_{5} / i$. With the identities mentioned in chapter A. 1 we can eliminate the $\gamma_{5}$ 's and simplify the expressions like $\left(\gamma_{\nu} \ldots \gamma^{\nu}\right)$ :

$$
\begin{align*}
& a \int_{q}^{\Lambda} \frac{b}{i}\left\{\left[-4 \delta_{\alpha \epsilon} i q_{+}^{\alpha} q_{-}^{\epsilon}-2 \gamma_{\alpha} q_{+}^{\alpha}+2 \gamma_{\epsilon} q_{-}^{\epsilon} M-4 i M^{2}\right] \cdot E+\left[-2\left(\gamma_{\epsilon} \gamma_{\beta} \gamma_{\alpha}\right) q_{+}^{\alpha} P^{\beta} M^{-1}\right.\right. \\
&\left.\left.-4 \delta_{\alpha \beta} i q_{+}^{\alpha} P^{\beta}+4 \delta_{\beta \epsilon} i P^{\beta} q_{-}^{\epsilon}+2 \gamma_{\beta} P^{\beta} M\right] \cdot F\right\}=E-i \gamma_{\beta} \hat{P}^{\beta} F \tag{A.31}
\end{align*}
$$

Now we take the Dirac trace and divide the whole equation by 4 . Since the relation $\delta_{\alpha \beta} v^{\alpha} w^{\beta}=v \cdot w$ holds, we get

$$
\begin{equation*}
E=4 a \int_{q} b\left\{\left[-\left(q_{-} \cdot q_{+}\right)-M^{2}\right] \cdot E-\left[\left(q_{+} \cdot P\right)-\left(q_{-} \cdot P\right)\right] \cdot F\right\} \tag{А.32}
\end{equation*}
$$

That looks like a matrix multiplication expression like $E=\mathcal{K}_{E E} \cdot E+\mathcal{K}_{E F} \cdot F$. Before we go on with this expression, we derive an analogue expression for the matrix elements $\mathcal{K}_{F E}$ and $\mathcal{K}_{F F}$.
We start with Eq. (A.30) and isolate $F$. Since $P^{2}=-m_{0^{-}}^{2}$ we can multiply $-M \not P^{2} / m_{0^{-}}^{2}$ from the left side:

$$
\begin{align*}
-a \int_{q}^{\Lambda} \frac{b M}{m_{0^{-}}^{2}} \operatorname{tr} & \left\{\left[-\left(\gamma_{\delta} \gamma_{5} \gamma_{\nu} \gamma_{\alpha} \gamma_{5} \gamma_{\epsilon} \gamma^{\nu}\right) i P^{\delta} q_{+}^{\alpha} q_{-}^{\epsilon}+\left(\gamma_{\delta} \gamma_{5} \gamma_{\nu} \gamma_{\alpha} \gamma_{5} \gamma^{\nu}\right) P^{\delta} q_{+}^{\alpha} M E\right.\right. \\
& \left.-\left(\gamma_{\delta} \gamma_{5} \gamma_{\nu} \gamma_{5} \gamma_{\epsilon} \gamma^{\nu}\right) P^{\delta} q_{-}^{\epsilon} M E+\left(\gamma_{\delta} \gamma_{5} \gamma_{\nu} \gamma_{\alpha} \gamma_{5} \gamma^{\nu}\right) P^{\delta} M^{2} i E\right] \cdot E \\
& +\left[-\left(\gamma_{\delta} \gamma_{5} \gamma_{\nu} \gamma_{\alpha} \gamma_{5} \gamma_{\beta} \gamma_{\epsilon} \gamma^{\nu}\right) P^{\delta} q_{+}^{\alpha} P^{\beta} q_{-}^{\epsilon} M^{-1}-\left(\gamma_{\delta} \gamma_{5} \gamma_{\nu} \gamma_{\alpha} \gamma_{5} \gamma_{\beta} \gamma^{\nu}\right) P^{\delta} q_{+}^{\alpha} P^{\beta} i\right. \\
& \left.\left.-\left(\gamma_{\delta} \gamma_{5} \gamma_{\nu} \gamma_{5} \gamma_{\beta} \gamma_{\epsilon} \gamma^{\nu}\right) P^{\delta} P^{\beta} q_{-}^{\epsilon} i+\left(\gamma_{\delta} \gamma_{5} \gamma_{\nu} \gamma_{5} \gamma_{\beta} \gamma^{\nu}\right) P^{\delta} P^{\beta} M\right] \cdot F\right\} \\
& =-\frac{P i M E}{m_{0^{-}}^{2}}+F \tag{A.33}
\end{align*}
$$

Equivalent to the derivation of the other matrix elements, we use the rules for Dirac matrices to simplify the equation:

$$
\begin{align*}
-a \int_{q}^{\Lambda} \frac{b M}{m_{0^{-}}^{2}} \operatorname{tr} & \left\{\left[-\left(\gamma_{\delta} \gamma_{\nu} \gamma_{\alpha} \gamma_{\epsilon} \gamma^{\nu}\right) i P^{\delta} q_{+}^{\alpha} q_{-}^{\epsilon}-2\left(\gamma_{\delta} \gamma_{\alpha}\right) P^{\delta} q_{+}^{\alpha} M+2\left(\gamma_{\delta} \gamma_{\epsilon}\right) P^{\delta} q_{-}^{\epsilon} M\right.\right. \\
& \left.-\left(\gamma_{\delta} \gamma_{\nu} \gamma^{\nu}\right) P_{\delta} M^{2} i\right] \cdot E+\left[2\left(\gamma_{\delta} \gamma_{\epsilon} \gamma_{\beta} \gamma_{\alpha}\right) P^{\delta} q_{+}^{\alpha} P^{\beta} q_{-}^{\epsilon} M^{-1}\right. \\
& -\left(\gamma_{\delta} \gamma_{\nu} \gamma_{\alpha} \gamma_{\beta} \gamma^{\nu}\right) P^{\delta} q_{+}^{\alpha} P^{\beta} i+\left(\gamma_{\delta} \gamma_{\nu} \gamma_{\beta} \gamma_{\epsilon} \gamma^{\nu}\right) P^{\delta} P^{\beta} q_{-}^{\epsilon} i \\
& \left.\left.+2\left(\gamma_{\delta} \gamma_{\beta}\right) P^{\delta} P^{\beta} M\right] \cdot F\right\}=-\frac{P i M E}{m_{0^{-}}^{2}}+F \tag{A.34}
\end{align*}
$$

Now, in analogy to the other matrix elements, we take the Dirac trace with respect to the corresponding trace rules and divide the equation by 4 :

$$
\begin{align*}
F=a \int_{q}^{\Lambda} b\left\{\frac{2 M^{2}}{m_{0^{-}}^{2}}\left[\left(q_{+} \cdot P\right)-\left(q_{-} \cdot P\right)\right] \cdot E\right. & +\left[2 M^{2}-\frac{2}{m_{0^{-}}^{2}} \cdot\left(2\left(q_{+} \cdot P\right)\left(q_{-} \cdot P\right)\right.\right. \\
+ & \left.\left.\left.m_{0^{-}}^{2}\left(q_{+} \cdot q_{-}\right)\right)\right] \cdot F\right\} \tag{A.35}
\end{align*}
$$

Here, we can easily see the matrix multiplication structure with the matrix elements $\mathcal{K}_{F E}$ and $\mathcal{K}_{F F}$.
The momenta $q_{ \pm}$are given by

$$
\begin{equation*}
q_{+}=q+\eta P \quad \text { and } \quad q_{-}=q+(\eta-1) P, \tag{A.36}
\end{equation*}
$$

whereas $P=\left(i m_{0^{-}}, 0,0,0\right)$.
Hence, we can express the scalar products that appear in the products of $q_{ \pm}$and $P$ with the aid of the angle $\psi$ with $(q \cdot P)=i m_{0^{-}} \cdot|q| \cdot \cos \psi$. With that, one can derive the explicit expressions for $q_{+}^{2}, q_{-}^{2}, q_{+} q_{-}, q_{+} P$ and $q_{-} P$ in which the term $\cos \psi$ occurs. We define $z=\cos \psi$ to shorten the expressions in the integral a little bit.
With that knowledge we can solve the integrals for two angles trivially in hyperspherical coordinates (cf. A.1) and define

$$
\mathcal{N}=\frac{1}{3 m_{G}^{2} \pi^{3}}
$$

such that we can substitute

$$
a \int_{q}^{\Lambda} \rightarrow \mathcal{N} \iint_{q, z}^{\Lambda} .
$$

Finally, with $\left(q_{+} \cdot P\right)-\left(q_{-} \cdot P\right)=P^{2}=-m_{0^{-}}^{2}$, the matrix elements are determined by

$$
\begin{align*}
& \mathcal{K}_{E E}=-4 \cdot \mathcal{N} \iint_{q, z}^{\Lambda} \frac{\left(q_{+} \cdot q_{-}\right)+M^{2}}{\left(q_{+}^{2}+M^{2}\right)\left(q_{-}^{2}+M^{2}\right)}  \tag{A.37}\\
& \mathcal{K}_{E F}=4 m_{0^{-}}^{2} \cdot \mathcal{N} \iint_{q, z}^{\Lambda} \frac{1}{\left(q_{+}^{2}+M^{2}\right)\left(q_{-}^{2}+M^{2}\right)}  \tag{A.38}\\
& \mathcal{K}_{F E}=-2 M^{2} \cdot \mathcal{N} \iint_{q, z}^{\Lambda} \frac{1}{\left(q_{+}^{2}+M^{2}\right)\left(q_{-}^{2}+M^{2}\right)}  \tag{A.39}\\
& \mathcal{K}_{F F}=2 \cdot \mathcal{N} \iint_{q, z}^{\Lambda} \frac{M^{2}-m_{0^{-}}^{-2} \cdot\left(2\left(q_{+} \cdot P\right)\left(q_{-} \cdot P\right)+m_{0^{-}}^{2}\left(q_{+} \cdot q_{-}\right)\right)}{\left(q_{+}^{2}+M^{2}\right)\left(q_{-}^{2}+M^{2}\right)} \tag{A.40}
\end{align*}
$$

and solve the homogeneous BSE,

$$
\binom{E_{0^{-}}}{F_{0^{-}}}=\left(\begin{array}{ll}
\mathcal{K}_{E E} & \mathcal{K}_{E F}  \tag{A.41}\\
\mathcal{K}_{F E} & \mathcal{K}_{F F}
\end{array}\right)\binom{E_{0^{-}}}{F_{0^{-}}} .
$$

This derivation is exemplary for other derivations. Thus, similar calculations will not be derivated in this scale.

## Relations (3.4)

We start from Eq. (3.2) and create the trace. Because all gamma matrices are traceless and a product of an odd number of gamma matrices is traceless, too, the expression shortens. Thereby we have to consider, that $A \sim \gamma$ for any $A$. It follows

$$
\begin{equation*}
4 B\left(p^{2}\right)=4 m+\frac{4}{3 m_{G}^{2}} \int \frac{\mathrm{~d}^{4} q}{(2 \pi)^{4}} \operatorname{tr}_{\gamma}\left({ }_{\mu \mu}\right)\left[\frac{B\left(q^{2}\right)}{q^{2} A^{2}\left(q^{2}\right)+B^{2}\left(q^{2}\right)}\right] . \tag{A.42}
\end{equation*}
$$

With $\operatorname{tr}_{\gamma}(\mu \mu)=16$ and a division by 4 :

$$
\begin{equation*}
B\left(p^{2}\right)=m+\frac{1}{3 m_{G}^{2}} \int \frac{\mathrm{~d}^{4} q}{\pi^{4}}\left[\frac{B\left(q^{2}\right)}{q^{2} A^{2}\left(q^{2}\right)+B^{2}\left(q^{2}\right)}\right] \tag{A.43}
\end{equation*}
$$

Integrating over the angular coordinates after substituting $q \rightarrow q^{2}$ delivers an additional factor $\pi^{2}$, hence

$$
\begin{equation*}
B\left(p^{2}\right)=m+\frac{1}{3 \pi^{2} m_{G}^{2}} \int \mathrm{~d} q^{2}\left[\frac{q^{2} B\left(q^{2}\right)}{q^{2} A^{2}\left(q^{2}\right)+B^{2}\left(q^{2}\right)}\right] \tag{A.44}
\end{equation*}
$$

The factor $q^{2}$ results from the Jacobi-determinante.
For the other relation we start from Eq. (3.3). Every quantity $\not p \not p$ becomes $p^{2}$ (see A.1). The factor $\not p q$ is a little bit different and can be handled like a standard Euclidean scalar product that contains the angle $\chi$ between the two vectors. In this setting, we demand
that $p$ is parallel to the $x_{0}$-axis, so we can write Eq. (3.3) as

$$
\begin{equation*}
4 p^{2} A\left(p^{2}\right)=4 p^{2}-\frac{4}{3 m_{G}^{2}} \int \frac{\mathrm{~d}^{4} q}{\pi^{4}}\left[\frac{-p q \cos (\chi) A\left(q^{2}\right)}{q^{2} A^{2}\left(q^{2}\right)+B^{2}\left(q^{2}\right)}\right] \tag{A.45}
\end{equation*}
$$

The traces have already been created in a similar way than in the relation before with $\operatorname{tr}_{\gamma}(\mu \mu)=16$.
The integral on the right hand side is an odd function of $\chi$ and has to be integrated symmetric from -1 to 1 , thus is vanishing. Dividing the whole equation by $4 p^{2}$ delivers the wanted relation:

$$
\begin{equation*}
A\left(p^{2}\right)=1 \tag{A.46}
\end{equation*}
$$

## Relation (3.7)

This derivation is done quite straightforward with $M \rightarrow M \mathbb{1}$ :

$$
\begin{align*}
-\langle\overline{\mathrm{q}} \mathrm{q}\rangle & =\mathcal{N} \cdot \operatorname{tr} \int_{q}^{\Lambda} S_{\text {chiral }}(q) \\
& =\mathcal{N} \int_{q}^{\Lambda} \cdot \frac{-i \cdot \operatorname{tr} q q+4 M}{q^{2}+M^{2}} \\
& =\frac{\mathcal{N}}{4 \pi^{2}} \int_{0}^{\Lambda} \mathrm{d} q^{2} \frac{q^{2} M}{q^{2}+M^{2}} \tag{A.47}
\end{align*}
$$

In the last step we integrated over the angular coordinates, which yields (with the substitution $q \rightarrow q^{2}$ ) a factor of $\pi^{2}$.

## Relation (3.14)

We start with Eq. (2.33):

$$
\begin{equation*}
1=\frac{\mathrm{d}}{\mathrm{~d} P^{2}} \operatorname{tr} \int_{q}^{\Lambda} \bar{\Gamma}(q, K) S\left(q_{+}\right) \Gamma(q, K) S\left(q_{-}\right) \tag{A.48}
\end{equation*}
$$

With

$$
\begin{gathered}
\Gamma=\gamma_{5}\left(i E+\gamma_{\mu} P^{\mu} M^{-1} F\right) \quad ; \quad \bar{\Gamma}=\gamma_{5}\left(i E-\gamma_{\mu} P^{\mu} M^{-1} F\right) \\
\text { and } \quad S\left(q_{ \pm}\right)=-i \gamma_{\mu} q_{ \pm}^{\mu}+M / q_{ \pm}^{2}+M^{2}
\end{gathered}
$$

we proceed in an analogous way as we did for relation (2.56). We write this sum out in full and apply the rules for the gamma matrices until we arrive at this point:

$$
\begin{align*}
1=4 \frac{\mathrm{~d}}{\mathrm{~d} P^{2}} \int_{q}^{\Lambda} & \left\{\left[-M^{2}-\left(q_{+} \cdot q_{-}\right)\right] \cdot E^{2}+\left[2\left(\left(q_{-} \cdot K\right)-\left(q_{+} \cdot K\right)\right)\right] \cdot E F\right. \\
& \left.+\left[K^{2}+M^{-2}\left(2\left(q_{+} \cdot K\right)\left(q_{-} \cdot K\right)-K^{2}\left(q_{+} \cdot q_{-}\right)\right)\right] \cdot F^{2}\right\} \\
& \times\left[\left(q_{+}^{2}+M^{2}\right)\left(q_{-}^{2}+M^{2}\right)\right]^{-1} \tag{A.49}
\end{align*}
$$

Note that the trace over Dirac-, color- and flavour-indices had been taken. Due to the colorand flavour-space-conventions in A.1, we obtain $\operatorname{tr}_{\mathrm{cf}}=6$. Although the differentiation with respect to $P^{2}$ is possible, it is more intuitively accessible to differentiate with respect to $P$, or alternatively the $m_{\pi}$-parametrization as a real value. Therefore we can substitute $\mathrm{d} / \mathrm{d} P^{2} \rightarrow 1 / 2 P \cdot \mathrm{~d} / \mathrm{d} P$ and combine this with the factors that come out of the integral after integrating two angles trivially and, with $\left(q_{+} \cdot K\right)-\left(q_{-} \cdot K\right)=P \cdot K$, get the wanted equation:

$$
\begin{align*}
1=\frac{3}{P \pi^{3}} \frac{\mathrm{~d}}{\mathrm{~d} P} \iint_{q, z}^{\Lambda} & \left\{\left[-M^{2}-\left(q_{+} \cdot q_{-}\right)\right] \cdot E^{2}-2(P \cdot K) \cdot E F\right. \\
& \left.+\left[K^{2}+M^{-2} \cdot\left(2\left(q_{+} \cdot K\right)\left(q_{-} \cdot K\right)-K^{2}\left(q_{+} \cdot q_{-}\right)\right)\right] \cdot F^{2}\right\} \\
& \times\left[\left(q_{+}^{2}+M^{2}\right)\left(q_{-}^{2}+M^{2}\right)\right]^{-1} \tag{A.50}
\end{align*}
$$

An easy solution technique is to use the connection between $E$ and $F$, w.l.o.g. $F=\alpha E$ and set $E=1 / N$ with a normalization constant $N$. multiplying the whole equation by $N^{2}$ yields an equation $N^{2}=\mathcal{S}(M, P, \alpha)$. The right side can be computed and we obtain the normalized amplitudes.

## Relation (3.25)

The normalization condition reads (cf. [11], Eq. (39)):

$$
\begin{equation*}
\frac{1}{E_{\rho}^{2}}=-\left.9 m_{G}^{2} \frac{\mathrm{~d}}{\mathrm{~d} P^{2}} K_{\gamma}\left(P^{2}\right)\right|_{P^{2}=-m_{\rho}^{2}} \tag{A.51}
\end{equation*}
$$

For the term $K_{\gamma}$ we use the corresponding definitions to bring the expression back into a form which just includes the quantities we are working with in this thesis:

$$
\begin{gather*}
K_{\gamma}\left(P^{2}\right):=\frac{1}{3 \pi^{2} m_{G}^{2}} \int_{0}^{1} \mathrm{~d} \alpha \alpha(1-\alpha) P^{2} \overline{\mathcal{C}}_{1}^{i u}(\omega)  \tag{A.52}\\
\overline{\mathcal{C}}_{1}^{i u}(\omega):=\frac{\mathcal{C}_{1}^{i u}(\omega)}{\omega} ; \quad \mathcal{C}_{1}^{i u}(\omega):=-\omega \frac{\mathrm{d}}{\mathrm{~d} \omega} \mathcal{C}^{i u}(\omega)  \tag{A.53}\\
\Rightarrow \overline{\mathcal{C}}_{1}^{i u}(\omega)=-\frac{\mathrm{d}}{\mathrm{~d} \omega} \mathcal{C}^{i u}(\omega)  \tag{A.54}\\
\mathcal{C}^{i u}(\omega):=\int_{0}^{\infty} \mathrm{d} q^{2} q^{2} \frac{\mathrm{e}^{-\left(q^{2}+\omega\right) \tau_{u v}^{2}}-\mathrm{e}^{-\left(q^{2}+\omega\right) \tau_{i r}^{2}}}{q^{2}+\omega}  \tag{A.55}\\
\omega:=M^{2}+\alpha(1-\alpha) P^{2} \tag{A.56}
\end{gather*}
$$

Therefore:

$$
\begin{align*}
\frac{1}{E_{\rho}^{2}} & =\frac{3}{\pi^{2}} \frac{\mathrm{~d}}{\mathrm{~d} P^{2}} \int_{0}^{1} \mathrm{~d} \alpha \alpha(1-\alpha) P^{2} \frac{\mathrm{~d}}{\mathrm{~d} \omega} \int_{0}^{\infty} \mathrm{d} q^{2} q^{2} \frac{\mathrm{e}^{-\left(q^{2}+\omega\right) \tau_{u v}^{2}}-\mathrm{e}^{-\left(q^{2}+\omega\right) \tau_{i r}^{2}}}{q^{2}+\omega} \\
& =\frac{3}{\pi^{2}} \int_{0}^{1} \mathrm{~d} \alpha \alpha(1-\alpha) \int_{0}^{\infty} \mathrm{d} q^{2} q^{2} \frac{\mathrm{~d}}{\mathrm{~d} P^{2}} P^{2} \frac{\mathrm{~d}}{\mathrm{~d} \omega} \frac{\mathrm{e}^{-\left(q^{2}+\omega\right) \tau_{u v}^{2}}-\mathrm{e}^{-\left(q^{2}+\omega\right) \tau_{i r}^{2}}}{q^{2}+\omega} \tag{A.57}
\end{align*}
$$

## Relation (3.34)

Plugging in the propagators and vertices into Eq. (3.28) yields the following expression:

$$
\begin{equation*}
\Lambda_{\mu}=\operatorname{tr}_{\mathrm{sc}} \int_{k}^{\Lambda} \frac{\left(-i \gamma_{\alpha} q^{\alpha}+M\right) \gamma_{5}\left(-i \gamma_{\beta} q_{+}^{\beta}+M\right)\left(\gamma_{\mu}-\hat{Q}_{\delta} \gamma^{\delta} \hat{Q}_{\mu}\right)\left(-i \gamma_{\epsilon} q_{-}^{\epsilon}+M\right) \gamma_{5} E_{\pi}^{2} E_{\rho}}{\left(q^{2}+M^{2}\right)\left(q_{+}^{2}+M^{2}\right)\left(q_{-}^{2}+M^{2}\right)} \tag{A.58}
\end{equation*}
$$

Thereby, we used the imaginary part of the pion BSA like it's done in Ref. [16]. After applying the rules for gamma matrices we get

$$
\begin{align*}
\Lambda_{\mu}(Q, P)=4 i \cdot \operatorname{tr}_{\mathrm{c}} \int_{k}^{\Lambda} & {\left[M^{2}\left[q_{\mu}-(q \cdot \hat{Q}) \hat{Q}_{\mu}-q_{+\mu}+\left(q_{+} \cdot \hat{Q}\right) \hat{Q}_{\mu}-q_{-\mu}+\left(q_{-} \cdot \hat{Q}\right) \hat{Q}_{\mu}\right]\right.} \\
& -\left(q \cdot q_{+}\right) q_{-\mu}+\left(q_{+} \cdot q_{-}\right) q_{\mu}+\left(q_{+} q_{-}\right) q_{\mu}-\left(q \cdot q_{-}\right) q_{+\mu} \\
& \left.+\left(q \cdot q_{+}\right)\left(q_{-} \cdot \hat{Q}\right) \hat{Q}_{\mu}-(q \cdot \hat{Q})\left(q_{+} \cdot q_{-}\right) \hat{Q}_{\mu}+\left(q \cdot q_{-}\right)\left(q_{+} \cdot \hat{Q}\right) \hat{Q}_{\mu}\right] \\
& \times\left[\left(q^{2}+M^{2}\right)\left(q_{+}^{2}+M^{2}\right)\left(q_{-}^{2}+M^{2}\right)\right]^{-1} \cdot E_{\pi}^{2} E_{\rho} \tag{A.59}
\end{align*}
$$

Now we contract the equation by multiplying $K^{\mu}=(0, a, 0,0)=P^{\mu}$ on both sides. Solving the integral analytically as far as possible brings in a factor of $1 / 8 \pi^{3}$. Due to the fact that $Q$ and $K$ are perpendicular, every term containing a $\hat{Q}_{\mu}$ vanishes and we get the wanted expression:

$$
\begin{align*}
\Lambda(Q, P) \cdot P=\frac{i}{2 \pi^{3}} \cdot \operatorname{tr}_{\mathrm{c}} \iiint_{k, z, y}^{\Lambda} & {\left[M^{2}\left[(q \cdot K)-\left(q_{+} \cdot K\right)-\left(q_{-} \cdot K\right)\right]-\left(q \cdot q_{+}\right)\left(q_{-} \cdot K\right)\right.} \\
& \left.-\left(q \cdot q_{-}\right)\left(q_{+} \cdot K\right)+\left(q_{+} \cdot q_{-}\right)(q \cdot K)\right] \cdot E_{\pi}^{2} \cdot E_{\rho} \\
& \times\left.\left[\left(q^{2}+M^{2}\right)\left(q_{+}^{2}+M^{2}\right)\left(q_{-}^{2}+M^{2}\right)\right]\right|_{K_{\mu}=(0, a, 0,0)} \tag{A.60}
\end{align*}
$$

For $P \rightarrow-P$ we have to switch the sign of every $P$ contained in $q_{( \pm)}$. The scalar products are evaluated an analogous way as before. The additional direction accompanied by the vector $P$ demands us to consider an additional angle, e.g.

$$
k \cdot K=|k| a \cdot \sin (\psi) \cos (\theta)=|k| a \cdot \sqrt{1-z^{2}} y
$$

The integration happens in the way mentioned in chapter A.1.2.
For the sake of completeness, the scalar products are given by

$$
\begin{gathered}
q \cdot P=k P \sqrt{1-z^{2}} y+P^{2} / 2, \\
q_{ \pm} \cdot P=k P \sqrt{1-z^{2}} y-P^{2} / 2, \\
q \cdot q_{ \pm}=k^{2}-P^{2} / 4 \pm k Q z / 2, \\
q_{+} \cdot q_{-}=k^{2}-k P \sqrt{1-z^{2}} y+P^{2} / 4-Q^{2} / 4, \\
q_{ \pm}^{2}=k^{2}-k P \sqrt{1-z^{2}} y \pm k Q z+P^{2} / 4+Q^{2} / 4 \\
\text { and } \\
q^{2}=k^{2}+k P \sqrt{1-z^{2}} y+P^{2} / 4 .
\end{gathered}
$$

## A. 3 Numerical Methods

Several equations that have been solved in this thesis can just be solved numerically. Especially the self-consistency problems and the other integrals are not solvable analytically, hence we have to apply methods that give us a good approximation of the solution. Any of the computations in this thesis had been programmed in Fortran 90 using the Intel Compiler 17.0 for Windows and macOS that is covered in the Intel 64 Visual Studio environment.

## A.3.1 Gauß-Legendre Integration

One method to exploit integrals is the Gauß-Legendre method. A numerical method to solve integrals, is the well-known bar method, in which the integral is approximated by a summation of bars that have the same width. The idea is to calculate the function value and use the corresponding value of the domain of definition as a sampling point such that

$$
\begin{equation*}
\int_{a}^{b} \mathrm{~d} x f(x) \rightarrow \sum_{i=1}^{n} d \cdot f\left(x_{i}\right) \tag{A.61}
\end{equation*}
$$

with $x_{i}=a+\frac{2 i-1}{2 n} \cdot(b-a)$ and $d=\frac{b-a}{n}$.
Integrating with the Gauß-Legendre method abandons the idea of equally broad bars and turns it into a sum in which the distances of various sampling points are different and the function values get a specific weight factor to achieve a higher accuracy:

$$
\begin{equation*}
\int_{a}^{b} \mathrm{~d} x f(x) \rightarrow \sum_{i=1}^{n} w_{i} f\left(x_{i}\right) \tag{A.62}
\end{equation*}
$$

A subroutine that calculates the abscissas and the associated weights had been taken from [17].

## A.3.2 Root-Finding Methods

In this thesis, two root-finding methods are applied, the bisection method and the Newton method.
The bisection method is an intuitive method to find a simple root of a scalar function $f \in \mathcal{C}^{0}$.* The root has to be located in a guess domain $[a, b]$. It is absolutely necessary that $\operatorname{sgn} f(a) \neq \operatorname{sgn} f(b)$ such that we can rename $a$ and $b$, w.l.o.g. with $f(a)<0$ and $f(b)>0$, as $a \rightarrow x_{-}$and $b \rightarrow x_{+}$. We define a new quantity, $\bar{x}=x_{+}+x_{-} / 2$. If $f(\bar{x})>0$, we set $\bar{x} \rightarrow x_{+}$and if $f(\bar{x})<0$, we set $\bar{x} \rightarrow x_{-}$. We repeat this procedure up to a tolerance range $\varepsilon$ with $|f(\bar{x})| \leq \varepsilon / 2$ and set $\bar{x}$ as an approximate root of $f$. This method converges for any function that satisfies the properties mentioned in the beginning.
The other method that is applied in this thesis, the Newton's method, is similary intuitive and usually converges quicker than the bisection method. We fix a start value $x_{s}$ and

[^23]determine the derivative of the function $f^{\prime}\left(x_{s}\right)$, in case of need numerically. Then we construct a tangent of the function at this point and determine the root $\bar{x}$ of this tangent. We redefine this quantity as our start value, $\bar{x} \rightarrow x_{s}$, and repeat this procedure up to a tolerance range $\varepsilon$ like it's done by the bisection method. Unfortunately this method diverges for lots of functions, hence it is not applicable in every situation. In contrast to the bisection method, it leads, in some circumstances, to multiple roots.

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## Eigenständigkeitserklärung

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Datum:

Unterschrift:


[^0]:    *We use natural, god-given units, so $\hbar=c=1$.

[^1]:    ${ }^{*}$ with $q_{i}$ as a generalized coordinate
    ${ }^{\dagger} \ldots$ that resonates with the principle of least action.
    ${ }^{\ddagger}$ We apply Euclidean conventions: $\left\{\gamma_{\mu}, \gamma_{\nu}\right\}=2 \delta_{\mu \nu}$ and $\gamma_{\mu}=\gamma_{\mu}^{\dagger}$. A four vector $p_{\mu}$ is spacelike, iff $p^{2}>0$.
    ${ }^{\S}$ positive-energy, on-shell

[^2]:    ${ }^{*} C$ signifies the Color.
    ${ }^{\dagger}$ A hypothetic massive vector field would bring in a mass term that is not invariant under local transformations.

[^3]:    *The photons as the exchange bosons of the electromagnetic interaction can not interact with each other because of the missing charge.
    ${ }^{\dagger}$ In other words: Is the magnetization vector $\left.\mathbf{M}^{i}\right|_{T \lesssim T_{c}}$ well defined, starting from a temperature $T>T_{c}$ ?
    ${ }^{\ddagger} \mathbf{M}$ denotes the magnetization.

[^4]:    ${ }^{*}$ That follows directly from Eq. (2.13) and the anticommutator relation $\left\{\gamma^{5}, \not D\right\}=0$.
    ${ }^{\dagger}$ One can show that $U(N)=S U(N) \times U(1)$.

[^5]:    *Because of their non-zero mass they are often called Pseudo-Goldstone-bosons.
    ${ }^{\dagger}$ The origin of this equation stems from Fig. 3.

[^6]:    *here: the current quark mass $m_{c}$
    ${ }^{\dagger} \ldots$ which is nothing else than $1 / 4 \cdot \lambda_{\mu} \lambda^{\mu}=4 / 3$, since $\lambda^{a}$ are the eight Gell-Mann matrices.

[^7]:    *For the explicit derivation, see A.2.

[^8]:    * $\Lambda$ as a cutoff parameter takes care of the convergence of the integral.

[^9]:    * $1,-i \not p,-i \not p$ and $[\not p, \not p]$.
    ${ }^{\dagger}$ An extra factor $(p \cdot P)$ is implemented due to the meson's charge conjugation properties. [12]

[^10]:    *The quark condensate will be introduced in greater detail in chapter 3.1.1, Eq. (3.7).

[^11]:    ${ }^{*}$ The derivation of Eq. $(2.56)$ and the matrix elements $\mathcal{K}_{i j}$ can be consulted in A.2.

[^12]:    ${ }^{*} \gamma_{\mathrm{T}}^{\mu}=\mathrm{T}^{\mu \nu} \gamma_{\nu}=\gamma^{\mu}-\hat{P}^{\mu} \hat{P}$, making use of the transverse projector $\mathrm{T}^{\mu \nu}=\delta^{\mu \nu}-\hat{P}^{\mu} \hat{P}^{\nu}$ with $\hat{P}^{\mu}=(\mathbf{0}, 1)$. An analogous longitudinal projector $\mathrm{L}^{\mu \nu}$ exists, so the operators T and L satisfy the completeness relation $\mathrm{T}^{\mu \nu}+\mathrm{L}^{\mu \nu}=\mathbb{1}$. Moreover the relation $P_{\mu} \gamma_{\mathrm{T}}^{\mu}=0$ holds. [13], [11]

[^13]:    *Hyperspherical coordinates are defined in chapter A.1.2.
    ${ }^{\dagger}$ The explicit calculations up to (3.5) can be consulted in A.2.
    $\ddagger$ One can justify the divergence by computing $\lim _{q^{2} \rightarrow \infty}\left(\frac{q^{2} M}{q^{2}+M^{2}}\right) \simeq M>0$.

[^14]:    *see A.2.
    ${ }^{\dagger}$ One can show this by looking at the symmetries of the QCD Lagrangian and the vacuum expectation value in Weyl representation. The left- and the right-handed spinors will mix, which implies that $\langle\overline{\mathrm{q}} \mathrm{q}\rangle \neq 0$.
    ${ }^{\ddagger}$ data in MeV .
    ${ }^{\S}$ That means: $\varepsilon$ is set as the maximum, absolute difference between the both sides of Eq. (3.5).

[^15]:    *... up to a factor of normalization ...

[^16]:    ${ }^{*} m_{\mathrm{u}} \simeq 2.3 \mathrm{MeV}, m_{\mathrm{d}} \simeq 4.8 \mathrm{MeV}$

[^17]:    *This Feynman diagram is also valid treating the $\rho$-meson in RL-truncation.
    ${ }^{\dagger}$ In fact we calculate the properties of the pion, so starting from this point, we write $m_{\pi}$ instead of $m_{0-}$.

[^18]:    ${ }^{*}$ The derivation of these expressions is implemented in an analogous way to the one of (3.14).

[^19]:    *In the mentioned publications the condition looks different because of several notations which had been introduced by the authors. Especially the factor $\alpha$ is a Feynman parameter. The equivalence of both equations is shown in A.2. For time reasons, it was not possible to check the differences between 3.25 and 2.48 within this thesis and will be done in further research.

[^20]:    ${ }^{*}\left[K_{\gamma}\right]=1,\left[m_{\rho}\right]=\mathrm{GeV},\left[E_{\rho}\right]=1 \Rightarrow\left[f_{\rho}\right]=\mathrm{GeV}^{-1}$, but $f_{\rho}$ has to have the dimension of an energy.
    ${ }^{\dagger} 0.130 \mathrm{GeV}$.

[^21]:    *Strictly speaking, both amplitudes differ by flavour properties, but since we demand isospin symmetry, the physical properties of the u - and the d-quark are identical, so is $\Lambda_{\mu}$ for both flavours.
    ${ }^{\dagger}$ This expression was derivated with respect to the imaginary parts of the pion BSA, like it's done in Ref. [16]. Furthermore, the explicit derivation can be found in A.2.

[^22]:    *Strictly speaking we obtain an imaginary value for $g$, namely $9.65 i$. One can see that this imaginary unit cannot vanish in Eq. (3.28) because of the structure of the kernel. This unit will be excluded due to convention.
    ${ }^{\dagger}$ The "model-exact" values are given by the $(8,4)$ case, if we consider the complete Dirac decomposition.

[^23]:    *Furthermore the domain must be continuous, too.

