JUSTUS-LIEBIG-
UNIVERSITÄT
GIESSEN

FB. 07 - Institut für theoretische Physik

Master thesis

# Curvature dependence of quantum gravity

**Krümmungsabhängigkeit der Quantengravitation**

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September 25, 2018

# Contents

# 1 Introduction

The standard model of particle physics can be considered one of the most successful theories of modern physics. It describes the three fundamental forces of quantum electrodynamics, the weak interaction and quantum chromodynamics in one combined framework, which is quantum field theory. In this theory we define quantum fields which propagate and interact in spacetime, and excitations of these quantum fields manifest themselves as the elementary particles that we can observe. The interactions of these quantum fields are local excitations, that happen at given spacetime points, which makes the theory a microscopic theory. The quantum fields are not build from anything smaller and are truly thought to be fundamental. The predictions that this theory can make are phenomenal, as virtually all of them can be verified experimentally with very good precision. On the other hand we have general relativity, which describes the last missing force of nature we know, gravity. The predictions of this theory are also well established physical facts, ranging from black holes and planetary orbits to gravitational waves and gravitational time dilation, just to name a few examples. Gravity as described by general relativity manifests itself as the curvature of space and time, which is governed by a set of field equations that relates the geometry of spacetime to its matter content. Gravity is a very weak force and we really only notice it at macroscopic scales, like for example when an apple is attracted by earths gravity or the moon by the earth, or on the other hand we notice gravity when we have a lot of matter concentrated in very small regions like the center of black holes. Gravity at macroscopic scales can be combined with quantum field theory very nicely, as we simply have to quantize our fields not on a flat spacetime but rather on the curved spacetime that we get from solving Einsteins equations. If we however consider very strong gravitational fields and try to combine them with quantum mechanics, we run into a problem. Consider the following situation. You want to probe a quantum field in a region of space of size $\Delta x$. This means of course, we need an uncertainty in space that is smaller than this size $\Delta x$ we want to measure. However from the uncertainty principle, we know that in order to have a very small spatial uncertainty, we will have a very large uncertainty in momentum that is located in this patch of space

$$(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2 \geq \frac{\hbar^2}{4(\Delta x)^2} \tag{1.1}$$

$$\Rightarrow \langle p^2 \rangle \geq \frac{\hbar^2}{4(\Delta x)^2} \tag{1.2}$$

If we now identify this momentum with a source of gravity which is analog to the mass $M^2c^2 \sim \langle p^2 \rangle$ we know from general relativity that this mass will have a Schwarzschild radius $r_s = 2G_N M/c^2$. The Schwarzschild radius is a quantity which is obtained when solving the vacuum Einstein equations of general relativity with a spherically symmetric ansatz for the spatial part of spacetime around an object of mass $M$. It defines an event horizon, at which point the gravitational attraction is so strong that not even light is able to escape [1]. Because of this, we will not be able to probe structures smaller than this radius since no information can reach us from inside this radius, so we know that $\Delta x$ has to be larger than $r_s$ which means that

$$\Delta x \geq 2G_N M/c^2 \tag{1.3}$$

$$\tag{1.4}$$

and when using $M \geq \frac{\hbar}{2c\Delta x}$ we find that

$$\Delta x \geq \sqrt{\frac{\hbar G_N}{c^3}} \ . \tag{1.5}$$

What this means is that there has to be a minimal length which can be probed, at least when using general relativity and quantum mechanics as they stand now. This minimal length scale is called the Planck length which is only defined by the three fundamental constants $\hbar$, $c$ and $G_N$ which are characteristic for a relativistic quantum theory and general relativity respectively. These three constants can also be used to define a Planck energy

$$l_p = \sqrt{\frac{\hbar G_N}{c^3}} \approx 1.6 \times 10^{-35}\ m \tag{1.6}$$

$$E_p = \sqrt{\frac{\hbar c^5}{G_N}} \approx 1.2 \times 10^{19}\ GeV \tag{1.7}$$

and it is likely that at these scales gravitational and quantum mechanical effects become equally strong so it is expected that classical general relativity and the standard model can not be described in a distinct manner. So what we found is that classical general relativity when naively combined with quantum mechanics really only makes sense down to a certain minimal length scale, or equivalently up to a certain energy scale, which means naively quantized general relativity does not have an ultraviolet completion. New physics is expected to occur at these scales at which a theory of quantum gravity becomes truly necessary. Of course those kinds of energies are inaccessible for us on earth and so there is no urge to find a theory which can explain experimental data in this regime. For all scales that are in range, general relativity and the standard model work out just fine. However it is very unsatisfying to know that our current theories can not be fundamental and will break down at some point. The final goal in physics should clearly be a genuinely fundamental theory that explains all physical processes and is not self contradictory in any way, and finding a theory of quantum gravity will probably play a crucial role in this endeavor. In this thesis I want to give some conceptual ideas of how such a theory could look like, and especially study one approach which is called the asymptotic safety scenario. Asymptotic safety is a generalization of asymptotic freedom, which provides an ultraviolet completion for quantum chromodynamics. Instead of a theory in which the coupling strength go to zero at for large energies, in asymptotic safety they will tend towards a fixed point.

## 1.1 Outline

Section 2 will serve as an introductory section, in which we present some of the concepts of general relativity that we will need later on. We will introduce the notion of parallel transport in order to understand curved spacetimes. After that we will have a look at Einsteins field equations which classically govern the dynamics of such curved spacetimes. We will end this section by a short discussion of maximally symmetric spaces, which are the spaces we will work on later throughout this thesis.
In section 3, we will have a first peek at some theories for quantum gravity, starting with an argument for why gravity is not perturbatively renoramlizable if quantized in the standard quantum field theoretical sense. Then we will turn away from this standard quantization and explore the basic concepts of causal dynamical triangulations, loop quantum gravity, string theory and quantum graphity, which are all very different attempts with the same goal, quantum gravity.
In section 4 the main topic of this thesis will be introduced, the asymptotic safety scenario, whose goal is to find a non perturbative renormalization for quantum gravity. We will present the functional renormalization group in general and then apply it to gravity, where we will have to make use of the background field method for reasons explained later. In subsection 4.4 we will talk about how we can still render our theory background independent even though an explicit background metric we utilize the background field method. We will build upon a vertex expansion of the effective action, that was previously established in [2–5], to obtain the beta function for a system of three coupling constants, the graviton mass parameter $\mu$, the gravitational coupling $g$ and the level three cosmological coupling $\lambda_3$ that parameterize the two and three point vertices. After that we will extend this purely gravitational setting by introducing minimally coupled scalar fields as an additional source for gravity.
Then finally in section section 5 we will discuss the results we obtained from the previously given setup. We will study the background dependence of the couplings at the ultraviolet fixed point, and analyze how the results are impacted by the introduction of the additional scalar degrees of freedom. We will also look for solutions to the equations of motion, which due to the background field method there are two of, the background equation of motion and the quantum equation of motion. We will use these solutions to obtain the physical background space on which we can then evaluate the couplings. By making use of the background equation of motion we can obtain the fixed point background potential $f(r)$ that parameterizes the effective action, and from which the background gravitational and cosmological couplings are calculated.
In the appendices we will give some further technicalities that we need for our calculations. In Appendix A we will explain how we can perform functional derivatives to obtain as an example the graviton propagator from the effective action. In Appendix B we will explain how we can perform the traces over the Laplacian operator on positively and negatively curved maximally symmetric spaces, and how we approximated the action of uncontracted covariant derivatives by an approximate momentum space, constructed analogously to [3].

# 2 General relativity

## 2.1 Newtonian gravity and the equivalence principle

Until the beginning of the 20th century Newtonian gravity was the dominant theory describing the interaction between massive bodies. It provided the missing link between falling objects, like apples for example, here on earth and the elliptic orbits of planets around the sun, which are of course as we know now both effects of the gravitational pull of massive objects. Newtonian gravity is essentially given by the same force law as the electrostatic interaction of charged particles, so let us take a look at electrostatics first [6]. We know that any charge distribution $\rho$ creates an electrostatic potential around it given by Poisson's equation

$$\vec{\nabla}^2\phi(\vec{x}) = -\frac{\rho(\vec{x})}{\epsilon_0} \tag{2.1}$$

which then gives rise to an electric field

$$\vec{E}(\vec{x}) = -\vec{\nabla}\phi(\vec{x}) \tag{2.2}$$

The kinematics of other charges $q$ with mass $m$ moving it that field are then given by

$$\vec{F} = m\ddot{\vec{x}} = q\vec{E} = -q\vec{\nabla}\phi. \tag{2.3}$$

For the simple example of an electric field created by a point charge $Q$ at $\vec{x}_0 = 0$ the solution to Poisson's equation reads

$$\phi(\vec{x}) = \frac{Q}{4\pi\epsilon_0|\vec{x}|} \tag{2.4}$$

giving rise to Coulomb's force law

$$\vec{F} = \frac{qQ}{4\pi\epsilon_0|\vec{x}|^3}\vec{x}\ . \tag{2.5}$$

Newtons gravitational force on an object of mass $m$ in a gravitational field created by a mass $M$ has the same form and reads

$$\vec{F} = -G_N\frac{mM}{|\vec{x}|^3}\vec{x} \tag{2.6}$$

with Newtons constant $G_N$ and can be obtained in general from a gravitational potential satisfying the same Laplace equation

$$\vec{\nabla}^2\phi = 4\pi G_N\rho \tag{2.7}$$

where now $\rho$ is not the charge but the mass distribution. We can also define a gravitational field analogously to the electric field

$$\vec{g} = \frac{\vec{F}}{m} \ . \tag{2.8}$$

The classical theory of gravity is therefore completely analogous to the electrostatic interaction of charged particles however one can note something very peculiar. We defined our gravitational field by dividing the force by the mass $m$ of the object on which the force acts, which means that

$$\vec{g} = \frac{m\ddot{\vec{x}}}{m} = \ddot{\vec{x}} \ . \tag{2.9}$$

Hence all objects, no matter their mass will be accelerated in such a force field in the same way. This is known as the equivalence of inertial and gravitational mass, and eventually led Einstein together with his theory of special relativity to a new theory of gravitation called general relativity. We can also read this equivalence principle as the equivalence of gravitation on the left hand side and acceleration on the right hand side, which is an equivalence that Einstein took very seriously leading him to several thought experiments. He noticed that in general it is not possible to tell the difference between sitting in a resting frame on the surface of a massive planet, being attracted by its gravitational pull, and sitting in an accelerated reference frame in free space with no gravitational field being around. Lets consider the situation when resting on the surface of a planet, we only feel the downwards pull, because the surface is stopping us from falling down further. If we were floating in space and gravity would pull us towards the planet, we would not notice it until we hit the surface. In our own frame of reference there would be no gravity present. Consequently in the other situation, when being accelerated in free space we could turn off the effects of gravity by adding gravitational pull in the opposite direction. This equivalence of gravity and acceleration is known as the string equivalence principle.

This led Einstein to generalize his ideas about inertial frames from special relativity. There we are always looking at global inertial frames, that only move at constant speeds relative to other frames because in these frames all physical laws behave the same. The generalization is that if we want to include gravitation, we need to look at local inertial frames. At each point in space we can then find a reference frame in which we don't feel any gravity and therefore our ideas of inertial frames apply. However since we can not find a global inertial frame when being in a gravitational field, this means that globally we can not describe spacetime by the metric tensor we know from special relativity $\eta_{\mu\nu}$, as it only applies locally. The global structure of spacetime has to be described by a different metric $g_{\mu\nu}$ which at each point in space can be transformed into $\eta_{\mu\nu}$ but as soon as we move away from this point it takes a different form.

These ideas transformed the Newtonian theory of gravitational forces into a theory that does not describe gravity as a force but rather a geometric effect and the inability to describe spacetime as globally flat. This theory is called general relativity.

In the following we will not concentrate so much on how particles move in such a generalized curved spacetime. Our main concern will be the curvature of spacetime itself and the dynamics of $g_{\mu\nu}$ so we will motivate the concept of curvature in terms of parallel transport of a vector and then directly dive into the Einstein field equations and the symmetries of general relativity which will be important for our quantum theory. We will also define the concept of maximally symmetric spaces, as these will be the background spacetimes we will be working on later.

## 2.2 Curvature

We have seen that it is not sufficient to describe physics in terms of a flat spacetime and we have to generalize our globally flat metric $\eta_{\mu\nu}$ to a locally flat but globally curved metric $g_{\mu\nu}(x^\alpha)$ which will also be the quantity that carries information about gravity.
To understand what curvature means let us take a look at a simple example of a curved space, the two dimensional surface of a three dimensional ball and compare it to two dimensional flat space. To begin let us consider two points in flat space and attach two parallel vectors to these points. We can now at each point define a curve that will get traced out if we push the vectors at these points always in the direction they are facing. This will result in two parallel lines that will never meet, as we would expect of parallel lines.
Now let us do the same thing for two points on the sphere. We set the position of these points such that they both lie on the equator and we attach parallel vectors both facing in the direction of the north pole. If we now trace out the curves we get when we again push the vectors in the direction they are facing, we will find that the curves start out parallel but as we get closer to the north pole the lines will in fact become closer and closer to each other until they finally meet at the north pole. Imagine two ants walking on these lines. At every point of their journey everything around them would look like they are walking on a flat plane because they can not see that globally the sphere is curved. They would walk on these lines and notice that for some reason their parallel lines are attracted in some mysterious way, as if there was a force between them. In our four dimensions this is the same situation. We can not directly detect the curvature of our four dimensional spacetime, and only if we were to embed our spacetime in a higher dimensional space we could see that globally our universe is not flat.

### 2.2.1 Parallel transport

In the following I will assume some of the basic concepts of tensor calculus, and a more profound introduction to this topic can be found in [1].
There is a way to measure the intrinsic curvature of space and it has to do with the same concept that we used to construct our converging parallel lines. We said that we would displace the vectors we attached to the points always in the direction that they are facing. Now we demand a similar but more general thing, we want our vector to not change in direction and length when we displace it along some curve, which just means our vector,

let's call it $A$, will be a constant along the curve. It will always be parallel to itself while being transported, which was also the case previously, when we displaced it in the direction it was facing. If we use Cartesian coordinates for our plane we can write our vector in terms of orthonormal basis vectors along the $x$ and $y$ axis $A = A^x e_x + A^y e_y = A^\mu e_\mu$. Let us parametrize the curve we want to displace the vector along by a parameter $s$ then we can write the change of our vector as

$$0 = \frac{dA(s)}{ds} = \frac{dx^\mu}{ds}\frac{\partial A}{\partial x^\mu} \tag{2.10}$$

where now $\frac{dx^\mu}{ds}$ is the tangent vector of the curve. If now express $A$ in terms of its basis vectors we can find a formula to compute the change of each component of $A$. Considering that our basis vectors are constant in flat space, we find

$$0 = \frac{dx^\mu}{ds}\frac{\partial A^\alpha}{\partial x^\mu} e_\alpha \ . \tag{2.11}$$

So the change of the components of $A$ in the direction of $e_\alpha$ is given by

$$0 = \frac{dx^\mu}{ds}\frac{\partial A^\alpha}{\partial x^\mu} \tag{2.12}$$

To be even more concrete we can also set our curve to be aligned with the $x$ axis $x(s) = s e_x$ and we find

$$\frac{\partial A^\alpha}{\partial x} = 0 \tag{2.13}$$

so the components of $A$ are not supposed to change when transporting it along this straight line, which makes perfect sense. We could also have rewritten this equation as

$$\frac{dA^\alpha}{ds} = 0 \tag{2.14}$$

in which case we see that along any curve, not only specifically one along the $x$ axis, the components of $A$ stay unchanged. However in the case of our spherical surface, one crucial thing in the calculation changes, the basis vectors, which we used to express $A$ are now position dependent. We can not find a constant global coordinate system on the sphere and therefore the basis vectors will also change along any curve and we have $e_\alpha = e_\alpha(s)$. Performing the same steps as before we find

$$0 = \frac{dA}{ds} = \frac{dx^\mu}{ds}\left(\frac{\partial A^\alpha}{\partial x^\mu} e_\alpha + \frac{\partial e_\alpha}{\partial x^\mu} A^\alpha\right) \tag{2.15}$$

where the last term explicitly represents the change in basis vectors along the curve. When projected in the direction of the basis vectors the expression reads

$$0 = \frac{dx^\mu}{ds}\left(\frac{\partial A^\alpha}{\partial x^\mu} + \Gamma^\alpha{}_{\mu\nu} A^\nu\right) = \frac{dx^\mu}{ds}\nabla_\mu A^\alpha \tag{2.16}$$

where we have defined two things, $\Gamma$ and $\nabla$. ${\Gamma^\alpha}_{\mu\nu}$ are is called a connection and represents exactly the change in basis vectors

$$ {\Gamma^\alpha}_{\mu\nu} A^\nu = \frac{de_\nu}{dx^\mu} A^\nu e^\alpha \tag{2.17} $$

It is called a connection because in someway it provides a method to connect two different neighboring tangent spaces on our surface and compare vector components between the two in order to formulate a meaningful derivative, which brings us to the second quantity, the covariant derivative. It reflects the actual change of the components of a vector, considering also the change in basis vectors

$$ \frac{dA}{dx^\mu} e^\alpha = \nabla_\mu A^\alpha = \frac{\partial A^\alpha}{\partial x^\mu} + {\Gamma^\alpha}_{\mu\nu} A^\nu \ . \tag{2.18} $$

All in all we have found that if we want to displace our vector $A$ along the curve parameterized by $s$ its components have to satisfy the condition

$$ 0 = \frac{dx^\mu}{ds} \nabla_\mu A^\alpha = \frac{dA^\alpha}{ds} + \frac{dx^\mu}{ds} {\Gamma^\alpha}_{\mu\nu} A^\nu \ . \tag{2.19} $$

We can also formulate this condition for tensors of higher rank in which case every new index will introduce another term involving the connection $\Gamma$. We can find explicit expressions for the connection $\Gamma$ if we remember the requirement we gave in the beginning, we do not want our displaced vectors to change in length $A^2 = g_{\mu\nu} A^\mu A^\nu$. This can only be achieved by requiring $\nabla_\alpha g_{\mu\nu} = 0$. If our covariant derivative satisfies this condition we called it metric compatible and we can write it as

$$ {\Gamma^\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left( \partial_\mu g_{\beta\nu} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu} \right) \ . \tag{2.20} $$

In this case we call the connection a Christoffel symbol.

### 2.2.2 The curvature tensor

So how can we now use all this to measure the intrinsic curvature of our sphere? Let us start with flat space again, if we transport our vector along any curve we have seen that its components will not change, so if we transport it along a closed loop, back to its starting point, we will end up with the same vector we started with. Now lets think about the transport of a vector on the sphere which is shown in Figure 1. We start at point $P_1$ facing in the direction of the equator. Then we transport it to point $P_2$ then $P_3$ and then back to the starting point $P_1$. What we see is that the direction in which our vector faces has changed by 90 degrees, this is only possible in curved spaces and so we can use this fact to measure the intrinsic curvature.
To make this measurement more precise we will consider an infinitesimal loop as shown in Figure 2.

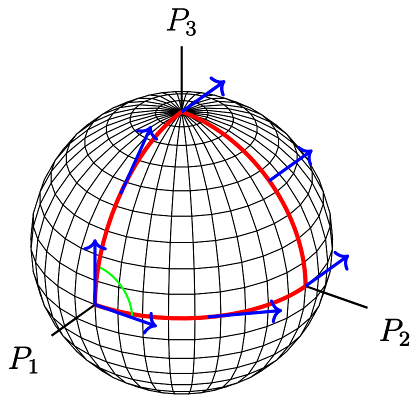



Figure 1: Parallel transport of a vector along the red curve on a spherical surface. After transportation the vector will be rotated relative to its starting orientation.

We want to quantify the total change of the components of a vector $A$ when we transported it along a parallelogram, which is spanned by the vectors $a_\mu$ and $b_\mu$, whose absolute value is considered to be small, by

$$\delta A^\mu = R^\mu{}_{\nu\alpha\beta} A^\nu a^\alpha b^\beta \tag{2.21}$$

If we perform this transport and neglects all contributions cubic in the small vectors spanning the parallelogram, we find the following expression for the so called Riemann curvature tensor [1]

$$R^\mu{}_{\beta\sigma\rho} = \partial_\sigma \Gamma^\mu{}_{\beta\rho} - \partial_\rho \Gamma^\mu{}_{\beta\sigma} + \Gamma^\mu{}_{\nu\sigma}\Gamma^\nu{}_{\beta\rho} - \Gamma^\mu{}_{\nu\rho}\Gamma^\nu{}_{\beta\sigma} \ . \tag{2.22}$$

With the help of this tensor we can measure the curvature in the various directions of our space, and we know that if it vanishes globally then there is no curvature present.
Let us define symbols for complete symmetrization or antisymmetrization so we can compactly write its symmetries. We define

$$T_{[\mu_1 \ldots \mu_n]} = \frac{1}{n!} \sum_\sigma \operatorname{sgn}(\sigma) T_{\mu_{\sigma(1)} \cdots \mu_{\sigma(n)}} \tag{2.23}$$

$$T_{(\mu_1 \ldots \mu_n)} = \frac{1}{n!} \sum_\sigma T_{\mu_{\sigma(1)} \cdots \mu_{\sigma(n)}} \tag{2.24}$$

with a sum over all possible permutations $\sigma$. The sign of $\sigma$ is determined by $\operatorname{sgn}(\sigma) = (-1)^m$, with the number of transpositions $m$ needed for the permutation. For example in the case of three indices one has

$$T_{[\mu\nu\sigma]} = \frac{1}{6} \left( T_{\mu\nu\sigma} - T_{\nu\mu\sigma} + T_{\nu\sigma\mu} - T_{\mu\sigma\nu} + T_{\sigma\mu\nu} - T_{\sigma\nu\mu} \right) \ . \tag{2.25}$$

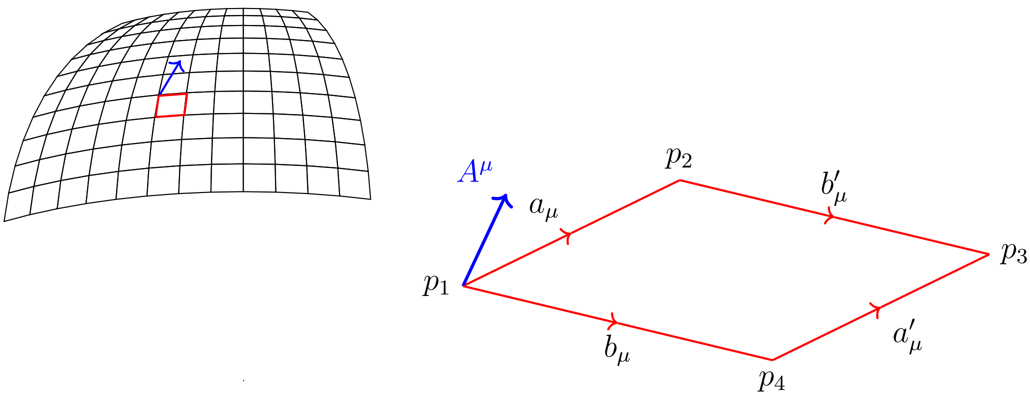



Figure 2: Parallel transport of a vector along an infinitesimal parallelogram spanned by the vectors $a_\mu$ and $b_\mu$ as the definition of the Riemann curvature tensor.

The Riemann tensor $R_{\mu\nu\sigma\rho} = g_{\alpha\mu} R^\alpha{}_{\nu\sigma\rho}$ then has the following symmetry relations

$$R_{\mu\nu\sigma\rho} = -R_{\nu\mu\sigma\rho} \tag{2.26}$$
$$R_{\mu\nu\sigma\rho} = -R_{\mu\nu\rho\sigma} \tag{2.27}$$
$$R_{\mu\nu\sigma\rho} = R_{\sigma\rho\mu\nu} \tag{2.28}$$
$$R_{\mu[\nu\sigma\rho]} = 0 \tag{2.29}$$
$$\nabla_{[\alpha} R_{\mu\nu]\sigma\rho} = 0 \ . \tag{2.30}$$

Due to these symmetries there exists only one independent non zero contraction of the Riemann tensor which is called the Ricci tensor

$$R^\mu{}_{\alpha\mu\beta} = R_{\alpha\beta} \tag{2.31}$$
$$R_\mu{}^\alpha{}_{\nu\alpha} = R^\alpha{}_{\mu\alpha\nu} = R_{\mu\nu} \tag{2.32}$$
$$R_{\alpha\beta\mu}{}^\mu = 0 \tag{2.33}$$
$$R^\mu{}_{\mu\alpha\beta} = 0 \tag{2.34}$$

This tensor is symmetric in its indices

$$R_{\alpha\beta} = R^\mu{}_{\alpha\mu\beta} = R_{\mu\beta}{}^\mu{}_\alpha = R_{\beta\alpha} \tag{2.35}$$

and its trace is called the curvature scalar

$$R = g^{\mu\nu} R_{\mu\nu} = R^\mu{}_\mu \ . \tag{2.36}$$

The commutator of covariant derivatives is also related to curvature, for example when differentiating vector components, the commutator is given by

$$[\nabla_\mu, \nabla_\nu] A^\beta = R^\beta{}_{\gamma\mu\nu} A^\gamma \tag{2.37}$$

which can be seen by plugging in the expression for $\nabla_\mu$, given earlier. Since the covariant derivative acts differently on tensors of different type, the commutator will also look different. For example for scalars the covariant derivative is simply the ordinary derivative which makes the commutator zero and for tensors of rank two an additional term including the curvature tensor comes up

$$[\nabla_\mu, \nabla_\nu]T^{\alpha\beta} = R^\alpha{}_{\gamma\mu\nu}T^{\gamma\beta} + R^\beta{}_{\gamma\mu\nu}T^{\alpha\gamma} \ . \tag{2.38}$$

These commutation rules will especially be important in later sections when we will use them to symmetrize products of covariant derivatives according to

$$\nabla_\mu\nabla_\nu = \frac{1}{2}\left(\nabla_\mu\nabla_\nu + \nabla_\nu\nabla_\mu\right) + \frac{1}{2}[\nabla_\mu, \nabla_\nu] \ . \tag{2.39}$$

## 2.3 Symmetries of general relativity and the Einstein field equations

We want to describe gravitational phenomena not by some force acting on masses, because the idea in general relativity is freely falling objects are force free and following their natural paths through a curved spacetime. The geometry of spacetime, which is encoded in the metric tensor $g_{\mu\nu}$ tells objects how to move. This alone is not that much different to special relativity, since the only difference is that instead of a flat spacetime, we would have to formulate our physical laws in this general curved spacetime. We could formulate this theory in such a way that it is invariant under general coordinate transformations, or diffeomorphism, just by replacing the flat metric $\eta_{\mu\nu}$ by the general metric $g_{\mu\nu}$ and replace partial derivatives with covariant ones and we would have included the possibility of gravitational effects on our objects in this theory.

However, the interesting part of general relativity is how the curvature of spacetime itself is generated. We know from Newtonian gravity that matter creates a gravitational potential around it that tells the other masses how they move in this potential. In general relativity, as we said, spacetime itself defines how objects move, so matter somehow has to tell spacetime how to curve. This however means that the metric tensor becomes a dynamical quantity that is the solution of some equations of motion that determine its relation to the present matter content. In special relativity such dynamics of the spacetime are not present and there is no way we could influence the geometry of space. One could view the metric of special relativity as a background field that is non-dynamic. In general relativity there are no non-dynamic quantities and therefore no background structures. Hence we call general relativity a background independent theory. All structures and quantities in this theory are generated by some dynamical laws that only relate them to one another and do not refer to any given background structures.

Later when we explore some possibilities for a theory of quantum gravity this requirement will again be very important, as not all of the approaches respect it. In fact the theory we will formulate later in section 4 heavily relies on a background metric and our main task in this thesis will be to study the dependence of this theory on the background metric.

So what we need to obtain the dynamics for our metric tensor is, just as in any other theory, an action functional that when varied with respect to the metric gives us equations of motion. For field theories that are formulated in flat space this functional takes the form

$$S = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi) \tag{2.40}$$

with a Lagrangian that depends on the field $\phi$ itself and derivatives thereof. For general relativity we want an action functional that is invariant under coordinate transformations so we already know we will have to modify our integration by a factor $\sqrt{g} = \sqrt{\det g_{\mu\nu}}$ and replace partial derivatives by covariant ones. Also we want a Lagrangian that depends on the metric $g_{\mu\nu}$ and its covariant derivative. But since we already know that the metric is covariantly constant, this dependence will drop out and we find

$$S = \int d^4x \sqrt{g} \mathcal{L}(g_{\mu\nu}) \ . \tag{2.41}$$

We also know that the Lagrangian is a scalar quantity, so we need to construct scalars from $g_{\mu\nu}$. Contracting the metric with itself will only yield constants, which we can of course include in the action. Another scalar we can construct is the Ricci scalar $R$, and so the simplest non trivial action we could write down is

$$S_{\text{EH}} = \frac{1}{16\pi G_N} \int d^4x \sqrt{g} \left(2\Lambda - R\right) \tag{2.42}$$

which is called the Einstein Hilbert action, where $G_N$ is the gravitational coupling, known from Newtonian gravity, and $\Lambda$ is known as the cosmological constant. Since we have not included any matter coupling, this will produce vacuum field equations for the metric which read

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0. \tag{2.43}$$

If we remember the definition of the Ricci scalar and tensor we see that we have second order partial differential equations, which even though we have not even introduced any matter to the system are already highly non linear. This non linearity is due to the fact that the metric itself is a dynamical field which also has momentum and energy, which consequently serve as sources of gravity. This is a very different situation to for example electro dynamics where electric and magnetic fields are not charged on their own and therefore do not interact with one another. They follow the superposition principle while the gravitational field clearly does not due to this self interaction. We can couple other fields to gravity by simply including their corresponding Lagrangian next to the gravitational one

$$S = \int d^4x \sqrt{g} \left(\mathcal{L}_{\text{EH}} + \mathcal{L}_{\text{matter}}\right) \ . \tag{2.44}$$

Varying this action with respect to the matter fields results in modified equations for their dynamics that respect the curvature of spacetime and therefore include the effects of gravity, while a variation with respect to the metric gives the Einstein field equations that couple the gravitational field to the matter fields

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu} \tag{2.45}$$

where $T_{\mu\nu}$ is the stress energy tensor of $\mathcal{L}_{\text{matter}}$ and is given by

$$T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta \sqrt{g} \mathcal{L}_{\text{matter}}}{\delta g^{\mu\nu}} \ . \tag{2.46}$$

We will later couple non interacting scalar fields to gravity just in this way.
Finally let us talk about the symmetry of general relativity that we mentioned earlier, which is the invariance under coordinate transformations. Let us therefore take a look at how the metric $g_{\mu\nu}$ transforms under a coordinate transformation $x'^{\mu} = x^{\mu} - \xi^{\mu}$. We want to compare the metric in the old coordinates $g_{\mu\nu}(x)$ to the transformed metric, but also evaluated at the old coordinates

$$\delta_\xi g_{\mu\nu} = g'_{\mu\nu}(x) - g_{\mu\nu}(x) \tag{2.47}$$

So let us first calculate the transformed metric evaluated at the new coordinates

$$\begin{aligned} g'_{\mu\nu}(x') &= \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) \\ &= \left( \delta^\alpha_\mu + \frac{\partial \xi^\alpha}{\partial x'^\mu} \right) \left( \delta^\beta_\nu + \frac{\partial \xi^\beta}{\partial x'^\nu} \right) g_{\alpha\beta}(x) \\ &= g_{\mu\nu}(x) + \frac{\partial \xi^\alpha}{\partial x'^\mu} g_{\alpha\nu}(x) + \frac{\partial \xi^\beta}{\partial x'^\nu} g_{\mu\beta}(x) + \mathcal{O}(\xi^2) \end{aligned}$$

from which we can conclude by expanding

$$g'_{\mu\nu}(x') = g'_{\mu\nu}(x) - \frac{\partial g_{\mu\nu}(x)}{\partial x^\lambda} \xi^\lambda \tag{2.48}$$

and

$$\frac{\partial \xi^\alpha}{\partial x'^\mu} = \frac{\partial \xi^\alpha}{\partial x^\nu} \frac{\partial x^\nu}{\partial x'^\mu} = \frac{\partial \xi^\alpha}{\partial x^\nu} \frac{\partial (x'^\nu + \xi^\nu)}{\partial x'^\mu} \tag{2.49}$$

$$= \frac{\partial \xi^\alpha}{\partial x^\nu} \left( \delta^\nu_\mu - \frac{\partial \xi^\nu}{\partial x'^\mu} \right) = \frac{\partial \xi^\alpha}{\partial x^\mu} + \mathcal{O}(\xi^2) \tag{2.50}$$

that

$$\delta_\xi g_{\mu\nu} = g'_{\mu\nu}(x) - g_{\mu\nu}(x) = \frac{\partial g_{\mu\nu}(x)}{\partial x^\lambda} \xi^\lambda + \frac{\partial \xi^\alpha}{\partial x^\mu} g_{\alpha\nu}(x) + \frac{\partial \xi^\beta}{\partial x^\nu} g_{\mu\beta}(x) + \mathcal{O}(\xi^2) \ . \tag{2.51}$$

which we can then again rewrite as

$$\begin{aligned}\delta_\xi g_{\mu\nu} &= (\partial_\alpha g_{\mu\nu})\,\xi^\alpha + g_{\mu\alpha}\nabla_\nu\xi^\alpha + g_{\sigma\nu}\nabla_\mu\xi^\alpha - (\Gamma_{\mu\nu\alpha} + \Gamma_{\nu\mu\alpha})\,\xi^\alpha \\ &= (\partial_\alpha g_{\mu\nu})\,\xi^\alpha + \nabla_\nu\xi_\mu + \nabla_\mu\xi_\nu - (\partial_\alpha g_{\mu\nu})\,\xi^\alpha \\ &= \nabla_\nu\xi_\mu + \nabla_\mu\xi_\nu = \mathcal{L}_\xi g_{\mu\nu}\end{aligned} \tag{2.52}$$

where $\mathcal{L}_\xi g_{\mu\nu}$ is the Lie derivative of $g_{\mu\nu}$ along the field $\xi$. We therefore find that the metric $g_{\mu\nu}$ is invariant under a coordinate transformation if its Lie derivative along in the direction of this transformation vanishes. The vectors $\xi$ are then called Killing vectors. Later when formulating a quantum theory of gravity using the path integral approach, we will have to keep this invariance in mind, as we will have to impose a gauge fixing condition to prevent an over counting of equivalent metrics.

## 2.4 Maximally symmetric spaces

We will base all following calculations on a maximally symmetric background spacetime, or rather, as we are dealing with euclidean quantum gravity, on maximally symmetric spaces. The term maximally symmetric means that space has the same curvature everywhere and in every direction which makes the Ricci scalar the only remaining value that has to be determined to completely specify the curvature [7].
The Ricci curvature tensor and the Riemann tensor can be related to the Ricci scalar for a $d$- dimensional maximally symmetric space according to

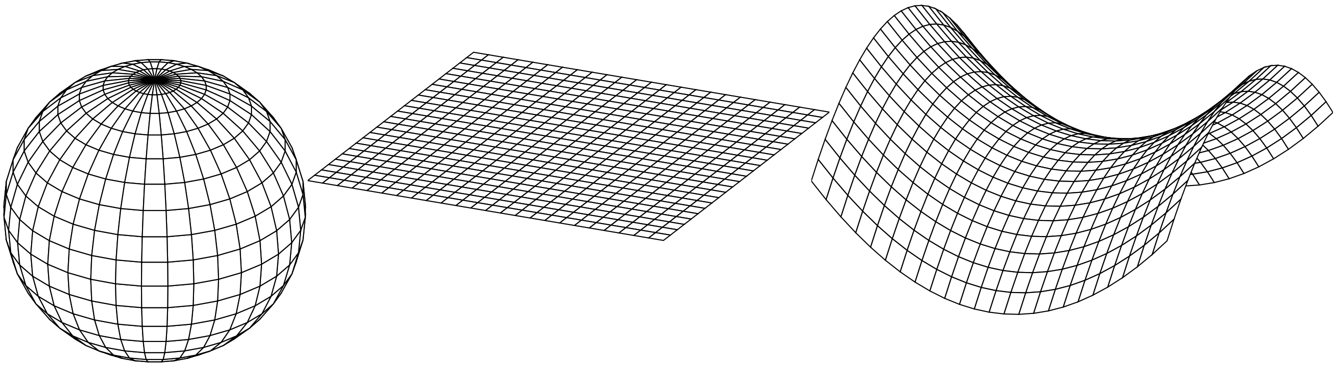
$$R^{\mu\nu} = \frac{R}{d} g^{\mu\nu} \tag{2.53}$$

$$R^{\mu\nu\rho\sigma} = \frac{R}{d(d-1)}\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right)\ , \tag{2.54}$$

by assuming, since they are maximally symmetric that they also have a maximal set of $d + \frac{d(d-1)}{2}$ independent Killing vector fields that represent the $d$ translational symmetries and the $\frac{d(d-1)}{2}$ rotational symmetries of such spaces. Analyzing these vector fields, by making use of commutation relations of the covariant derivatives will yield the desired relations.
One can also show by the same method that on these spaces the Ricci scalar is constant everywhere

$$\partial_\mu R(x) = 0 = \nabla_\mu R(x) \tag{2.55}$$

and we can use its value to identify three different types of maximally symmetric spaces. When $R = 0$ the space is just a flat manifold because its curvature vanishes everywhere. When $R > 0$ one is looking at a finite $d-$dimensional sphere whose radius can be related to the curvature scalar, and when $R < 0$ one has an infinite hyperpolic space. When working with a Lorentzian signature these are called Minkowski, de Sitter and anti de



(a) Spherical space. (b) Flat space. (c) Hyperbolic space.

Figure 3: The three different types of maximally symmetric two dimensional spaces, embedded in three dimensional flat space are shown. The sphere has a positive curvature, the hyperbolic space has negative curvature and the flat space has vanishing curvature.

Sitter spacetime respectively. The two dimensional examples of these spaces are shown in figure 3, where they were embedded in three dimensional flat space.
When dealing with a positive curvature scalar, we can also relate its value to the radius $r$ and therefore also to the volume $V$ of the $d-$sphere

$$R = \frac{d(d-1)}{r^2} \ . \tag{2.56}$$

In the four dimensional case one finds

$$R = \frac{12}{r^2} \tag{2.57}$$

so the volume of the four sphere in terms of the Ricci scalar reads

$$V = \frac{8}{3}\pi^2 r^4 = \frac{384\pi^2}{R^2} \ . \tag{2.58}$$

We will need the Laplacian spectra on these three types of spaces later, which will be different when acting on scalars, vectors or tensors respectively. Generally we can note that in hyperbolic and in flat space we have infinite volumes and therefore we expect eigenvalues which can be labeled by a continuous parameter $\sigma$. Their multiplicities will then be given by spectral densities $\rho(\sigma)$. For a positive Ricci scalar on the other hand we have a finite volume, so the eigenvalues will have a discrete label $l$, with multiplicities $m_l$. We can also expect the eigenvalues to grow if we increase the curvature of space. To see this, consider a one dimensional example: the Laplacian on a circle. If we embed our circle in two dimensional space we can write

$$-\partial_\mu \partial^\mu \psi(x,y) = \lambda^2 \psi(x,y) \tag{2.59}$$

or in polar coordinates

$$-\frac{1}{r}\partial_r\left(r\partial_r\psi(r,\theta)\right)-\frac{1}{r^2}\partial_\theta\partial_\theta\psi(r,\theta)=\lambda^2\psi(r,\theta) \tag{2.60}$$

If we restrict ourselves to fixed values of $r$ we obtain the laplacian spectrum on the circle

$$-\frac{1}{r^2}\partial_\theta\partial_\theta\psi(r,\theta)=\lambda^2\psi(r,\theta) \tag{2.61}$$

which is solved by eigenfunctions $\psi(r,\theta)=e^{i\theta p}$ and therefore $\lambda^2=\frac{p^2}{r^2}$ and as we see the eigenvalues grow if we shrink the radius $r$, which in this case is proportional to the one dimensional volume of our circle. Therefore a smaller volume leads to larger eigenvalues. This is also true in higher dimensions and since our smaller volume corresponds to larger curvature, this means that the Laplacian eigenvalues grow with curvature.
In Appendix B the different spectra for positive and negative curvature are listed.
Finally let us consider the classical Einstein field equations on maximally symmetric backgrounds. We can write the Ricci tensor as $R_{\mu\nu}=\frac{R}{4}g_{\mu\nu}$ and therefore find

$$\frac{1}{4}Rg_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}+\Lambda g_{\mu\nu}=0 \tag{2.62}$$

$$\Leftrightarrow R=4\Lambda\ . \tag{2.63}$$

Therefore the value of the cosmological constant decides if we are dealing with a negatively or a positively curved space.

# 3 Ideas for quantum gravity

In the last section we had an insight into how the dynamics of spacetime are described by general relativity. The goal now is to understand why it is not possible to find a perturbatively renormalizable theory for quantum gravity in the usual quantum field theoretical sense, and what other possibilities there are for quantizing gravity. For that I firstly want to motivate why we can not define a predictive theory of quantum gravity using standard perturbation theory and then compare four very different examples of theories that people are working on today. These will be string theory, causal dynamical triangulations (CDT), loop quantum gravity (LQG) and quantum graphity.

## 3.1 The failure of perturbative quantum gravity

The standard model of particle physics is the most accurate theory we know to describe quantum phenomena. It utilizes the framework of quantum field theory together with the idea of renormalization to make predictions that are verified by experiment up to a lot of decimal places. In quantum electrodynamics we can make use of a perturbative expansion in terms of Feynman diagrams to find very good approximations of quantities like greens functions which can be used to calculate observables like cross sections. The perturbative expansion alone produces ultraviolet divergences which can be tamed by a procedure called perturbative renormalization. The idea is that the coupling constants we find for example in the Lagrangian of our theory are not observable quantities and we can redefine their values in such a way that they essentially absorb the divergences and we end up with finite observables. At each order of such an expansion we have to find counter terms that we add to the definition of these unobservable couplings which then exactly cancel the divergences. If we try to treat quantum gravity just the in the same perturbative fashion we shall see that this renormalization procedure is not always possible. Let us start with the Einstein Hilbert action, for now without a cosmological constant term

$$S_{\mathrm{EH}} = \frac{1}{16\pi G_N}\int d^4x\sqrt{g}R. \tag{3.1}$$

For our perturbative expansion in terms of Feynman diagrams we will have to expand this action in terms of the dynamic variable which in this case is of course the metric. We will write it as $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ and expand in powers of $h$. We only want to motivate why the perturbative renormalization fails, so we will not care about the exact structure of each term in the expansion but only take a look at what kinds of vertices will appear in such an expansion. Expanding the Ricci scalar will yield second derivatives of $h_{\mu\nu}$ in each term so we will find vertices of the form

$$(\partial h)^2 \tag{3.2}$$
$$h(\partial h)^2 \tag{3.3}$$
$$... \tag{3.4}$$
$$h^{N-2}(\partial h)^2 \tag{3.5}$$
$$... \tag{3.6}$$

and therefore the action includes interaction vertices with arbitrary powers of the field $h$. The first of these terms is of course no vertex but the kinetic term which yields the propagator. The others $n-$point graviton vertices will always be proportional to two powers of momentum $p^2$, because of the two derivatives. At the moment the kinetic term $(\partial h)^2$ gets multiplied by a dimensionful coupling $\frac{1}{G_N}$ but in order to cast the action in standard quantum field theory terms, we will have to normalize it by rescaling $h$ to a field of mass dimension one by writing $h \to \sqrt{G_N}h$. The expanded action will then read, again not writing the explicit tensor structures,

$$S_{\mathrm{EH}} \propto \int d^4x\left((\partial h)^2 + \sqrt{G_N}h(\partial h)^2 + G_N h^2(\partial h)^2 + ... + \sqrt{G_N}^N h^N(\partial h)^2 + ...\right) \tag{3.7}$$

One can then go ahead and calculate the superficial degree of divergence for arbitrary $L-$loop diagrams in $d$ dimensions by counting the powers of momentum contained in each vertex $V$ which is $p^2$ and propagator $P$ which is $1/p^2$, of a diagram. Each loop $L$ will be associated with a momentum integration $\propto p^d$. Therefore we find the superficial degree of divergence to be

$$D = dL + 2V - 2P \ . \tag{3.8}$$

The number of vertices $V$ and the number of internal lines $P$ can be related to the number of loops $L$ by $L = 1 + P - V$ such that we find that the superficial degree of divergence only depends on the number of loops

$$D_L = d + (L-1)(d-2) \tag{3.9}$$

meaning that $L-$loop diagrams in four dimensions will at most diverge as

$$L - \text{loop-diagram} \sim \Lambda_{\text{UV}}^{4+2(L-1)} \tag{3.10}$$

where $\Lambda_{\text{UV}}$ would be some ultraviolet cutoff [8].
Since this is true for all $L-$loop diagrams, there will be infinitely many different divergent diagrams at each order since the action features infinitely many different vertices, which can be used to construct these diagrams. And even worse each loop order will have infinitely many new divergent diagrams. Together this leads to the fact that one would need infinitely many counter terms in the Lagrangian that would have to be fixed by an experiment in order to render the theory renormalizable. However then our theory would be non predictive. Let us compare this for example to quantum electrodynamics. There one only has to measure the charge and the mass of an electron in order to fix the counter terms that are needed to cancel all divergent diagrams [9].
The argument here was only based on the superficial degree of divergence, and if one was to actually calculate the divergences that occur at each loop order, one would find that for all on shell quantities, which are of course relevant for observables, the one loop divergences vanish, making the theory one loop renormalizable. At two loops, or when including matter in the action, the theory is again non renormalizable [8]. Ultimately the fact that gravity is perturbatively non renormalizable is linked to the dimensionality of Newtons coupling $G_N$ which is connected to the two powers of momentum in each vertex. It has a mass dimension of negative two, and its dimensionless counter part grows quadratically with energy. Later when we introduce the asymptotic safety scenario, we will allow Newtons coupling to scale differently with energy, in order to achieve a non perturbative renormalization of gravity.
We can also try to understand the non renormalizability physically. If we quantize gravity just like any other field, we have to postulate quantum fluctuations of spacetime which would correspond to the exchange particles of gravity, the gravitons. These gravitons have to have energy and momentum, which however due to the non linearity of the field equations serve as sources for gravity themselves, such that they produce more quantum excitations which again lead to more gravity. At some point there are so many excitations that there is enough energy to form a black hole and we are unable to exctract information about the excitations, making the theory unpredictive. In theories like quantum electrodynamics no such self interactions

## 3.2 Causal dynamical triangulations

Now that we have seen that perturbative quantum gravity is not really an option let us explore another type of theory. In low energy quantum chromodynamics we know that perturbation theory is also non applicable, and a lot of research is done using a lattice formulation which regularizes the theory by putting it on a grid with finite lattice spacing $a$. Due to this finite lattice spacing a cutoff is introduced to the theory which regularizes any divergences. The grid resembles a fixed flat spacetime and the dynamic variables are defined either on the lattice sites or on the links between the sites. This is of course no viable option for quantum gravity since as we discussed the spacetime itself is dynamic, so choosing a fixed grid to quantize gravity on would be inherently background dependent.

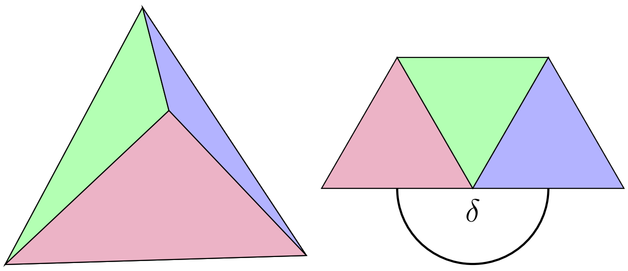



Figure 4: When considering the vertex at which three of the four triangles of a tetrahedron meet, one finds that their angle does not add up to $2\pi$ which means that there is curvature present at this vertex.

The theory of causal dynamical triangulations [10–12] performs this step from quantizing a theory on a fixed lattice to quantizing the lattice itself. It utilizes a discretized version of the Einstein Hilbert action and formulates a discrete path integral over different triangulations of spacetime. These triangulations are built from discrete four dimensional building blocks called simplices that have a side length $a$ that is taken to zero i order to perform the continuum limit. These simplices are the four dimensional generalizations of triangles in two dimensions and tetrahedra in three dimensions. To make sure that the triangulated spacetimes have a causal structure one has to cut spacetime into equal time slices and equip the simplices with timelike edges that connect two of these slices and spacelike edges that only lie in one slice. Gluing the simplices together in different ways, then defines the different triangulations which are summed over in a path integral. To obtain the amplitude for each of these triangulations one has to calculate the discretized Einstein Hilbert action, and therefore one also needs a notion of discretized curvature. To understand how this discrete version of curvature can be quantified let us consider a two dimensional example, where we glue three triangles together as depicted in Figure 4. The discrete curvature is then located at the vertex where the three triangles meet. By cutting one side open and flattening out the construction we see that the angles of each triangle do not add up to $2\pi$, so there is a deficit angle of $\delta = \pi$. If we were to glue six triangles together in the same way there would not be a deficit angle and therefore also no curvature at the vertex. Generalizing this idea to four dimensions gives a measure of the discrete curvature for each

configuration.
The resulting action functional, which has three free parameters corresponding to Newtons constant, the cosmological constant and an asymmetry parameter that accounts for different scaling of timelike and spacelike edges, is then used to generate configurations of the system, which depending on the values of the three parameters have very different properties. By varying these parameters a phase diagram of CDT was found where the different phases are distinguished by the different types of generated spacetime triangulations. The triangulations can be distinguished by comparing how the total three dimensional spatial volume varies between the different equal time slices, and what was found is that in one of the phases there exist spacetimes that have a finite three dimensional volume over a large range of time slices, while in the other phases the volume is either highly fluctuating or only extended over a very short time period.
This is a very important result, since it proves that CDT is able to reproduce four dimensional extended spacetimes that general relativity describes.

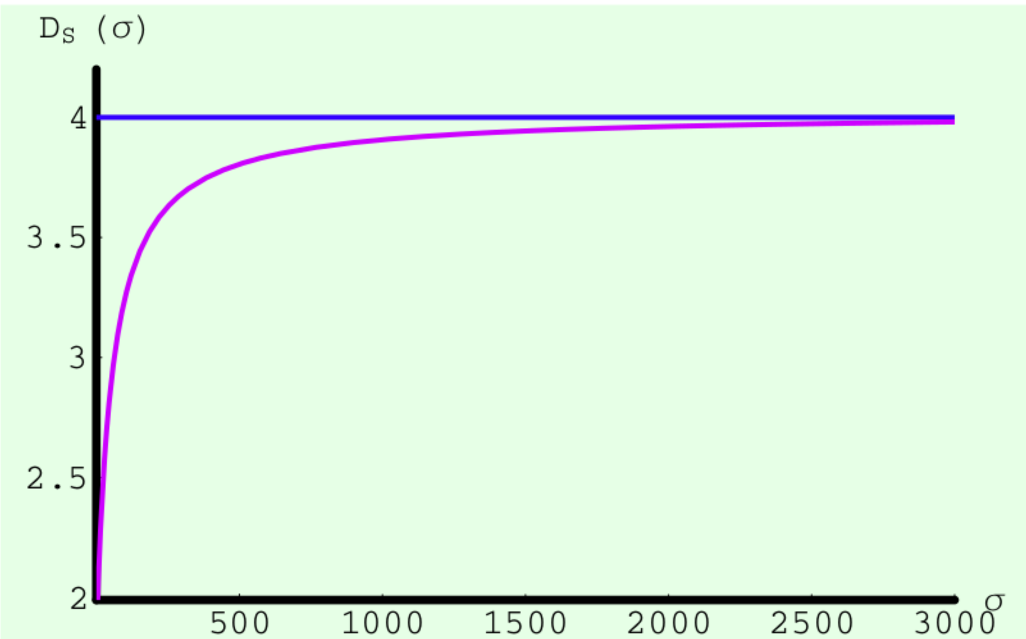



Figure 5: Spectral dimension $D(\sigma)$, obtained from a diffusion process with diffusion time $\sigma$. The blue line corresponds to the classical four dimensions known from general relativity. Figure taken from [11].

The configurations can be used to perform measurements of certain observables, one of which is the spectral dimension [11]. It can be obtained by considering a diffusion process with diffusion time $\sigma$. The process of how an initially localized quantity will spread with time according to the heat equation is sensitive to this dimensionality. In flat $d-$dimensional space the solution to this heat equation is given by

$$K(x, y, \sigma) = \frac{1}{(4\pi\sigma)^{d/2}} e^{-\frac{|x-y|^2}{4\sigma}} , \tag{3.11}$$

which generalizes to the heat kernel on Riemannian manifolds and its known heat kernel expansion, which we will also make use of later. Considering the trace of the heat kernel, again for flat spaces, gives an expression proportional to the total, formally infinite, volume $V$, and so dividing it by this volume gives some kind of return probability to start at point $x$ and end at the same point which is proportional to

$$\frac{1}{V} \int d^d x K(x, x, \sigma) \propto \frac{1}{\sigma^{d/2}} . \tag{3.12}$$

So setting up a generalized diffusion process on our quantum universe generated by our path integral is sensitive to the spectral dimension $D_S(\sigma)$ of this space. Small $\sigma$ probe small scales, while large $\sigma$ probe large scales as there is more time to wander around in the space. An analysis of this spectral dimension predicts it to reduce to a value close to $D_S(\sigma \to 0) = 2$ when going to very small scales, while for larger scales it converges to the usual four dimensions as shown in Figure 5. This dimensional reduction has also shown up in completely different approaches to quantum gravity for example when studying renormalization group flows.

## 3.3 Loop quantum gravity

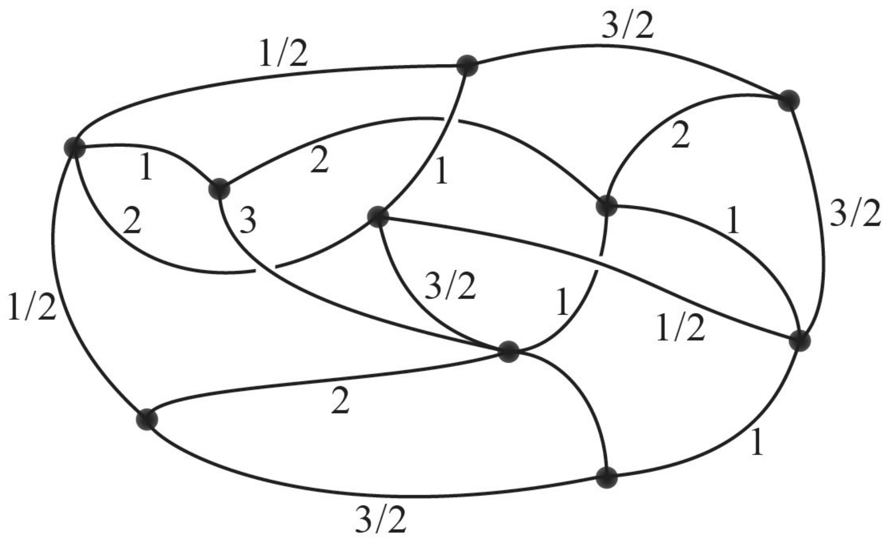



Figure 6: Each link between the nodes of a spin network is labeled by a spin $j_l$. Figure taken from [13].

Loop quantum gravity [14–16] aims at finding a direct quantization of the Einstein Hilbert action, but rather than using the metric as the dynamic variable, it uses a connection formulation in terms of Ashtekar variables [17]. It utilizes the Hamiltonian formalism in which space and time are treated separately, and so one decomposes the four dimensional spacetime into three dimensional spatial slices which are then labeled by a temporal parameter. In the three dimensional subspaces, the spatial metric $q^{ab}$ is written in terms of an orthonormal basis $e_i^a$ as

$$\det(q)q^{ab} = E_i^a E_j^b \delta^{ij} \tag{3.13}$$

where the $E_i^a$ are called triads. This reparameterization of the dynamic variables is however not unique since we could rotate the set of triads according to $O_j^i E_i^a$ with a rotation matrix $O \in SO(3)$. In loop quantum gravity this rotational invariance is represented by the group $SU(2)$ instead of $SO(3)$, which is a double covering of the latter. After introducing an $SU(2)$ connection $A_a^i$, whose conjugate momentum are the triads, we end up with the same variables as for an $SU(2)$ gauge theory, or Yang-Mills theory. What is different however are the dynamics of these variables. In Yang-Mills theory the action is given by the field strength tensor $F_{\mu\nu}^i$ which enters the action quadratically

$$S_{YM} \propto \int d^4x F_{\mu\nu}^i F^{i,\mu\nu} \tag{3.14}$$

and governs the dynamics of the gauge connection $A_\mu^i$. In gravity, after rewriting the action in terms of the new variables we find a theory which has no dynamics and is fully specified by a set of constraints, which implement the necessary symmetries in the theory. The Hamiltonian consists of a sum of three constraints, the first being an analog to Gauss' law in electrodynamics, which in this case is representing the $SU(2)$ invariance, that came from writing the metric degrees of freedom in terms of triads. The second constraint generates the invariance under spatial diffeomorphisms on the equal time slices while the last constraint, called the Hamiltonian constraint, represents the invariance under a reparameterization of the coordinate time that labels the slices. Together they combine to the four dimensional diffeomorphism invariance of general relativity when formulated in terms of metric degrees of freedom. The unconstrained phase space of general relativity is therefore the same as the unconstrained phase space of an $SU(2)$ Yang-Mills theory, and can hence be viewed as a different type of Yang-Mills theory which has additional constraints, and different dynamics. Since the total Hamiltonian only consists of constrains, there is no notion of actual time translation and the theory therefore has no dynamics, which is called the problem of time. It stems from the fact that a translation of the coordinate time is a symmetry of the theory, since general relativity is invariant under general coordinate transformations, and therefore the Hamiltonian does not generate time evolution.

The goal from here is to promote the new dynamic variables to operators and the constraints to operator equations that have to be satisfied when acting on the states that span the Hilbert space. The wavefunctions $\Psi(A)$ of this Hilbert space have to be invariant under $SU(2)$ gauge transformations to fulfill the Gauss constraint. Basic states that meet this criterion are Wilson loops, hence the name loop quantum gravity. Wilson loops $W_\gamma(A)$ are in some sense connected to the parallel transport of the connection $A$ around a closed loop $\gamma$, which we known in curved spacetimes measures the curvature. Here they can be used as a basis to span the wavefunctions

$$\Psi(A) = \sum_\gamma \Psi(\gamma) W_\gamma(A) \ . \tag{3.15}$$

The diffeomorphism constraint is then satisfied if the coefficients $\Psi(\gamma)$ are invariant under a reparameterization of the loop $\gamma$. Linking together such loops forms, what is called, a spin network state $|s\rangle$, consisting of several nodes which are connected by links, on which $SU(2)$ spins are located, see Figure 6.

What is left to do now, is to define the dynamics of such spin network states such that they also fulfill the Hamiltonian constraint. There are several ways in which this can be implemented. One is to explicitly construct the spin network states and act on them with the Hamiltonian, whose action then consists of adding further nodes and links to the network. Another way is to define a path integral which gives rise to an expansion that, instead of Feynman diagrams, features diagrams that consist of surfaces that meet at edges that meet at vertices. These diagrams are called spin foams, and they carry a spin on each of their surfaces. They represent the time evolution of spin network states. See Figure 7 for a simple vertex where at time $t = 0$ the spin network consists of three links meeting at a point and evolves into a more complicated spin network with an additional loop. The full

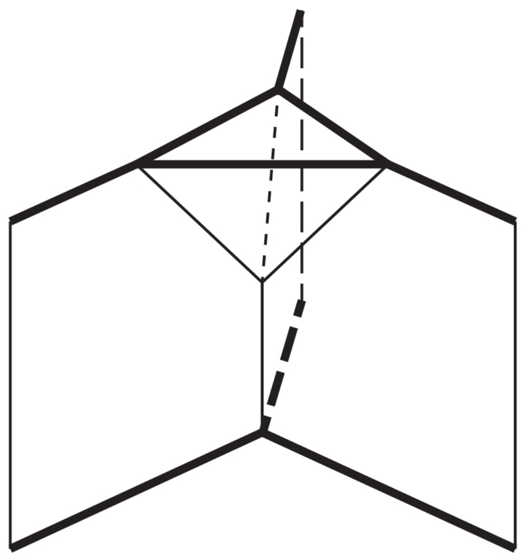


Figure 7: The diagram represents a simple vertex of a spin foam, which is the time evolution of a spin network with time going from the bottom up. Figure taken from [13]

dynamics of loop quantum gravity are not yet understood completely and are still a topic of research, as the theory is not yet fully compatible with general relativity. The theory, however, naturally features genuine background independence as the geometry of space and time itself is quantized without any reference to a background geometry. One of the interesting results from loop quantum gravity is the construction of geometric operators, which can act on spin network states, namely volume and area operators. These operators, have a discrete spectrum which means that loop quantum gravity predicts spacetime to be discrete at microscopic levels. The nodes of a spin network correspond to small finite chunks of space and they are connected by links representing finite areas [18]. The area of a surface is given by a sum over all links $l$ intersecting the surface as shown in Figure 8

$$A = \sum_{l} 8\pi l_p^2 \beta \sqrt{j_l(j_l+1)} \tag{3.16}$$

where $\beta$ is a free parameter that has to be fixed and $l_p$ is the Planck length.

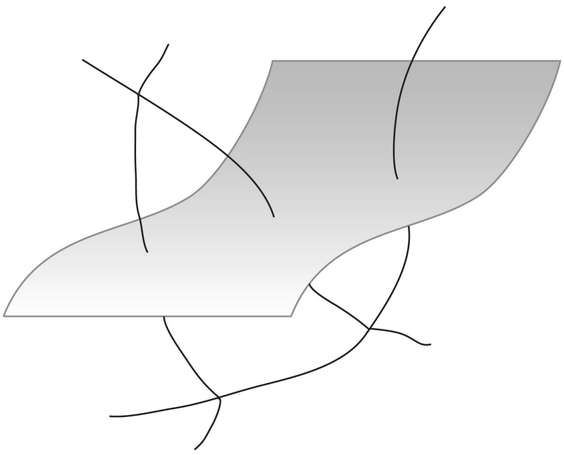


Figure 8: Each link intersecting the surface of interest contributes to the area discrete area $A$. Figure taken from [13].

## 3.4 String theory

Next I want to shortly introduce some concepts of String theory [19–21], which is a serious contender for a theory of everything, as it is able to describe the gauge theories known from the standard model and gravity in one combined framework. String theories main focus lies in the generalization of the dynamics of point particles. It describes the behavior of higher dimensional objects called p-branes. A 0-brane corresponds to a zero dimensional object i.e. the point particle, while for example a 1-brane would be a one dimensional object, called a string. There is however no restriction to 0- and 1-branes, and as most string theories are formulated in ten dimension, all branes of lower dimension than the background spacetime have their own dynamics. The necessity for ten dimensions comes from a requirement that no tachyonic or ghost states appear in the spectrum. These are states with negative mass squared or a negative norm respectively.

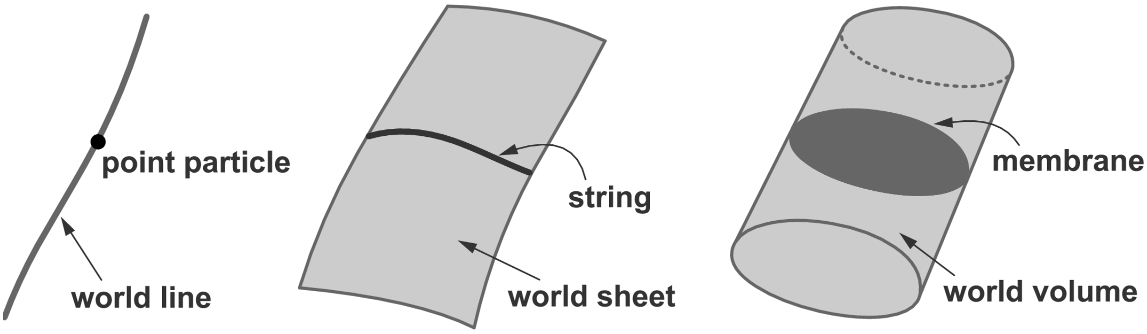



Figure 9: Generalization of world lines as the trajectories of point particles to world sheets for strings and $p+1$ dimensional world volumes for p-branes. Figure taken from [21]

Classically the motion of a point particle can be described by its position in spacetime $X^\mu(\tau)$, labeled by a proper time parameter $\tau$, which is governed by the relativistic action

$$S_0 \propto \int d\tau \sqrt{-g_{\mu\nu}\dot{X}^\mu(\tau)\dot{X}^\nu(\tau)}\ , \tag{3.17}$$

where $g_{\mu\nu}$ is the background metric. In this background spacetime the $X^\mu(\tau)$ traces out one dimensional time like world lines. To describe a (bosonic) string, or a 1-brane, an additional space like parameter $\sigma$ has to be added to account for the string being one dimensional in space and therefore tracing out two dimensional world sheets, with one space like and one time like direction as shown in figure 9.
Its generalized action can be written as

$$S_1 \propto \int d\tau d\sigma \sqrt{-\mathrm{det} G_{ab}} \tag{3.18}$$

where $G_{ab}$ is the induced metric on the two dimensional world sheet, given by

$$G_{ab} = \frac{\partial X^\mu}{\partial \sigma^a}\frac{\partial X^\nu}{\partial \sigma^b} g_{\mu\nu} \tag{3.19}$$

with $\sigma^0 = \tau$ and $\sigma^1 = \sigma$. This action can be generalized to arbitrary p-branes with dimension smaller than the background dimension. When varying this action one also has to worry about boundary conditions, which play a rather important role in string theory. For example a string can have three different boundary conditions on its ends. It could be closed, which means that its ends are connected to one another leading to periodic boundary conditions. If its ends are open there are two options, either one imposes Neumann boundary conditions, which means setting the $\sigma$ derivative at the ends to zero, or one sets the end positions of the string to be fixed, which are Dirichlet conditions. To include fermionic strings in the theory one adds, instead of bosonic fields $X^\mu(\tau,\sigma)$, two component spinor fields $\Psi^\mu(\tau,\sigma) = (\Psi^\mu_1(\tau,\sigma), \Psi^\mu_2(\tau,\sigma))$, called superstrings, which are also defined on the world sheet. These new variables now also have to obey certain boundary conditions, which are however different from the bosonic ones. Quantizing the variables $X^\mu$ and $\Psi^\mu$ gives rise to the so called Virasoro algebra for the creation and annihilation operators in this theory. One can then analyze the states and label them by their mass. What one finds is a supersymmetric spectrum of states which, most importantly for quantum gravity, includes a massless bosonic symmetric spin two tensor field $h_{\mu\nu}$. This is of course interpreted as the graviton and it is generated by a closed string excitation. The spectrum includes many more states whose nature depends on the choice of boundary conditions but, as mentioned before, is supersymmetric, meaning that every fermionic excitation has a corresponding bosonic super partner in the spectrum. In terms of gravity, what we are left with now is a background metric $g_{\mu\nu}$ and a graviton field $h_{\mu\nu}$ whose relation to one another is not very clear. General relativity requires a background independent description of space and time, which is not the case here, since the background metric is arbitrary at this point. However it is the background metric $g_{\mu\nu}$ which describes the curvature of the space in which the strings move, and therefore this metric describes the gravitational forces between objects. It should therefore somehow be constructed from the graviton field $h_{\mu\nu}$. This can be achieved by a perturbative expansion of the background metric around flat space where its first order correction is the graviton field

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ . \tag{3.20}$$

It is then possible to derive a path integral formulation which describes the motion of free strings in a background metric with interactions of the strings with the graviton, given as an expansion in terms of Feynman diagrams. But this expansion is of course only valid in the limit where metric fluctuations are small, so this is a low energy effective description. One can even write down a low energy action for the background metric itself, that to first order gives the ten dimensional Einstein Hilbert action, coupled to the other effective background fields of string theory. These other background fields are, just as the graviton, excitations that showed up in the string spectrum and are now interpreted not as string excitations, but as fields that obey effective equations of motion. They always consist of a dilaton field and a torsion field and, depending on the choice of boundary conditions, various other fields. It is also possible to construct string theories that give supersymmetric Yang-Mills theories as their low energy description. These theories include open strings whose ends are attached to the higher dimensional p-branes which carry the charges of the gauge theory and have dynamics of their own. As I mentioned the boundary conditions for the strings play a very important role, as the resulting theories contain fairly different string excitations. There are however dualities between the different theories and one can map one version to the other. As it turns out all of the different theories are given by certain limits of a more fundamental theory called M-theory, which is formulated in eleven spacetime dimensions. Compactifying one of the spatial dimensions to a circle yields the various 10 dimensional string theories from before. The radius of this eleventh dimension is related to the string coupling constant $g_s$, such that only in the large coupling limit the extra dimension becomes visible. This is also where the AdS/CFT correspondence comes into play, which is the correspondence between a non gravitational conformal field theory and a theory which includes gravity on an Anti de-Stitter background. The low energy limit of some string theories corresponds to a gravitational theory in a curved spacetime, while another limit corresponds to a conformal supersymmetric Yang-Mills theory, and since both theories are related in the ultraviolet by M-theory there is also a duality between the low energy limits, see figure 10. The string coupling is by the way the

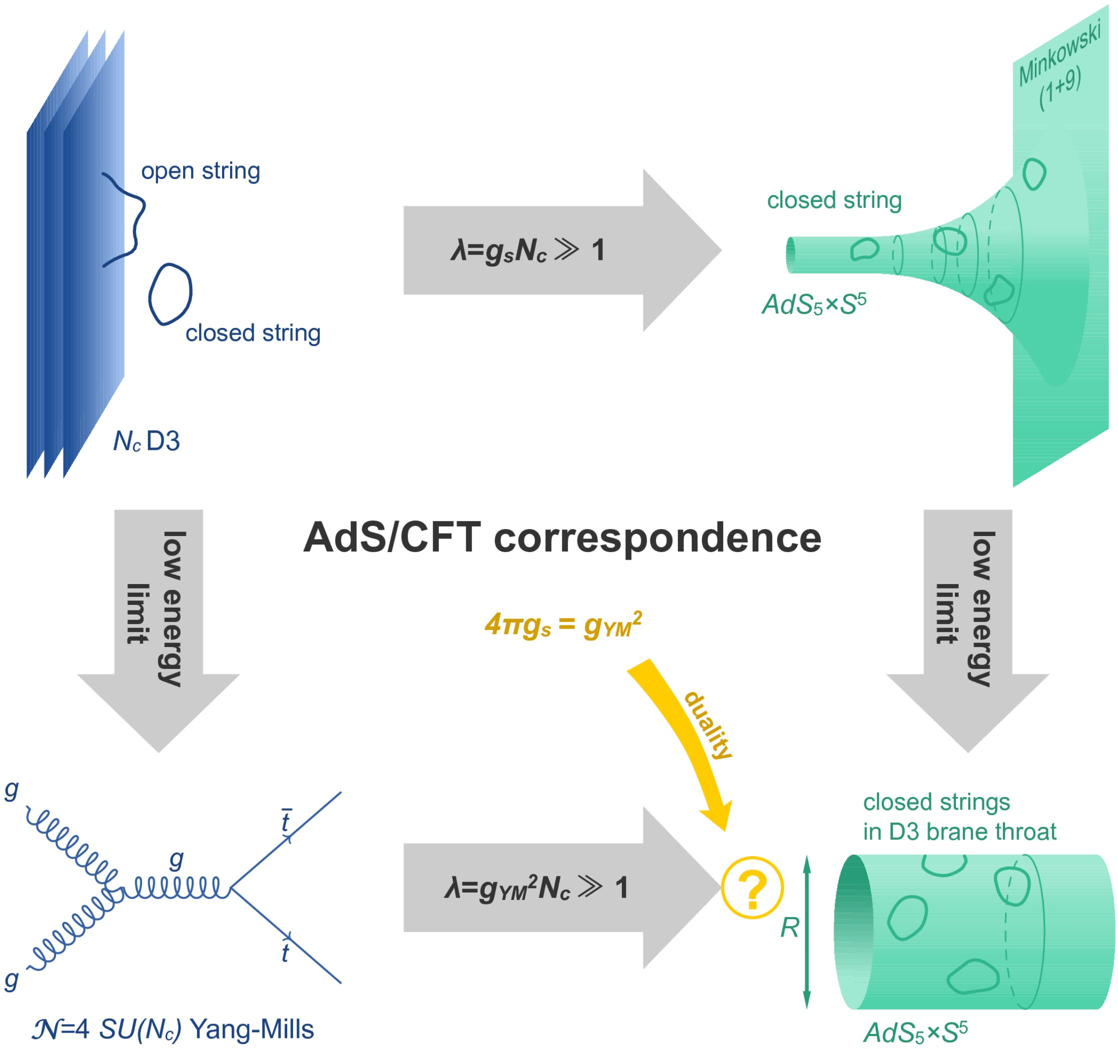



Figure 10: Diagrammatic depiction of the AdS/CFT correspondence. The large energy theories on the top half are related by S-duality which is a transformation from a weakly to a strongly coupled theory or vice versa. Taking the low energy limit of the weakling interacting theory on the top left, yields a super symmetric Yang-Mills theory on a flat background, while taking the low energy limit of the strongly interacting theory on the top right yields an effective string theory in a curved background, i.e. with gravity included. Figure taken from [22]

only free parameter of string theory which makes it really compelling as a potential theory of everything, compared to the many free parameters of the standard model whose origin are unexplained. All coupling constants of string theory are related to one another and

the masses of all excitations are calculable quantities.
So let us summarize: we have a string theory with only one free parameter, which has a massless spin two particle in the spectrum that, when interpreted as the graviton, gives in a low energy limit ten dimensional general relativity. What we also have are many more string excitations, which in the low energy limit are fields that are minimally coupled to gravity. However in order to interpret the massless spin two excitation as the graviton, an expansion around flat space is necessary, which of course raises concerns about background independence. There is however another formulation of string theory, called string field theory, which does not focus on describing and quantizing the strings on their worldsheets, but tries to describe them with respect to the background in which they live. This formulation of string theory does not rely on a perturbative expansion of the graviton around flat space, but gives a non-perturbative description of interacting strings. Another prediction from string theory is supersymmetry which has not been detected until now. The length scales at which the stringy properties become important are comparable to the Planck length which makes a direct measurement of them rather tricky. There still has to be a lot of effort put in to find observable quantities that could verify or, on the other hand, falsify the theory. But this problem does of course not only concern string theory but essentially all quantum theories of gravity as mentioned before already.

## 3.5 Quantum graphity

General relativity at its heart is a theory that deals with the geometry of spacetime, and we have in the previous sections seen several different attempts to deal with the conceptual problem we run into at the Planck scale. However all of the theories we have discussed before, in some sense still incorporate geometrical concepts. All of them somehow feature, in the case of string theory and CDT smooth, or in the case of LQG discrete, spacetimes. What I want to present now is the idea that geometry is only the low energy state of something deeper, that at high energies has no resemblance to what we would call a spacetime. It is the pre-geometric theory of quantum graphity [23]. As the name suggests this theory builds upon quantum mechanical graphs. The idea is that instead of a spacetime we only have vertices and edges that connect these vertices. The edges, in the simplest implementation of this model, are given by quantum mechanical systems that consist of an on state $|1\rangle$ and an off state $|0\rangle$. Each vertex is then connected to each other vertex by such an edge, and there is a Hamiltonian that governs the dynamics of these states. This system can then be in two very different phases. At high energies most of the edges are in the $|1\rangle$ state which means that every vertex really is connected to every other vertex. In this highly symmetric phase it is not possible to speak of different regions, areas or volumes since all vertices are neighbors. There are no local structures and all the vertices can influence each other.
Then, when lowering the energy, more and more of the edges are turned off, until at some point one reaches the ground state of the system in which every vertex is only connected to a small finite number of other vertices. This is the geometric phase of the model, where it becomes possible to define local structures, and the graph has similarities to a spacetime

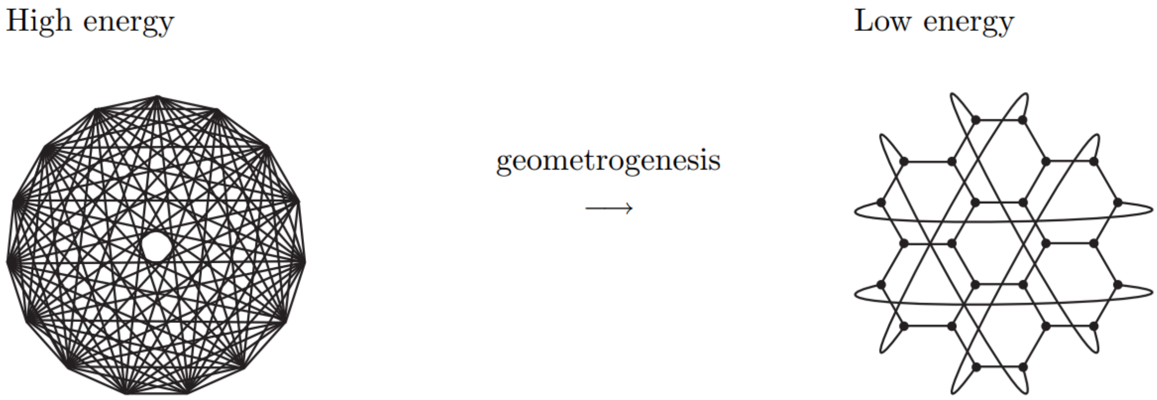



Figure 11: On the left hand side we see the fully connected high energy phase of quantum graphity which transitions down to the low energy phase shown on the right, where only turned on edges are shown. The low energy phase has a far smaller connectivity leading to a sense of locality. Figure taken from [23]

manifold. One can speak of different regions in the graph that somehow resemble a neighborhood of points which are not directly connected to other regions, and can therefore not directly influence each other. This transition is given the name of geometrogenesis, visually depicted in Figure 11. It is a phase transition, characterized by an order parameter, which is the average number of edges that are turned on at each vertex. During this phase transition the low energy geometry emerges from a purely quantum mechanical system.
What I have described above is of course only the simplest possible model one could imagine that features such a phase transition and it is not very clear how this all is really connected to general relativity at low energies. It merely demonstrates the idea that beyond the Planck scale we do not necessarily have to keep the concept of geometry.
There are extensions of this simple model, that attach further structures to the edges and vertices, and it is even possible to formulate the model in such a way that the low energy limit features effective particles that travel along the edges, and emerge from collective excitations of the underlying quantum states. So there is not only the concept of emergent spacetime but also the idea that matter emerges from the same quantum system at high energies as well.

# 4 Asymptotic safety and the functional renormalization group

We have now seen several possible directions in the search for a theory of quantum gravity. But the main topic of interest in this thesis will be none of the above. The approach we chose, namely asymptotic safety, is a lot more conservative, in that it tries to make sense of gravity in a quantum field theoretical manner. We have seen earlier that as far as perturbation theory goes, there is no chance of achieving this goal so we will have to resort to non-perturbative methods which in our case will be the functional renormalization group. The problem with perturbative quantum gravity was that we could not absorb all ultra violet divergences in the theory without rendering it completely unpredictive. The asymptotic safety scenario makes it possible to define a non perturbative theory of quantum gravity and, if certain criteria are fulfilled, make it renormalizable and predictive. In the next subsection we will give an overview of the general ideas of asymptotic safety and after that the needed theoretical concepts, that are the basis of all calculations in this thesis, will be explained. In subsection 4.4 we will return to the previously discussed topic of background independence in quantum gravity. This topic serves as a central motivation for this thesis, as we will try to formulate the asymptotic safety scenario without specifying a background metric to get a sense for the curvature dependence of quantum gravity. In this framework we will also explore the effects that standard model matter can have on our theory by using free scalar fields and study their interplay with the curvature dependence. After laying down all the foundations that we need, we will discuss the results, we obtained from our calculations in section 5.

## 4.1 Asymptotic safety

We have seen that the perturbative renormalization of quantum gravity essentially fails because of the mass dimension of Newtons coupling which is $[G_N] = -2$. This lead to additional divergences at each loop order which needed the introduction of further counter terms. Since all these counter terms would have to be tuned exactly to cancel the divergences this makes the theory unpredictive due to the infinitely many new free parameters. The idea behind the asymptotic safety scenario is now to make these perturbatively non renormalizable coupling constants dependent on a momentum scale $k$ and study their behavior in the ultraviolet (UV) regime. One starts with an action functional containing all possible diffeomorphism invariant operators which are parameterized by dimensionless, scale dependent coupling constants $g_{i,k}$. As we shall see in the next section we can then formulate an exact renormalization group equation that determines the scaling properties for each of these couplings. A theory is then given by a renormalization group trajectory through this so called theory space, spanned by the infinitely many couplings $g_{i,k}$. There are several possible outcomes that can occur when we compute the ultraviolet limit $k \to \infty$, from which two are already realized in the standard model. The first outcome is that the couplings tend to infinity for a finite or infinite renormalization group scale $k$, which corresponds to the Landau poles of the electro-weak sector of the standard model. If this is the case we know that the theory has to be incomplete in the UV, and has to be modified to be a fundamental theory. The scales at which the Landau poles occur are however many magnitudes greater than the Planck scale so one could expect that by including gravity in the theory these poles could possibly go away, which is however still unclear. The next possibility is that the coupling constants tend to zero, in which case we call the theory asymptotically free. This is also realized in the QCD sector of the standard model, where we know that at large energies it is possible to employ perturbative methods due to the smallness of the couplings. Finally the last possibility is that at some point in the renormalization group trajectory all couplings stop running and tend to a non trivial fixed point. Then we would call the theory asymptotically safe [24]. This fixed point can be found by deriving the beta functions for each of the couplings, which determine their scaling behavior

$$k\partial_k g_{i,k} = \beta_{g_{i,k}}[g_{j,k}] \ . \tag{4.1}$$

The beta function for one coupling $g_{i,k}$ can in general depend on the values of each of the other couplings $g_{j,k}$ such that we have an infinite system of first order differential equations. To determine the fixed point we have to find the value of each coupling $g^*_{i,k}$ such that the beta functions vanish

$$k\partial_k g^*_{i,k} = \beta_{g_{i,k}}[g^*_{j,k}] = 0 \ . \tag{4.2}$$

Finding such a fixed point in the renormalization group flow is the first step in order to realize an asymptotically safe theory. The second step is to actually look for trajectories that end up in this fixed point. We can define all the points in theory space that, if we use them as initial conditions, will flow towards this fixed point. We call this subspace the UV critical surface. Especially important is the dimensionality of this subspace. If it were infinite dimensional, then no matter what initial conditions we chose, we would always end up at the fixed point and to pick out the single physical theory corresponds to fixing infinitely many initial conditions, in which case again our thoery is unpredictive. However if this UV critical surface is a finite dimensional subspace of theory space, then we would only have to fix finitely many couplings to specify the initial conditions, since all infinitely many other couplings are automatically given by the requirement that we start on the UV critical surface. So our second requirement for an asymptotically safe theory is that we need a finite dimensional UV critical surface. The dimensionality of this surface also determines how many parameters we have to fix in order to pick out one specific theory, so a smaller dimension makes the theory more predictive. One can study the UV critical surface by linearizing the beta functions near the fixed point and check which couplings run into the fixed points and which are repelled. The dimensionality of the UV critical surface is then given by the number of couplings that are attracted to the fixed point.
In general it is not possible to track the flow of all infinitely many coupling constants so one has to find a suitable truncation and only study a subspace of theory space that will hopefully be a good representation of the full infinite dimensional flow. If one found a fixed point in a given truncation this does not automatically mean that the fixed point is present in general, however we will not focus on such consistency checks.
In the following we will only focus on the first criterion, i.e. that there exists a fixed point for a certain truncation that will be specified in subsection 4.5. Also we will not study the dimensionality of the UV critical surface, but concentrate on the background curvature dependence of the dimensionless couplings at the fixed point.

## 4.2 Exact renormalization group equation (ERGE)

We study the fixed point structure and curvature dependence of quantum gravity with the help of an exact renormalization group equation (ERGE) for the effective average action $\Gamma_k$. The scale parameter $k$ is used to label the momentum scales down to which we have integrated out quantum fluctuations in an effective theory sense. Hence the limit $k \to \infty$ means we have not integrated out any momenta and our effective average action is given by the microscopic action $S$ including all high energy degrees of freedom

$$\Gamma_{k\to\infty} = S \tag{4.3}$$

while the limit $k \to 0$ corresponds to the case where all fluctuations are integrated out, so we can describe the theory solely by the standard effective action $\Gamma$, the generating functional of one particle irreducible n-point functions. The exact renormalization group equation will interpolate between those two limits which we will call the infrared (IR) and the ultraviolet (UV) limit. Let us collectively label all (bosonic) quantum fields by a generalized super field $\hat{\varphi}$ and their corresponding source fields by $J$. Then the generating functional of time orderer correlation functions is given by

$$Z[J] = \int \mathcal{D}\hat{\varphi} e^{-S[\hat{\varphi}]+J\cdot\hat{\varphi}} = e^{W[J]} \tag{4.4}$$

where we have used the following notation

$$J \cdot \hat{\varphi} = \int d^4x J(x)\hat{\varphi}(x) \ . \tag{4.5}$$

The effective action is then given by a legendre transform of the Schwinger functional $W[J]$

$$\Gamma[\phi] = -W[J] + J \cdot \phi \ . \tag{4.6}$$

which switches variables from the sources $J$ to the classical fields

$$\phi = \langle\hat{\varphi}\rangle_J = \frac{\delta W[J]}{\delta J} = \frac{1}{Z[J]}\int \mathcal{D}\hat{\varphi}\ \hat{\varphi} e^{-S[\hat{\varphi}]+J\cdot\hat{\varphi}} \ . \tag{4.7}$$

From the definition of $\Gamma[\phi]$ we see that functional differentiation with respect to $\phi$ gives back the sources $J$

$$\frac{\delta\Gamma[\phi]}{\delta\phi} = J \ , \tag{4.8}$$

which leads to a very nice property of the effective action. If we want to find the on shell vacuum expectation value of the fields $\hat{\varphi}$, we can solve effective equations of motion by setting $J = 0$

$$\left.\frac{\delta\Gamma[\phi]}{\delta\phi}\right|_{\phi=\phi_{EoM}=\langle\hat{\varphi}\rangle_{J=0}} = 0, \tag{4.9}$$

analogously to the equations of motion for a classical field theory with classical action $S[\phi]$

$$\frac{\delta S[\phi]}{\delta \phi} = 0 \ . \tag{4.10}$$

To promote the effective action to the scale dependent effective average action $\Gamma_k$ we will add a quadratic cutoff action

$$\Delta S_k[\hat{\varphi}] = \frac{1}{2}\hat{\varphi} \cdot R_k \hat{\varphi} \tag{4.11}$$

to the action in the exponent of Equation 4.4 and subtract it from Equation 4.6

$$Z[J] \to Z_k[J] = \int \mathcal{D}\hat{\varphi} e^{-S[\hat{\varphi}] - \Delta S_k[\hat{\varphi}] + J\cdot\hat{\varphi}} = e^{W_k[J]} \tag{4.12}$$

$$\Gamma[\phi] \to \Gamma_k[\phi] = -W_k[J] - \Delta S_k[\phi] + J \cdot \phi \ , \tag{4.13}$$

again with

$$\phi = \langle \hat{\varphi} \rangle_{J,k} = \frac{\delta W[J]_k}{\delta J} \ . \tag{4.14}$$

and

$$\frac{\delta(\Gamma[\phi] + \Delta S_k[\phi])}{\delta \phi} = J \ . \tag{4.15}$$

The cutoff action acts as a scale dependent mass term, with a mass given by the regulator function $R_k$. The regulator function is supposed to suppress those momentum modes $p^2$ that are smaller than the cutoff scale $k^2$, such that at scale $k$ all low momentum modes essentially decouple from the high energy modes, because they have a large effective mass. $R_k$ has to depend on some kinetic operator $\Delta$ with eigenvalues $p^2$, most conveniently the one already present in the action $S$. In order to obtain the desired limits we then know that the regulator has to fulfill the following conditions:

$$\lim_{k\to\infty} R_k(p^2) \to \infty \tag{4.16}$$

$$\lim_{k\to 0} R_k(p^2) \to 0 \tag{4.17}$$

$$\lim_{p\to 0} R_k(p^2) > 0 \ . \tag{4.18}$$

The first condition ensures that in the UV limit all modes are given a huge mass such that nothing in the theory fluctuates and all modes are suppressed. The second condition gives the correct infrared limit, in which no modes are suppressed and are fully integrated out in the path integral. The last condition says that the regulator should stay finite at a fixed scale $k$ for vanishing momentum $p$ such that all infrared modes are regulated, even those with arbitrarily small momenta, which is especially important when dealing with massless modes. In Figure 12 we can see two examples for such a regulator function, an exponential regulator and what is called the optimized regulator [25]

$$R_k(p^2)^{\text{exp.}} = k^2 e^{-(p/k)^4} \tag{4.19}$$
$$R_k(p^2)^{\text{opt.}} = k^2(1-(p/k)^2)\theta(1-(p/k)^2)\ . \tag{4.20}$$

As discussed before the regulator suppresses modes with $p^2 < k^2$, which can especially be seen for the optimized regulator, which is identically zero for $p^2 > k^2$. The exponential regulator on the other hand follows a smooth curve which only goes to zero for $p^2 \to \infty$.

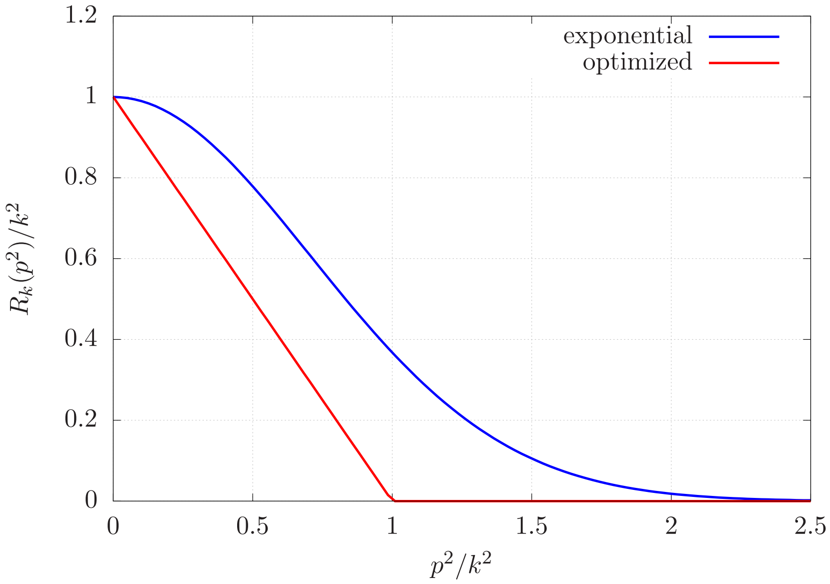



Figure 12: Visualization of example regulator functions. The exponential regulator, shown in blue, will be used for our calculations later.

Now we can simply take the derivative of $\Gamma_k$ with respect to $k$ to find an exact renormalization group equation, called the Wetterich equation [26].

$$\dot{\Gamma}_k = k\frac{d\Gamma}{dk} = -k\frac{dW[J]}{dk} - k\frac{d\Delta S_k[\phi]}{dk} \tag{4.21}$$
$$-k\frac{dW[J]}{dk} = \langle k\frac{d\Delta S_k[\hat{\varphi}]}{dk}\rangle_{J,k} = \frac{1}{2}\langle\hat{\varphi}\cdot\dot{R}_k\hat{\varphi}\rangle_{J,k} = \frac{1}{2}\text{Tr}(\langle\hat{\varphi}\hat{\varphi}\rangle_{J,k}\dot{R}_k) \tag{4.22}$$
$$-k\frac{d\Delta S_k[\phi]}{dk} = -\frac{1}{2}\phi\cdot\dot{R}_k\phi = -\langle\hat{\varphi}\rangle_{J,k}\cdot\dot{R}_k\langle\hat{\varphi}\rangle_{J,k} = -\frac{1}{2}\text{Tr}(\langle\hat{\varphi}\rangle_{J,k}\langle\hat{\varphi}\rangle_{J,k}\dot{R}_k) \tag{4.23}$$
$$\Rightarrow \dot{\Gamma}_k = \frac{1}{2}\text{Tr}\left[(\langle\hat{\varphi}\hat{\varphi}\rangle_{J,k} - \langle\hat{\varphi}\rangle_{J,k}\langle\hat{\varphi}\rangle_{J,k})\dot{R}_k\right] \tag{4.24}$$
$$= \frac{1}{2}\text{Tr}\left[\left(\Gamma_k^{(2)} + R_k\right)^{-1}\dot{R}_k\right]. \tag{4.25}$$

In the last step we have written the full regularized propagator in terms of the inverse regularized two point function

$$G_{J,k} = \langle\hat{\varphi}\hat{\varphi}\rangle_{J,k} - \langle\hat{\varphi}\rangle_{J,k}\langle\hat{\varphi}\rangle_{J,k} = \left(\frac{\delta^2(\Gamma_k[\phi] + \Delta S_k[\phi])}{\delta\phi\delta\phi}\right)^{-1} = \left(\Gamma_k^{(2)} + R_k\right)^{-1} \tag{4.26}$$

which can easily be checked. We are left with an exact partial differential equation for the effective average action with respect to the fields $\phi$ and the scale $k$. This short derivation is only valid for bosonic fields and differs at some point when dealing with anticommuting fermionic fields. Conjugate fermionic fields $\psi$ and $\bar{\psi}$ will enter the trace in the final result with a factor of two and an additional minus sign

$$\dot{\Gamma}_k[\phi,\psi,\bar{\psi}] = \frac{1}{2}\mathrm{Tr}\left[\left(\Gamma_k^{(2\phi)} + R_k\right)^{-1}\dot{R}_k\right] - \mathrm{Tr}\left[\left(\Gamma_k^{(\psi\bar{\psi})} + R_k\right)^{-1}\dot{R}_k\right] . \tag{4.27}$$

In Figure 13 we have shown examples of massless regularized propagators of the form

$$G_k(p^2) = \frac{1}{p^2 + R_k(p^2)} \tag{4.28}$$

using the regulators from before. We can see that momentum modes smaller than $k^2$ are suppressed, while in the limit $k \to 0$ we recover the original massless propagator which diverges at $p^2 = 0$.

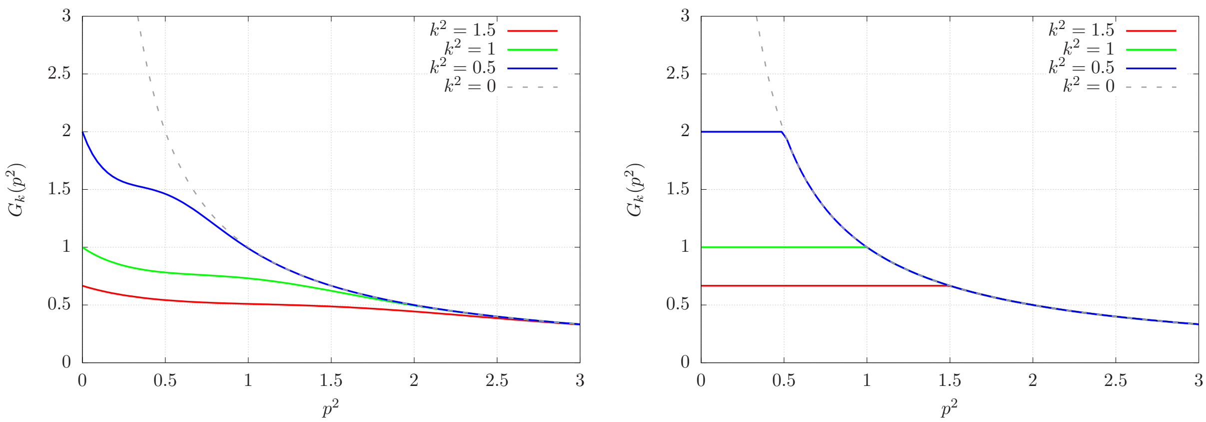



Figure 13: Visualization of regularized propagators $G_k$ for different scales $k^2$ using the regulator functions from before. In the left panel we see the exponential regulator, which also suppresses modes that are larger than $k^2$, since it decays smoothly beyond $p^2 = k^2$ and in the left panel we see the optimized regulator, which only cuts momentum modes strictly smaller than $k^2$. The limit $k \to 0$ recovers the unregularized propagator where no modes are suppressed in both cases.

## 4.3 ERGE for gravity

In the previous section we have defined an exact renormalization group equation for the effective average action which interpolates between the microscopic action in the UV and the effective action in the IR. In this section we want to specifically look at the ERGE for quantum gravity, which we will have to formulate a little bit differently. To see why, we have to get back to the definition of our cutoff action in terms of a regulator function $R_k$. We said that we want to distinguish modes with momenta smaller than $k$ from those with momenta larger than $k$. The different modes were characterized by eigenvalues of some kinetic operator acting on the fields. However because we are dealing with gravity, our dynamic variable will be the metric. This is of course problematic since we need a metric to define our kinetic operator. The metric defines the spacetime we are working on, and therefore determines length and energy scales. Thus we run into trouble with keeping our theory background independent. We have already noted that we can not chose a metric a priori, which would be analog to postulating a solution to the equations of motion before solving them. Besides this conceptual reasoning we also have a technical reason why the previous ansatz is poblematic. Say we would define a kinetic operator $\Delta_g$ with respect to the undetermined metric $g_{\mu\nu}$ which carries our degrees of freedom. If we write down our cutoff action as before

$$\Delta S_k[g_{\mu\nu}] = \frac{1}{2}\int d^4x\sqrt{g}g_{\mu\nu}R_k^{\mu\nu\rho\sigma}[\Delta_g]g_{\rho\sigma} \ , \tag{4.29}$$

we see that this is no longer quadratic in our dynamic field and therefore can not directly be interpreted as a mass term. We would not even know if we end up in the correct IR and UV limits.

### 4.3.1 Background field method

The solution to the problems given above is that we have to make use of the background field method. We will split the full quantum metric into a background and fluctuation part

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \tag{4.30}$$

and write down a modified version of the effective average action $\bar{\Gamma}_k[\bar{g}, h]$ which gives back the original effective action when setting the fluctuation metric to zero, which means we identify the background metric with the full metric [27]

$$\bar{\Gamma}_k[\bar{g}, h = 0] = \Gamma[\bar{g}] = \Gamma[g] \ . \tag{4.31}$$

In any intermediate steps we will use the background metric to define our reference scale $k$ so we will write the cutoff action as

$$\Delta S_k[\bar{g}, h] = \int d^4x\sqrt{\bar{g}}h_{\mu\nu}R_k^{\mu\nu\rho\sigma}[\Delta_{\bar{g}}]h_{\rho\sigma} \ . \tag{4.32}$$

This procedure however comes with the obvious caveat that we now have to deal not with a single metric but two metrics $\bar{g}$ and $h$. On top of this we know that general relativity has a gauge symmetry that we will have to deal with, by using the Faddeev-Popov procedure. We want our theory to be invariant under general coordinate transformations $x^\mu \to x^\mu - \xi^\mu$, which will induce a change in the full metric $g_{\mu\nu}$ given by its Lie derivative in the direction of $\xi$

$$\delta_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi \mu = \mathcal{L}_\xi g_{\mu\nu}. \tag{4.33}$$

To prevent an over counting of these gauge equivalent metrics we have to introduce a gauge fixing action $S_{\text{gf}}$ and a corresponding ghost action $S_{\text{gh}}$ to the exponent of our path integral. There is however a choice in how to perform this gauge fixing, since we have split the metric. We can choose to distribute the gauge transformation as

$$\delta_\xi^{(B)} \bar{g}_{\mu\nu} = \mathcal{L}_\xi \bar{g}_{\mu\nu} \tag{4.34}$$

$$\delta_\xi^{(B)} h_{\mu\nu} = \mathcal{L}_\xi h_{\mu\nu} \tag{4.35}$$

which is called a background gauge transformation, or we can perform a quantum gauge transformation

$$\delta_\xi^{(Q)} \bar{g}_{\mu\nu} = 0 \tag{4.36}$$

$$\delta_\xi^{(Q)} h_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} \ . \tag{4.37}$$

In the Fadeev-Popov procedure we would like to break the quantum gauge invariance, but keep the action invariant under background gauge transformations. This can be achieved by choosing a gauge fixing action which is quadratic in $h_{\mu\nu}$, and therefore only utilizes background covariant derivatives $\bar{\nabla}_\mu$ instead of the full covariant derivatives $\nabla_\mu$. This choice is called a background gauge, since it keeps the whole theory invariant under background gauge transformations. This however again introduces, just like the cutoff action, a separate dependence on the two metrics.

It is now time to write down the complete definition of our modified background quantities and especially of our modified effective action. Writing the sources collectively as $J = \{J^h_{\mu\nu}, J^c_\mu, J^{\bar{c}}_\nu\}$ we have

$$\begin{aligned} \bar{Z}_k[J, \bar{g}_{\mu\nu}] &= \int \mathcal{D}\hat{h}\mathcal{D}\hat{c}\mathcal{D}\hat{\bar{c}}\ e^{-S[\bar{g}+\hat{h}]-S_{\text{gf}}[\bar{g},\hat{h}]-S_{\text{gh}}[\bar{g},\hat{h},\hat{\bar{c}},\hat{c}]-\Delta S_k[\bar{g},\hat{h},\hat{c},\hat{\bar{c}}]+\int d^4x\sqrt{\bar{g}}\left(J^h_{\mu\nu}\hat{h}^{\mu\nu}+J^c_\mu \hat{c}^\mu+\hat{\bar{c}}^\mu J^{\bar{c}}_\mu\right)} \\ &= e^{\bar{W}_k[J,\bar{g}]} \ . \end{aligned} \tag{4.38}$$

The first thing to point out is that we only integrate over the fluctuation fields and not the background metric $\bar{g}$ in the path integral. The next thing to note is that the only term which respects the original split $\hat{g} = \bar{g} + \hat{h}$ is the classical action, while the gauge fixing, the ghost and the cutoff action break this split. Now we can define our classical fields

$\phi = \{h_{\mu\nu}, c_\mu, \bar{c}_\mu\}$

$$h^{\mu\nu} = \langle \hat{h}_{\mu\nu} \rangle_{J,\bar{g},k} = \frac{1}{\sqrt{\bar{g}}} \frac{\delta W_k[J,\bar{g}]}{\delta J^h_{\mu\nu}} \tag{4.39}$$

$$\tag{4.40}$$

and analogously for the ghost fields. Then our modified effective action reads

$$\bar{\Gamma}_k[\bar{g},\phi] = -\bar{W}_k[J,\bar{g}] - \Delta S_k[\phi,\bar{g}] + J \cdot \phi \tag{4.41}$$

where now

$$J \cdot \phi = \int d^4x \sqrt{\bar{g}} \left( J^h_{\mu\nu} h^{\mu\nu} + J^c_\mu c^\mu + \bar{c}^\mu J^{\bar{c}}_\mu \right) \ , \tag{4.42}$$

One can then show that, when taking the fluctuation fields to zero, we recover the original effective average action [27]

$$\bar{\Gamma}_k[\phi = 0, \bar{g}] = \Gamma_k[\bar{g}] \tag{4.43}$$

and therefore it is meaningful to study the flow of the modified effective action, which satisfies the following renormalization group equation

$$\dot{\bar{\Gamma}}_k[\phi,\bar{g}] = \frac{1}{2} \mathrm{Tr} \left[ \left( \bar{\Gamma}^{(hh)}_k[\phi,\bar{g}] + R^h_k \right)^{-1} \dot{R}^h_k \right] - \mathrm{Tr} \left[ \left( \bar{\Gamma}^{(\bar{c}c)}_k[\phi,\bar{g}] + R^c_k \right)^{-1} \dot{R}^c_k \right] \ . \tag{4.44}$$

From now on we will only work with the modified effective average action so the bar will be suppressed. We can still talk about both quantities if we keep Equation 4.43 in mind. The new quantities will be denoted by

$$\bar{\Gamma}_k[\phi,\bar{g}] \to \Gamma_k[\phi,\bar{g}] \tag{4.45}$$

$$\Gamma_k[g] \to \Gamma_k[\phi = 0,\bar{g}] \ . \tag{4.46}$$

## 4.4 Background (in-)dependence of quantum gravity and equations of motion

As we have already discussed general relativity is a background independent theory, and we want our quantum theory also to independent of any background metric. However in the last section we worked in background field method, which introduced an explicit dependence of our theory on a background metric $\bar{g}_{\mu\nu}$, such that our effective action is now a functional of two metrics $\Gamma_k[h,\bar{g}]$. If we want our theory to be background independent this means that there has to be relation that effectively reduces these two separate dependences to a single dependence, so let us determine this relation. We therefore take a look at the functional derivative of $\Gamma_{k=0}[h,\bar{g}] = \Gamma$ firstly with respect to $\bar{g}$ and then with respect to $h$. If the $\Gamma$ would depend on the combination $\bar{g}+h$ then the difference of these two derivatives should vanish. This difference is called the Nielsen identity or split Ward identity [28], and to find it lets first consider the variation of $\Gamma$ under a change of $\bar{g}$, which is given by

$$\begin{aligned}\frac{\delta\Gamma}{\delta\bar{g}} = -\frac{\delta W}{\delta\bar{g}} = \frac{1}{Z}\int \mathcal{D}\hat{h}\mathcal{D}\hat{c}\mathcal{D}\hat{\bar{c}}\left(\frac{\delta S}{\delta\bar{g}}\right) e^{-S[\hat{h},\hat{c},\hat{\bar{c}},\bar{g}]+\int d^4x\sqrt{\bar{g}}\left(J^{\mu\nu}\hat{h}_{\mu\nu}+J^c_\mu\hat{c}^\mu+\hat{\bar{c}}^\mu J^{\bar{c}}_\mu\right)}\\ = \left\langle\frac{\delta S}{\delta\bar{g}}\right\rangle_J\end{aligned} \tag{4.47}$$

where we now take $S$ to contain the classical action, the gauge fixing action and the ghost action.
For the variation of $\Gamma$ under a change in $h$ lets consider the generating functional $Z$. The integral should stay invariant under a change in $\hat{h}$ and so one finds

$$\begin{aligned}0 &= \int \mathcal{D}\hat{h}\mathcal{D}\hat{c}\mathcal{D}\hat{\bar{c}}\left(\frac{\delta}{\delta\hat{h}}\right) e^{-S[\hat{h},\hat{c},\hat{\bar{c}},\bar{g}]+\int d^4x\sqrt{\bar{g}}\left(J^{\mu\nu}\hat{h}_{\mu\nu}+J^c_\mu\hat{c}^\mu+\hat{\bar{c}}^\mu J^{\bar{c}}_\mu\right)}\\ &= \int \mathcal{D}\hat{h}\mathcal{D}\hat{c}\mathcal{D}\hat{\bar{c}}\left(-\frac{\delta S}{\delta\hat{h}}+J\sqrt{\bar{g}}\right) e^{-S[\hat{h},\hat{c},\hat{\bar{c}},\bar{g}]+\int d^4x\sqrt{\bar{g}}\left(J^{\mu\nu}\hat{h}_{\mu\nu}+J^c_\mu\hat{c}^\mu+\hat{\bar{c}}^\mu J^{\bar{c}}_\mu\right)}\\ \Leftrightarrow \frac{\delta\Gamma}{\delta h} &= \left\langle\frac{\delta S}{\delta\hat{h}}\right\rangle_J\end{aligned} \tag{4.48}$$

which means the Nielsen identity is given by

$$\frac{\delta\Gamma}{\delta\bar{g}_{\mu\nu}} - \frac{\delta\Gamma}{\delta h_{\mu\nu}} = \left\langle\left[\frac{\delta}{\delta\bar{g}_{\mu\nu}} - \frac{\delta}{\delta\hat{h}_{\mu\nu}}\right](S_{gf}+S_{gh})\right\rangle_J . \tag{4.49}$$

because the contribution from the classical action cancels out, since it is indeed a function of $\bar{g}+h$, while the gauge fixing, and ghost action are not.
The identity implies that for any observables we calculate from $\Gamma$, the two dependences automatically reduce to a single dependence on $\bar{g}+h$ since the right hand side vanishes onshell as we take $J=0$. So at renormalization scale $k=0$ we know that if $\Gamma$ satisfies the Nielsen identity, it is actually background independent even though it depends on two metrics. From the Nielsen identity we can also infer information about the equations of motion at vanishing scale $k=0$. From the last section we know that we recover the true effective action by setting the fluctuation fields to zero, and therefore the equation of motion that determines the expectation value of $\bar{g}_{\mu\nu}$ reads

$$\frac{\delta\Gamma[h=0,\bar{g}]}{\delta\bar{g}} = 0 . \tag{4.50}$$

However due to the Nielsen identity we find that we can also use

$$\left.\frac{\delta\Gamma[h,\bar{g}]}{\delta h}\right|_{h=0} = 0 \tag{4.51}$$

which is called the quantum equation of motion, to obtain the same solution we obtained from the background equation of motion, since $\Gamma$ depends on the combination $\bar{g}+h$ on shell. Let us now take a look at the case when $k \neq 0$ and we have an additional background dependence due to the regulator function $R_k$, which adds a further term to the then called modified Nielsen identity. Changing the background metric induces a change in the regulator function captured by

$$\frac{\delta\Gamma}{\delta\bar{g}_{\mu\nu}} = \frac{\delta\Gamma}{\delta\bar{g}_{\mu\nu}}\bigg|_{R_k=\text{const.}} + \int d^4y \frac{1}{\sqrt{\bar{g}(x)}} \frac{\delta R_k(y)\sqrt{\bar{g}(y)}}{\delta\bar{g}_{\mu\nu}(x)} \frac{\delta\Gamma}{\delta R_k(y)}. \tag{4.52}$$

The variation of $\Gamma$ with respect to $R_k$ can be performed and yields

$$\begin{aligned}\int d^4x \frac{\delta\Gamma}{\delta R_k} &= -\frac{\delta W_k}{\delta R_k} - \frac{\delta\Delta S_k}{\delta R_k} = \frac{1}{2}\int d^4x\sqrt{\bar{g}}\left(\langle hh\rangle) - \langle h\rangle\langle h\rangle\right) \\ &= \frac{1}{2}\mathrm{Tr}G_k\end{aligned} \tag{4.53}$$

So all in all the modified Nielsen identity reads

$$\frac{\delta\Gamma}{\delta\bar{g}_{\mu\nu}} - \frac{\delta\Gamma}{\delta h_{\mu\nu}} = \left\langle \left[\frac{\delta}{\delta\bar{g}_{\mu\nu}} - \frac{\delta}{\delta\hat{h}_{\mu\nu}}\right](S_{gf}+S_{gh})\right\rangle_J + \frac{1}{2}\mathrm{Tr}\left[\frac{1}{\sqrt{\bar{g}}}\frac{\delta\sqrt{\bar{g}}R_k[\bar{g}]G_k}{\delta\bar{g}_{\mu\nu}}\right]. \tag{4.54}$$

This difference will not vanish on shell which leaves us with the observation that at finite $k$ the background and quantum equations of motion will not give the same solution $\bar{g}_{\text{EoM}}$. To calculate observables, however we need this expectation value, so we have to wonder which equation of motion gives the physically correct solution at finite scale $k$. We found that for in the physical limit $k \to 0$ the solutions are the same and the introduction of a regulator directly modified the background equation of motion

$$\frac{\delta\Gamma}{\delta\bar{g}_{\mu\nu}} \propto \frac{1}{2}\mathrm{Tr}\left[\frac{1}{\sqrt{\bar{g}}}\frac{\delta\sqrt{\bar{g}}R_k[\bar{g}]G_k}{\delta\bar{g}_{\mu\nu}}\right] \tag{4.55}$$

so we suspect that at finite renormalization group scale we should use the quantum equation of motion to determine $\bar{g}_{\text{EoM}}$. The idea how we can implement background independence for observables at finite scale $k$ is therefore to work with a general background metric $\bar{g}$ and in the end determine the physical background by solving the quantum equations of motion. There are also other approaches to implementing background independence, for example one could for each scale $k$ require the Nielsen identities to be satisfied, which reduces the dependence on two metrics to just one and therefore automatically forces the theory to be background independent. A third approach would be to choose a specific background first, and then solve the equations of motion in an expansion around the previously chosen background.

In the following we will go with the first approach, that is we will calculate background dependent correlation functions, write down the quantum equations of motion and then

determine the expectation value for $\bar{g}_{\mu\nu}$. We will then use this expectation value to evaluate the correlation functions on the physical background.
We will also take a look at the background equation of motion and confirm the result that indeed it will not lead to the same solutions as the quantum equation of motion at finite $k$.

## 4.5 Vertex expansion

Our goal in this section is to introduce a way to study the explicit dependence of fluctuation correlation functions, which are given by functional differentiation of the effective action with respect to the fluctuation metric, on the background metric $\bar{g}_{\mu\nu}$. This means we will not employ the background field approximation at first but rather disentangle the dependences of the effective action $\Gamma_k$ on its metric arguments. For better notation let us first define the fluctuation super field $\phi$ consisting of the fluctuation metric and the two ghost fields. The effective action is a functional of the background metric and this combined super field. In order to disentangle the background dependence we expand our effective action in a scheme called vertex expansion, which comes down to an expansion of $\Gamma_k[\bar{g}, \phi]$ in powers of fluctuation fields $\phi$, leaving the background metric $\bar{g}$ fixed

$$\Gamma_k[\bar{g}, \phi] = \sum_{n=0}^{\infty} \sum_{\phi_i \in \phi} \frac{1}{n!} \Gamma_k^{(\phi_1 \dots \phi_n)}[\bar{g}, 0] \phi_1 \dots \phi_n \ . \tag{4.56}$$

Here we have defined the vertex or $n$-point functions $\Gamma_k^{(\phi_1 \dots \phi_n)}[\bar{g}, 0]$ which are given by functional differentiation of $\Gamma_k$ with respect to the fields $\phi_1 \dots \phi_n$ with $\phi_i$ being some element of the fluctuation super field $\phi$, evaluated at vanishing $\phi$

$$\Gamma^{(\phi_1 \dots \phi_n)}[\bar{g}, 0] = \left. \frac{\delta^n \Gamma[\bar{g}, \phi]}{\delta\phi_1 \dots \delta\phi_2} \right|_{\phi=0} \tag{4.57}$$

Depending on the kinds of fields $\phi_i$ these are objects with different numbers of indices. For example the graviton two point function $\Gamma_k^{(hh)}[\bar{g}, h]$ would have four open indices, two coming from each differentiation with respect to $h_{\mu\nu}$

$$\Gamma_k^{(hh),\mu\nu\alpha\beta}[\bar{g}, 0] = \left. \frac{\delta^2 \Gamma[\bar{g}, \phi]}{\delta h_{\mu\nu} \delta h_{\alpha\beta}} \right|_{\phi=0} \ . \tag{4.58}$$

This kind of ansatz was previously used in this context in [2–5].
If we wish to include matter fields $\varphi$ in our system, they will naturally join in with the other fluctuation fields in $\phi$ and result in the same expansion, only also featuring vertex functions including the matter field.
In the previous section we found that our exact renormalization group equation for the effective action using the background field method is given by

$$\dot{\Gamma}_k[\phi, \bar{g}] = \frac{1}{2} \mathrm{Tr} \left[ \left( \Gamma_k^{(hh)}[\phi, \bar{g}] + R_k^h \right)^{-1} \dot{R}_k^h \right] - \mathrm{Tr} \left[ \left( \Gamma_k^{(\bar{c}c)}[\phi, \bar{g}] + R_k^c \right)^{-1} \dot{R}_k^c \right] \ , \tag{4.59}$$

which can be used to find the flow equations for the vertex functions that were just defined before. Functionally differentiating both sides with respect to $\phi_i..\phi_n$ and then setting $\phi = 0$ results in flow equations for the corresponding vertex functions $\Gamma_k^{(\phi_i...\phi_1)}$. Here we focus only on the flow of the graviton two and three point functions, which defines our current truncation. Diagrammatically we can write these flows as shown in Figure 14
Here we represented open indices on the right hand side by red double lines and graviton propagators by black double lines. The dashed lines correspond to ghost propagators and the crosses denote regulator insertions $\partial_t R_k = \dot{R}_k$ of the respective graviton and ghost regulators. If we include scalar matter fields to the system, as we will do later, there will be additional diagrams on the right, which are similar to the diagrams in Figure 14 that include graviton loops but with the loop variable replaced by the scalar fields. To handle the right hand side notationally we define the dimensionless flows

$$\partial_t \Gamma_k^{(\phi_1...\phi_n)} = k^2 \mathrm{Flow}^{\phi_1...\phi_n} \ . \tag{4.64}$$

We now have a coupled system of flow equations for the $n$-point vertex functions which could theoretically be extend up to arbitrary order. At each order $n$ the flow of the $n$-point vertex depends on higher order vertices such it can never be closed and we have to chose a truncation. In our truncation we include flows up to the three point function which on the right hand side features the four and five point functions. This means we have to decide how to incorporate these higher order vertex functions, which we are unable to calculate. But first let us define our parametrization of the vertex functions which we will use as an ansatz. This ansatz will be the same as was used in [3], so that we can extend the results they obtained to a more general setting, while still being able to compare the results they already obtained. In general, every graviton $n$-point function is given by a set of background gauge invariant operators multiplied by dimensionfull coupling constants. From this infinite set of possible operators at each level $n$ we focus on the classical Einstein Hilbert tensor structures which can be obtained by functionally differentiating the gauge fixed Einstein Hilbert action

$$S_{\mathrm{EH}} = \frac{1}{16\pi G_N} \int d^4x \sqrt{g}\,(2\Lambda - R) + S_{\mathrm{gf}} + S_{\mathrm{gh}} \tag{4.65}$$

This means we will always have operators that maximally contain two derivatives, as the Einstein Hilbert action is only linear in curvature. The fluctuation metric at this point has mass dimension zero, so we rescale it by

$$h_{\mu\nu} \to \sqrt{Z_h G_N} h_{\mu\nu} \tag{4.66}$$

to turn it into a field of mass dimension one. Here we introduced the graviton wavefunction renormalization $Z_h$. At each level we then introduce separate scale and curvature dependent level $n$ coupling constants $G_N \to G_n(\bar{R})$ and $\Lambda \to \Lambda_n(\bar{R})$ to parametrize the tensor structures obtained from $S_{\mathrm{EH}}$

$$\Gamma_k^{(\phi_1...\phi_n)}(\vec{p}, \bar{R}, \Lambda_n(\bar{R}), G_n(\bar{R})) = S_{\mathrm{EH}}^{(\phi_1...\phi_n)}(\vec{p}, \bar{R}, \Lambda_n(\bar{R}), G_n(\bar{R})) \ . \tag{4.67}$$

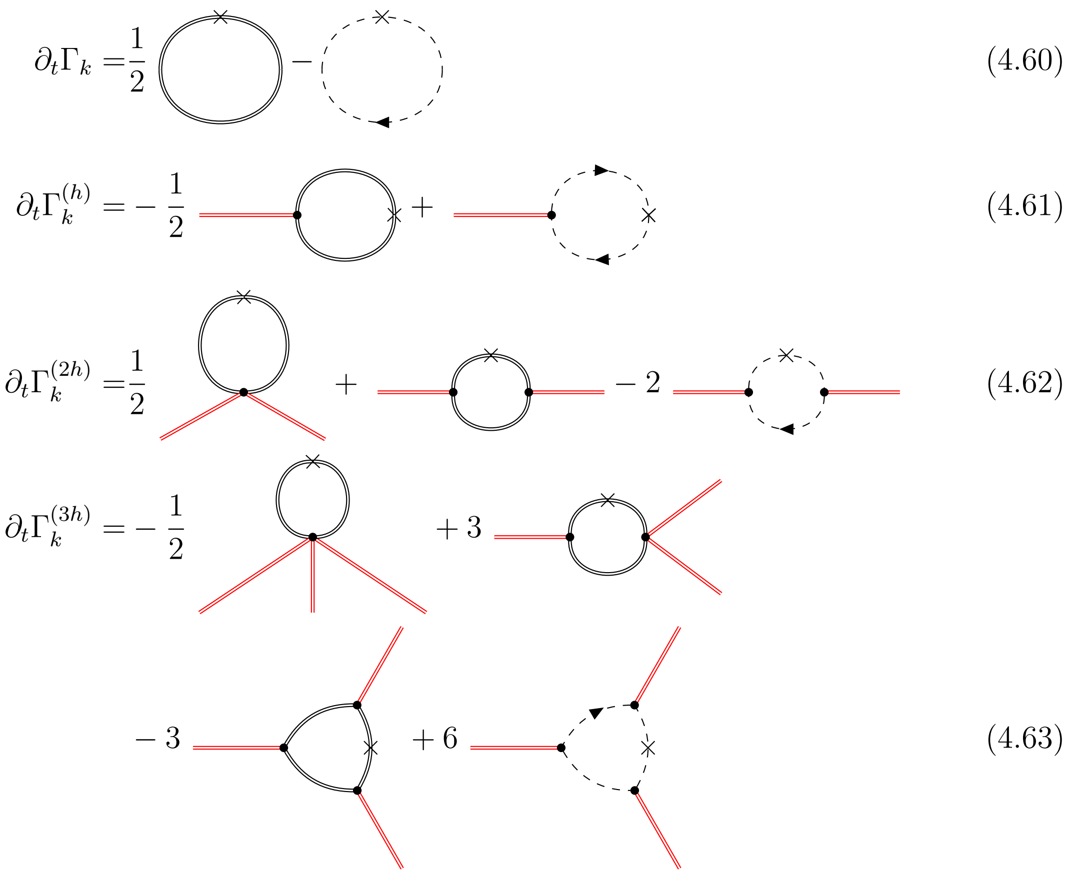



Figure 14: Diagrammatic representation for the flow equations $\partial_t \Gamma^{(n)}$. Black double lines represent graviton propagators, black dashed lines ghost propagators and red double lines stand for uncontracted indices at the vertices, where no propagators are attached. The crosses represent insertions of $\dot{R}_k$ and $\dot{R}_k^{(gh)}$ for gravitons and ghosts respectively. The different diagrams can be labeled as tadpole diagrams (both diagrams in $\partial_t \Gamma_k^{(h)}$), fish diagrams (first diagram in $\partial_t \Gamma_k^{(2h)}$), self energy diagrams (second and third in $\partial_t \Gamma_k^{(2h)}$) and star diagrams (third and fourth in $\partial_t \Gamma_k^{(3h)}$). The diagrams were created using the tikz-feynman package [29]

The vector $\vec{p}$ represents the set of spectral values, or in some sense momenta, that are assigned to the external legs of each diagram. We will mostly study the case where this external momentum is chosen to be $\vec{p} = 0$. The full momentum configurations for the diagrams in Flow$^{hhh}$ are given in Appendix B.
Now we can think about how we should close the flow, because we have several possibilities to do so. The simplest possibility would be to neglect higher order couplings all together and set $\Lambda_n = G_n = 0$ for $n > 3$. What we will do instead is to only set $\Lambda_5$ equal to zero while $G_5$ and $G_4$ are identified with $G_3$ and $\Lambda_4$ is identified with $\Lambda_3$. This scheme resulted in the most stable solution. We also set the wavefunction renormalization constant of all fields to $Z_{\phi_i} = 1$ such that the anomalous dimension is zero

$$\eta_{\phi_i} = -\partial_t \ln Z_{\phi_i} = 0 \tag{4.68}$$

Finally, to conclude our ansatz we need to specify a gauge fixing action $S_{\mathrm{gf}}$ which, by the standard Fadeev-Popov procedure, gives rise to a corresponding ghost action $S_{gh}$. We employ a linear gauge condition of the form

$$F_\mu = \bar{\nabla}^\nu h_{\mu\nu} - \frac{1+\beta}{4} \bar{\nabla}_\mu h^\nu{}_\nu \tag{4.69}$$

which gives rise to

$$S_{gf} = \frac{1}{2\alpha} \int d^4x \sqrt{\bar{g}} F^\mu F_\mu \ . \tag{4.70}$$

The gauge parameters $\alpha$ and $\beta$ are in general also scale dependent and obey their own flow equations. We are however not interested in the evolution of these parameters so we chose $\alpha \to 0$ which can be shown to be a fixed point at which $\partial_t \alpha_k^* = \partial_t \beta_k^* = 0$ regardless of the value for $\beta^*$ and we can therefore choose it also to be zero [30].
Corresponding to this gauge fixing condition we have a Fadeev-Popov operator

$$\mathcal{M}_{\mu\nu} = \frac{\delta F_\mu [h_{\mu\nu} + \delta_\xi h_{\mu\nu}]}{\delta \xi^\nu} = \bar{\nabla}^\rho (g_{\mu\nu} \nabla_\rho + g_{\rho\nu} \nabla_\mu) - \frac{1+\beta}{2} \bar{\nabla}_\mu \nabla_\nu \tag{4.71}$$

where $\delta h_{\mu\nu}$ is given by the gauge transformation

$$\delta h_{\mu\nu} = \nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu \ . \tag{4.72}$$

In the next section we will take a look at how we can extract beta functions for the level $n$ coupling constants from the vertex flows

$$\partial_t \Gamma_k^{(\phi_1 \ldots \phi_n)} = k^2 \mathrm{Flow}^{\phi_1 \ldots \phi_n}(p^2) \ . \tag{4.73}$$

### 4.5.1 Dimensionless beta functions

When studying asymptotic safety, we are normally interested in dimensionless coupling constants, because it is their beta functions that have to vanish at the fixed point. So let us define the dimensionless versions of the couplings in our truncation and also a dimensionless version of the Ricci scalar

$$
\begin{aligned}
r &= \frac{\bar{R}}{k^2} & \mu(r) &= -\frac{2\Lambda_2}{k^2} \\
g(r) &= g_3(r) = G_3(r)k^2 & \lambda_3(r) &= \frac{\Lambda_3}{k^2}\ .
\end{aligned} \tag{4.74}
$$

We will call $\mu(r)$ the graviton mass parameter since it defines the momentum independent part of the graviton two point function or the inverse propagator, which normally corresponds to a mass. In the following we will only focus on the transverse traceless part of the vertices, which can be obtained by an appropriate projection procedure, which will be explained shortly. The transverse traceless projection of the graviton two point function, see Appendix A, reveals that its momentum independent part is determined by an effective mass parameter which, next to its implicit dependence on curvature in $\mu(r)$ also has an explicit curvature dependence such that we define the effective graviton mass parameter

$$
\mu_{\text{eff}}(r) = \mu(r) + \frac{2}{3}r\ . \tag{4.75}
$$

We can also define an effective analog of $\lambda_3(r)$ which parametrizes the momentum independent part of the three point vertex.

$$
\lambda_{3,\text{eff}}(r) = \lambda_3(r) + \frac{1}{12}r\ . \tag{4.76}
$$

Let us go through the projection process now. We consider the $n$-point vertices after replacing the occurring operators, like $\bar{\nabla}^\mu$ or $\bar{\nabla}^2$, by their respective spectral values according to the procedure described in Appendix B. If we take a look at the two point function we see that it has four open indices, and one free external spectral value $p^2$ due to momentum conservation. We can construct a transverse traceless projector $\Pi^{\text{TT}}_{\mu\nu\alpha\beta}$ [8] and contract it with

$$
\partial_t \Gamma^{(hh)}_{k,\mu\nu\alpha\beta} = k^2 \text{Flow}^{hh}_{\mu\nu\alpha\beta}\ (p^2). \tag{4.77}
$$

to obtain

$$
\frac{1}{32\pi}\partial_t\left(k^2\left(p^2 + \mu(r) + \frac{2}{3}r\right)\right) = k^2\text{Flow}^{hh}_{\text{TT}}(p^2) \tag{4.78}
$$

Where $\text{Flow}^{hh}_{\text{TT}}$ is the projected right hand side of the flow equation. Here we can nicely see how $\mu_{\text{eff}}$ indeed parametrizes the momentum independent part of the two point function.

Now the beta function for $\mu$ can be extracted by setting the external momentum $p^2$ to zero. Rearranging the terms gives

$$\partial_t \mu(r) = -2\mu(r) + 2r\mu'(r) + 32\pi \mathrm{Flow}^{hh}_{\mathrm{TT}}(0) \tag{4.79}$$

in which $\mu'(r)$ denotes a derivative with respect to $r$.
To gain access to the beta functions for $g$ and $\lambda_3$ the three point vertex $\Gamma^{(3h)}$ is contracted with the following dimensionless tensors, which are the classical Einstein Hilbert vertices with each leg projected onto its transverse traceless part.

$$\Pi_G^{\alpha\beta\mu\nu\sigma\rho} = \frac{1}{\Lambda} S^{(3h)}_{EH,TT}{}^{\alpha\beta\mu\nu\sigma\rho}(\Lambda = 0, r = 0) \tag{4.80}$$

$$\Pi_\Lambda^{\alpha\beta\mu\nu\sigma\rho} = \frac{1}{p^2} S^{(3h)}_{EH,TT}{}^{\alpha\beta\mu\nu\sigma\rho}(p^2 = 0, r = 0). \tag{4.81}$$

This procedure leads to the equations

$$\partial_t \left( k^2 \sqrt{g} \left( \frac{5}{2304\pi^2} r + \frac{5}{192\pi^2} \lambda_3 - \frac{9}{4096\pi^2} p^2 \right) \right) = k^2 \mathrm{Flow}^{(3h)}_\Lambda(p^2) \tag{4.82}$$

$$\partial_t \left( k^2 \sqrt{g} \left( -\frac{3}{16384\pi^2} r - \frac{9}{4096\pi^2} \lambda_3 + \frac{171}{32767\pi^2} p^2 \right) \right) = k^2 \mathrm{Flow}^{(3h)}_G(p^2) \tag{4.83}$$

from which the beta functions can be extracted

$$\partial_t \lambda_3 = -2\lambda_3 + 2r\lambda_3' + \frac{2g - \partial_t g + 2rg'}{2g} \left( \lambda_3 + \frac{r}{12} \right) + \frac{3(32\pi)^2}{80\sqrt{g}} \mathrm{Flow}^{(3h)}_\Lambda(0) \tag{4.84}$$

$$\partial_t g = 2g + 2rg' + \frac{64(32\pi)^2\sqrt{g}}{171} \partial_{p^2} \mathrm{Flow}^{(3h)}_G(p^2) \Big|_{p^2=0} \tag{4.85}$$

Here we can find the definition of $\lambda_{3,\mathrm{eff}}$ again, which parametrizes the momentum independent parts of both projections

$$\frac{5}{192\pi^2} \lambda_3 + \frac{5}{2304\pi^2} r = \frac{5}{192\pi^2} \lambda_{3,\mathrm{eff}} \tag{4.86}$$

$$-\frac{3}{16384\pi^2} \lambda_3 - \frac{9}{4096\pi^2} r = -\frac{9}{4096\pi^2} \lambda_{3,\mathrm{eff}} \ . \tag{4.87}$$

At this point I would like to talk a little bit about the computational effort that goes into the derivation of the flow equations. At first what one has to do is to calculate functional derivatives of the Einstein Hilbert action up to fifth order. Already the three point function with symmetrized legs has around 500 terms, so an evaluation by hand is a hopeless task. In order to calculate the vertices we therefore make use of the Mathematica package xPert [31] which is a submodule of the bigger package xAct. The xAct packages allow to define a metric and a manifold on which we can define our tensor quantities right inside Mathematica. The package makes symbolic manipulations of these tensors very easy. The xPert package is a submodule which specifies on metric perturbation theory, which we can use to calculate the functional derivatives. Using this package we can expand the Einstein Hilbert action up to arbitrary order in the fluctuation metric $h_{\mu\nu}$. At this point we can use a built in function to obtain functional derivatives of this expansion, which gets rid of all possible total derivatives in the expression automatically. After that we have to use our symmetrization rule for covariant derivatives given in Appendix B to extract most of the curvature dependence before replacing the derivative operators with Laplacian spectral values. At this point we have to symmetrize the open indices and spectral values at the external legs to finally obtain the symmetrized $n$-point vertex functions. When they are at hand, we can turn to computing the diagrams contained in $\mathrm{Flow}^{(\phi_1 \dots \phi_n)}$, which consist of various contractions of vertex functions with propagators and regulators. Here we pair vertex functions with at least around 500 terms to graviton or ghost propagators that themselves also contain multiple terms. And the projection described above makes the effort even larger. It turns out that these contractions are so large that not even xAct can handle them in a feasible amount of time, which is why we resort to calculate the diagrams using the symbolic manipulation system FORM. The contracted diagrams now depend on the couplings that parametrize each vertex, the background curvature $r$ and the internal and external spectral values. The external ones are dealt with as described before in the derivation of the flow equations. The internal spectral values are the equivalent of loop momenta and we have to perform the spectral sums if $r > 0$, or spectral integrals if $r < 0$, as described in B.

## 4.6 Inclusion of scalar matter fields

So far what we have described is a pure gravity system. As we know our universe is filled with matter, the source of gravity, so are better talking about a theory that is compatible with standard model matter, i.e. fermions, gauge bosons and scalar fields. In this thesis we will only explore the simplest kind of matter, the scalar fields, and study its effect on the fixed point couplings and the equations of motion.
The simplest possible setting consists of $N_s$ non interacting massless scalar fields $\varphi_n$, where $n = 1 \dots N_s$ labels their flavor, which means that our ansatz for the effective average action will be extended by a kinetic term

$$S_{\mathrm{scalar}} = \frac{1}{2} \int d^4x \sqrt{g} \nabla^\mu \varphi_n \nabla_\mu \varphi_n \tag{4.88}$$

where we will disregard the scaling of the scalar fields i.e. we set the anomalous dimension $\eta_\varphi$ to zero. This action is the direct generalization of a generic scalar kinetic term in flat space. The difference to is that we use covariant instead of partial derivatives and that we include the factor of $\sqrt{g}$ which makes the action invariant under coordinate transformations. Adding this action to our previous ansatz for the effective action will introduce another term in the ERGE from before

$$\partial_t \Gamma_k[\bar{g},\phi] = \frac{1}{2}\mathrm{Tr}\left[\frac{\partial_t R_k[\bar{g}]}{\Gamma^{hh}[\bar{g},\phi]+R_k[\bar{g}]}\right] - \mathrm{Tr}\left[\frac{\partial_t R_k[\bar{g}]}{\Gamma^{\bar{c}c}[\bar{g},\phi]+R_k[\bar{g}]}\right] \tag{4.89}$$

$$+\frac{1}{2}\mathrm{Tr}\left[\frac{\partial_t R_k[\bar{g}]}{\Gamma^{\varphi_n\varphi_n}[\bar{g},\phi]+R_k[\bar{g}]}\right] \tag{4.90}$$

where now the super field $\phi$ also includes the scalar fields. Since the scalar action does not mix the flavors of $\varphi_n$, each field will just give the same contribution to the flow, so we can also just consider one field $\varphi$ and multiply its contribution by $N_s$

$$\frac{1}{2}\mathrm{Tr}\left[\frac{\partial_t R_k[\bar{g}]}{\Gamma^{\varphi_n\varphi_n}[\bar{g},\phi]+R_k[\bar{g}]}\right] = \frac{N_s}{2}\mathrm{Tr}\left[\frac{\partial_t R_k[\bar{g}]}{\Gamma^{\varphi\varphi}[\bar{g},\phi]+R_k[\bar{g}]}\right] \; . \tag{4.91}$$

Due to the extra term our vertex expansion of the ERGE will have new diagrams on the right hand side. These diagrams will include the scalar propagator $(\Gamma^{\varphi\varphi}[\bar{g},0]+R_k)^{-1}$ and the vertices $\Gamma^{\varphi\varphi h}[\bar{g},0]$, $\Gamma^{\varphi\varphi hh}[\bar{g},0]$ and $\Gamma^{\varphi\varphi hhh}[\bar{g},0]$, which can be obtained by expanding $S_{\text{scalar}}$ in powers of $h$ up to cubic order. All scalar vertices are not explicitly curvature dependent, however there is an implicit dependence on $\bar{R}$ due to the background Laplacian $\bar{\nabla}^2$ whose scalar eigenvalues are curvature dependent.
The fixed point couplings $g^*(r)$, $\mu^*(r)$ and $\lambda_3^*(r)$ will be effected by these new diagrams and depend on the number of scalar fields $N_s$. We will study if the fixed point will exist for an arbitrary number of scalar fields or if we can find some upper bound from which we can not find a fixed point of the ERGE. We will also study the effect of scalar fields on the equations of motion, they depend on the number of scalar fields in two ways. Firstly, the equations of motion depend on the graviton propagator, which itself depends on the graviton mass parameter $\mu(r)$, and as we just noticed, $\mu(r)$ depends on the number of scalar fields. The other effect is more direct. Since we study the equations of motion with the help of the ERGE, there will also be additional diagrams on the right hand side of $\partial_t\Gamma[\bar{g},0]$ and $\partial_t\Gamma^{(h)}[\bar{g},0]$. These new contributions will also be proportional to $N_s$. We will check for solutions of both the background and quantum equation of motion depending on the number of scalar fields, and again try to find a bound for $N_s$, above which no solution can be found.

# 5 Results

In this section we will discuss what we obtained from our calculations in the framework introduced in the previous section. They build upon established results, for positive curvature in a pure gravity setting, from [3] and will mostly be presented as they are here in an upcoming paper [32]. We will only be interested in fixed point quantities, i.e. we present the system of background dependent fixed point couplings $g^*(r)$, $\mu^*(r)$ and $\lambda_3^*(r)$ or rather their effective analogues

$$g^*_{\text{eff}}(r) = g^*(r) \tag{5.1}$$

$$\mu^*_{\text{eff}}(r) = \mu^*(r) + \frac{2}{3}r \tag{5.2}$$

$$\lambda^*_{3,\text{eff}}(r) = \lambda^*_3(r) + \frac{1}{12}r \ . \tag{5.3}$$

Due to our approximation for the action of covariant derivatives, explained in Appendix B, we will restrict ourselves to a curvature domain $r \in [-2, 2]$ for most calculations, for which we think our approximation is valid. We will also analyze the effect of introducing $N_s$ scalar matter fields to our system and check if this spoils the existence of our fixed point. After that we will turn our attention to the background and quantum equations of motion, which we expect to have different solutions based on the fact that at non zero scale $k$ we have non trivial Nielsen-identities as explained in subsection 4.4.

## 5.1 Curvature dependent couplings

### 5.1.1 Pure gravity system

To find the fixed point functions, we set the beta functions in Equation 4.79 and Equation 4.82 equal to zero, such that we end up with ordinary differential equations in the background curvature $r$. The fixed point couplings in the limit $r \to 0$ serve as our initial conditions, which we calculate using a heat kernel expansion, explained in Appendix B. This is necessary as neither our trace evaluation method for positive nor that for negative curvature include this limit explicitly. Hence the heat kernel expansion makes it possible to shift the initial values away from $r = 0$ to a small $r_{\text{start}}$ where we can connect these methods to the flat limit. Evaluating the expansion up to first order gives

$$\begin{pmatrix} g^*(r) \\ \mu^*(r) \\ \lambda^*_3(r) \end{pmatrix} = \begin{pmatrix} 0.598 \\ -0.381 \\ -0.116 \end{pmatrix} + r \begin{pmatrix} -0.435 \\ -0.134 \\ -0.716 \end{pmatrix} + \mathcal{O}(r^2) \, . \tag{5.4}$$

For our starting value we chose $r_{\text{start}} = \pm 0.03$, which is not too large such that our linear approximation would fail and also not too small, in which case our spectral sums and integrals would fail to produce reliable results.
The full curvature dependent fixed point functions are shown in Figure 15. For positive curvature the calculations coincide with the results found in [3], and here we extended the calculation to negative curvatures.
We can directly see the benefit of plotting the effective couplings, as $\mu^*_{\text{eff}}(r)$ and $\lambda^*_{3,\text{eff}}(r)$ are always close to their flat space values, which means that the explicit curvature dependence of the momentum independent parts of the two and three point function almost perfectly cancels the implicit one, such that they are almost curvature independent. Ignoring the curvature dependence of the non kinetic parts of the two and three point functions $\mu_{\text{eff}}$ and $\lambda_{3,\text{eff}}$ and setting them equal to their flat values is therefore not such a bad approximation and will be used later, when solving the quantum equation of motion.

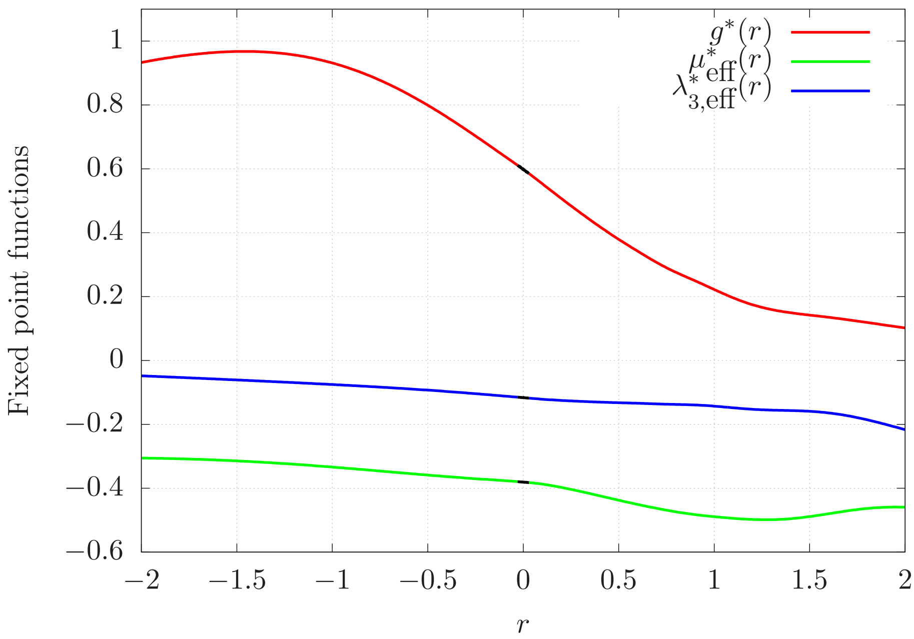



Figure 15: Shown are the background curvature dependent fixed point couplings $g^*$, $\mu^*_{\text{eff}}$ and $\lambda^*_{3,\text{eff}}$ for positive and negative curvature. The flat space initial conditions, shown in black, are determined by a first order heat kernel expansion.

### 5.1.2 Scalar matter

We will now study, how the fixed point functions behave under the effects of $N_s$ scalar matter fields. In Figure 16 we have plotted the zeroth and first order heat kernel coefficients, which we again use to translate our initial conditions away from $r = 0$. We see that the flat fixed point, shown in the left panel only changes mildly for $N_s \leq 30$. Especially $\lambda_{3,\text{eff}}$ seems to be almost unaffected. The gravitational coupling $g_0$ grows slowly for increasing number of scalar fields, while the graviton mass parameter tends to zero, however it looks like, for larger numbers of scalar fields it will go beyond zero and turn positive. The reason why we could only show the coefficients up to $N_s = 30$ scalar fields becomes apparent when looking at the right panel in Figure 16. The first order coefficient of $g$ grows towards larger and larger negative values until if finally diverges at around $N_s \approx 32.2$. Regardless of the divergence in the first order coefficient, a fixed point at zeroth order can actually be found for up to $N_s \approx 38.5$, but for larger $N_s$ this fixed point also vanishes.
The full curvature dependence of the three couplings is shown in Figure 17, Figure 18 and Figure 19. The line printed in darker red corresponds to the case when $N_s = 0$ while the other curves show the results up to $N_s = 16$ in steps of $\Delta N_s = 1$.
Let us first take a look at the curvature dependence of the gravitational coupling $g$. We see that for positive curvature the scalar fields have almost no effect at all and the curvature dependence of $g$ is qualitatively the same for all $N_s$. For negative $r$ however the coupling

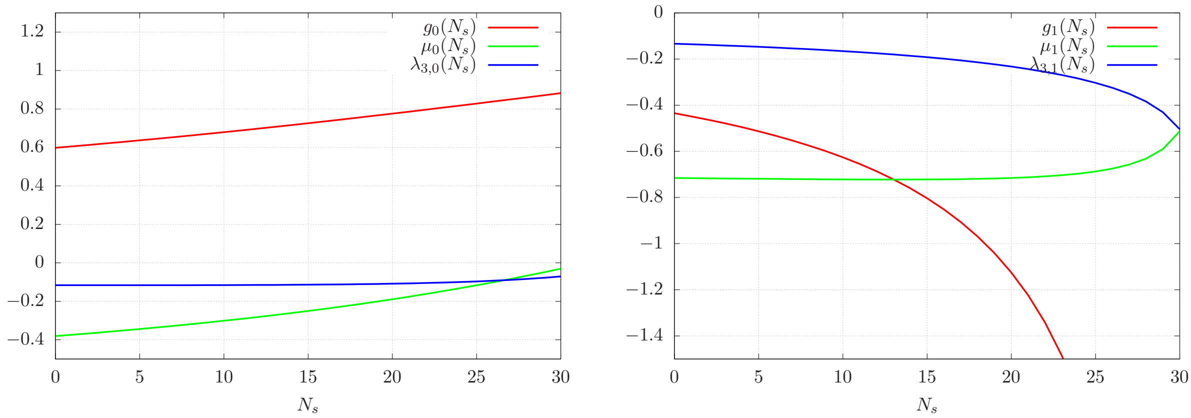



Figure 16: Shown are the zeroth and first order coefficients from the heat kernel expansion of the fixed point couplings $g^*$, $\mu^*$ and $\lambda_3^*$, dependent on the number of scalar fields $N_s$.

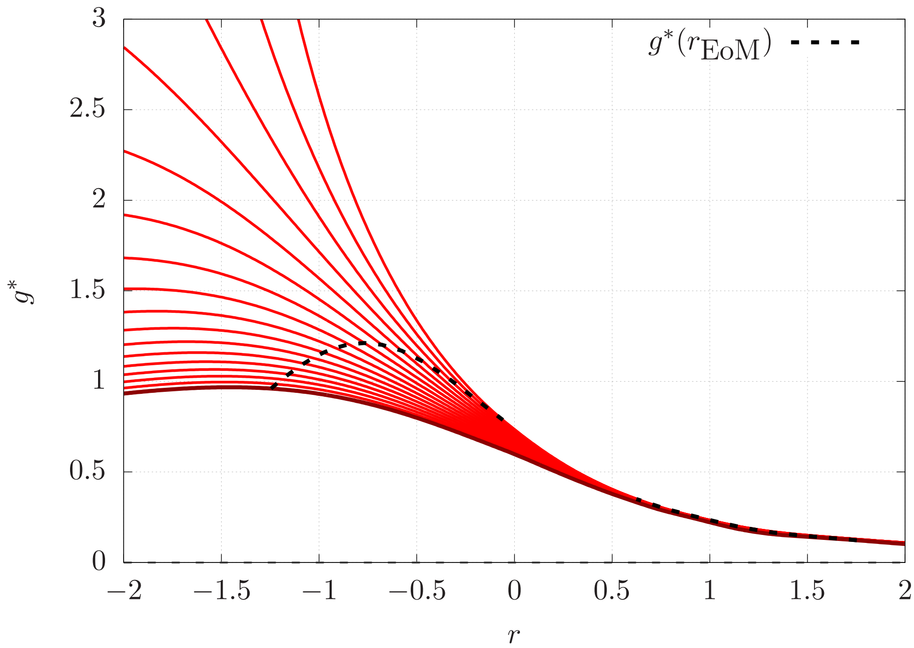



Figure 17: The figure shows the curvature dependence of the fixedpoint function $g^*(r)$ for different number of scalar fields. The darker line corresponds to the $N_s = 0$ case while the other lines stand for increasing number of scalar fields in steps of $\Delta N_s = 1$ up to $N_s = 16$. The black line corresponds to the value of $g^*(r)$ when evaluated at the solutions to the quantum equation of motion $r_{\mathrm{EoM}}$.

diverges at a finite value of $r$, which is also the case why we could only display the results up to $N_s = 16$. This was the maximal number of scalar fields for which we found a solution of $g(r)$ in the whole investigated curvature regime. However we should take these results

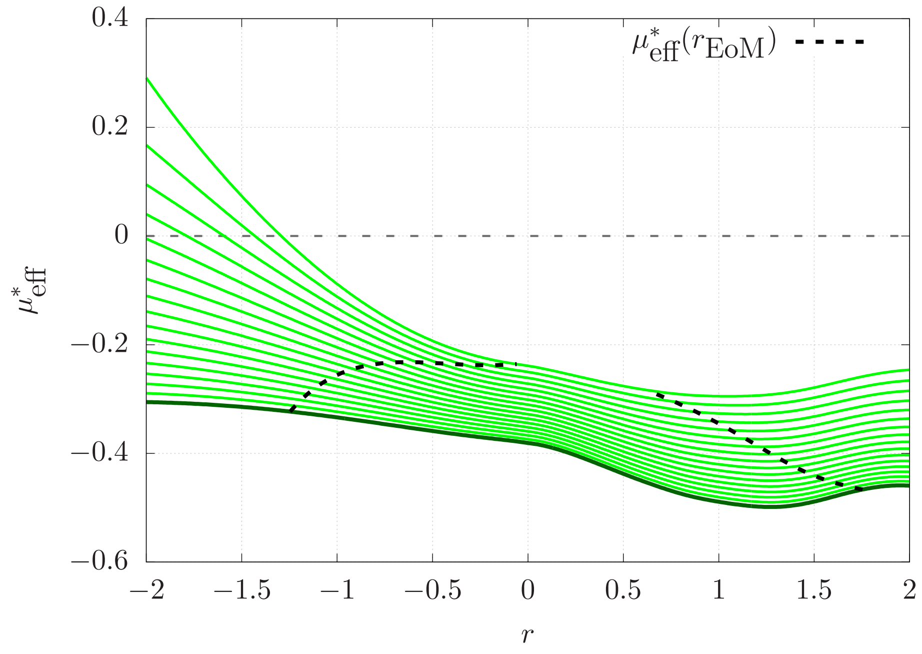



Figure 18: The figure shows the curvature dependence of the fixedpoint function $\mu^*_{\text{eff}}(r)$ for different number of scalar fields. The darker line corresponds to the $N_s = 0$ case while the other lines stand for increasing number of scalar fields in steps of $\Delta N_s = 1$ up to $N_s = 16$.The black line corresponds to the value of $\mu^*_{\text{eff}}(r)$ when evaluated at the solutions to the quantum equation of motion $r_{\text{EoM}}$.

with a grain of salt, as we completely neglected the flow and curvature dependence of the wavefunction renormalization. It is very likely that including it would have helped in finding a finite solution for $g(r)$. Generally however we still expect $g$ to grow for large negative values of $r$ based on what we see for smaller numbers of scalar fields.
The effective graviton mass parameter $\mu_{\text{eff}}$ just seems to be shifted up both at positive and negative $r$, which can especially be seen at positive curvatures. For large negative curvature it seems like large numbers of scalar fields result again in the onset of a divergent behavior, which is however most likely triggered by the previously noted large values of $g$ in this regime. The approximation $\mu_{\text{eff}} = \text{const.}$ should therefore still not be too bad.
When looking at the curvature dependence of $\lambda_{3,\text{eff}}$ we find that when including more scalar fields the approximation $\lambda_{3,\text{eff}} = \text{conts.}$ does not hold anymore. The curve seems to be rotated downwards, such that at positive curvature the coupling tends towards smaller values while at negative curvature the values become larger.

## 5.2 Background equation of motion

In this section we have a look at the background equation of motion, which can be studied by looking at the effective average action at vanishing fluctuation field $\Gamma^*_k[\phi = 0, \bar{g}]$. We

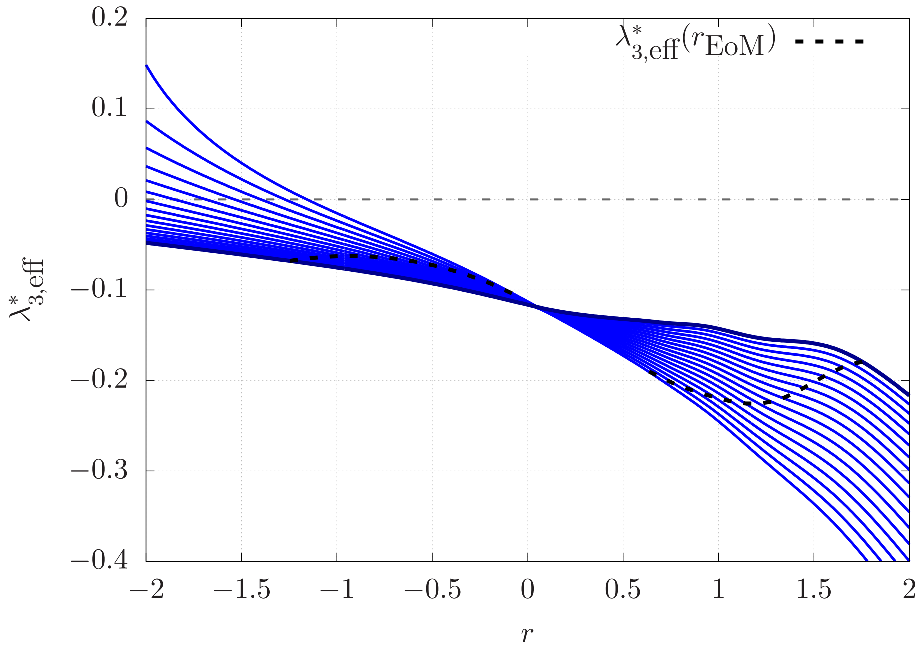



Figure 19: The figure shows the curvature dependence of the fixedpoint function $\lambda^*_{3,\text{eff}}(r)$ for different number of scalar fields. The darker line corresponds to the $N_s = 0$ case while the other lines stand for increasing number of scalar fields in steps of $\Delta N_s = 1$ up to $N_s = 16$.The black line corresponds to the value of $\lambda^*_{3,\text{eff}}(r)$ when evaluated at the solutions to the quantum equation of motion $r_{\text{EoM}}$.

parameterize $\Gamma_k$ by the dimensionless background potential

$$\Gamma_k[\phi = 0, \bar{g}] = \int d^4x \sqrt{\bar{g}} k^4 f(r) \tag{5.5}$$

If we consider the scale derivative of $\Gamma_k[\phi = 0, \bar{g}]$ we can use the right hand side of the zeroth order term in our vertex expansion

$$\dot{\Gamma}_k[\phi = 0, \bar{g}] = \frac{1}{2}\text{Tr}\left[\left(\Gamma_k^{(hh)}[\phi, \bar{g}] + R_k^h\right)^{-1} \dot{R}_k^h\right]\Big|_{\phi=0} - \text{Tr}\left[\left(\Gamma_k^{(\bar{c}c)}[\phi, \bar{g}] + R_k^c\right)^{-1} \dot{R}_k^c\right]\Big|_{\phi=0} . \tag{5.6}$$

which, if we restrict ourselves to the fixed point, is just a function of $r$ and the fixed point coupling $\mu^*(r)$ which parameterizes the previously calculated fixed point fluctuation two point function

$$\Gamma_k^{*(hh)}[\phi, \bar{g}]\big|_{\phi=0} . \tag{5.7}$$

On the left hand side we can plug in our parameterization of $\Gamma_k[\phi = 0, \bar{g}]$ and directly perform the scale derivative

$$\dot{\Gamma}_k[\phi = 0, \bar{g}] = \int d^4x\sqrt{\bar{g}}\left(4k^4 f(r) + k^4 \dot{f}(r) + k^4 f'(r)\partial_t r\right) \tag{5.8}$$

$$= \int d^4x\sqrt{\bar{g}}\left(4k^4 f(r) + k^4 \dot{f}(r) - 2rk^4 f'(r)\right) \tag{5.9}$$

$$\tag{5.10}$$

such that at the fixed point we find

$$\dot{\Gamma}^*_k[\phi = 0, \bar{g}] = k^4 V\left(4f^*(r) - 2rf^{*'}(r)\right) \tag{5.11}$$

where $V$ for positive curvature corresponds to the volume of the four sphere $V = \frac{385\pi^2}{k^4r^2}$ and for negative curvature to the formally infinite volume of hyperbolic space, which however cancels out with the same factor on the right hand side. What we finally get is a differential equation of the form

$$f^{*'}(r) = \mathcal{F}(f^*(r), \mu^*(r), r) \tag{5.12}$$

which we can solve to obtain $f^*(r)$. To find solutions to the background equation of motion we have to consider the functional derivative of $\Gamma[0, \bar{g}]$ with respect to $\bar{g}_{\mu\nu}$

$$\frac{\delta}{\delta\bar{g}_{\mu\nu}}\sqrt{\bar{g}}f(r) = \sqrt{\bar{g}}\left(\frac{r}{4}f'(r) - \frac{1}{2}f(r)\right)\bar{g}^{\mu\nu} = \sqrt{\bar{g}}\bar{f}_1(r)\bar{g}^{\mu\nu} = 0\ . \tag{5.13}$$

In Figure 20 we plotted the function $\bar{f}_1^*(r)$, for the different numbers of scalar fields. The zero crossings of this function would correspond to solutions to the background equation of motion, but as we can see there are none. This is however not directly a problem since, we already said that the quantum equation of motion, which we will discuss in the upcoming section, should be used to calculate the background metric $\bar{g}_{\text{EoM}}$. We can check the stability of this result, by treating $\mu^*_{\text{eff}} = \text{const.}$ as a free input parameter rather than using the exact solution for $\mu^*_(r)$. We only find solutions in the pure gravity system and for positive values of $\mu^*_{\text{eff}} > 0.26$, which, given our current calculations are far away from the computed value of $\mu_{\text{eff}}$ which is always negative. What we also observe is that $\bar{f}_1^*(r)$ diverges for finite positive $r$, because we run into a pole of the graviton fluctuation propagator in Equation 5.6. The fact that the background equation of motion does not have a solution for positive curvatures in our framework was already established in [3], here we extended this calculation and verified that this result also holds for negative curvatures.
However we can still use the background potential $f^*(r)$ to obtain values for the background gravitational and cosmological couplings $g_0^*$ and $\lambda_0^*$ by expanding it up to first order in curvature

$$f^*(r) \approx f_0 + f_1 r \tag{5.14}$$

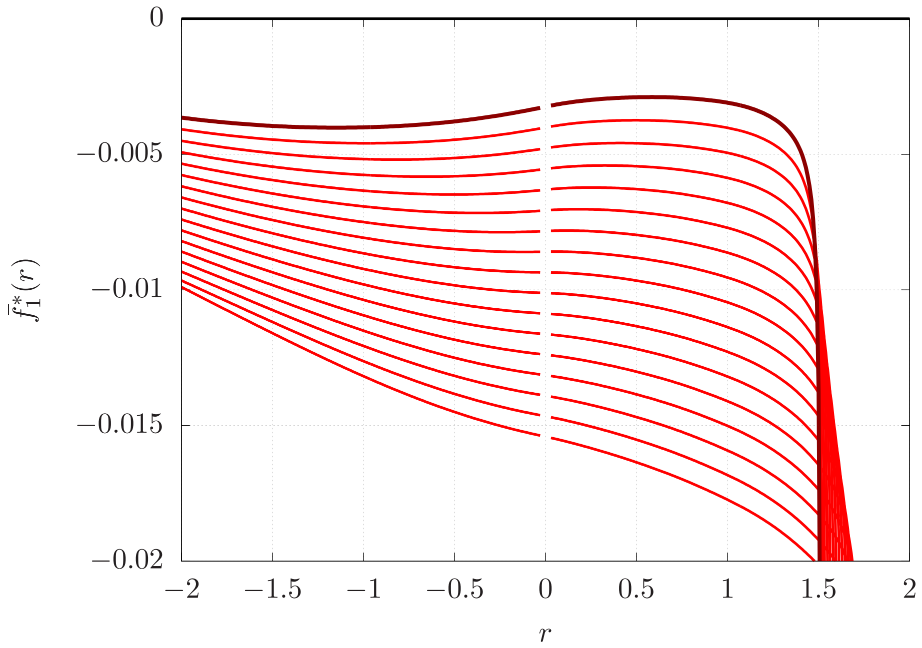



Figure 20: The figure shows the fixedpoint function $\bar{f}_1^*(r)$ for different number of scalar fields. The darker line corresponds to the $N_s = 0$ case while the other lines stand for increasing number of scalar fields in steps of $\Delta N_s = 1$ up to $N_s = 16$. The zero crossings of this function would correspond to solutions of the background equation of motion, but as seen in the figure, there are none.

We can then compare the zeroth and first order coefficients to the Einstein Hilbert action to obtain

$$g_0^* = -\frac{1}{16\pi f_1} \tag{5.15}$$

$$\lambda_0^* = -\frac{f_0}{2f_1} \tag{5.16}$$

In Figure 21 we can see the background couplings, depending on $N_s$, and we observe that both couplings seem to diverge around $N_s \approx 7$. This divergence is not present in the fluctuation couplings $\mu$, $g$ and $\lambda_3$, which coincides with the results found in [5]. However the pole they found for the background couplings was located at far larger $N_s > 60$, so we conclude that the observed location of the pole most likely depends on the choice of gauge and regulator.

## 5.3 Quantum equation of motion

We found that the background equation of motion has no solutions in the investigated curvature regime for all numbers of $N_s$ under consideration. In previous sections we already

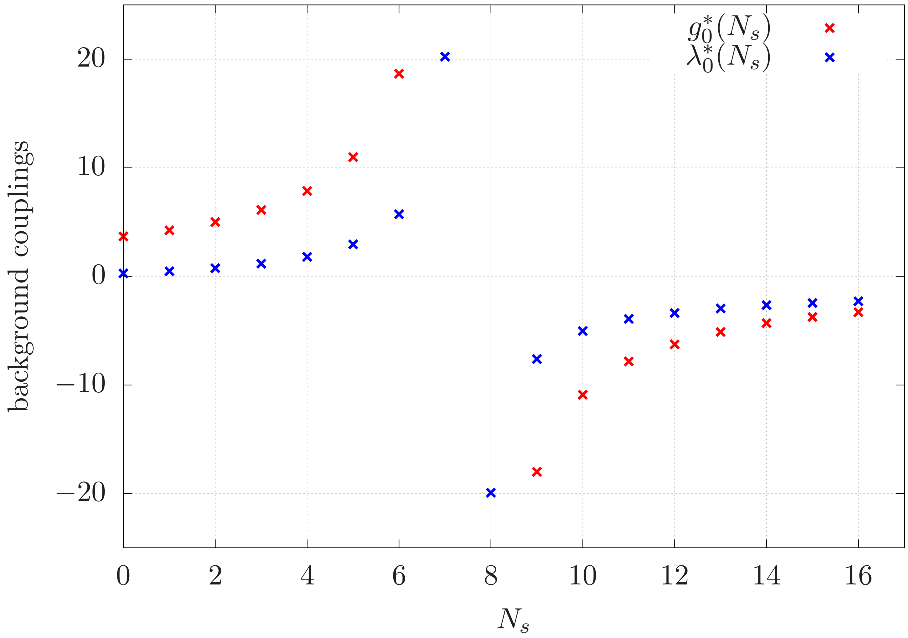



Figure 21: We plot the background couplings, that were obtained by expanding the $f(r)$ potential up to first order in $r$. We observe a divergent behavior around $N_s \approx 7$, after which both couplings turn negative.

mentioned that we should use the quantum equation of motion to gain insides into the physical solution for $\bar{g}_{\mu\nu}$ if $k > 0$, which is of course the case here. The quantum equation of motion requires us to find the zeros of

$$\frac{\delta \Gamma_k[\phi, \bar{g}]}{\delta h_{\mu\nu}} = 0 \tag{5.17}$$

Due to our restriction to maximally symmetric backgrounds we can conclude that there can not be a first order operator involving the transverse traceless part of $h_{\mu\nu}$. We can see this by assuming there is a first order coefficient $f_1^{\mu\nu} h_{\mu\nu}$. Since there can be no tensor structures involving the Ricci or Riemann tensors $r^{\mu\nu}$ and $r^{\mu\nu\alpha\beta}$ in $f_1^{\mu\nu}$ we can reparameterize the coefficient as $f_1^{\mu\nu} = \tilde{f}_1 \bar{g}_{\mu\nu}$ which directly implies a projection of $h_{\mu\nu}$ onto its trace part $h$, and there is no coefficient proportional to $h_{\mu\nu}^{\mathrm{TT}}$.
Hence we can restrict our attention to the trace part of the graviton one point function and look for its zeros

$$\left. \frac{\delta \Gamma[\phi, \bar{g}]}{\delta h} \right|_{h=0} = 0 \ . \tag{5.18}$$

Therefore we consider the flow equation for $\Gamma_k^{(h)}[\phi = 0, \bar{g}]$ and project it onto its trace component to obtain the right hand side of our fixed point equation. We parameterize the one point function in a similar way as we did for the effective action to study the background equation of motion. We again make use of a dimensionless function $f_1(r)$

$$\Gamma_k^{(h)}[\phi = 0, \bar{g}] = \int d^4x \sqrt{\bar{g}} k^3 f_1(r) \tag{5.19}$$

to obtain the left hand side of the fixed point equation

$$\dot{\Gamma}_k^*[\phi = 0, \bar{g}] = \int d^4x \left(3k^3 f_1^*(r) - 2 f_1^{*\prime}(r) r\right) \tag{5.20}$$

$$= k^2 \mathrm{Flow}_{\mathrm{Tr}}^h \; . \tag{5.21}$$

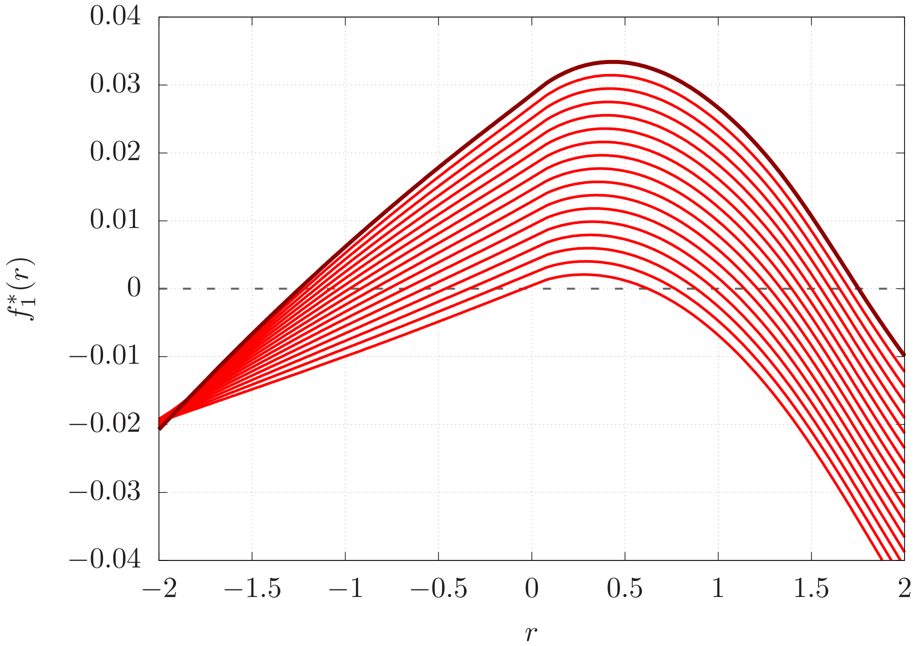



Figure 22: This figure shows the fixed point function $f_1^*(r)$ for various $N_s < 17$, using the full solution of $\mu_{\mathrm{eff}}(r)$ that was shown in Figure 18. Again the darker line represents the $N_s = 0$ case and the other lines display the results for increasing number of scalar fields in steps of $\Delta N_s = 1$. The zeros of $f_1^*(r)$ correspond to solutions of the quantum equation of motion, which give the $r_{\mathrm{EoM}}$.

The solutions to this differential equation for various $N_s$ are shown in Figure 22, where again the zero crossing solve the equation of motion. For each number of scalar fields we considered, we found two solutions, one of which is always located at negative $r$ and the other at positive $r$. For increasing number of scalar fields the absolute values of $r$ where the solutions are located decrease, meaning that large numbers of $N_s$ result in an expectation value of $g_{\mu\nu}$ that corresponds to a less curved space than what we get in the pure gravity system. Unfortunately we have no solution to the equation of motion for $N_s = 17$ since

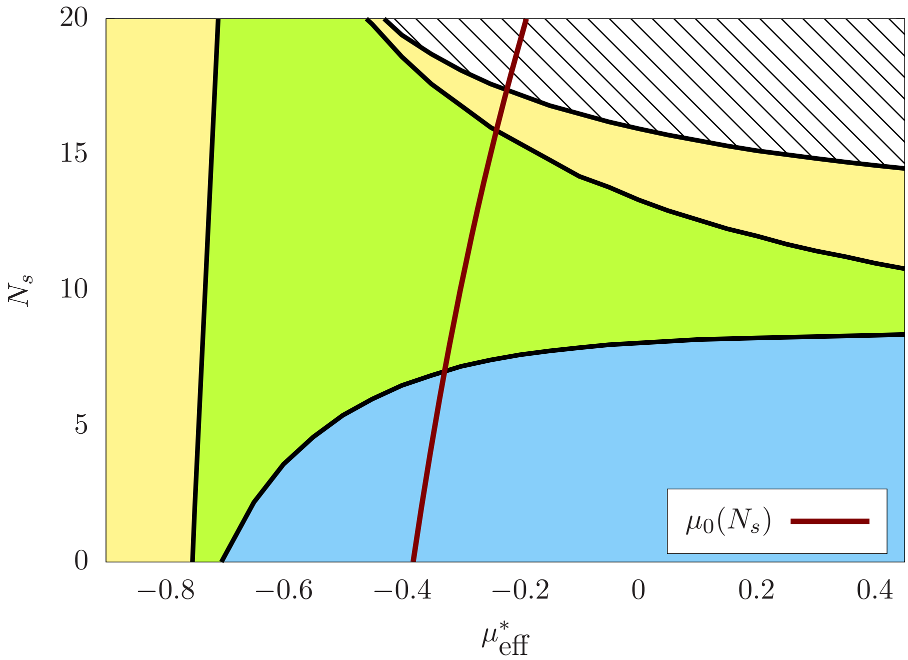



Figure 23: Here we display the roots of $f_1^*(r)$ for various inputs of $N_s$ and $\mu^*_{\text{eff}} = \text{const.}$. The white area corresponds to the case when the quantum equation of motion has no solution. The beige, blue and green areas stand for one positive, one negative or both a positive and a negative solution respectively. The dark red line shows the actual value of $\mu_0^*(N_s)$.

the gravitational coupling showed a divergent behavior for $N_s > 16$, and so we could also not determine a solution for $\mu^*(r)$ which is needed to compute $f_1^*(r)$. However it seems that for large numbers of scalar fields the solutions to the quantum equation of motion vanish just at that value of $N_s$ above which we could not find a fixed point in the whole curvature regime. This could be a coincidence or a hint, that there might exist a maximum number of scalar fields in general above which the asymptotic safety scenario can not be realized anymore. Luckily the number of scalar fields included in the standard model is four which correspond to the components of the Higgs field, and is therefore much smaller than this critical value of $N_s \approx 16$ we found here. However if we take this result seriously this would be a restriction for possible extensions of the standard model that include more scalar degrees of freedom. We also have to note that this upper bound will probably be modified when we include other field types like fermions or vector fields in the system. Calculations from single field approximations also give hints for various bounds for the number of degrees of freedom for each field type [8], which however differ from the results we obtained here. Nevertheless, qualitatively they also predict upper bounds on $N_s$, for asymptotically safe theories of gravity.

Again we can check stability of these results by treating $\mu^*_{\text{eff}} = \text{const.}$ as a free input parameter in the quantum equation of motion. In Figure 23 we depicted the various outcomes for different combinations of $\mu^*_{\text{eff}}$ and $N_s$. In [3], the approximation $\mu^*_{\text{eff}} = \text{const.}$ was already used to establish the existence of solutions for positive curvature, while they used a heat kernel approximation at negative $r$. Here we extended these calculations to negative curvatures while not relying on the heat kernel for very small $r$. We also generalized the results by including scalar fields. The beige and blue areas represent the regimes in which we found solutions only for positive or negative $r$ respectively, while the green area is the regime where two solutions at positive and negative $r$ were present.

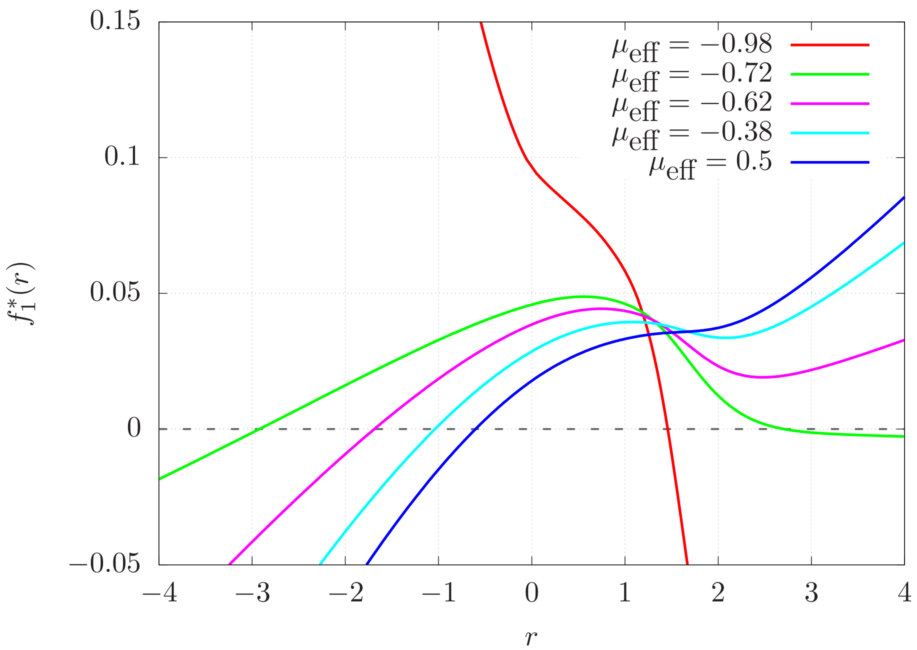



Figure 24: This plot shows the fixed point function $f_1^*(r)$ if we use $\mu_{\text{eff}}(r) = \text{const.}$ as an input for the quantum equation of motion. We see a different behavior from the exact case what we observed in the exact case.

The shaded area corresponds to a regime where no solutions were obtained. In red we depicted the values of for the effective coupling when we use the flat space results $\mu^*_{\text{eff}} = \mu^*_0$. We observe that for values of $\mu^*_{\text{eff}}$ greater than about $-0.4$ there exists an upper bound on $N_s$, which is shifted to larger $N_s$ if we decrease $\mu^*_{\text{eff}}$. Since the flat space values $\mu_0$ increase for large $N_s$ we run into this regime with our solution. However the quantitative value of $\mu^*_{\text{eff}}$ could change if we increase the order of our truncation or choose to use a different gauge or regulator function. So it might very well be that this upper bound only showed up because of the choices we made for our system. What we also observe is that, when using a constant $\mu^*_{\text{eff}}$, there is a regime at small $N_s$ where only a solution for negative curvatures exist. We have demonstrated this behavior in Figure 24, where we took $N_s = 0$ and plotted $f_1^*(r)$ for various constant inputs of $\mu^*_{\text{eff}}$. If we take a look at the case $\mu^*_{\text{eff}} = -0.38$ which is approximately the calculated value of $\mu^*_0$ for $N_s = 0$, we see that instead of tending towards negative values the solution bends upwards, which is a qualitatively different behavior from

what we found when using the exact $\mu^*(r)$. This is probably due to the slight bend of $\mu^*(r)$ we see in Figure 15 at positive curvature, opposed to the constant input we used here.
If we take a look back at Figures 17-18 we can understand the black lines we mentioned earlier, which are the couplings evaluated on the solutions of the equation of motion that we found here. We see that the couplings evaluated on the negative branch of our solutions always have a maximum at intermediate values of $N_s$. For the positive branch $\lambda^*_{3,\text{eff}}(r_{\text{EoM}})$ shows also shows a minimum for intermediate $N_s$, while the absolute value of $g^*(r_{\text{EoM}})$ increases and the absolute value of the effective graviton mass parameter decreases towards larger $N_s$.

# 6 Conclusion

For many decades now theoretical phyicists have been working on new ideas and concepts of how we can make sense of gravity in a combined framework with the other forces. The naive approach of quantizing gravity as any other field theory fails due to the non-renormalizablity we encounter in perturbative quantum gravity, which stems from the fact that if we try to resolve very small length scales, gravity becomes so strong that we encounter a breakdown of the smooth geometry we know from general relativity. This gives a hint that general relativity can merely be considered an effective low energy description of our world, and probably has to be replaced by some deeper fundamental theory we have not found yet. We have discussed some of the very different theories people came up with over the years, like CDT, LQG, string theory or even more exotic theories that completely get rid of our notion of space and time at the Planck scale. In this thesis we explored one of the less exotic approaches, that is formulated in the framework of quantum field theory, but instead of defining it perturbatively the asymptotic safety scenario aims at a non perturbative renormalization of gravity. The machinery used in this theory is the functional renormalization group which is also proven to be useful in many other areas of physics, like QCD for which an ultraviolet completion in terms of asymptotic freedom could be found. We want to keep our asymptotically safe theory of gravity to behave like general relativity at low energies, and we argued that we need a background independent framework which does not postulate any background spacetime to do so. Trying to implement the functional renormalization group for gravity however, relies on the background field method in order to make sense. This method explicitly introduced a background metric to our theory, and rendered the effective action at finite scale $k \neq 0$ background dependent. We resolved this by disentangling the dependence on the fluctuation metric and the background metric through a vertex expansion of the effective average action. With the help of this expansion we could study the renormalization group flow for the $n-$ point fluctuation correlation functions. In our ansatz we truncated and closed the flow of the otherwise infinite dimensional tower of flow equations at third order. We parameterized the two and three point functions by the classical Einstein Hilbert tensor structures and restricted our calculations to maximally symmetric spaces, collapsing the curvature dependence of these correlation functions to a simple dependence solely on the Ricci scalar. We then studied the curvature dependent fixed point functions of the graviton mass parameter $\mu(r)$, the gravitational coupling $g(r)$ and $\lambda_3(r)$ which parameterized the momentum independent part of our three point function. After solving the fixed point equations for a pure gravity system, we also introduced non interacting scalar fields and studied how the fixed point solutions change in this extended system. We could not find a fixed point in the entire curvature regime when the number of scalar fields exceeded $N_s = 16$, because the gravitational coupling showed a divergent behavior for large negative curvatures. We then used the curvature dependent two point function to study the background and quantum equations of motion in order to obtain the physical background $\bar{g}_{\mathrm{EoM}}$. In the physical limit $k \to 0$, the solutions to both of these equations agree, while at finite scales $k \neq 0$ the quantum equation of motion should be used to find $\bar{g}_{\mathrm{EoM}}$ because the background equation of motion explicitly gets modified by the introduction of a regulator. We found that while the background equation of motion in our system had no solutions at all, the quantum equation of motion showed solutions at negative and positive curvature for $N_s < 17$. For larger numbers of scalar fields we found that these solutions vanish so we concluded that this could be a hint that the asymptotic safety scenario is only valid up to some upper bound, whose numerical value however can depend on the choice of regulator, gauge, truncation and other field types that one could include in the system. We also calculated the fixed point background potential $f(r)$ that parameterizes the effective action at vanishing fluctuation field, from which we obtained values for the background gravitational and cosmological couplings.

Background independence for the gravitational functional renormalization group is a very delicate issue, since as opposed to other field theories it is mandatory in gravity to make use of the background field method to even define the flow equations. A lot of work still has to be done to properly resolve this issue, like performing similar calculations on more general backgrounds than we used here, or extending the truncation of the vertex expansion. Also one has to include more general tensor structures in the $n-$point functions than just the classical ones, for example higher derivative terms or possibly non local terms. In our framework we also only studied the correlation functions at vanishing external momentum which is also something one could improve on, if one is interested in the full momentum dependent vertices.

Eventually, to truly determine if the asymptotic safety scenario is a viable candidate for a theory of quantum gravity, one would have to include the whole standard model Lagrangian and look for non trivial fixed point in this very large system. However until this problem can possibly be tackled there is a long way with hard work ahead and maybe another theory for quantum gravity will prove to be correct in the meantime. One way or the other, quantum gravity is a very interesting open topic of modern physics and finding a solution to it will, no matter which theory will be correct in the end, provide many more insights to the phyiscal world we live in and strengthen our understanding of nature and probably even revolutionize our current ideas of space and time.

# Appendices

## A Graviton propagator

In this section I want to focus on the regularized graviton propagator which is needed for the evaluation of each diagram contained in the expressions $\mathrm{Flow}^{(\phi_1 \dots \phi_n)}$ in subsection 4.5, and also for calculating the background and quantum equation of motion in section 5. The needed propagator $G_{k,h}$ is defined by the inverse of the second functional derivative of the effective action $\Gamma_k$ to which we add the regulator function $R_{k,h}$

$$G_{k,h} = \frac{1}{\Gamma^{(2h)} + R_{k,h}} \ . \tag{A.1}$$

The graviton two point function $\Gamma^{(2h)}$ is, as all other vertex functions, in our ansatz given by the second functional derivative of the Einstein Hilbert action

$$S_{\mathrm{EH}} = \frac{1}{16\pi G_N} \int d^4x \sqrt{g} \, (2\Lambda - R) + S_{\mathrm{gf}} + S_{\mathrm{gh}} \tag{A.2}$$

with the gravitational and cosmological couplings $G_N$ and $\Lambda$ replaced by their order two counter parts. Due to our previous rescaling of $h \to \sqrt{G_N} h$, the factor of $\frac{1}{G_N}$ in $S_{\mathrm{EH}}$ will cancel out in the two point function and we are only left with $\Lambda_2$ which we express by the graviton mass parameter $\mu = -\frac{2\Lambda_2}{k^2}$ later. The two and one point functions are the only ones for which it is feasible to perform the calculation by hand so let us start by calculating $\Gamma^{(h)}$ from which we can go on and calculate $\Gamma^{(2h)}$. In the following we will suppress all spatial arguments and integrations for notational reasons. Normally we would have

$$\frac{\delta}{\delta h_{\mu\nu}(y)} \int d^4x h_{\alpha\beta}(x) = \frac{1}{2} \int d^4x \left( \bar{g}_{\alpha\mu}(x) \bar{g}_{\beta\nu}(x) + \bar{g}_{\alpha\nu}(x) \bar{g}_{\beta\mu}(x) \right) \delta(x-y) \tag{A.3}$$

$$= \frac{1}{2} \left( \bar{g}_{\alpha\mu}(y) \bar{g}_{\beta\nu}(y) + \bar{g}_{\alpha\nu}(y) \bar{g}_{\beta\mu}(y) \right) \tag{A.4}$$

but we will abbreviate this notation by just writing

$$\frac{\delta}{\delta h_{\mu\nu}} h_{\alpha\beta} = \frac{1}{2} \left( \bar{g}_{\alpha\mu} \bar{g}_{\beta\nu} + \bar{g}_{\alpha\nu} \bar{g}_{\beta\mu} \right) \ . \tag{A.5}$$

We can first of all focus on the momentum independent part and expand it in powers of $h$. For that purpose let us switch to a matrix notation in which we denote matrices with an tilde so $\tilde{g}$ is the matrix with components $g_{\mu\nu}$ etc. What we have to do is to calculate the functional derivative of $\sqrt{\det \tilde{g}}$. For that purpose we can make use of an identity relating the trace to the determinant of a matrix $M$, which reads

$$\det e^M = e^{\mathrm{tr} M} \ . \tag{A.6}$$

Why this holds can be seen by remembering that the determinant is given by the product of eigenvalues of $M$ and the eigenvalues of $e^M$ are given by $e^\lambda_n$ where $\lambda_n$ are the eigenvalues of $M$. By combining the product of exponentials we, for a $d$ dimensional matrix, arrive at

$$\det e^M = \prod_{n=1}^{d} e^{\lambda_n} = e^{\sum_{n=1}^{d}\lambda_n} = e^{\mathrm{tr}M} \tag{A.7}$$

If we now consider $g = \det \tilde{g}$ we find

$$\frac{\delta \det \tilde{g}}{\delta h_{\mu\nu}} = \frac{\delta \det e^{\ln \tilde{g}}}{\delta h_{\mu\nu}} = \frac{\delta e^{\mathrm{tr}\ln \tilde{g}}}{\delta h_{\mu\nu}} = e^{\mathrm{tr}\ln \tilde{g}}\mathrm{tr}\left(\tilde{g}^{-1}\frac{\delta \tilde{g}}{\delta h_{\mu\nu}}\right) = \det \tilde{g} g^{\mu\nu} \tag{A.8}$$

Using this result we can expand $\sqrt{\det \tilde{g}} = \sqrt{g}$ up to arbitrary order

$$\sqrt{g} = \sqrt{\bar{g}}\left(1 + \frac{1}{2}h + \frac{1}{8}h^2 - \frac{1}{4}h^{\mu\nu}h_{\mu\nu} + ...\right) \ . \tag{A.9}$$

Next we have to deal with second term, containing the Ricci scalar. For that purpose we write the variation as

$$\delta R = g^{\mu\nu}\delta R_{\mu\nu} + R_{\mu\nu}\delta g^{\mu\nu} \tag{A.10}$$

and since we already know the functional derivative of the second term with respect to $h_{\mu\nu}$, we will only have to deal with $\delta R_{\mu\nu}$. An easy way to obtain this variation is to go straight with the definition of the Riemann tensor and rewrite the variation it in terms of variations of the Christoffel symbols $\delta\Gamma^{\alpha}{}_{\mu\nu}$. In this way we find

$$\delta R_{\mu\nu} = \nabla_\alpha\left(\delta\Gamma^{\alpha}{}_{\mu\nu}\right) - \nabla_\nu\left(\delta\Gamma^{\alpha}{}_{\alpha\mu}\right) \tag{A.11}$$

which, as we realize, is a total derivative and will therefore not contribute to the one point function. We find

$$\Gamma^h[\bar{g}, h=0] = \frac{1}{16\pi\sqrt{G_1}}\left(\bar{R}_{\mu\nu} - \frac{1}{2}\bar{R}\bar{g}_{\mu\nu} + \Lambda_1\bar{g}_{\mu\nu}\right) \tag{A.12}$$

to be the one point function, which is of course just the equation of motion for standard general relativity, since we chose the Einstein Hilbert action for our ansatz. Using similar techniques as before we can also find an expression for the two point function, however now we have to pay attention that we also include the gauge fixing action. We are working in the limit $\alpha \to 0$, which we can however only take after inverting $\Gamma^{(2h)}$, setting $\beta = 0$ can be done right in the beginning. After some calculation we find

$$\begin{aligned}\Gamma^{2h}[\bar{g}, h=0] =& \frac{1}{32\pi\alpha}\frac{\delta^2}{\delta h_{\alpha\beta}\delta h_{\mu\nu}}\Big(2\alpha\Lambda_2 h_{\mu\nu}h^{\mu\nu} - h(\alpha\Lambda_2 h + (\alpha+2\pi)\bar{\nabla}^2 h - (8\pi+\alpha)\bar{\nabla}_\mu\bar{\nabla}_\nu h^{\mu\nu}) \\ &+ h^{\mu\nu}\left[\alpha(\bar{\nabla}^2 h_{\mu\nu} - 2\bar{\nabla}_\sigma\bar{\nabla}_\nu h_\mu{}^\sigma) - 32\pi\bar{\nabla}_\nu\bar{\nabla}_\sigma h_\mu{}^\sigma + (8\pi+\alpha)\bar{\nabla}_\mu\bar{\nabla}_\nu h\right]\Big)\Big|_{h=0}\end{aligned} \tag{A.13}$$

where we have first of all expanded $S_{\text{EH}}$ up to order $h^2$ and then discarded all terms that are not quadratic in $h$ as they are zero after functionally differentiating twice or because we set $h = 0$. The functional derivatives of the quadratic part still have to be performed, but let us remember that we are interested in the inverse of $\Gamma^{2h}$, so we have to find a way to invert this expression properly. This is most easily achieved by using a York-decomposition [33] of the fluctuation metric and change to new variables

$$h_{\mu\nu} \to \left\{h^{\text{TT}}_{\mu\nu}, h, \xi_\mu, \sigma\right\} \tag{A.14}$$

$$h_{\mu\nu} = h^{\text{TT}}_{\mu\nu} + \frac{1}{4}\bar{g}_{\mu\nu} h + 2\bar{\nabla}_{(\mu}\xi_{\nu)} + \left(\bar{\nabla}_\mu\bar{\nabla}_\nu - \frac{\bar{g}_{\mu\nu}}{4}\bar{\nabla}^2\right)\sigma \tag{A.15}$$

Where $h^{\text{TT}}_{\mu\nu}$ is transverse and traceless and therefore satisfies $\bar{g}^{\mu\nu}h^{\text{TT}}_{\mu\nu} = 0 = \bar{\nabla}^\mu h^{\text{TT}}_{\mu\nu}$, $\xi_\mu$ is transverse and satisfies $\bar{\nabla}^\mu\xi_\mu = 0$, $h$ is just the trace $h = \bar{g}^{\mu\nu}h_{\mu\nu}$ and $\sigma$ is a scalar longitudinal degree of freedom. Since this transformation is non linear, we will introduce a Jacobian to the path integral, however instead of calculating the Jacobian we can also just rescale the fields $\xi_\mu$ and $\sigma$ according to

$$\xi_\mu \to \frac{1}{\sqrt{-\bar{\nabla}^2 - \frac{\bar{R}}{4}}}\xi_\mu \tag{A.16}$$

$$\sigma \to \frac{1}{\sqrt{\left(-\bar{\nabla}^2\right)^2 + \bar{\nabla}^2\frac{\bar{R}}{3}}}\sigma \tag{A.17}$$

which will exactly cancel the Jacobian. The zero modes of the operators that were used for the rescaling are not problematic, because these modes are exactly given by

$$\bar{\nabla}_{(\mu}\xi_{\nu)} = 0 \tag{A.18}$$

and

$$\left(\bar{\nabla}_\mu\bar{\nabla}_\nu - \frac{\bar{g}_{\mu\nu}}{4}\bar{\nabla}^2\right)\sigma = 0 \tag{A.19}$$

and so they do not contribute to $h_{\mu\nu}$ anyway. We can now write the two point function as a sum

$$\Gamma^{2h} = \sum_{i,j\in\{h^{\text{TT}},h,\xi,\sigma\}} \Gamma^{(ij)}\Pi^{ij} \tag{A.20}$$

with appropriate projection operators $\Pi^{ij}$ for the different combinations of modes. Luckily the mixed terms vanish on maximally symmetric spaces such that our two point function is given by

$$\Gamma^{2h} = \sum_{i\in\{h^{\text{TT}},h,\xi,\sigma\}} \Gamma^{(ii)}\Pi^{i} \tag{A.21}$$

The projection operators $\Pi^i$ are orthogonal to each other such that the inversion of $\Gamma^{(2h)}$ is achieved by inverting the modes $\Gamma^{(ii)}$ separately. They are given by

$$\Pi^{tt}_{\mu\nu\sigma\rho}(q) = \frac{1}{2}(\Pi^t_{\mu\sigma}\Pi^t_{\rho\nu} + \Pi^t_{\mu\rho}\Pi^t_{\sigma\nu}) - \frac{1}{3}\Pi^t_{\mu\nu}\Pi^t_{\sigma\rho} \tag{A.22}$$

$$\Pi^{tr}_{\mu\nu\sigma\rho}(q) = \frac{1}{4}\bar{g}_{\mu\nu}\bar{g}_{\sigma\rho} \tag{A.23}$$

$$\Pi^{v}_{\mu\nu\sigma\rho}(q) = \frac{1}{2}(\Pi^t_{\mu\sigma}\Pi^l_{\nu\rho} + \Pi^t_{\mu\rho}\Pi^l_{\nu\sigma} + \Pi^t_{\nu\sigma}\Pi^l_{\mu\rho} + \Pi^t_{\nu\rho}\Pi^l_{\mu\sigma}) \tag{A.24}$$

$$\Pi^{ll}_{\mu\nu\sigma\rho}(q) = \frac{1}{3}\Pi^t_{\mu\nu}\Pi^t_{\sigma\rho} + \Pi^l_{\mu\nu}\Pi^l_{\sigma\rho} - \Pi^{tr}_{\mu\nu\sigma\rho} \tag{A.25}$$

with

$$\Pi^t_{\mu\nu}(q) = \bar{g}_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \tag{A.26}$$

$$\Pi^l_{\mu\nu}(q) = \frac{q_\mu q_\nu}{q^2} \tag{A.27}$$

and can be found in [8]. When added they sum up to

$$(\Pi^{tt} + \Pi^{tr} + \Pi^v + \Pi^{ll})^{\mu\nu}{}_{\sigma\rho}h^{\sigma\rho} = \frac{1}{2}\left(\delta^\mu_\sigma\delta^\nu_\rho + \delta^\nu_\sigma\delta^\mu_\rho\right)h^{\sigma\rho} = \frac{1}{2}(h^{\mu\nu} + h^{\nu\mu}) = h^{\mu\nu} \tag{A.28}$$

which is the identity on the space of symmetric rank two tensors.
Now we only have to add the regulator function, which we choose to be proportional to the two point function itself

$$R_k = \Gamma^{2h}\Big|_{\Lambda=\bar{R}=0} \cdot r_k(-\bar{\nabla}^2) \tag{A.29}$$

and we can write for the regularized propagator

$$(G_{k,h})_{\mathrm{TT}} = \frac{32\pi}{-\bar{\nabla}^2(1 + r_k(-\bar{\nabla}^2)) + k^2\left(\mu + \frac{2}{3}r\right)} \tag{A.30}$$

$$(G_{k,h})_{\mathrm{Tr}} = \frac{-32\pi\frac{8}{3}}{-\bar{\nabla}^2(1 + r_k(-\bar{\nabla}^2)) + \frac{2}{3}\mu k^2} \tag{A.31}$$

which are the transverse traceless and the trace mode of the propagator. All other modes vanish due to the limit $\alpha \to 0$. This situation is however different if we want to compute the product $G_k\partial_t R_k$, again because $R_k$ is proportional to the two point function such that factors of $\alpha$ will cancel out and the other modes will not vanish in the limit $\alpha \to 0$. Instead one finds the following expressions

$$(G_{k,h}\partial_t R_k)_{\mathrm{TT}} = \frac{-\bar{\nabla}^2 \partial_t r_k(-\bar{\nabla}^2)}{-\bar{\nabla}^2(1 + r_k(-\bar{\nabla}^2)) + k^2\left(\mu + \frac{2}{3}r\right)} \tag{A.32}$$

$$(G_{k,h}\partial_t R_k)_{\mathrm{Tr}} = \frac{-\bar{\nabla}^2 \partial_t r_k(-\bar{\nabla}^2)}{-\bar{\nabla}^2(1 + r_k(-\bar{\nabla}^2)) + \frac{2}{3}\mu k^2} \tag{A.33}$$

$$(G_{k,h}\partial_t R_k)_{\xi} = \frac{-\bar{\nabla}^2 \partial_t r_k(-\bar{\nabla}^2)}{-\bar{\nabla}^2(1 + r_k(-\bar{\nabla}^2)) - \frac{1}{4}rk^2} \tag{A.34}$$

$$(G_{k,h}\partial_t R_k)_{\sigma} = \frac{-\bar{\nabla}^2 \partial_t r_k(-\bar{\nabla}^2)}{-\bar{\nabla}^2(1 + r_k(-\bar{\nabla}^2)) - \frac{1}{3}rk^2} \tag{A.35}$$

Similarly one can derive the ghost and scalar propagators. For the ghost we perform a transverse decomposition

$$c_\mu = c_\mu^T + \bar{\nabla}_\mu \eta \tag{A.36}$$

and rescale $\eta \to \frac{1}{-\bar{\nabla}^2}\eta$, while for the scalar field $\phi$ there is nothing to decompose as it only has one degree of freedom.
Their corresponding propagators read

$$(G_{k,\mathrm{gh}})_T = \frac{1}{-\bar{\nabla}^2(1 + r_k(-\bar{\nabla}^2)) - \frac{1}{4}rk^2} \tag{A.37}$$

$$(G_{k,\mathrm{gh}})_\eta = \frac{\frac{2}{3}}{-\bar{\nabla}^2(1 + r_k(-\bar{\nabla}^2)) - \frac{1}{3}rk^2} \tag{A.38}$$

$$G_\phi = \frac{1}{-\bar{\nabla}^2(1 + r_k(-\bar{\nabla}^2))} \tag{A.39}$$

For all results we chose an exponential regulator of the form

$$r_k(p^2) = k^2 \frac{e^{-\left(\frac{p^2}{k^2}\right)^2}}{p^2} \tag{A.40}$$

where $p^2$ are the eigenvalues of the scalar background Laplacian $-\bar{\nabla}^2$.

# B Functional traces

The evaluation of the projected flow equations involves functional traces over functionals involving the background Laplacian $\bar{\Delta} = -\bar{\nabla}^2$ and in general background covariant derivatives $\bar{\nabla}_\mu$ of the form

$$\mathrm{Tr} f(\bar{\Delta}, \bar{\nabla}_\mu) = \int d^4x \sqrt{\bar{g}}\, \langle x| \, f(\bar{\Delta}, \bar{\nabla}_\mu) \, |x\rangle \ . \tag{B.1}$$

The problem is that on general curved backgrounds the covariant derivative does not commute with the Laplacian such that no common eigensystem for both operators can be found, which is the case in flat space, where the eigenfunctions of both operators are just plane waves. To get around this problem we expand $f(\bar{\Delta}, \bar{\nabla}_\mu)$ in eigenfunctions of the Laplace operator and approximate the action of uncontracted covariant derivatives in the following way. Firstly, we symmetrize products of covariant derivatives such that the anti-symmetric part introduces further curvature dependent terms. And then we approximate the symmetric part by using scalar Laplacian eigenvalues $p^2 = p_\mu p^\mu$, corresponding to

$$\bar{\nabla}_\mu \bar{\nabla}^\mu h^{\alpha\beta}(p^2) = -p^2 h^{\alpha\beta}(p^2) \tag{B.2}$$

$$\begin{aligned}\bar{\nabla}_\mu \bar{\nabla}_\nu h^{\alpha\beta}(p^2) &= \left(\frac{1}{2}\left\{\bar{\nabla}_\mu, \bar{\nabla}_\nu\right\} + \frac{1}{2}\left[\bar{\nabla}_\mu, \bar{\nabla}_\nu\right]\right) h^{\alpha\beta}(p^2) \\ &= -p_\mu p_\nu h^{\alpha\beta}(p^2) + \bar{R}^\alpha{}_{\gamma\mu\nu} h^{\gamma\beta}(p^2) + \bar{R}^\beta{}_{\gamma\mu\nu} h^{\alpha\gamma}(p^2)\end{aligned} \tag{B.3}$$

$$\begin{aligned}\left(\bar{\nabla}^\mu h^{\alpha\beta}(p_1^2)\right)\left(\bar{\nabla}_\mu h^{\sigma\rho}(p_2^2)\right) &= -p_1^\mu p_{2\mu} h^{\alpha\beta}(p_1^2) h^{\sigma\rho}(p_2^2) \\ &= -\sqrt{p_1^2}\sqrt{p_2^2}\cos(\theta_{1,2}) h^{\alpha\beta}(p_1^2) h^{\sigma\rho}(p_2^2)\end{aligned} \tag{B.4}$$

where $\theta_{1,2}$ is the angle between $\vec{p}_1$ and $\vec{p}_2$. By using this approximation it becomes possible to separate the angular dependence of the trace such that we can split the trace in a sum over Laplacian eigenvalues and an angular integral which we approximate by the flat four dimensional solid angle integral $\int d\Omega$

$$\mathrm{Tr} \to \sum_{p^2} \int d\Omega \ . \tag{B.5}$$

Since this approximation mimics flat space, it is exact in the limit $\bar{R} \to 0$ and for larger $\bar{R}$ it becomes more inaccurate. The momentum configurations for the different diagrams are shown in figure 25. The angles $\theta_n$ between the external momenta are chosen to be symmetric, which leads to $\theta_n = \frac{2\pi}{n}$ for $n$ external lines. In this case we are looking at three momenta so the angle is $\theta = \frac{2\pi}{3}$. Additionally their absolute values are all chosen to be $|\vec{p}_1| = |\vec{p}_2| = |\vec{p}_1 + \vec{p}_2| = p$. The scalar products between them are then given by

$$\begin{aligned}\vec{p}_1 \cdot \vec{p}_2 = p^2 \cos\frac{2\pi}{3} &= -\frac{1}{2}p^2 \\ \vec{p}_1 \cdot (-\vec{p}_1 - \vec{p}_2) &= -\frac{1}{2}p^2 \\ \vec{p}_2 \cdot (-\vec{p}_1 - \vec{p}_2) &= -\frac{1}{2}p^2\end{aligned} \tag{B.6}$$

Since there is a loop momentum $\vec{q}$ in every diagram, whose scalar product with the external momenta shows up, the angular integral is not trivial. For four dimensional spherical coordinates there are three angles $\phi_1 \in [0, \pi]$, $\phi_2 \in [0, \pi]$, $\phi_3 \in [0, 2\pi]$ with a volume element

$$d\Omega = \sin^2\phi_1 \sin\phi_2 d\phi_1 d\phi_2 d\phi_3 \ . \tag{B.7}$$

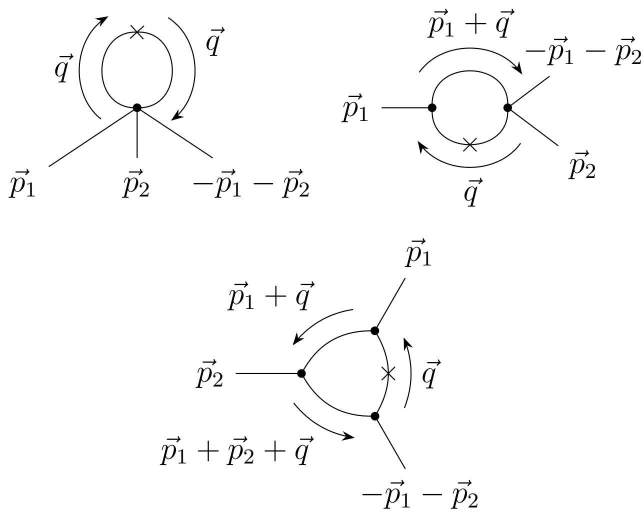



Figure 25: The momentum configuration in the different diagrams for $\partial_t \Gamma^{(3h)}$ is shown, external lines are always labeled with ingoing momenta. One external momentum was already eliminated by momentum conservation $\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0$.

Lets write the general form of a vector $\vec{q}$ in spherical coordinates such that the scalar products between $\vec{p}_1$, $\vec{p}_2$ and $\vec{q}$ can be determined

$$\vec{q} = \begin{pmatrix} q \sin\phi_1 \cos\phi_2 \\ q \sin\phi_1 \sin\phi_2 \cos\phi_3 \\ q \sin\phi_1 \sin\phi_2 \sin\phi_3 \\ q \cos\phi_1 \end{pmatrix} . \tag{B.8}$$

If we chose for example the vector $\vec{p}_1$ such that its scalar product with $\vec{q}$ is given by $\vec{p}_1 \cdot \vec{q} = pq \cos\phi_1$ we can immediately determine the vector $\vec{p}_2$ in such a way that only the angles $\phi_1$ and $\phi_2$ come up in the scalar product $\vec{p}_2 \cdot \vec{q}$ which then reads

$$\vec{p}_2 \cdot \vec{q} = -\frac{1}{2} pq \left( \cos\phi_1 + \sqrt{3} \sin\phi_1 \cos\phi_2 \right) \tag{B.9}$$

with

$$\vec{p}_2 = \begin{pmatrix} \frac{-\sqrt{3}}{2} p \\ 0 \\ 0 \\ -\frac{1}{2} p \end{pmatrix} . \tag{B.10}$$

A variable substitution $\cos\phi_1 = x$ and $\cos\phi_2 = y$ with $x, y \in [-1, 1]$ then gives a volume element

$$d\Omega = 2\pi \sqrt{1 - x^2}\ dx\ dy \tag{B.11}$$

in which the $\phi_3$ integration was already performed. The scalar products are then given by

$$\vec{p}_1 \cdot \vec{q} = pqx \tag{B.12}$$

$$\vec{p}_2 \cdot \vec{q} = -\frac{1}{2}pq(x + \sqrt{3 - 3x^2}y)\ . \tag{B.13}$$

When the angular integration was performed for all diagrams, all that is left to do is a sum over all the eigenvalues of the Laplacian. Here one has to distinguish two different cases, a spherical and a hyperbolic space.

## B.1 Positive curvature

Maximally symmetric spaces with positive scalar curvature correspond to four spheres, which have a finite volume and therefore the Laplace operator has a discrete spectrum

$$\sum_{p^2} \to \frac{1}{V}\sum_{l=0}^{\infty} m(l) \tag{B.14}$$

where $m(l)$ is the multiplicity of the $l^{th}$ eigenvalue $p^2 \to \lambda(l)$. $V$ is the volume of the four sphere $V = \frac{8}{3}\pi^2 R_s^4 = \frac{384\pi^2}{R^2} = \frac{384\pi^2}{k^4 r^2}$, which serves as a normalization constant such that for $R \to 0$ we obtain the correct flat space limit. The spectra for different kinds of fields the Laplace operator can act on [8] are shown in Table 1.

| | $\lambda(l)$ | $m(l)$ |
|---|---|---|
| scalar | $\frac{l(l+3)}{12}r$ | $\frac{(2l+3)(l+2)!}{6l!}$ |
| transverse vector | $\frac{l(l+3)-1}{12}r$ | $\frac{l(l+3)(2l+3)(l+1)!}{2(l+1)!}$ |
| transverse traceless tensor | $\frac{l(l+3)-2}{12}r$ | $\frac{10(l+4)(l-1)(2l+3)(l+1)!}{12(l+1)!}$ |

Table 1: Dimensionless eigenvalues and multiplicities of the Laplace operator acting on different fields.

Since the eigenvalues $\lambda(l)$ are proportional to $r$ we are unable to calculate the limit $r = 0$. For smaller values of $r$ we need to include more and more terms in the sum until for $r = 0$ we would need infinitely many terms which is unfeasible in a numeric calculation.

## B.2 Negative curvature

Negatively curved maximally symmetric spaces are unbounded such that they have an infinite volume and therefore a continuous spectrum of the Laplace operator. The Laplacian eigenvalues $p^2 \to \lambda$ can be labeled by a continuous parameter $\sigma$ which runs from $\sigma = 0$ to $\sigma = \infty$. The spectral integral is weighted with the spectral density $\rho(\sigma)$ and reads

$$\sum_{p^2} \to \frac{1}{3V}\int_0^{\infty} \rho(\sigma) \tag{B.15}$$

where again the factor $\frac{1}{3V}$ ensures the correct flat space limit. The eigenvalues $\lambda(\sigma)$ and their spectral density $\rho(\sigma)$ are given by [8]

$$\lambda(\sigma) = \frac{|r|}{12}\left(\sigma^2 + s + \frac{9}{4}\right) \tag{B.16}$$

$$\rho(\sigma) = (2s+1)\left[\sigma^2 + \left(s + \frac{1}{2}\right)^2\right]\sigma \tanh(\pi\sigma) \tag{B.17}$$

depending on the parameter $s$ which labels scalars ($s = 0$), transverse vectors ($s = 1$) and transverse traceless tensors ($s = 2$). Here again the flat space limit $r = 0$ has to be excluded as the eigenvalues $\lambda(\sigma)$ are proportional to $r$.

## B.3 Matching around flat space

Since both of the previous spectra do not include the case $r = 0$, we have to calculate this limit using a different approach, namely a heat kernel approximation [8], which is valid in a small curvature regime around $r = 0$. It consists of an expansion in powers of scalar curvature of the form

$$\mathrm{Tr} f(\bar{\Delta}) \approx \frac{1}{(4\pi)^2}\sum_{n=0}^{\infty} Q_{2-n}[f] a_n r^n \tag{B.18}$$

where the $Q_n[f]$ functionals are given by

$$Q_n[f] = \frac{1}{\Gamma(n)}\int_0^\infty d\lambda \lambda^{n-1} f(\lambda, r) \tag{B.19}$$

for $n > 0$ and by

$$Q_n[f] = (-1)^n f^{(-n)}(0) \tag{B.20}$$

for $n \leq 0$. The coefficients $a_n$ can be obtained for example from an expansion of the spectral integral from before around $r = 0$ or equally well from the spectral sum at positive curvature when using the Euler Maclaurin formula to approximate the sum by an integral. The $a_n$ are different for the various types of fields. When acting on scalars the coefficients up to third order in $r$ are given in Table 2.

| $n$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $a_n$ | 1 | $\frac{1}{6}$ | $\frac{29}{2160}$ | $\frac{37}{54432}$ |

Table 2: Heat kernel coefficients for the scalar Laplacian and positive $r$ up to third order, obtained by calculating the Euler-Maclaurin formula for the spectral sum. They coincide with the heat kernel coefficients for negative $r$ which can be checked by expanding the spectral integral in powers of $r$.

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Hiermit versichere ich, die vorgelegte Thesis selbstständig und ohne unerlaubte fremde Hilfe und nur mit den Hilfen angefertigt zu haben, die ich in der Thesis angegeben habe. Alle Textstellen, die wörtlich oder sinngemäß aus veröffentlichten Schriften entnommen sind, und alle Angaben die auf mündlichen Auskünften beruhen, sind als solche kenntlich gemacht. Bei den von mir durchgeführten und in der Thesis erwähnten Untersuchungen habe ich die Grundsätze gute wissenschaftlicher Praxis, wie sie in der ‚Satzung der Justus-Liebig-Universität zur Sicherung guter wissenschaftlicher Praxis' niedergelegt sind, eingehalten. Gemäß § 25 Abs. 6 der Allgemeinen Bestimmungen für modularisierte Studiengänge dulde ich eine Überprüfung der Thesis mittels Anti-Plagiatssoftware.

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