# Masterthesis 

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# "Meson spectrum from functional methods beyond rainbow ladder" 

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# "Das Spektrum der Mesonen aus funktionalen <br> Methoden jenseits der Regenbogen-Leiter-Näherung" 

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#### Abstract

In this thesis a novel approach to construct an expression for the quark self-energy from a Bethe-Salpeter kernel is presented. We will show that this approach satisfies the axialvector Ward-Takahashi identity. We will use this approach to calculate the quark propagator and solve the corresponding Bethe-Salpeter equation. Furthermore, we will investigate, how different terms inside the Bethe-Salpeter kernel affect the spectrum of light scalar and pseudoscalar mesons.

\section*{Zusammenfassung}

In dieser Arbeit wird ein neuartiger Ansatz gezeigt, mit dem ein Ausdruck für die Quark Selbstenergie aus einem Bethe-Salpeter Kernel berechnet werden kann. Wir werden zeigen, dass dieser Ansatz die Axialvektor Ward-Takahashi Identität erfüllt. Wir werden diesen Ansatz verwenden, um den Quarkpropagator zu berechnen, und die Bethe-Salpeter Gleichung lösen. Weiterhin wird untersucht, wie sich unterschiedliche Terme im Bethe-Salpeter Kernel auf das Spektrum leichter skalarer und pseudoskalarer Mesonen auswirkt.


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## Chapter 1

## Introduction

The theory which describes the strong interaction between hadrons and their constituents is called Quantumchromodynamics, or QCD for short. On a quantum field theory level its main components are $N_{f}$ massive fermion fields, called quarks, and eight $S U(3)$ gauge fields, called gluons. When taking a look at the spectrum of light hadrons and comparing it to the masses of the bare quarks, there are a few things that seem odd at a first glance. First of all, it seems odd, that the masses of the light hadrons are so much higher than the bare masses of their quark constituents. In fact, the masses of the proton and neutron are about 100 times larger than the combined masses of their quark content [1]. Knowing this, it should come as a surprise, that the light pseudoscalar $\left(J^{P C}=0^{-+}\right)$mesons, in particular the pions, kaons and the $\eta$, are considerably lighter compared to the light baryons. These phenomena are consequences of the symmetries of the underlying theory. In particular, they occur due to an effect called spontaneous symmetry breaking. It is the process, which is responsible for the light masses of the pseudoscalar mesons. It also makes the quarks obtain an effective mass due to self interaction, which is much higher than their bare mass.
Other than the light pseudoscalar mesons, one can also form a multiplet of light scalar $\left(J^{P}=0^{++}\right)$mesons. This light meson nonet has been under debate for a long time. Especially the experimental situation appeared questionable. In fact, it has only established its place in the Review of Particle Physics [1] in the last century. One of the reasons for this is that it seems incompatible with a $q \bar{q}$ picture of mesons. If they were ordinary $q \bar{q}$ states, the isosinglet $\sigma / f_{0}(500)$ and the isotriplet $a_{0}(980)$ were mass-degenerate, which is not the case [2]. Also, in the nonrelativistic quark model the scalar mesons are $p$-waves, carrying orbital angular momentum. Therefore, their masses should lie above 1 GeV like the axialvector and tensor mesons with quantum numbers $J^{P C}=1^{+-}, 1^{++}$and $2^{++}$. Instead, the lightest scalar meson found in nature, the $f_{0}(500)$ has a mass of $400-550 \mathrm{MeV}[1]$. Furthermore, the mass ordering of the light scalar mesons is not compatible with a
$q \bar{q}$ bound state picture [3]. Modern research suggests, that the light scalar meson nonet are no conventional $q \bar{q}$ states, but rather bound states of four (anti-)quarks ( $q q \overline{q q}$ ), so called tetraquarks [4].
In this thesis we will use fundamental equations of QCD to see how meson masses are generated dynamically. In particular, we will use the Bethe-Salpeter formalism in order to calculate meson masses. This formalism leads to bound states of quarks and antiquarks which have to satisfy certain conditions by the symmetries of the underlying quantum field theory. For the underlying equation of this formalism, the Bethe-Salpeter equation (BSE), to be solved we first need to solve equations for the quark propagator. It is obtained by solving the Dyson-Schwinger equations (DSE), a set of self-consistent equations derived from QCD. Since the Dyson-Schwinger equations cannot be solved in general, a truncation scheme has to be employed to simplify the interaction between quarks and gluons. We will explicitly see how this works by applying the Rainbow-Ladder truncation. Then we will use a novel approach to go beyond the Rainbow-Ladder truncation, which we will call the Kernel-first truncation. The idea behind that approach is to construct an expression for the quark self-energy from the quark-antiquark scattering kernel used in the BSE. This approach satisfies the axialvector Ward-Takahashi identity (AVWTI), thus preserving the effects of spontaneous symmetry breaking. In particular, we will add further terms to the scattering kernel and see, what their effects on the spectrum of scalar and pseudoscalar mesons are. We will then modify the parameters in the model used for the correction terms and investigate their influence on the mass spectrum. After we get an idea of how they change the masses, we will tune them in a way so that the Bethe-Salpeter equation for the pseudoscalar meson reproduces the physical properties of the lightest pseudoscalar found in nature, the pion. Meanwhile, we will see how the parameters can be tuned, so that the scalar mass goes above 1 GeV . This is done, because, as discussed above, the lightest scalar mesons are better described as tetraquarks. Hence, we will investigate, if the scalar mass will go up as far as to reach the mass of the next lightest scalar meson, the $a_{0}(1450)$.

## Chapter 2

## Technical foundations

### 2.1 Symmetries

To understand the concept of spontaneous symmetry breaking, it is useful to first think about what a symmetry actually is. For our purposes we will use the following definition:

A symmetry is a transformation, which when applied to the fundamental degrees of freedom of a theory leaves the action unchanged.

In field theories, such as QCD, the fundamental degrees of freedom mentioned above are the fields themselves. In the case of QCD they are the quark and gluon fields. The transformations can be split into two different kind of transformations, discrete and continuous ones. Discrete transformations are classified by the fact, that they form a group, which has a countable amount of elements. These include, but are not limited to, spatial inversion, permutation and charge conjugation. Continuous transformations also form a group, but in contrast to discrete transformations the group has an uncountable amount of elements, which can be continuously transformed into each other by varying a set of parameters. Such transformations include Poincaré transformations, $U(N)$ transformations and $S O(N)$ transformations. Next we shall see how symmetries actually affect the physics of a theory.

### 2.1.1 Consequences of symmetries

On top of oftentimes simplifying calculations, symmetries also manifest themselves in the actual physics of a theory. The most notable consequence of a continuous symmetry is Noether's theorem. It states that for every continuous symmetry of a field theory there is a conserved current $\partial_{\mu} j^{\mu}=0$. An important example for this is that space-time symmetries, i.e. invariance under spatial rotations and
space-time translations, imply the conservation of energy, momentum and angular momentum. Other symmetries also have conservation laws attached to them, some of which we will take a closer look at later.

### 2.1.2 Symmetry breaking

Since symmetries play an important role in our understanding of filed theories, it is also interesting to look at symmetries, that are not exact, but broken in some way.

## Explicit symmetry breaking

The most obvious way to break a symmetry is by adding a term to the action, which is not invariant under the symmetry transformation. One such case is a mass term breaking the axial $U(1)_{A}$ and $S U(N)_{A}$ symmetries of a theory with $N$ flavours of fermions in it. This is also the case in QCD, as we will see later. Symmetries do not have to be broken explicitly though. They can be broken in different, more subtle ways.

## Anomalous symmetry breaking

One such way is via anomalous symmetry breaking. It only occurs in quantum field theories, when quantum fluctuations break an otherwise fine symmetry of the classical action. This manifests itself through the fact, that such symmetries are broken by the introduction of an ultraviolet regulator and that the symmetry is not restored when the regulator is removed after renormalization. This results in a non-zero divergence of the current $j^{\mu}$, which does not vanish when the regulator is removed, so the current is not conserved. A famous example for this is the $U(1)_{A}$ symmetry of a gauge theory with fermions [5]. As mentioned above, a mass term breaks this symmetry explicitly, but even if the fermions are massless, the symmetry is broken anomalously.

## Spontaneous symmetry breaking

Another way to break a symmetry is spontaneous symmetry breaking. We call a symmetry spontaneously broken, if it is a symmetry of the action, but not of the ground state of the theory. Note, that this is not the same as anomalous symmetry breaking. While anomalies occur due to quantum fluctuations and the need of a regulator in quantum field theories, spontaneous symmetry breaking can also appear outside of quantum field theories. The prime example for this is a ferromagnet under the critical temperature $T_{c}$. At these temperatures the ground state of the ferromagnet has a non zero magnetization along some axis. Therefore, the
$S O(3)$ symmetry of the action is broken down to an $S O(2)$ symmetry, representing rotations along its magnetization axis.
Spontaneous symmetry breaking is particularly interesting, since it has direct consequences on physical observables. Most notably, every spontaneously broken continuous symmetry leads to one massless boson appearing in the spectrum of the theory. This is called Goldstone's theorem, the massless bosons are called Goldstone bosons.
While this theorem can be proven in general [6], it is helpful to find a way to intuitively think about this result. If we assume a theory with a spontaneously broken continuous symmetry, this theory will have a continuum of degenerate ground states, which can be transformed into each other via that symmetry transformation. Thus, the ground states can be labelled by a set of parameters corresponding to the parameters of the transformation. We will write them as $\theta$. Given a ground state $|\theta\rangle$, a transformation into a different ground state $\left|\theta^{\prime}\right\rangle$ requires no additional energy. Therefore, it can be transmitted by excitations with arbitrary small energies. This implies, that there is no mass gap in the spectrum, so there have to be massless particles, which correspond to these excitations.
This already hints at the low masses of the pseudoscalar mesons in the QCD spectrum. Later we will see, that they are in fact Goldstone bosons of a spontaneously broken symmetry. But in order to understand, which symmetry this is and why their masses are not exactly zero in reality, we have to take a closer look at the symmetries of QCD.

### 2.1.3 Symmetries of QCD

We now want to investigate the symmetries of QCD. Therefore, we take a look at the QCD action, which is the simplest local $S U(3)$ gauge theory with fermions we can write down. In 4 -dimensional euclidean space-time ${ }^{1}$ it is given by [5]

$$
\begin{equation*}
S_{\mathrm{QCD}}=\int \mathrm{d}^{4} x\left[\sum_{i=1}^{N_{f}} \bar{\psi}_{i}\left(\not D+m_{i}\right) \psi_{i}+\frac{1}{4} F_{\mu \nu}^{a} F_{a}^{\mu \nu}\right] . \tag{2.1}
\end{equation*}
$$

Here the $\psi_{i}$ and $\bar{\psi}_{i}$ are the quark fields and their conjugate fields, $\not D=\not \partial+i g_{A} \mathscr{A}$ is the covariant derivative and $F_{\mu \nu}^{a}$ is the gluon field strength tensor. By construction, this action is invariant under a local $S U(3)$ gauge transformation. This action also has a set of global symmetries. We want to focus on the symmetries of the quark fields.

[^0]First of all, the action is also invariant under a global $U(1)_{V}$ transformation of the quark fields

$$
\begin{equation*}
\psi_{i}(x) \rightarrow e^{i \theta} \psi_{i}(x) \tag{2.2}
\end{equation*}
$$

where $\theta$ is a real parameter. The conservation law connected with this symmetry is the conservation of baryon number, which is an unbroken symmetry even in the full quantum field theory. If the quark masses $m_{i}$ are zero, the action also has an axial $U(1)_{A}$ symmetry, which differs from the $U(1)_{V}$ symmetry by insertion of a $\gamma^{5}$ matrix in the argument of the exponential

$$
\begin{equation*}
\psi_{i}(x) \rightarrow e^{i \gamma^{5} \theta} \psi_{i}(x) \tag{2.3}
\end{equation*}
$$

This symmetry is explicitly broken by the quark masses, since the mass term $m_{i} \bar{\psi}_{i} \psi_{i}$ is not invariant under this transformation but picks up a factor of $\exp \left(2 i \gamma^{5} \theta\right)$. Even in the chiral limit when the quark masses are set to zero, the symmetry is broken anomalously as mentioned in the previous section.
The QCD action has additional symmetries under the condition, that there are $N \leq N_{f}$ flavours of quarks, which have the same mass $m_{c}$. In that case, the action is also invariant under an $S U(N)$ transformation which mixes these flavours

$$
\begin{equation*}
\psi_{i}(x) \rightarrow\left(U_{l}\right)_{i j} \psi_{j}(x), \tag{2.4}
\end{equation*}
$$

with $l \in\{V, A\}$ labeling the symmetry as vector or axial symmetry. $U_{V}=$ $\exp \left(i \theta^{a} \tau^{a}\right)$ is an $S U(N)_{V}$ matrix, the $\tau^{a}$ are the $N(N-1) / 2$ generators of the $S U(N)$ group and the $\theta^{a}$ are the corresponding parameters. If additionally $m_{c}=0$, there is also the axial $S U(N)_{A}$ symmetry, which again corresponds to an insertion of $\gamma^{5}$ in the argument of the exponential, $U_{A}=\exp \left(i \gamma^{5} \theta^{a} \tau^{a}\right)$. In total these global symmetries can be summarized as

$$
\begin{equation*}
S U(N)_{V} \otimes S U(N)_{A} \otimes U(1)_{V} \otimes U(1)_{A}, \tag{2.5}
\end{equation*}
$$

In the real world, the quark masses are all different and nonzero, therefore all the symmetries except for the $U(1)_{V}$ are broken explicitly by mass terms. If the quark masses and their differences are small compared to $\Lambda_{\mathrm{QCD}}$, which is the case for up-, down- and, arguably, strange quarks these symmetries are good approximate symmetries though.

### 2.1.4 The pion as Goldstone boson

In the chiral limit the $S U(N)_{A}$ symmetry is actually broken spontaneously. By Goldstone's theorem we therefore expect massless Goldstone bosons to appear. In fact, the spontaneous symmetry breaking results in the pion mass being zero for $N=2$ flavours of massless quarks and the kaon and $\eta$ having zero mass additionally to the pion for $N=3$. This can be seen in numerical calculations [7], but it can also be shown analytically. Since the real world does have quark masses, the $S U(N)_{A}$ symmetry is broken explicitly, therefore the pion mass is not zero. The relation between the pion mass and the quark mass $m_{c}$ is called the Gell-Mann-Oakes-Renner relation [8]

$$
\begin{equation*}
f_{\pi}^{2} m_{\pi}^{2}=-2 m_{c}\langle\bar{\psi} \psi\rangle / N+O\left(m_{c}^{2}\right), \tag{2.6}
\end{equation*}
$$

with $f_{\pi}$ being the decay constant of the pion. $\langle\bar{\psi} \psi\rangle$ is called the chiral quark condensate. A non-zero value of $\langle\bar{\psi} \psi\rangle$ indicates, that the chiral $S U(N)_{A}$ symmetry is spontaneously broken. From this relation we can directly see, that in the chiral limit either the mass or the decay constant of the pion has to be zero. One can further show that if chiral symmetry is spontaneously broken the decay constant has to be non zero [8]. Therefore, the mass of the pion is indeed zero in the chiral limit, making it the Goldstone boson we expect.
Another consequence of the relation (2.6) is, that for low quark masses the mass of the pion grows proportionally to the square root of the quark mass $m_{c}$. This explains, why in the real world the pion has a non-zero, but light mass.

### 2.2 The Dyson-Schwinger equations

The Dyson-Schwinger equations (DSEs) are an infinite set of coupled integral equations that describe the propagation of the particles of a quantum field theory. In this thesis we will focus on the DSE for the quark in QCD. If we express the quark DSE for the full, "dressed", quark propagator $S(p)$ in terms of the bare propagator $S_{0}(p)$ and the quark self-energy $\Sigma(p)$, it has the form [9]

$$
\begin{equation*}
S^{-1}(p)=Z_{2} S_{0}^{-1}(p)-\Sigma(p) \tag{2.7}
\end{equation*}
$$

The DSE expressed this way is also known as the gap equation. In order to solve this equation, we need to know $\Sigma(p)$. It is usually derived from the effective action of QCD, which results in the expression [9]


Figure 2.1: The gap equation expressed in Feynman diagrams. Straight lines with an arrow correspond to a free quark propagator, lines with an blob correspond to a dressed quark propagator. The springy line with a blob on it is a dressed gluon propagator. The vertex with a solid blob on it is a dressed quark-gluon vertex.

$$
\begin{equation*}
\Sigma(p)=-Z_{1 F}\left(\mu^{2}, \Lambda^{2}\right) g^{2} C_{f} \int \mathrm{~d}^{4} q \gamma_{\mu} S(q) \Gamma_{\nu}(q, p) D^{\mu \nu}(p-q), \tag{2.8}
\end{equation*}
$$

where $g$ is the coupling strength from the QCD action, $C_{f}$ is a factor obtained from taking traces of the colour Gell-Mann matrices, which is equal to $4 / 3$ for a $S U(3)$ gauge theory, $D^{\mu \nu}$ is the gluon propagator and $\Gamma_{\nu}(q, p)$ is a fully dressed quark-gluon vertex. ${ }^{2} Z_{1 F}\left(\mu^{2}, \Lambda^{2}\right)$ is a renormalization constant, which we will take a closer look at in chapter 3 when talking about truncation schemes. Using this expression for the quark self-energy we can express the gap equation diagrammatically as it is seen in figure 2.1.
The gluon propagator and quark-gluon vertex also satisfy their own set of DysonSchwinger equations. In these equations higher order vertices appear, which again satisfy their own set of DSEs [10]. This results in an infinite tower of coupled integral equations, which has to be truncated in some way in order to get a finite set of equations, which can be solved self-consistently. Before we apply a truncation to this expression of the self-energy, we will take a look at the formalism we will use to describe bound states.

### 2.3 The Bethe-Salpeter equation

The goal of this thesis is to calculate properties of light scalar and pseudoscalar mesons. Therefore, we need to develop a formalism, which can be used to describe bound states of quarks and antiquarks. In particular, we will focus on states consisting of a single quark and a single antiquark with no net colour charge.

[^1]The formalism we are going to use for this is the Bethe-Salpeter formalism. The idea behind this formalism is, to start with the exact equation for the scattering $T$-matrix in QCD, which is related to the full 4-point Green's function $G$ via [11]

$$
\begin{equation*}
G=G_{0}+G_{0} T G_{0} . \tag{2.9}
\end{equation*}
$$

Here $G_{0}$ denotes the free 4-point Green's function. The $T$-matrix satisfies a Dyson equation itself

$$
\begin{equation*}
T=K+K G_{0} T \tag{2.10}
\end{equation*}
$$

where $K$ is the quark-antiquark scattering kernel. Bound states between quarks and antiquarks occur, by definition, at poles of the $T$-matrix. Their mass is given by $m^{2}=-P^{2}$, where $P^{\mu}$ is the position of the pole. Therefore, we can describe the $T$-matrix close to bound states with the ansatz

$$
\begin{equation*}
T \propto \frac{\Gamma \bar{\Gamma}}{P^{2}+m^{2}} . \tag{2.11}
\end{equation*}
$$

Here we introduced the so called Bethe-Salpeter amplitude (BSA) $\Gamma$ and the conjugate amplitude $\bar{\Gamma}$. Plugging this ansatz into eq. (2.10), multiplying both sides with $\left(P^{2}+m^{2}\right)$ and then evaluating both sides on-shell, i.e. at $P^{2}=-m^{2}$, we get

$$
\begin{equation*}
\Gamma \bar{\Gamma}=K G_{0} \Gamma \bar{\Gamma} \tag{2.12}
\end{equation*}
$$

If we now multiply both sides from the right with the multiplicative inverse of $\bar{\Gamma}$, we get

$$
\begin{equation*}
\Gamma=K G_{0} \Gamma . \tag{2.13}
\end{equation*}
$$

This equation is known as the Bethe-Salpeter equation (BSE). It can be seen diagrammatically in fig. 2.2. The structure of this equation is that of an eigenvalue equation of the matrix $K G_{0}$. In order to see this more easily, we can explicitly write an eigenvalue $\lambda$ into eq. (2.13)

$$
\begin{equation*}
\left(K G_{0}\right) \cdot \Gamma=\lambda \cdot \Gamma . \tag{2.14}
\end{equation*}
$$



Figure 2.2: The Bethe-Salpeter equation in diagrammatic form. Here, $\Gamma$ denotes the Bethe-Salpeter amplitude, $K$ is the scattering kernel and lines with a blob are fully dressed quark propagators.

Thus, physical bound states are characterized by an eigenvalue of $\lambda=1$. If we plug in the explicit expression for the free 4 -point Green's function, we can write the BSE as [12]

$$
\begin{equation*}
[\Gamma(p, P)]_{t u}=\int \mathrm{d}^{4} q\left[S\left(q_{+}\right) \Gamma(q, P) S\left(q_{-}\right)\right]_{s r} K_{t u}^{r s}(q, k, P) \tag{2.15}
\end{equation*}
$$

The indices $r, s, t$ and $u$ are Dirac indices, which denote in which order tensor structures have to be multiplied. The momentum argument $p$ is the relative momentum, $P$ is the absolute momentum of the meson, and $q_{ \pm}=q \pm \frac{1}{2} P$ are the momenta of the quark and the antiquark. In general, the total momentum $P$ of the meson can split between the momentum in an antisymmetric fashion, with a routing parameter $\eta$ ranging between 0 and 1 and $q_{ \pm}=q+\eta_{ \pm} P$ with $\eta_{+}=\eta$ and $\eta_{-}=\eta-1$. But since all equations we are using are Lorentz invariant, the solutions of the BSE are Lorentz invariant as well. ${ }^{3}$ We will thus use a symmetric momentum routing, splitting the total momentum of the meson equally to both constituents, i.e. $\eta=0.5$.

### 2.4 The Ward-Takahashi identities

The concrete form of the BSE depends on the choice of the scattering kernel $K$. The kernel $K$ can not be chosen arbitrarily though. In order to conserve the features of spontaneous symmetry breaking in QCD, in particular the pion being a massless Goldstone boson in the chiral limit, the scattering kernel $K$ and the quark self-energy $K$ must satisfy a set of relations, known as the Ward-Takahashi

[^2]

Figure 2.3: The axialvector Ward-Takahashi identity in diagrammatic form. The crossed boxes represent an injection of a $\gamma^{5}$ matrix.
identities (WTIs). Therefore, every truncation we apply to the kernel and the selfenergy must not violate the WTIs. The identity, which is most important to us, is the axialvector Ward-Takahashi identity (AVWTI). It is of particular interest, since it ensures, that the effects of the spontaneous chiral symmetry breaking are conserved. In its explicit form the AVWTI is given by [8]

$$
\begin{equation*}
P_{\mu} \Gamma_{5}^{\mu}(p, P)+2 m_{c} \Gamma_{5}(p, P)=S^{-1}\left(p_{+}\right) i \gamma^{5}+i \gamma^{5} S^{-1}\left(p_{-}\right) \tag{2.16}
\end{equation*}
$$

where the notation $p_{ \pm}$follows the same convention as $q_{ \pm}$introduced earlier. $\Gamma_{5}$ and $\Gamma_{5}^{\mu}$ are the pseudoscalar and axialvector Bethe-Salpeter amplitudes respectively. However, for our purpose it is more useful to write the AVWTI in a different way, which can be derived from (2.16). This way it reads [13]

$$
\begin{equation*}
\Sigma\left(p_{+}\right) \gamma^{5}+\gamma^{5} \Sigma\left(p_{-}\right)=-\int \mathrm{d}^{4} q K(p, q, P)\left(\gamma^{5} S\left(q_{-}\right)+S\left(q_{+}\right) \gamma^{5}\right) \tag{2.17}
\end{equation*}
$$

This relation is illustrated diagrammatically in fig. 2.3. Conceptually, the AVWTI balances the amount of binding energy provided by the scattering kernel with the effective mass produced by the quark self-energy, such that they exactly cancel in the chiral limit, resulting in a massless bound state. We will see this in practice when numerically solving the Bethe-Salpeter equation in section 3.4.1.

## Chapter 3

## The Rainbow-Ladder truncation

We now want to take a look at a well established truncation scheme, which is the so-called Rainbow-Ladder truncation. The idea behind the Rainbow-Ladder truncation is to replace the fully dressed quark-gluon vertex in the quark selfenergy by a bare vertex, while replacing the full gluon propagator by a free one with a modeled dressing function. In order not to violate the AVWTI, we need to truncate the scattering kernel in the BSE as well. It will only consist of a one gluon exchange, dressed with the same dressing function as the gluon in the self-energy. In this chapter, we will use this truncation scheme to solve the quark DSE and the BSE for the scalar and pseudoscalar channel. Later we will then find a way to go beyond the Rainbow-Ladder truncation by constructing an expression for the quark self-energy from the scattering kernel.

### 3.1 Truncating the quark DSE

We will start by applying the Rainbow-Ladder truncation to the quark DSE. To do this, we start with the expression for the quark self-energy, eq. (2.8). The first piece, we want to take a closer look at, is the gluon propagator $D^{\mu \nu}(k)$. In Landau gauge, the dressed gluon propagator is given by [14]

$$
\begin{equation*}
D^{\mu \nu}(k)=T^{\mu \nu}(k) \frac{Z\left(k^{2}\right)}{k^{2}}, \tag{3.1}
\end{equation*}
$$

where $T^{\mu \nu}(k)=\left(\delta^{\mu \nu}-k^{\mu} k^{\nu} / k^{2}\right)$ is a transverse projector. The function $Z\left(k^{2}\right)$ is a dressing function, which contains the nonperturbative properties of the gluon propagator. The gluon propagator is a well known object by now. It's dressing function is well known from functional methods [14] as well as from lattice calculations [15]. The nonperturbative structure of the quark-gluon vertex on the
other hand is not as well understood yet. In general, the full quark-gluon vertex is composed out of twelve independent tensor structures, which are all possible combinations of three independent four vectors and four scalars [7]

$$
\begin{equation*}
\Gamma^{\mu}(p, q) \in\left\{\gamma^{\mu}, p^{\mu}, q^{\mu}\right\} \otimes\left\{\mathbb{1}, \not p, q,[\not p, q]_{-}\right\} . \tag{3.2}
\end{equation*}
$$

Combining these structures together, we can decompose the quark-gluon vertex in the following way

$$
\begin{equation*}
\Gamma^{\mu}=i g\left(\sum_{i=1}^{4} \lambda_{i} L_{i}^{\mu}+\sum_{i=1}^{8} \tau_{i} T_{i}^{\mu}\right) . \tag{3.3}
\end{equation*}
$$

In the Rainbow-Ladder truncation, only the leading structure, $L_{1}^{\mu}=\gamma^{\mu}$ is used. To be more precise, the full quark-gluon vertex is replaced by

$$
\begin{equation*}
\Gamma_{\mathrm{RL}}^{\mu}(p, q)=Z_{1 F} \gamma^{\mu} \lambda_{1}\left(k^{2}\right), \tag{3.4}
\end{equation*}
$$

with $k=p-q$. We can now plug all of this into the quark self-energy, eq. (2.8), to get

$$
\begin{equation*}
\Sigma(p)=-C_{f} Z_{1 F}^{2} g^{2} \int \mathrm{~d}^{4} q \gamma_{\mu} S(q) \gamma_{\nu} T^{\mu \nu} \frac{\lambda_{1}\left(k^{2}\right) Z\left(k^{2}\right)}{k^{2}} \tag{3.5}
\end{equation*}
$$

We will further combine the dressing functions of the gluon propagator and the leading structure of the quark-gluon vertex by introducing the abbreviation

$$
\begin{equation*}
\alpha\left(k^{2}\right):=\left(\frac{\tilde{Z}_{1}}{\tilde{Z}_{3}}\right)^{2} \frac{g^{2}}{4 \pi} Z\left(k^{2}\right) \lambda_{1}\left(k^{2}\right) \tag{3.6}
\end{equation*}
$$

The renormalization constants $\tilde{Z}_{1}$ and $\tilde{Z}_{3}$, that appear in this expression, are no independent factors, but they are related to $Z_{1 F}$ and $Z_{2}$ via Slavnov-Taylor identities (STIs). The STIs relate renormalization constants of different parts of the QCD action (2.1) with each other. They can be used to find, that $Z_{1 F}, Z_{2}, \tilde{Z}_{1}$ and $\tilde{Z}_{3}$ satisfy the relation

$$
\begin{equation*}
Z_{1 F}^{2}=Z_{2}^{2}\left(\frac{\tilde{Z}_{1}}{\tilde{Z}_{3}}\right)^{2} \tag{3.7}
\end{equation*}
$$



Figure 3.1: Diagrammatic illustration of the DSE for the quark propagator in Rainbow-Ladder truncation. The high order diagrams closely resemble rainbows, which inspired the name of the truncation scheme.

For a derivation of this expression, section B. 1 can be consulted. If we combine this expression with definition (3.6) and eq. (3.5), we get

$$
\begin{equation*}
\Sigma(p)=-C_{f} Z_{2}^{2} 4 \pi \int \mathrm{~d}^{4} q \gamma_{\mu} S(q) \gamma_{\nu} T^{\mu \nu} \frac{\alpha\left(k^{2}\right)}{k^{2}} . \tag{3.8}
\end{equation*}
$$

oxford To see, where the name of the Rainbow-Ladder truncation comes from, we can iterate the truncated DSE a few times and see, how the gluon propagators in the higher order diagrams start to look like a rainbow. This is illustrated in fig. 3.1. Before we go on and talk about solution strategies for the quark DSE, we will show, how the Rainbow-Ladder truncation affects the BSE.

### 3.2 Truncating the BSE

In order to apply the Rainbow-Ladder truncation to the BSE, we need to define the scattering kernel $K$ first. Therefore, we start by using an expression for the kernel, which is analogous to the quark self-energy

$$
\begin{equation*}
K_{a b c d}(p, q, P)=-C_{f} Z_{1 F} g^{2} D_{\mu \nu}(k) \gamma_{a b}^{\mu} \Gamma_{c d}^{\nu}(p, q) . \tag{3.9}
\end{equation*}
$$

If we now again replace the full quark-gluon vertex by its leading structure, introduce $\alpha\left(k^{2}\right)$ as in eq. (3.6) and use the STIs (3.7), we can bring the kernel to the form

$$
\begin{equation*}
K_{a b c d}(p, q, P)=-C_{f} Z_{2}^{2} 4 \pi T_{\mu \nu}(k) \gamma_{a b}^{\mu} \gamma_{c d}^{\nu} \frac{\alpha\left(k^{2}\right)}{k^{2}} \tag{3.10}
\end{equation*}
$$

If we plug this expression into eq. (2.15), we get for the truncated form of the BSE

$$
\begin{equation*}
\Gamma(p, P)=-C_{f} Z_{2}^{2} 4 \pi \int \mathrm{~d}^{4} q \gamma^{\mu} S\left(q_{+}\right) \Gamma(q, P) S\left(q_{-}\right) \gamma^{\nu} T_{\mu \nu}(k) \frac{\alpha\left(k^{2}\right)}{k^{2}} \tag{3.11}
\end{equation*}
$$

The expression (3.10) already looks similar to the quark self-energy in the rainbowladder truncation, eq. (3.8). It needs to be shown though, that these expressions indeed satisfy the AVWTI. To verify this, we start by plugging the expression for $\Sigma(p)$ into the left-hand side of the AVWTI, eq. (2.17). Ignoring factors, which both $\Sigma$ and $K$ have in common, such as $C_{f}, Z_{2}^{2}$ and $4 \pi$, the left-hand side now reads

$$
\begin{equation*}
\int \mathrm{d} q^{4} \Delta_{\mu \nu}\left(p_{+}-q\right) \gamma^{\mu} S(q) \gamma^{\nu} \gamma^{5}+\Delta_{\mu \nu}\left(p_{-}-q\right) \gamma^{5} \gamma^{\mu} S(q) \gamma^{\nu} \tag{3.12}
\end{equation*}
$$

Here we have introduced the abbreviation $\Delta_{\mu \nu}(p)=T_{\mu \nu}(p) \alpha(p) / p^{2}$. If we now plug eq. (3.10) into the right-hand side of eq. (2.17), it reads

$$
\begin{equation*}
-\int \mathrm{d} q^{4} \Delta(p-q) \gamma^{\mu} \gamma^{5} S\left(q_{-}\right) \gamma^{\nu}+\Delta(p-q) \gamma^{\mu} S\left(q_{+}\right) \gamma^{5} \gamma^{\nu} \tag{3.13}
\end{equation*}
$$

Since we are already assuming, that all divergent integrals are regulated in a Lorentz invariant way, we can safely shift the integration variable on the right hand side by a constant. In the first term of the integral we shift the integration variable by $q \rightarrow q+P / 2$ and in the second term we shift it by $q \rightarrow q-P / 2$. This leaves us with

$$
\begin{equation*}
-\int \mathrm{d} q^{4} \Delta\left(p_{-}-q\right) \gamma^{\mu} \gamma^{5} S(q) \gamma^{\nu}+\Delta\left(p_{+}-q\right) \gamma^{\mu} S(q) \gamma^{5} \gamma^{\nu} \tag{3.14}
\end{equation*}
$$

Since $\gamma^{5}$ anticommutes with all occurrences of $\gamma^{\mu}$, we can change the order of each neighboring $\gamma^{\mu} \gamma^{5}$ pair at the cost of an additional minus sign. After doing this it is clear, that the left-hand side and the right-hand side of the equation are indeed equal. Therefore, the Rainbow-Ladder truncation does in fact satisfy the AVWTI. Thus, we expect the pseudoscalar meson (pion) to be massless in the chiral limit when we use the Rainbow-Ladder truncation to solve the BSE. We will verify this explicitly by numerically calculating the mass of the pion in section 3.4 . We will also see, that the pion is no longer massless if the AVWTI is explicitly violated.

### 3.3 Solving the truncated quark Dyson-Schwinger equation

We now want to discuss a solution strategy for the truncated quark DSE. We start by recalling the gap-equation (2.7) and plugging in the truncated quark selfenergy (3.8). If we do this, we end up with

$$
\begin{equation*}
S(p)^{-1}=Z_{2}\left(i \not p+Z_{m} m_{c}\right)+C_{f} Z_{2}^{2} 4 \pi \int \mathrm{~d}^{4} q \gamma_{\mu} S(q) \gamma_{\nu} T^{\mu \nu} \frac{\alpha\left(k^{2}\right)}{k^{2}}, \tag{3.15}
\end{equation*}
$$

where we have already replaced the free quark propagator $S_{0}(p)$ by its exact perturbative form. $Z_{m}$ is the mass renormalization constant. To solve this equation, we use the fact, that the inverse quark propagator $S(p)^{-1}$ can in general be decomposed into a scalar and a vector part [16]

$$
\begin{equation*}
S(p)^{-1}=i \not p A\left(p^{2}\right)+B\left(p^{2}\right) \tag{3.16}
\end{equation*}
$$

with two dressing functions $A\left(p^{2}\right)$ and $B\left(p^{2}\right)$, which contain all nonperturbative properties of the full propagator. This means that if we manage to determine $A\left(p^{2}\right)$ and $B\left(p^{2}\right)$ from eq. (3.15) we have full knowledge of the quark propagator. To determine $A\left(p^{2}\right)$ and $B\left(p^{2}\right)$, we project onto the scalar or vector part of the expression respectively by multiplying both sides of eq. (3.15) with a projector to the respective part and taking the trace over the remaining Dirac indices. The technicalities of this are elaborated in more detail in sec. B.2. After doing this, we end up with two separate equations for $A\left(p^{2}\right)$ and $B\left(p^{2}\right)$

$$
\begin{align*}
& A\left(p^{2}\right)=Z_{2}+\Sigma_{A}\left(p^{2}\right)  \tag{3.17}\\
& B\left(p^{2}\right)=Z_{2} Z_{m} m_{c}+\Sigma_{B}\left(p^{2}\right) \tag{3.18}
\end{align*}
$$

with

$$
\begin{align*}
& \Sigma_{A}\left(p^{2}\right)=Z_{2}^{2} \frac{16 \pi}{3} \frac{1}{p^{2}} \int \mathrm{~d}^{4} q \frac{A\left(q^{2}\right)}{q^{2} A^{2}\left(q^{2}\right)+B^{2}\left(q^{2}\right)} \frac{\alpha\left(k^{2}\right)}{k^{2}}\left(p \cdot q+\frac{(p \cdot k)(q \cdot k)}{k^{2}}\right)  \tag{3.19}\\
& \Sigma_{B}\left(p^{2}\right)=Z_{2}^{2} \frac{16 \pi}{3} \int \mathrm{~d}^{4} q \frac{B\left(q^{2}\right)}{q^{2} A^{2}\left(q^{2}\right)+B^{2}\left(q^{2}\right)} \frac{\alpha\left(k^{2}\right)}{k^{2}}(3) . \tag{3.20}
\end{align*}
$$

In order to solve these two equations, we have to specify a model for $\alpha\left(k^{2}\right)$. We are going to use a model, which consists of an infrared part and a separate ultraviolet
part. The ultraviolet part is motivated by the perturbative behaviour of the gluon propagator at high momenta. To be consistent with one-loop perturbative QCD, $\alpha\left(k^{2}\right)$ must approach the asymptotic behaviour of QCD's running coupling [11]

$$
\begin{equation*}
\alpha\left(k^{2}\right) \xrightarrow{k^{2} \rightarrow \infty} \frac{\pi \gamma_{m}}{\ln k^{2} / \Lambda_{\mathrm{QCD}}^{2}}, \tag{3.21}
\end{equation*}
$$

where $\gamma_{m}=12 /\left(11 N_{c}-2 N_{f}\right)$ is the anomalous dimension of the quark propagator. In our calculations we will use $N_{f}=4$, therefore $\gamma_{m}=12 / 25$. For the infrared part several models have been employed in the past. The one we are going to use has first been used by Maris and Tandy and reads [17]

$$
\begin{align*}
\alpha\left(k^{2}\right) & =\alpha_{\mathrm{IR}}\left(k^{2}\right)+\alpha_{\mathrm{UV}}\left(k^{2}\right)  \tag{3.22}\\
\alpha_{\mathrm{IR}}\left(k^{2}\right) & =\pi \eta^{7}\left(\frac{k^{2}}{\Lambda^{2}}\right)^{2} e^{-\eta^{2} k^{2} / \Lambda^{2}}  \tag{3.23}\\
\alpha_{\mathrm{UV}}\left(k^{2}\right) & =\frac{\pi \gamma_{m}\left(1-e^{-k^{2} / \Lambda_{0}^{2}}\right)}{\ln \sqrt{e^{2}-1+\left(1+k^{2} / \Lambda_{\mathrm{QCD}}^{2}\right)^{2}}} . \tag{3.24}
\end{align*}
$$

The ultraviolet term is constructed in such a way, that for $k^{2} \rightarrow \infty$ it produces the correct asymptotic behaviour (3.21) while going to zero for $k^{2} \rightarrow 0$. We use $\Lambda_{0}=$ 1.0 GeV as scale in the ultraviolet part and $\Lambda_{\mathrm{QCD}}=0.234 \mathrm{GeV}$ for the interaction scale of QCD. Note however, that we are using a different parametrization of the infrared part compared to [17]. In this version, the infrared part is parametrized by a dimensionless parameter $\eta$, which determines the strength of the interaction, and an interaction width $\Lambda$. Fig. 3.2 shows $\alpha\left(k^{2}\right)$ for several values of $\eta$ and $\Lambda$. Having specified $\alpha\left(k^{2}\right)$, equations (3.17) can be solved numerically using an iterative method. The details about all the numerical methods used throughout this thesis can be found in appendix C. Before presenting the results of these calculations, we have to take a closer look at the renormalization of the quark DSE.

### 3.3.1 Renormalization of the quark DSE

We now want to talk more about renormalization in the context of the quark DSE. Up to this point we have neglected the cutoff and renormalization point dependencies of the quark propagator and the bare quark mass $m_{c}$. Since QCD is multiplicatively renormalizable, renormalized and unrenormalited dressing functions and quark mass obey the relations [18]


Figure 3.2: Upper panel: The model used for $\alpha\left(k^{2}\right)$ for several values of $\eta$. Lower panel: The model used for $\alpha\left(k^{2}\right)$ for several values of $\Lambda$.

$$
\begin{align*}
A\left(p^{2}, \mu^{2}\right) Z_{2}^{-1}\left(\mu^{2}, \Lambda^{2}\right) & =A^{0}\left(p^{2}, \Lambda^{2}\right)  \tag{3.25}\\
B\left(p^{2}, \mu^{2}\right) Z_{2}^{-1}\left(\mu^{2}, \Lambda^{2}\right) & =B^{0}\left(p^{2}, \Lambda^{2}\right)  \tag{3.26}\\
m_{c}\left(\mu^{2}\right) Z_{m}\left(\mu^{2}, \Lambda^{2}\right) & =m_{c}^{0}\left(\Lambda^{2}\right) \tag{3.27}
\end{align*}
$$

where $\mu$ is the renormalization point and $\Lambda$ is the UV cutoff. The cutoff dependent expressions on the right-hand side of these relations are the unrenormalized version of the ones appearing on the right and are thus denoted with a superscript zero. Since the right-hand side of all of these expressions do not depend on the renormalization point, a finite change in the renormalization point from $\mu$ to $\nu$ is described by

$$
\begin{equation*}
A\left(p^{2}, \nu^{2}\right)=A\left(p^{2}, \mu^{2}\right) \frac{Z_{2}\left(p^{2}, \nu^{2}\right)}{Z_{2}\left(p^{2}, \mu^{2}\right)} \tag{3.28}
\end{equation*}
$$

for the dressing function $A\left(p^{2}\right)$ and similar relations for other renormalization point dependent quantities. In order to determine the values of $Z_{2}$ and $Z_{m}$ for a given cutoff and renormalization point, we first have to specify a renormalization condition. This is done by requiring the dressing functions to have a fixed value at the renormalization point. We are going to employ the natural choice

$$
\begin{align*}
& A\left(\mu^{2}, \mu^{2}\right) \stackrel{!}{=} 1  \tag{3.29}\\
& B\left(\mu^{2}, \mu^{2}\right) \stackrel{!}{=} m_{c} \tag{3.30}
\end{align*}
$$

If we apply this condition to equation (3.17), we can rearrange them to find

$$
\begin{align*}
Z_{2}\left(\mu^{2}, \Lambda^{2}\right) & =\frac{1}{1+\Sigma_{A}\left(\mu^{2}, \Lambda^{2}\right)}  \tag{3.31}\\
Z_{m}\left(\mu^{2}, \Lambda^{2}\right) & =\frac{1}{Z_{2}\left(\mu^{2}, \Lambda^{2}\right)}-\frac{Z_{2}\left(\mu^{2}, \Lambda^{2}\right) \Sigma_{B}\left(\mu^{2}, \Lambda^{2}\right)}{m_{c}\left(\mu^{2}\right)} . \tag{3.32}
\end{align*}
$$

Note, that $\Sigma_{A}$ and $\Sigma_{B}$ still depend on $\Lambda$, since the integrals appearing in eqs. (3.19) are still divergent and need a cutoff.

### 3.3.2 Numerical results

We now want to take a look at explicit solutions of the quark DSE. These are obtained by numerically solving the coupled set of equations (3.17). All of these calculations are done with a renormalization point of $\mu=19 \mathrm{GeV}$ and a UV cutoff of $\Lambda_{\mathrm{UV}}=10^{3} \mathrm{GeV} .{ }^{1}$ For the parameters $\eta$ and $\Lambda$ in the effective coupling (3.22) we are using the values $\eta=1.8$ and $\Lambda=0.72 \mathrm{GeV}$. Using these parameters, the results for the dressing functions are shown in fig. 3.3.
Another way of writing the dressed quark propagator (3.16) is by expressing it in terms of two other dressing functions, the wave function $Z\left(p^{2}\right)$ and the effective mass function $M\left(p^{2}\right)$. In this notation the dressed quark propagator can be written as

[^3]

Figure 3.3: Left panel: Numerical results for $A\left(p^{2}\right)$. The upper line (dashed) is calculated for a bare quark mass of $m_{c}=3.7 \mathrm{MeV}$, the lower line (solid) is calculated in the chiral limit, $m_{c}=0$.
Right panel: Numerical results for $B\left(p^{2}\right)$. The upper line (dashed) is calculated for a bare quark mass of $m_{c}=3.7 \mathrm{MeV}$, the lower line (solid) is calculated in the chiral limit, $m_{c}=0$. Note, that the right panel uses a logarithmic scale on the y -axis while the left panel does not.

$$
\begin{equation*}
S(p)=Z\left(p^{2}\right) \frac{1}{\not p+M\left(p^{2}\right)} . \tag{3.33}
\end{equation*}
$$

Therefore we can express $Z\left(p^{2}\right)$ and $M\left(p^{2}\right)$ in terms of $A\left(p^{2}\right)$ and $B\left(p^{2}\right)$

$$
\begin{align*}
Z\left(p^{2}\right) & =\frac{1}{A\left(p^{2}\right)}  \tag{3.34}\\
M\left(p^{2}\right) & =\frac{B\left(p^{2}\right)}{A\left(p^{2}\right)} \tag{3.35}
\end{align*}
$$

Since the integrand in $\Sigma_{A}\left(p^{2}\right)$, eq. (3.19) is always positive, the value of $A\left(p^{2}\right)$ is greater than zero. Thus, these functions are always well-defined. The numerical results for $Z\left(p^{2}\right)$ and $M\left(p^{2}\right)$ can be found in fig. 3.4. From that we see, that the wave function goes to $1 / Z_{2}$ in the UV, which is indeed the free renormalized quark. Hence, this result describes an asymptotically free quark. In the infrared


Figure 3.4: Left panel: Numerical results for $Z\left(p^{2}\right)$. The lower line (dashed) is calculated for a bare quark mass of $m_{c}=3.7 \mathrm{MeV}$, the upper line (solid) is calculated in the chiral limit, $m_{c}=0$.
Right panel: Numerical results for $M\left(p^{2}\right)$. The upper line (dashed) is calculated for a bare quark mass of $m_{c}=3.7 \mathrm{MeV}$, the lower line (solid) is calculated in the chiral limit, $m_{c}=0$. Note, that the right panel uses a logarithmic scale on the $y$-axis while the left panel does not.
the wave function drops below one, indicating that the quark is no longer free, but surrounded by a cloud of gluons.
As stated before, the function $M\left(p^{2}\right)$ can be interpreted as the effective mass function of the quark. Note, that its value in the infrared, i.e. for low values of the momentum argument, is higher than the bare quark mass by several orders of magnitude. In particular, for a bare quark mass of 3.7 MeV the effective quark mass in the infrared is equal to $M(0)=488 \mathrm{MeV}$. Even in the chiral limit the quark obtains an effective mass of 475 MeV . This phenomenon is known as dynamic mass generation, since the high effective mass is generated by the quark self-interaction via an effective one gluon exchange in the Rainbow-Ladder truncation. It occurs as a consequence of the chiral symmetry of QCD being broken spontaneously [19]. This is also part of the reason, why bound states, such as baryons and mesons, are so much heavier than their bare quark constituents. To describe bound states precisely though, interactions between multiple (anti-)quarks have to be taken into consideration. In the chiral limit the effective quark mass goes to zero in the UV limit though, which is a consequence of QCD being asymptotically free.
During the numerical calculations, the values of $Z_{2}$ and $Z_{m}$ have also been cal-
culated. They are shown for different renormalization points and different values of $m_{c}$ in fig. 3.5. One can see, that $Z_{2}$ barely even changes if $m_{c}$ is increased. The value of $Z_{m}$ on the other hand seems to drop off drastically for small current quark masses. At first, this seems logical when we consider, how we calculate $Z_{m}$, as equation (3.31) indeed goes to $-\infty$ in the chiral limit. However, this is not a physical result but rather an artifact of the numerical method used. In fact, $Z_{m}$ should also be nearly constant, just like $Z_{2}$. We do not get the correct result numerically, because eq. (3.31) is no longer correct in the chiral limit. This has to do with $B\left(p^{2}\right)$ having a different asymptotic ultraviolet behaviour in the chiral limit than when there is a non-zero current quark mass, as can be seen in fig. 3.5. Thus, we would need to go to the limit $\mu \rightarrow \infty$ if we want to calculate $Z_{m}$ in the chiral limit numerically [20]. However, since $Z_{m}$ only slightly varies with changing $m_{c}$, good results can be obtained by fixing it to a value, which is calculated at larger bare quark masses, i.e. $Z_{m}=0.69$. The dependency on the renormalization point is more noticeable for both renormalization constants. In the limit $\mu \rightarrow \Lambda_{\mathrm{UV}}$ their values go to 1 , which is again consistent with eq. (3.25).


Figure 3.5: Upper left panel: Numerical results for $Z_{2}$ as a function of $m_{c}$. A UV cutoff of $\Lambda_{\mathrm{UV}}=10^{3} \mathrm{GeV}$ is used. The lines correspond to renormalization points of $13 \mathrm{GeV}, 19 \mathrm{GeV}$ and 25 GeV , with the bottommost line (solid) being $\mu=13 \mathrm{GeV}$.
Upper right panel: Numerical results for $Z_{m}$ as a function of $m_{c}$. The same UV cutoff of $\Lambda_{\mathrm{UV}}=10^{3} \mathrm{GeV}$ is used. Also, the same values for $\mu$ are used, again with the lowest line (solid) corresponding to $\mu=13 \mathrm{GeV}$.
Lower panels: Numerical results for $Z_{2}$ and $Z_{m}$ as a function of $\mu$. Here the bare quark mass been fixed to $m_{c}=3.7 \mathrm{MeV}$.

### 3.4 Solving the truncated meson Bethe-Salpeter equation

Now, that we know the dressed quark propagator $S(p)$, we can use the results to solve the BSE for quark-antiquark bound states. In section 3.2 we have already specified the scattering kernel $K$, eq. (3.10), and we have shown, that it satisfies the AVWTI. Since we want to solve the BSE for several different meson channels, we first have to specify the transformation properties of these channels under parity transformation and charge conjugation. In this thesis we will focus on the scalar $\left(J^{P C}=0^{++}\right)$and pseudoscalar $\left(J^{P C}=0^{-+}\right)$channels. In general, we can decompose the Bethe-Salpeter amplitudes into several tensor structures multiplied with dressing functions, in similar fashion as we did with the quarkgluon vertex and the dressed quark propagator. In particular, we are going to use the decomposition [21]

$$
\begin{align*}
\Gamma_{\mathrm{S}}(p, P) & =\mathbb{1}\left(E(p, P)-i(p \cdot P) \not P F(p, P)-i \not p G(p, P)+[\not p, \not p]_{-} H(p, P)\right)  \tag{3.36}\\
\Gamma_{\mathrm{PS}}(p, P) & =\gamma^{5}\left(E(p, P)-i \not P F(p, P)-i(p \cdot P) \not p G(p, P)+[\not p, \not p]_{-} H(p, P)\right) . \tag{3.37}
\end{align*}
$$

The subscript labels S and PS stand for the scalar and pseudoscalar amplitude respectively. Note that the amplitudes differ by a factor of $\gamma^{5}$ to account for parity. Additionally, the amplitudes differ by a factor of $(p \cdot P)$ being present at different tensorstructures. This is done to ensure that all parts of the amplitude are even under the transformation $p \cdot P \rightarrow-p \cdot P$. Analogous to the quark propagator, we need to know the dressing functions $E, F, G$ and $H$ to know the full amplitude. We can again project onto the individual dressing functions by plugging the decomposition of the amplitude into the truncated Bethe-Salpeter equation (3.11), multiplying both sides with projectors and taking the trace over the open Dirac indices. The details of this process are again elaborated in section B. 2 in the appendix. Doing this leaves us with four coupled integral equations for the dressing functions $E$ through $H$. This system of equations can be written as ${ }^{2}$

$$
\left(\begin{array}{l}
E(p)  \tag{3.38}\\
F(p) \\
G(p) \\
H(p)
\end{array}\right)=\left(\begin{array}{llll}
\kappa_{E E}(p, q) & \kappa_{E F}(p, q) & \kappa_{E G}(p, q) & \kappa_{E H}(p, q) \\
\kappa_{F E}(p, q) & \kappa_{F F}(p, q) & \kappa_{F G}(p, q) & \kappa_{F H}(p, q) \\
\kappa_{G E}(p, q) & \kappa_{G F}(p, q) & \kappa_{G G}(p, q) & \kappa_{G H}(p, q) \\
\kappa_{H E}(p, q) & \kappa_{H F}(p, q) & \kappa_{H G}(p, q) & \kappa_{H H}(p, q)
\end{array}\right) \cdot\left(\begin{array}{c}
E(q) \\
F(q) \\
G(q) \\
H(q)
\end{array}\right)
$$

[^4]where $q$ is the integration variable from the integrals, which appear inside the $\kappa$-functions. As explained in section 2.3, physical bound states are those, where the eigenvalue of the $\kappa$-matrix on the right-hand side is equal to 1 . Hence, we can solve equation (3.38) by finding a value for $m$, such that this condition is met. ${ }^{3}$ The dressing functions are then the eigenvector of the $\kappa$-matrix to the eigenvalue 1. Equation (3.38) can technically be solved iteratively in a similar fashion to the coupled equations for the dressing functions in the quark propagator, eqs. (3.17) and (3.19). However, there is a more efficient way to solve this eigenvalue equation. Since we are solving the BSE numerically on a finite momentum grid, we can interpret it as an eigenvalue equation of a $4 N$ dimensional square matrix, where $N$ is the number of sites in the momentum grid. Therefore, we can efficiently find the eigenvalue of the $\kappa$-matrix for each value of $m$. This gives us the eigenvalue as a function of the meson mass, i.e. $\lambda(m)$. We then need to find the value of $m$, where this function intersects 1 , or equivalently where $1-\lambda(m)$ is zero. A root finding algorithm, such as the false position method (regula falsi) can be used for this [22]. Once the physical mass is found, we can easily calculate the 4 N dimensional eigenvector of the $\kappa$-matrix. The first $N$ components of it yield the value of $E(p)$ on the sites of the momentum grid, the next $N$ components contain those of $F(p)$ and so on.

### 3.4.1 Numerical Results

We first want to take a look at how the masses of the scalar and pseudoscalar mesons depend on the bare quark mass $m_{c}$. For the pseudoscalar meson (pion) we recall, that, if chiral symmetry is spontaneously broken, the Gell-Mann-OakesRenner relation (2.6) should hold. Since we have shown in section 3.2, that the Rainbow-Ladder truncation satisfies the AVWIT, we expect a behaviour of the form $m_{\pi} \propto \sqrt{m_{c}}$. Especially, we expect $m_{\pi}$ to be zero in the chrial limit. To confirm, whether this is true in our numeric calculations, the masses of the mesons have been calculated for several values of $m_{c}$. The results of this can be seen in fig. 3.6. Fitting a function of the form $m_{\pi}=\sqrt{a m_{c}}+b$ to this result gives the best parameters as $a=5.64 \mathrm{GeV}$ and $b=-2.8 \mathrm{MeV}$. This fit has only been conducted on quark masses up to 20 MeV , since the Gell-Mann-Oakes-Renner relation is only valid for small masses. At higher masses higher order terms have to be included as well. This can be seen in the plot, as for high quark masses the fit no longer matches the numerical data. These results show, that for a bare quark mass of $m_{c}=3.7 \mathrm{MeV}$ the mass of the pseudoscalar meson is 141 MeV , which only differs from the actual physical value of 139.57 MeV [1] by about $1 \%$. We also see, that

[^5]

Figure 3.6: The pseudoscalar mass as a function of $m_{c}$. Note, how the mass goes to zero in the chiral limit and reaches its physical value at around $m_{c}=3.7 \mathrm{MeV}$. The solid line shows a fit to a function of the form $f(m)=\sqrt{a m}+b$.
the mass of the pseudoscalar meson is zero in the chiral limit, as we expected. This behaviour indicates, that the pseudoscalar meson is indeed the Goldstone boson indicating spontaneous symmetry breaking.
In order to show, that this is not just a coincidence but in fact a result of spontaneous chiral symmetry breaking, we can calculate the pseudoscalar mass when the AVWTI is explicitly violated. This can easily be done by choosing different values for the parameters $\eta$ and $\Lambda$ in equation (3.22) for the quark self-energy in the DSE and the scattering kernel in the BSE. The results of this can be seen in fig. 3.7. In these calculations we have fixed the values for $\eta$ and $\Lambda$ used in the quark DSE and varied their values in the BSE. In the DSE, the values $\eta=1.8$ and $\Lambda=0.72 \mathrm{GeV}$ have been used. Therefore, these points on the $x$-axes of the plots represent the point, where the AVWTI is satisfied. One can clearly see, that the mass drastically increases, when the AVWTI is violated, and goes to zero when the AVWTI is restored.


Figure 3.7: These plots show the pseudoscalar mass in the chiral limit, when the AVWTI is violated. The values for $\eta$ (upper panel) and $\Lambda$ (lower panel) shown on the horizontal axes of the plots are the ones used in the scattering kernel in the BSE. In the DSE the values $\eta=1.8$ and $\Lambda=0.72 \mathrm{GeV}$ are used. At these points on the horizontal axes the AVWTI is no longer violated and the mass drops to zero.


Figure 3.8: The scalar meson mass as a function of $m_{c}$. Note, that the scalar meson does not become massless in the chiral limit, but still carries a mass of around 683 MeV .

We can also calculate the scalar mass from the BSE as a function of $m_{c}$. The results of this can be seen in fig. 3.8. Note, that the scalar meson has a much higher mass than its pseudoscalar counterpart. It also does not become massless in the chiral limit, but still has a dynamically generated mass. This indicates, that it is not a Goldstone boson like the pseudoscalar meson. Thus, it does not come as a surprise, that it also does not satisfy the Gell-Mann-Oakes-Renner relation (2.6). It is also important to note, that the scalar mass is less sensitive to changes in $m_{c}$, especially when compared to the pseudoscalar mass close to the chiral limit. For quark masses higher than the ones included in this plot the scalar mass exceeds 1 GeV . This leads to some technical difficulties in the calculations, which we will discuss in more detail later. As mentioned in the introduction, in the nonrelativistic quark model the scalar meson mass lies above 1 GeV . At a bare quark mass of 3.7 MeV , which produces a good result for the pion mass, the scalar mass is 712 MeV , which is far below the 1 GeV threshold. Therefore, the Rainbow-Ladder truncation struggles to reproduce this result. We will later see how going beyond the Rainbow-Ladder truncation can improve the results in this regard.
It is also interesting to ask, how much the individual tensor structures of the Bethe-Salpeter amplitudes effect the meson masses. Therefore, these calculations

| Structures | pseudoscalar mass | scalar mass |
| :--- | :--- | :--- |
| E | 122 | 779 |
| E, F | 144 | 791 |
| E, F, G | 146 | 788 |
| E, F, G, H | 141 | 712 |

Table 3.1: The masses of the pseudoscalar and scalar mesons, calculated with different tensor structures in use. They have been calculated with $m_{c}=3.7 \mathrm{MeV}$, as this reproduces the physical pion mass. All of these masses are given in MeV .
have been repeated with several combinations of tensor structures in use. To be more precise, we start with only the leading tensor structure in place. ${ }^{4}$ We then add one structure after the other and see, how the results for the meson masses change. The results of this procedure can be seen in table 3.1. From this we can deduce, which of the tensor structures are the most relevant for the meson mass in each channel. For the pseudoscalar channel we see, that the E and F structures are the biggest contributors to the mass with F increasing it by around $17 \%$, while G and H each only contribute a few percent each. In the scalar channel however, the H structure in particular has a big influence on the mass, changing it by around $10 \%$, while F and G only contribute little to the mass.
In all the calculations done so far, the Bethe-Salpeter amplitudes (3.36) have been calculated as well. The results for the dressing functions $E(p)$ through $H(p)$ can be found in fig. 3.9. We can see that in both channels $E(p)$ is the dominant dressing function. Note however, that the amplitude is calculated as the eigenvector of a matrix and thus so far only known up to a constant factor. This factor has to be fixed by the separate normalization condition [23]

$$
\begin{equation*}
\left(\frac{\mathrm{d} \log \left(\lambda\left(P^{2}\right)\right)}{\mathrm{d} P^{2}}\right)^{-1}=3 \operatorname{tr} \int \mathrm{~d}^{4} q \bar{\Gamma}(q,-P) S\left(q_{+}\right) \Gamma(q, P) S\left(q_{-}\right) \tag{3.39}
\end{equation*}
$$

We will for now not normalize the amplitude, but we can still compare the asymptotic behaviour of $E(p)$ in the pseudoscalar channel and the quark's scalar dressing function $B\left(p^{2}\right)$ in the chiral limit. In fig. 3.10 we can see, that they match each other up to a multiplicative factor. This not just a coincidence, but can be shown analytically. ${ }^{5}$

[^6]

Figure 3.9: The dressing functions of the pseudoscalar (left panel) and scalar (right panel) Bethe-Salpeter amplitude at $m_{c}=3.7 \mathrm{MeV}$.


Figure 3.10: The dressing functions $B\left(p^{2}\right)$ and $E(p)$ on a logarithmic scale. The third line is $B\left(p^{2}\right)$ multiplied by a factor, such that it aligns with $E(p)$. Both functions have been evaluated in the chiral limit.

## Chapter 4

## Beyond the Rainbow-Ladder truncation

### 4.1 Motivation

### 4.1.1 Problems of the Rainbow-Ladder truncation

The Rainbow-Ladder truncation has already been used a lot in previous research. It has been applied to a multitude of problems, ranging from the analytic structure of the quark propagator [24] over the mass spectrum of light pseudoscalar and vector mesons [17], heavy quarkonia spectra [25], baryon studies [8] and all the way to the study of exotic states, such as tetraquarks [26]. While it has shown good results in some aspects, such as light vector and pseudoscalar ground state masses and the baryon spectrum, it fails to describe some other phenomena, such as light scalar meson masses, the decay of the $\rho$ meson, excited mesons and exotic states. This is a consequence of the drastic simplification done to the quark-gluon vertex and quark-antiquark scattering kernel. Therefore, it is necessary to include more complicated interactions into the calculations in order to correctly describe such phenomena.

### 4.1.2 Previous research

There have already been a number of efforts to go beyond the Rainbow-Ladder truncation using functional methods. For an overview, the review [8] can be consulted. Most of them can be summarized in that they include the Dyson-Schwinger equations for the gluon propagator, the quark-gluon vertex and other vertices as well. They can as well be derived by expanding the effective action of QCD into $n$ particle irreducible vacuum diagrams and taking functional derivatives thereof. An example of this done with the 3PI effective action can be seen in [10]. In these
calculations many more dressing functions appear than in a Rainbow-Ladder calculation, which all satisfy coupled integral equations, which again take way more effort and technicalities to solve. ${ }^{1}$ The corresponding Bethe-Salpeter scattering kernels are derived by taking second derivatives of the $n$ PI effective action [27]. Hence, there is an incentive to find a way to go beyond the Rainbow-Ladder truncation in a manner, that still allows for easy and efficient calculations.

### 4.2 Introducing the Kernel-first truncation

We now want to take a look at a different approach to go beyond the RainbowLadder truncation. In order to see the motivation behind this approach, note that the derivation the Dyson-Schwinger equations contains taking first order functional derivatives of the effective action and that a consistent Bethe-Salpeter scattering kernel can be constructed by taking the second order functional derivative. In general, it is not possible however to take a functional derivative of the quark self-energy (2.8) to obtain a scattering kernel which satisfies the AVWTI. It does not work, because to arrive at equation (2.8), one needs to evaluate the expression at the physical point after taking the first functional derivative, which leads to certain stationarity conditions [14]. To see how this happens, we can compare the general expressions for both quantities

$$
\begin{align*}
\Sigma\left(S_{0}\right) & =\left.\frac{\delta \Gamma[\hat{\phi}]}{\delta S}\right|_{\hat{\phi}=\phi}  \tag{4.1}\\
-K_{q \bar{q}} & =\left.\frac{\delta^{2} \Gamma[\hat{\phi}]}{\delta S \delta S}\right|_{\hat{\phi}=\phi} \neq \frac{\delta \Sigma\left(S_{0}\right)}{\delta S_{0}} . \tag{4.2}
\end{align*}
$$

Thus, if we wanted to construct a scattering kernel from equation (2.8), we would need to apply the chain rule of differentiation and take functional derivatives of the quark-gluon vertex as well. Doing so leads to a four-quark-gluon five-point function appearing. This leads to a whole new set of coupled integral equations for this five-point function, as illustrated in fig. 4.1.
Another way of preserving chiral symmetry would be to begin with a quarkantiquark scattering kernel and then construct a quark self-energy in such a way, that the AVWTI is not violated. The advantage of this approach is that it leads to a condition, which is nearly trivial to satisfy. The downside however is that the self-energy is no longer the same derived from the effective action. Thus, we will derive an equation for the quark propagator, which is technically no longer a

[^7]

Figure 4.1: Coupled integral equation for the four-quak-gluon five-point function, which arises from taking the functional derivative of the quark-gluon vertex.

Dyson-Schwinger equation. Since they are both derived from functional methods and have a very similar structure, we will still use the name DSE synonymously for it.

### 4.2.1 Contructing the quark self-energy

The method used in this section is based on unpublished work by R. Williams and C. S. Fischer [28]. To construct a quark self-energy from a given scattering kernel $K$, we start with Goldstone's theorem. In the Bethe-Salpeter equation it is shown through

$$
\begin{equation*}
\frac{\delta^{2} \Gamma}{\delta S \delta S}\left\{\gamma^{5}, S\right\}=0 \tag{4.3}
\end{equation*}
$$

The first term is the Bethe-Salpeter operator, $S_{a b}^{-1} \otimes S_{c d}^{-1}-K_{a b c d}$, while the second term is the analog of the Bethe-Salpeter wave function. Evaluating this expression we find ${ }^{2}$

$$
\begin{equation*}
\left[S^{-1}(p)\right]_{a b}\left\{\gamma^{5}, S(p)\right\}_{b c}\left[S^{-1}(p)\right]_{c d}=\int \mathrm{d}^{4} q K_{a b c d}(p, q, P)\left\{\gamma^{5}, S(q)\right\}_{b c} \tag{4.4}
\end{equation*}
$$

We can now again decompose the quark propagator into a vector and a scalar part according to (3.16). We can then use the fact that the vector part anticommutes with all occurrences of the $\gamma^{5}$ matrix to get

$$
\begin{equation*}
2 \gamma_{a d}^{5} B\left(p^{2}\right)=\int \mathrm{d}^{4} q K_{a b c d}(p, q, P)\left\{\gamma^{5}, S(q)\right\}_{b c} . \tag{4.5}
\end{equation*}
$$

[^8]If we multiply by $\gamma^{5}$ from the left and take the trace over the uncontracted dirac indices, we can separate $B\left(p^{2}\right)$ from this equation. Doing so yields

$$
\begin{equation*}
B\left(p^{2}\right)=\frac{1}{8} \operatorname{tr}\left(\gamma_{a b}^{5} \int \mathrm{~d}^{4} q K_{b c d e}(p, q, P)\left\{\gamma^{5}, S(q)\right\}_{c d}\right), \tag{4.6}
\end{equation*}
$$

which replaces, as the consequence of introducing a constraint equation, the usual result for the scalar dressing function from the equations of motion. Note that we may also include a current quark mass $m_{c}$. Doing so does not change the fact that the equation remains in satisfaction of the AVWTI. Therefore, the scalar part of the gap equation becomes

$$
\begin{equation*}
B\left(p^{2}\right)=Z_{2} Z_{m} m_{c}-\frac{1}{8} \operatorname{tr}\left[\gamma_{a b}^{5} \int \mathrm{~d}^{4} q K_{b c d e}(p, q, P)\left\{\gamma^{5}, S(q)\right\}_{c d}\right] \tag{4.7}
\end{equation*}
$$

We can once more use the anticommutator rules of the $\gamma$ matrices to further simplify the right-hand side of that equation, i.e.

$$
\begin{equation*}
\left\{\gamma^{5}, S(p)\right\}=\left\{\gamma^{5},-i \not p A\left(p^{2}\right)+B\left(p^{2}\right)\right\} d\left(p^{2}\right)=2 \gamma^{5} B\left(p^{2}\right) d\left(p^{2}\right) \tag{4.8}
\end{equation*}
$$

with $d\left(p^{2}\right)=1 /\left(p^{2} A^{2}\left(p^{2}\right)+B^{2}\left(p^{2}\right)\right)$. Thus equation (4.7) simplifies further to

$$
\begin{equation*}
B\left(p^{2}\right)=Z_{2} Z_{m} m_{c}-\frac{1}{4} \operatorname{tr}\left[\gamma_{a b}^{5} \int \mathrm{~d}^{4} q K_{b c d e}(p, q, P) \gamma_{c d}^{5} B\left(q^{2}\right) d\left(q^{2}\right)\right] . \tag{4.9}
\end{equation*}
$$

Note, that this construction just gives a constraint on the self-energy corrections of $B\left(p^{2}\right)$. The self-energy used for $A\left(p^{2}\right)$ does not need to be constrained in order to preserve the AVWTI and thus chiral symmetry breaking. Therefore, we are free to choose a self-energy for $A\left(p^{2}\right)$, it is sensible though to include similar corrections to it as well, as we will see later.

### 4.2.2 Specifying a kernel

Now that we have derived the self-energy corrections needed to satisfy the AVWTI given any kernel $K$, we can see, how they look when using an explicit kernel.

## Rainbow-Ladder kernel

As a first consistency test, we will show, that this construction leads to the DSE obtained from the Rainbow-Ladder truncation, eq. (3.19), when we start with a
single bare gluon exchange in the scattering kernel. Recall, that a single bare gluon exchange kernel is given by

$$
\begin{equation*}
K_{a b c d}(p, q, P)=-C_{f} Z_{2}^{2} 4 \pi\left[\gamma^{\mu}\right]_{a b}\left[\gamma^{\nu}\right]_{c d} D_{\mu \nu}(k) \tag{4.10}
\end{equation*}
$$

with $D_{\mu \nu}(k)$ being the gluon propagator and again using the notation $k=p-q$. If we plug this expression into equation (4.9), we get

$$
\begin{align*}
B\left(p^{2}\right) & =Z_{2} Z_{m} m_{c}-Z_{2}^{2} C_{f} \frac{1}{4} \operatorname{tr}\left[\gamma^{5} \int \mathrm{~d}^{4} q \gamma^{\mu} \gamma^{5} \gamma^{\nu} D_{\mu \nu}(k) B\left(q^{2}\right) d\left(q^{2}\right)\right]  \tag{4.11}\\
& =Z_{2} Z_{m} m_{c}+Z_{2}^{2} \frac{16 \pi}{3} \int \mathrm{~d}^{4} q \frac{B\left(q^{2}\right)}{q^{2} A^{2}\left(q^{2}\right)+B^{2}\left(q^{2}\right)} \frac{\alpha\left(k^{2}\right)}{k^{2}}(3) \tag{4.12}
\end{align*}
$$

which is indeed the same as eq. (3.19). Hence, we see that the Kernel-first truncation is consistent with the Rainbow-Ladder truncation. Numerical results for this truncation are presented in section 3.3.2 and 3.4.1.

## Dressed ladder exchange

One kernel, which we want to discuss in more detail in this thesis is a kernel consisting of a ladder exchange between two fully dressed quark-gluon vertices [10]

$$
\begin{equation*}
K_{a b c d}(p, q, P)=-g^{2} C_{F} D_{\mu \nu}(k)\left[\Gamma^{\mu}\left(p_{-}, q_{-}\right)\right]_{a b}\left[\Gamma^{\nu}\left(p_{+}, q_{+}\right)\right]_{c d} . \tag{4.13}
\end{equation*}
$$

Recall, that a fully dressed quark-gluon vertex can be decomposed into twelve tensor structures, as can be seen in eq. (3.3). While the Rainbow-Ladder truncation only uses the perturbatively leading structure

$$
\begin{equation*}
L_{\mu}^{1}(p, q)=\gamma_{\mu} \tag{4.14}
\end{equation*}
$$

other structures contribute to certain amounts to the results as well. In this work we will not work with the full vertex, but only include another component, which is the scalar piece [7]

$$
\begin{equation*}
L_{\mu}^{3}(p, q)=i l_{\mu}=i(p+q)_{\mu} . \tag{4.15}
\end{equation*}
$$

Here we introduced the momentum four vector $l_{\mu}:=(p+q)_{\mu}$, which is the sum of incoming and outgoing quark momenta. Since the quark-gluon vertex has to
be dimensionless, $l_{\mu}$ has to be properly normalized by it dividing by its norm. In contrast to the leading structure $L_{\mu}^{1}$, this structure is not invariant under chiral transformations. It only appears, when chiral symmetry is dynamically broken and then provides significant enhancement of symmetry breaking effects in the quark DSE [29]. If we use both of these structures in the scattering kernel, it can be written as a sum of three parts [10]

$$
\begin{equation*}
K \propto K_{\gamma \gamma}+K_{\gamma p}+K_{p p} . \tag{4.16}
\end{equation*}
$$

The first part $K_{\gamma \gamma}$ is the already well known Rainbow-Ladder kernel. The second part $K_{\gamma p}$ describes a single gluon exchange between a leading vertex structure and a scalar one, the third part $K_{p p}$ is a single gluon exchange between two scalar structures. These can be thought of as subleading correction terms to the RainbowLadder kernel and will also have an impact on the results for the quark propagator according to equation (4.9). Additionally, we can dress each of these exchanges with a separate dressing function. While these dressing functions could again be calculated from a Dyson-Schwinger equation for the quark-gluon vertex, we will employ dressing functions similar to the effective running coupling (3.22) for all three parts of the kernel. Written out explicitly the kernels are given by ${ }^{3}$

$$
\begin{align*}
& {\left[K_{\gamma p}\right]_{a b c d}(p, q, P)=-C_{F} Z_{2}^{2} 4 \pi\left[\gamma^{\mu}\right]_{a b} l^{\nu}[\mathbb{1}]_{c d} T_{\mu \nu}(k) \frac{Z_{\gamma p}\left(k^{2}\right)}{k^{2}}}  \tag{4.17}\\
& {\left[K_{p p}\right]_{a b c d}(p, q, P)=-C_{F} Z_{2}^{2} 4 \pi l^{\mu}[\mathbb{1}]_{a b} l^{\nu}[\mathbb{1}]_{c d} T_{\mu \nu}(k) \frac{Z_{p p}\left(k^{2}\right)}{k^{2}},} \tag{4.18}
\end{align*}
$$

with the dressing functions $Z_{\gamma p}$ and $Z_{p p}$ defined as

$$
\begin{align*}
& Z_{\gamma p}=\pi \eta_{\gamma p}^{7}\left(\frac{k^{2}}{\Lambda_{\gamma p}^{2}}\right)^{2} e^{-\eta_{\gamma p}^{2} k^{2} / \Lambda_{\gamma p}^{2}}+Z_{U V}\left(k^{2}\right)  \tag{4.19}\\
& Z_{p p}=\pi \eta_{p p}^{7}\left(\frac{k^{2}}{\Lambda_{p p}^{2}}\right)^{2} e^{-\eta_{p p}^{2} k^{2} / \Lambda_{p p}^{2}}+Z_{U V}\left(k^{2}\right) \tag{4.20}
\end{align*}
$$

The ultraviolet part is the same as in eq. (3.22). For the $K_{\gamma \gamma}$ part of the kernel, we will use the same parameters as we did before. For the parameters of the other two contributions will at first also use the same values as in the Rainbow-Ladder part and then later vary them to see, how they influence the results. This kernel can be seen diagrammatically in fig. 4.2.

[^9]

Figure 4.2: A diagrammatic illustration of the dressed ladder exchange kernel. The momentum $l^{\mu}$ at the vertices is again defined as the normalized sum of the incoming and outgoing momenta.

### 4.3 Solving the modified DSE

We now will investigate how the self-energy corrections affect the results of the quark DSE. As stated above, the Kernel-first truncation only gives us an explicit constraint on the equation for the scalar dressing function $B\left(p^{2}\right)$. The equation for the vector dressing function $A\left(p^{2}\right)$ is not constrained. We can therefore chose whether to include corrections to it or not. For the sake of consistency we are going to implement corrections similar to the kernel as well. In particular, we are going to start with the Rainbow-Ladder $(\gamma \gamma)$ expression and add $\gamma p$ and $p p$ terms in a canonical fashion as well. The explicit form of these terms are derived by taking the original quark self-energy (2.8) and replacing both vertices with fully dressed quark-gluon vertices. ${ }^{4}$ We then project onto the vector dressing function by multiplying both sides with a projector and taking the trace over Dirac indices as explained in section B.2. The upside of this procedure is, that we can systematically turn the corrections in the kernel on or off and see, how they each affect the results for $A\left(p^{2}\right)$ and $B\left(p^{2}\right)$ while ensuring, that the effects of dynamical chiral symmetry breaking are preserved.

[^10]
### 4.3.1 Numerical results

Next, we will present numerical results of the DSE with the correction terms in place. As mentioned above, we will separately turn the correction terms on and see what their effects are.

## $\gamma p$ corrections

The first correction we want to focus on is the $\gamma p$ correction. This correction to the kernel has a tensor structure of the form

$$
\begin{equation*}
\left[K_{\gamma p}\right]_{a b c d} \propto\left[\gamma^{\mu}\right]_{a b}[\mathbb{1}]_{c d} p^{\nu} D_{\mu \nu} \tag{4.21}
\end{equation*}
$$

with $p^{\nu}$ chosen to be the normalized sum of incoming and outgoing momenta. ${ }^{5}$ This expression contains exactly one occurrence of gamma matrices. Thus, if we plug it into equation (4.9), we are left with a trace over a product of three gamma matrices, which is zero. ${ }^{6}$ Even though this means that the $\gamma p$ correction to the selfenergy for $B\left(p^{2}\right)$ is zero, $B\left(p^{2}\right)$ still is affected by the corrections to the self-energy for $A\left(p^{2}\right)$, which do not vanish. Since the explicit expressions of these corrections are quite lengthy, we will not write them down explicitly here but instead focus on the results they produce. The numerical results for $A\left(p^{2}\right)$ and $B\left(p^{2}\right)$ alongside the mass function $M\left(p^{2}\right)$ and the wave function $Z\left(p^{2}\right)$ can be seen in fig. 4.3. In these calculations we have again used a current quark mass of $m_{c}=3.7 \mathrm{MeV}$. Note, that since we can still modify the dressing functions of the correction terms, we can change their sign, giving us the choice to either add or subtract the correction terms. For now, we will take a look at the results for both options, as both are shown in fig. 4.3. We will then later choose one option by taking a look at which option produces the best results in the pseudoscalar meson channel.

[^11]

Figure 4.3: Numerical results for the dressing functions $A\left(p^{2}\right)$ (upper left), $B\left(p^{2}\right)$ (upper right), the wave function $Z\left(p^{2}\right)$ (lower left) and the mass function $M\left(p^{2}\right)$ (lower right). Each plot shows the Rainbow-Ladder results together with the results for single correction terms with positive and negative signs respectively.

Taking a look at the results we see, that adding the $\gamma p$ correction term increases the vector dressing function $A\left(p^{2}\right)$ in the infrared significantly. The correction also smoothens out the little bump that the function had in the Rainbow-Ladder truncation around the 1 GeV scale, which can be seen in fig. 3.3. The scalar dressing function $B\left(p^{2}\right)$ gets decreased in the infrared by adding the correction. These results combined lead to the mass function $M\left(p^{2}\right)$ being lowered to a value of about 284 MeV . This is a nice result, as it is closer to an effective quark mass
which is compatible with a constituent quark model than the Rainbow-Ladder result of 488 MeV [30]. These results also manifest in the wave function being lowered even further to around 0.5 in the infrared, underlining the picture of an asymptotically free quark mentioned in section 3.3.
In contrast, when we subtract the correction term by giving its dressing function a negative sign, we see the exact opposite qualitative behaviour. The value of $A\left(p^{2}\right)$ now gets suppressed in the infrared, leading to an even bigger bump around the 1 GeV scale, while the value of $B\left(p^{2}\right)$ gets enhanced in the infrared. This leads to an effective quark mass of 1372 MeV and a wave function which is bigger than one in the infrared. Thus, we should not expect any promising results when subtracting the $\gamma p$ correction.

## $p p$ corrections

Next we will investigate, how the $p p$ correction to the scattering Kernel affects the DSE results. The tensor structure of this correction is of the form

$$
\begin{equation*}
\left[K_{p p}\right]_{a b c d} \propto[\mathbb{1}]_{a b}[\mathbb{1}]_{c d} p^{\mu} p^{\nu} D_{\mu \nu} . \tag{4.22}
\end{equation*}
$$

This time around, the correction to the equation for $B(p)^{2}$ contains a trace over an even number of gamma matrices and thus is not zero, but instead also gives a lengthy expression. Additionally, we obviously have a non-vanishing correction to the equation for $A\left(p^{2}\right)$ as well. We can again calculate both dressing functions with positive or negative signs relative to the $\gamma \gamma$ term, as long as we use the same sign convention for both $A\left(p^{2}\right)$ and $B\left(p^{2}\right)$. The results for the dressing functions, the wave function and $M\left(p^{2}\right)$ can also be seen in fig. 4.3. For the scalar dressing function $B\left(p^{2}\right)$ we get similar results compared to the $\gamma p$ corrections. In particular the value of $B\left(p^{2}\right)$ increases in the infrared when subtracting the correction and decreases when adding it. There is a new bump appearing at the 1 GeV scale though when using the positive sign. Subtracting the correction again leads to an increase in the infrared value of $B\left(p^{2}\right)$. The vector dressing function shows a completely different response to the $p p$ correction than to the $\gamma p$ correction. This time around, $A\left(p^{2}\right)$ has its infrared value decreased and its bump enhanced when adding the correction term. Subtracting it this time leads to an increase in the infrared and a small bump at the 1 GeV scale. Combined these results again lead to a lower effective mass of around 452 MeV with a positive sign in the correction term and a higher effective mass of 546 MeV with a negative sign. Within these results none really stick out as being unlikely physical results. We thus have to investigate their influence on the Bethe-Salpeter equation's results to see, which sign produces more promising results.

## Combining both corrections

Lastly, we will combine both corrections to the self-energy, i.e. using two full vertices in the kernel. We now also have four different ways of combining the signs of these correction terms, being

- $K_{++}=K_{\gamma \gamma}+K_{\gamma p}+K_{p p}$
- $K_{-+}=K_{\gamma \gamma}-K_{\gamma p}+K_{p p}$
- $K_{+-}=K_{\gamma \gamma}+K_{\gamma p}-K_{p p}$
- $K_{--}=K_{\gamma \gamma}-K_{\gamma p}-K_{p p}$.

We again calculate the quark propagator, the wave function and the effective mass function for all possible combinations. The results can be seen in fig. 4.4. These results already look reminiscent of the results with only one correction term in place. We again see that the effective mass and wave function have unlikely high values if we subtract the $\gamma p$ term. When focusing on the terms where the $\gamma p$ correction term is added, the effect of the $p p$ term is less prevalent though. In fact, the effective mass only changes by around 23 MeV when changing the sign of the $p p$ term, which is a change of less than $10 \%$. This is a way less severe effect compared to the influence the $\gamma p$ correction has on the effective mass. We thus will again rely on the BSE results to fix the sign.


Figure 4.4: Numerical results for the dressing functions $A\left(p^{2}\right)$ (upper left), $B\left(p^{2}\right)$ (upper right), the wave function $Z\left(p^{2}\right)$ (lower left) and the mass function $M\left(p^{2}\right)$ (lower right). Each plot shows the Rainbow-Ladder results together with the results for a combination of correction terms with positive and negative signs.

### 4.4 Solving the modified BSE

Now that we know the quark propagator with the corrections put in place, we can use the full kernel (4.13) to formulate and solve the Bethe-Salpeter equation. The general solution strategy stays the same as in the Rainbow-Ladder truncation; we decompose the Bethe-Salpeter amplitude into its tensor structures, which depend on the quantum numbers of the channel in question (pseudoscalar, scalar), dress each structure with dressing functions $E$ through $H$ and formulate the BSE as an eigenvalue equation. Therefore, the only thing that changes in comparison to equation (3.38) is, that the elements of the $\kappa$-matrix pick up additional terms, which are also derived by the projection procedure explained in section B.2. Similar to the corrections to the DSE, they become very lengthy, so we will not write them down explicitly here. Instead, we will again focus on the results of numerical calculations done with the modified BSE.

### 4.4.1 Numerical results

We will start by taking a look at how the corrections to the scattering kernel affect the masses of the scalar and pseudoscalar mesons. We will then see, which combination of signs looks the most promising in terms of increasing the difference between scalar and pseudoscalar masses, ideally pushing the scalar mass even over 1 GeV . We can then further modify the parameters in the infrared part of the dressings used in the correction terms to see how they influence the results. The results for the meson masses using all the different combinations of corrections can be seen in table 4.1. From these table a few things immediately stand out. First we see, that no physical solution to the Bethe-Salpeter with an eigenvalue of 1 can be found for some cases. All of these cases have in common, that they have a negative sign in front of the $p p$ correction term. There are two cases, where a solution could be found with a negative $p p$ term, but those were a very high pseudoscalar mass of over 200 MeV and an unrealistically low scalar mass of around 300 MeV . Additionally, in these cases only one of the two masses could be found, while the other could not be determined. Thus we can rule out a negative sign for the $p p$ correction from the BSE results, even though the results from the DSE did not explicitly rule such a possibility out. Surprisingly both signs in front of the $\gamma p$ term lead to similar results, even though one of them was favoured by the results of the DSE. A small difference can be noted however, being that the difference between the masses is a bit larger when adding the $\gamma p$ term instead of subtracting it. In fact the best result thus far is the one with positive signs in front of both corrections, having the largest overall scalar mass and a pseudoscalar mass only slightly below its physical value. This result is not too surprising, as this combination also produces promising solutions for the quark DSE and is in

| Kernel | pseudoscalar | scalar |
| :--- | :--- | :--- |
| $K_{\gamma \gamma}$ | 141 | 712 |
| $K_{\gamma \gamma}+K_{\gamma p}$ | 156 | 691 |
| $K_{\gamma \gamma}-K_{\gamma p}$ | 127 | 634 |
| $K_{\gamma \gamma}+K_{p p}$ | 122 | 798 |
| $K_{\gamma \gamma}-K_{p p}$ | 205 | - |
| $\longrightarrow$ | $K_{\gamma \gamma}+K_{\gamma p}+K_{p p}$ | 131 |
| $K_{\gamma \gamma}+K_{\gamma p}-K_{p p}$ | - | 832 |
| $K_{\gamma \gamma}-K_{\gamma p}+K_{p p}$ | 112 | 307 |
| $K_{\gamma \gamma}-K_{\gamma p}-K_{p p}$ | - | 808 |

Table 4.1: The masses of the scalar and pseudoscalar mesons calculated using the kernel combination specified in the first column. All masses are given in MeV and have been calculated with a current quark mass of $m_{c}=3.7 \mathrm{MeV}$. Fields marked with a dash signify, that for this combination of correction terms and channel no solution could be found, since the eigenvalue never crosses the $\lambda=1$ mark. The combination marked with an arrow gives the most promising results.
general the most natural way to employ the corrections to the scattering kernel. It is nice to see however, that the signs, which are suggested by the solutions of both DSE and BSE, are consistent with each other. Henceforth, we will be focusing on this exact combination of signs, as it looks the most promising.
In Fig. 4.5 the dependence of the meson masses on the current quark mass is shown with the kernel corrections in place. As we see, the mass of the pseudoscalar meson is again zero in the chiral limit, which indicates, that the Kernel-first truncation actually conserves spontaneous chiral symmetry breaking with the pion being the corresponding Goldstone boson. The pseudoscalar mass has again been fitted to a function of the form $m_{\pi}=\sqrt{a m_{c}}+b$ for quark masses up to 20 MeV as suggested by the Gell-Mann-Oakes-Renner relation (2.6). This can be seen in fig. 4.5. We see, that with the extended kernel in place, the relation still holds with $a=4.55 \mathrm{GeV}$ and $b=1.2 \mathrm{MeV}$. It is interesting to note, that here the actual values of the mass lie under the fit. In the Rainbow-Ladder case, the actual mass was higher than the fit to the Gell-Mann-Oakes-Renner relation.


Figure 4.5: The pseudoscalar mass as a function of the current quark mass. Again, we see that the mass goes to zero in the chiral limit and from there on has a square root like behaviour.


Figure 4.6: Left Panel: The dressing functions of the pseudoscalar (left panel) and scalar (right panel) Bethe-Salpeter amplitude at $m_{c}=3.7 \mathrm{MeV}$. This time the full kernel as given in (4.16) is used.

Furthermore, we can again calculate the Bethe-Salpeter amplitudes for both channels using the Kernel-first truncation. The results for the dressing functions $E(p)$ through $H(p)$ are shown in fig. 4.6. It is interesting to note, that the structures, which are contributing most to the mass of each meson ( $E$ and $F$ for the pseudoscalar, $E$ and $H$ for the scalar) still look similar to the results in the RainbowLadder truncation, while the other two structures have their values in the infrared lowered by a drastic amount, making them even negative with only a bump above zero around the 1 GeV mark for some structures.

### 4.4.2 Varying the parameters

Now that we know, how the correction terms affect the results for the quark propagator and the meson masses and amplitudes, we can further vary the parameters in their dressing functions and see, how they influence the results. Recall, that we so far used a dressing function, which was the same as the effective coupling used in the Rainbow-Ladder part of the kernel, see equation (3.22). This model is parametrized by two parameter $\eta$ and $\Lambda$, which were set to $\eta=1.8$ and $\Lambda=0.72 \mathrm{GeV}$ up to this point. For now, we will not use a completely different model for the dressing functions of the correction terms, but just modify the value of $\eta$ and $\Lambda$. To do this, we multiply both with a factor, which we can then vary over a range of values and see, how they change the meson masses. Since it is known, that $\Lambda$ does not have a significant influence on the meson mass but more on the decay constant [11], we will for now keep its value fixed. This means we define

$$
\begin{align*}
\eta_{\gamma p} & :=\alpha_{\gamma p} \eta  \tag{4.23}\\
\eta_{p p} & :=\alpha_{p p} \eta \tag{4.24}
\end{align*}
$$

with $\eta$ being fixed to 1.8. After finding a range for $\alpha_{\gamma p}$ and $\alpha_{p p}$, in which they produce meaningful results by trying out several values for $\alpha_{\gamma p}$ and $\alpha_{p p}$ first, the domain, where interesting results could be expected, was narrowed down to a rectangular area in the $\alpha_{\gamma_{p}}-\alpha_{p p}$ plane with the boundaries

$$
\begin{equation*}
\alpha_{\gamma p} \in[1.5,2] \quad \alpha_{p p} \in[0.5,1.5] . \tag{4.25}
\end{equation*}
$$

One thing to note is, that the area we are focusing on now does not include the unmodified case, i.e. the point $(1,1)$. The meson masses and their difference can be seen as a function of these two parameters in fig. 4.7 to 4.9. Note, that the mass of the scalar meson has error bars when it goes above the 1 GeV threshold. These denote the numerical error introduced by extrapolating the eigenvalue curve ${ }^{7}$ to

[^12]

Figure 4.7: The mass of the pseudoscalar meson as a function of $\alpha_{\gamma p}$ and $\alpha_{p p}$. The masses are given in MeV .
meson masses over this threshold. The reason we have to extrapolate the eigenvalue curve is that the quark propagator has complex conjugate poles in the complex momentum plane, which we would need to integrate over in order to get to higher meson masses [16]. Since we are integrating numerically though, we can not expect our integration method to correctly sample complex momenta around these poles correctly. However, we can calculate the eigenvalues up to around 1 GeV without running into trouble and then use an extrapolation method to estimate, where the physical mass lies. For detailed information on the method used and how the error bars are calculated, section C. 5 can be consulted.


Figure 4.8: The mass of the scalar meson as a function of $\alpha_{\gamma p}$ and $\alpha_{p p}$. The masses are given in MeV . The error bars on masses over 1 GeV have been omitted for the sake of readability.


Figure 4.9: The difference between the scalar and pseudoscalar mass as a function of $\alpha_{\gamma p}$ and $\alpha_{p p}$. At the point $(2,0.8)$ the difference has its maximum. At this point even lower end of the error bar is still above 1 GeV .

From fig. 4.9 we see, that the difference between the scalar and pseudoscalar mesons is at its highest value around the point $(2,0.8)$ in the $\alpha_{\gamma_{p}}-\alpha_{p p}$ plane. Since the scalar masses over 1 GeV have an error bar attached to them, we can not certainly identify a single point as a maximum. It could even be that the difference increases even further for $\alpha_{\gamma p}>2$. We are still going to assume, that the point $(2,0.8)$ indeed maximizes the difference between the meson masses. At this point the pseudoscalar mass is 188 MeV and the scalar mass is $(1301 \pm 103) \mathrm{MeV}$ with the difference being $(1133 \pm 103) \mathrm{MeV}$. Since the pseudoscalar mass is now higher than the physical pion mass of 139 MeV , we can lower the current quark mass we use as an input in order to rescale the mass, so that the pesudoscalar meson becomes indeed a physical pion. By doing so, we get a pseudoscalar mass which is equal around the physical pion mass of 139 MeV at $m_{c}=1.95 \mathrm{MeV}$. At this value of $m_{c}$, the
scalar mass becomes $(1306 \pm 108) \mathrm{MeV}$. This is $(475 \pm 108) \mathrm{MeV}$ heavier than the scalar mass without varied parameters and $(595 \pm 108) \mathrm{MeV}$ heavier than in the Rainbow-Ladder case. One thing to note is, that the scalar mass barely changes when $m_{c}$ is varied. In fact, the result for the lower value of $m_{c}$ is even slightly larger than for the original value. This makes sense, as the pion is a Goldstone boson and thus its mass is very sensitive to changes in $m_{c}$ close to the chiral limit. The scalar on the other hand is not massless in the chiral limit as discussed earlier. The change in mass of the scalar is smaller than the numerical uncertainty and should therefore not be alarming. A summary of these results is given is chapter 5 . Note, that we have not investigated, how the decay constant of the pseudoscalar meson has changed as a consequence of the corrections added to the kernel. In order to ensure, that the kernel used indeed describes a physical pion, we would have to calculate the decay constant $f_{\pi}$ and modify the values of $\Lambda_{\gamma p}$ and $\Lambda_{p p}$ in a way, that the physical value of $f_{\pi}=93 \mathrm{MeV}$ is obtained. However, this is not done within the scope of this thesis, but is a task for further research.

## Chapter 5

## Concluding remarks

Finally, we want to summarize all the results from the previous chapters. We started by talking about symmetries of quantum field theories in general and then focused on the chiral $S U(N)_{A}$ symmetry of QCD and that it is broken spontaneously. As a result, we saw that massless Goldstone bosons arise in the spectrum of the theory.
We then took a look at the Dyson-Schwinger equations, which are exact equations for Green's functions derived from the QCD effective action. Since they can not be solved exactly, we talked about how to truncate the Dyson-Schwinger equations and explicitly solved the quark Dyson-Schwinger equation within the Rainbow-Ladder truncation. In this truncation we used a simple model for the gluon propagator and the quark-gluon vertex. These calculations showed, how an effective quark mass is generated dynamically by the self interaction of the quark. In order to investigate bound states, we introduced the Bethe-Salpeter equation, which is an eigenvalue equation for a quark-antiquark scattering kernel. We saw, that this kernel has to obey a certain condition in order to preserve the effects of spontaneous chiral symmetry breaking, which is the so called axialvector WardTakahashi identity. An obvious choice was to use a kernel, which is compatible with the truncated quark DSE. With this kernel we then solved the Bethe-Salpeter equation for a pseudoscalar and a scalar meson. The solutions were the BetheSalpeter amplitudes. We also calculated how the mass of the meson changes, when parameters of the equations are varied. This revealed, that the pseudoscalar meson becomes indeed massless in the chiral limit, making it the Goldstone boson of the spontaneously broken chiral symmetry. Furthermore, the calculations showed how violating the AVWTI destroys the effects of the symmetry breaking.
Then, we summarized for which applications the Rainbow-Ladder truncation is well suited and for which it can not produce correct qualitative or quantitative results. We especially talked about why the light scalar mesons are still not fully understood and are a topic of ongoing research.

Lastly, we introduced a novel method to go beyond the Rainbow-Ladder truncation. It provides a simple scheme, which allows us to start with a quark-antiquark scattering kernel and derive a quark self-energy from that. This Kernel-first truncation has been derived from the axialvector Ward-Takahashi identity, so the constructed self-energy and the scattering kernel automatically satisfy the AVWTI. We showed explicitly, that this construction reproduces the Rainbow-Ladder truncated self-energy when starting with the corresponding scattering kernel. Using this method, we derived self-energy corrections from a more involved kernel than the Rainbow-Ladder one. These self-energy corrections and their effects on the quark propagator were then investigated further. They have then been applied to the meson Bethe-Salpeter equation to calculate the corrected meson masses. Finally, the parameters in the correction terms have been varied in a way, that the scalar meson mass is maximized. The pseudoscalar meson mass has been fixed to its physical value by adjusting the bare quark mass.

### 5.1 Summary of results

We now want to summarize the explicit numerical results for the meson spectrum from the numerical calculations done. We are going to compare the meson spectrum calculated from the Rainbow-Ladder truncation and the Kernel-first truncation to experimental results for the meson spectrum. This can be seen in fig. 5.1. It should not come as a surprise, that the pion mass of both truncation schemes match the experimental value very closely, since we used it to fix the parameters in our model. What is more interesting however, is that the Kernel-first truncation with modified parameters in the dressing functions of the correction terms produces a much higher scalar mass, which is even over 1 GeV at a value of $(1306 \pm 108) \mathrm{MeV}$. This result looks very promising, as this is much closer to the physical mass of the $a_{0}(1450)$ with a mass of $(1474 \pm 19) \mathrm{MeV}$, which is also a scalar meson which consists of light quarks. In order to fix the pion mass to its physical mass, we had to set the current quark mass to 1.95 MeV , which is also in the same order of magnitude as the PDG values for the $u$ - and $d$-quark masses.

### 5.2 Outlook

We have seen, that the Kernel-first truncation produces promising results for the light scalar mesons, though they do not match the experiment perfectly yet. This has multiple reasons. Firstly, we are still using a truncated quark-antiquark scattering kernel. The truncation we used for it is already more sophisticated than the Rainbow-Ladder truncation, but it is difficult to say how big the systematic


Figure 5.1: The calculated meson spectrum compared to experimental results [1]. The values, which are labeled as " RL " are results from the Rainbow-Ladder truncation, those labeled with "BRL" have been calculated with the Kernel-first truncation without varying any parameters, "BRL+" values are the results with the parameters varied as described in section 4.4.2.
error introduced by truncations really is. Secondly, we are only using two out of twelve tensor structures of the fully dressed quark-gluon vertices appearing in the kernel. As mentioned earlier, these tensor structures are usually the leading structures in the vertex, but the entire structure of the quark-gluon vertex is not fully understood yet. So, the influence subleading structures have on the results might not be negligible. Thirdly, we have just chosen a simple model for the dressing functions of the correction terms in the kernel. Using an effective coupling just like in the Rainbow-Ladder kernel is a natural choice, and we have managed to get a grasp on what the parameters in it do, but it does not necessarily resemble the actual dressing functions one would obtain from solving the full Dyson-Schwinger equation for the quark-gluon vertex. A more systematic approach would be to use dressing functions obtained from explicit calculations of the vertex. Another approach would be to fit the dressing functions to spectra of heavy quarkonia, such as charmonium or even bottomonium, since they can be described well by calculations in non-relativistic QCD [31]. Finally, some of the numerical methods used in these can be optimized further to narrow down the numerical error, especially
when extrapolating eigenvalue curves to high meson masses.
Nevertheless, this new method is at the very least an interesting new approach to improve the Rainbow-Ladder truncation. One of its advantages is, that it can be used to investigate, what the effects of individual structures appearing in a scattering kernel are. It is also very efficient in terms of computation time, as it only takes slightly longer than a Rainbow-Ladder calculation. There is still work to do in order to improve it and to see how viable it actually is. Once this work is done however, it can be applied to many areas of particle physics, where functional methods are used. These range from investigating analytic properties of the quark propagator to the full spectrum of light and heavy mesons, including vector, axialvector and tensor mesons and excited states of those. It can also be applied to the baryon spectrum, even though we do not have any evidence yet on whether it will produce good results there. Lastly, it can also be used to do tetraquark physics beyond the Rainbow-Ladder truncation, as most research done on tetraquarks so far has been done in the Rainbow-Ladder truncation [2].

## Appendix A

## Conventions and relations

Throughout this thesis we mostly follow the conventions used in [32]. For completeness' sake we are going to quote some important ones here.

## A. 1 Euclidean conventions

In all of the calculations in this thesis Euclidean conventions are applied, which means that the used metric is given by

$$
g_{\mu \nu}=g_{\mu}^{\nu}=g^{\mu \nu}=\left(\begin{array}{llll}
1 & 0 & 0 & 0  \tag{A.1}\\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right) .
$$

With that, scalar products of four vectors can be written as

$$
\begin{equation*}
a \cdot b=\sum_{\mu=0}^{3} a_{\mu} b_{\mu} \equiv a_{\mu} b^{\mu}=a^{\mu} b_{\mu} . \tag{A.2}
\end{equation*}
$$

Associated with the use of an Euclidean instead of a Minkowski metric the four momentum vector has been Wick rotated, such that

$$
\begin{equation*}
p^{\mu}:=\binom{i E}{\vec{p}} \tag{A.3}
\end{equation*}
$$

where $E=\sqrt{m^{2}+|\vec{p}|^{2}}$ is the Energy and $\vec{p}$ is the three momentum vector. In this convention, a four vector $p \in \mathbb{C} \times \mathbb{R}^{3}$ is spacelike, if $p^{2}>0$. A free fermion propagator in this convention takes the form [33]

$$
\begin{equation*}
S_{0}(p)=(i p p+m)^{-1} . \tag{A.4}
\end{equation*}
$$

## A. 2 Gamma matrices

The gamma matrices $\gamma_{\mu}(\mu \in\{0,1,2,3,5\})$, also sometimes called Dirac matrices, are a set of matrices, that obey a certain (anti-)commutator algebra. One possible representation of the matrices in Euclidean convention is via the Pauli matrices $\vec{\sigma}$ :

$$
\gamma_{0}=\left(\begin{array}{cc}
\mathbb{1} & 0  \tag{A.5}\\
0 & -\mathbb{1}
\end{array}\right), \quad \gamma_{j}=\left(\begin{array}{cc}
0 & i \sigma_{j} \\
-i \sigma_{j} & 0
\end{array}\right), \quad \gamma_{5}=\left(\begin{array}{ll}
0 & \mathbb{1} \\
\mathbb{1} & 0
\end{array}\right),
$$

where $\mathbb{1}=\mathbb{1}_{2 \times 2}$ and $j \in\{1,2,3\}$. In Euclidean metric the gamma matrices are hermitian

$$
\begin{equation*}
\gamma_{\mu}=\left(\gamma_{\mu}\right)^{\dagger} \tag{A.6}
\end{equation*}
$$

and obey the anticommutator rules of a Clifford algebra

$$
\begin{equation*}
\left\{\gamma_{\mu}, \gamma_{\nu}\right\}=2 \delta_{\mu \nu} \tag{A.7}
\end{equation*}
$$

## Helpful relations

In some derivations in this thesis traces of products of gamma matrices are evaluated. Therefore the following relations for products and traces of gamma matrices are used frequently.

## Products:

- $\left(\gamma_{5}\right)^{2}=\mathbb{1}$
- $\left\{\gamma_{\mu}, \gamma_{\nu}\right\}=2 \delta_{\mu \nu}$
- $\gamma_{j} \gamma^{j}=4 \cdot \mathbb{1}$
- $\gamma_{\mu} \gamma_{\nu} \gamma^{\mu}=-2 \gamma_{\nu}$
- $\gamma_{\mu} \gamma_{\nu} \gamma_{\rho} \gamma^{\mu}=4 \delta_{\nu \rho} \cdot \mathbb{1}$
- $\gamma_{\mu} \gamma_{\nu} \gamma_{\rho} \gamma_{\sigma} \gamma^{\mu}=-2 \gamma_{\sigma} \gamma_{\rho} \gamma_{\nu}$

Here again $j$ and $\mu$ can take the value $\{1,2,3\}$ and $\{0,1,2,3\}$ respectively. Additionally, the anticommutator of $\gamma_{5}$ and every other gamma matrix vanishes.

## Traces:

- $\operatorname{tr}\left(\gamma_{\mu}\right)=0$
- $\operatorname{tr}\left(\gamma_{\mu} \gamma^{\mu}\right)=16$
- $\operatorname{tr}\left(\gamma_{\mu} \gamma_{\nu}\right)=4 \delta_{\mu \nu}$
- $\operatorname{tr}\left(\gamma_{\mu} \gamma_{\nu} \gamma_{\rho} \gamma_{\sigma}\right)=4\left(\delta_{\mu \nu} \delta_{\rho \sigma}+\delta_{\mu \sigma} \delta_{\nu \rho}-\delta_{\mu \rho} \delta_{\nu \sigma}\right)$
- $\operatorname{tr}(\underbrace{\gamma_{\alpha} \gamma_{\beta} \ldots \gamma_{\omega}}_{\text {odd } \#})=0$


## A.2.1 Feynman slash notation

Another practical abbreviation is the so called Feynman slash notation. It stands for a scalar product between a four vector and a four vector containing the gamma matrices $\gamma_{1}$ to $\gamma_{3}$,

$$
\begin{equation*}
A:=\gamma_{\mu} A^{\mu} \tag{A.8}
\end{equation*}
$$

Due to the fact, that the gamma matrices' trace is zero, the trace of every slashed vector is zero as well,

$$
\begin{equation*}
\operatorname{tr}(\mathcal{A})=0 . \tag{A.9}
\end{equation*}
$$

Since the gamma matrices form a Clifford algebra, see eq. (A.7), the product of two slashed vectors is the same as the scalar product of the vectors times a unity matrix

$$
\begin{equation*}
A \cdot \mathbb{B}=(A \cdot B) \cdot \mathbb{1}_{4 \times 4} . \tag{A.10}
\end{equation*}
$$

## A. 3 Natural units

All equations and results in this thesis are calculated in a natural unit system. In this system we use the fact, that the speed of light $c$ and the Planck constant $\hbar$ are finite value greater than zero, but the physics don't change qualitatively if they are exchanged for another value. So for convenience we set them to $\hbar=c=1$. This also leads to the convenient situation that only powers of one unit, the unit of energy, are required to describe physical quantities. Some important examples
for that are given in table A.1. When converting back to SI units it is sufficient to multiply the value in natural units by a conversion factor, which is a power of $\hbar$ and $c$ in SI units. For example to convert a distance $x$ given in natural units to SI units, it needs to be multiplied by a factor of $\hbar c=197.327 \mathrm{MeV} \mathrm{fm}$ to be given in terms of meters.

| Quantity | SI | N.u. | Conversion factor |
| :--- | :---: | :---: | :---: |
| Energy | J | $\mathrm{eV}^{1}$ | 1 |
| Momentum | kg m s | $\mathrm{eV}^{1}$ | $c^{-1}$ |
| Mass | kg | $\mathrm{eV}^{1}$ | $c^{-2}$ |
| Time | s | $\mathrm{eV}^{-1}$ | $\hbar$ |
| Length | m | $\mathrm{eV}^{-1}$ | $\hbar c$ |
| Energy density | J m |  |  |
|  | $\mathrm{eV}^{4}$ | $(\hbar c)^{-3}$ |  |

Table A.1: Natural and SI units for some important quantities and the corresponding conversion factor $f$, such that $A_{S I}=f A_{\text {n.u }}$.

## A. 4 Hyperspherical coordinates

Throughout this thesis, there are several four-dimensional integrals to calculate. For convenience, they have been evaluated in hyperspherical coordinates, which are a generalization of three-dimensional spherical coordinates. A general, fourdimensional vector $x=\left(x_{0}, x_{1}, x_{2}, x_{3}\right)$ can be expressed as

$$
\begin{align*}
& x_{0}=r \cos \psi  \tag{A.11}\\
& x_{1}=r \sin \psi \cos \vartheta  \tag{A.12}\\
& x_{2}=r \sin \psi \sin \vartheta \cos \varphi  \tag{A.13}\\
& x_{3}=r \sin \psi \sin \vartheta \sin \varphi \tag{A.14}
\end{align*}
$$

Using this convention the Jacobi-determinant becomes

$$
\begin{equation*}
\mathrm{d}^{4} x=\mathrm{d} \varphi \mathrm{~d} \vartheta \sin \vartheta \mathrm{~d} \psi \sin ^{2} \psi \mathrm{~d} r r^{3} \tag{A.15}
\end{equation*}
$$

so that the integral of a function of $x$ can be rewritten as

$$
\begin{equation*}
\int_{\mathbb{R}^{4}} \mathrm{~d}^{4} x f(x)=\int_{0}^{2 \pi} \mathrm{~d} \varphi \int_{0}^{\pi} \mathrm{d} \vartheta \sin \vartheta \int_{0}^{\pi} \mathrm{d} \psi \sin ^{2} \psi \int_{0}^{\infty} \mathrm{d} r r^{3} f(r, \psi, \vartheta, \varphi) . \tag{A.16}
\end{equation*}
$$

Introducing the abbreviations $y=\cos \vartheta$ and $z=\cos \psi$ we can express eq. (A.16) in terms of $z$ and $y$

$$
\begin{equation*}
\int_{\mathbb{R}^{4}} f(x) \mathrm{d}^{4} x=\int_{0}^{2 \pi} \mathrm{~d} \varphi \int_{-1}^{1} \mathrm{~d} y \int_{-1}^{1} \mathrm{~d} z \sqrt{1-z^{2}} \int_{0}^{\infty} \mathrm{d} r r^{3} f(r, z, y, \varphi) . \tag{A.17}
\end{equation*}
$$

Furthermore, most of the dressing functions appearing throughout this thesis typically depend only on a squared momentum $p^{2}$. It is thus convenient to further transform the integration variable from $r$ to $r^{2}$. For a UV-regulated integral this means

$$
\begin{equation*}
\int_{0}^{\Lambda} \mathrm{d} r r^{3} f(r)=\int_{0}^{\Lambda^{2}} \mathrm{~d}\left(r^{2}\right) 2 r^{2} f\left(r^{2}\right) \tag{A.18}
\end{equation*}
$$

Lastly, most structure of the dressing functions lies in the infrared, i.e. momenta at or below 1 GeV . So, in order to improve the accuracy of our numerical integration, we can transform the integration variable to a logarithmic grid, enhancing the accuracy in the infrared. This results in an integral of the form

$$
\begin{equation*}
\int_{0}^{\Lambda^{2}} \mathrm{~d}\left(r^{2}\right) 2 r^{2} f\left(r^{2}\right)=\int_{0}^{2 \log \Lambda} \mathrm{~d}(\log (r)) 4 r^{4} f(\log (r)) . \tag{A.19}
\end{equation*}
$$

## A.4.1 Scalar products

In the integrals appearing in the quark DSE and meson BSE a number of scalar products between different momenta appear. The angles of those momenta follow the conventions

$$
\begin{align*}
P & \propto(0,0,0,1)  \tag{A.20}\\
p & \propto(0,0, \sin (\beta), \cos (\beta))  \tag{A.21}\\
q & \propto\left(\sqrt{1-y^{2}} \sqrt{1-z^{2}} \cos \varphi, \sqrt{1-y^{2}} \sqrt{1-z^{2}} \sin \varphi, y \sqrt{1-z^{2}}, z\right) \tag{A.22}
\end{align*}
$$

where $P$ is used to denote the total and $p$ is the relative momentum of the meson. The variable $q$ is the integration variable appearing in the integrals. The angle $\beta$ only slightly affects the outcome of the BSE, hence fix it in all calculations to the value of $\pi / 4$.

## Appendix B

## Derivations

## B. 1 Slavnov-Taylor identities

As mentioned in section 3.1, the renormalization constants appearing in the quark DSE and the Rainbow-Ladder truncation are not all independent, but satisfy the Slavnov-Taylor identities [18]

$$
\begin{align*}
Z_{1} & =Z_{g} Z_{3}^{3 / 2}  \tag{B.1}\\
\tilde{Z}_{1} & =Z_{g} \tilde{Z}_{3} Z_{3}^{1 / 2}  \tag{B.2}\\
Z_{1 F} & =Z_{g} Z_{3}^{1 / 2} Z_{2}  \tag{B.3}\\
Z_{4} & =Z_{g}^{2} Z_{3}^{2}  \tag{B.4}\\
\tilde{Z}_{4} & =Z_{g}^{2} \tilde{Z}_{3}^{2} . \tag{B.5}
\end{align*}
$$

In order to get to derive relation (3.7), we take the third Slavnov-Taylor indentity and square it to get

$$
\begin{equation*}
Z_{1 F}^{2}=Z_{g}^{2} Z_{3} Z_{2}^{2} . \tag{B.6}
\end{equation*}
$$

We can eliminate $Z_{3}$ from this equation, by taking the squared second STI

$$
\begin{equation*}
\tilde{Z}_{1}^{2}=Z_{g}^{2} \tilde{Z}_{3}^{2} Z_{3} \quad \Rightarrow \quad Z_{3}=\frac{\tilde{Z}_{1}^{2}}{Z_{g}^{2} \tilde{Z}_{3}^{2}} \tag{B.7}
\end{equation*}
$$

and plug it into (B.6). By doing so, we can cancel $Z_{g}^{2}$ and obtain

$$
\begin{equation*}
Z_{1 F}^{2}=Z_{2}^{2}\left(\frac{\tilde{Z}_{1}}{\tilde{Z}_{3}}\right)^{2} \tag{B.8}
\end{equation*}
$$

which is the relation we want.

## B. 2 Projecting onto tensor structures

Both the quark Dyson-Schwinger equation and the meson Bethe-Salpeter equation are equations for full $4 \times 4$ dimensional Dirac structures, making a total of 16 equations each. As discussed earlier though, we do not need to solve all 16 of these equations independently. Instead, we can decompose their solutions into a few independent Dirac structures, each multiplied with a dressing function. We will now take a closer look on how the full equations can be reduced to equations for those tensor structures.

## B.2.1 Dyson-Schwinger equation

Recall, that the Dyson-Schwinger equation can be written as

$$
\begin{equation*}
i p A\left(p^{2}\right)+B\left(p^{2}\right)=Z_{2} S_{0}(p)^{-1}-\Sigma(p) \tag{B.9}
\end{equation*}
$$

In order to separate this equation into two coupled equations for $A\left(p^{2}\right)$ and $B\left(p^{2}\right)$ respectively, we multiply it with the projectors $p_{A}$ or $p_{B}$ respectively and take the trace of the uncontracted Dirac indices. These projectors are given by

$$
\begin{align*}
& p_{A}=\frac{-i}{4 p^{2}} \not p  \tag{B.10}\\
& p_{B}=\frac{1}{4} \mathbb{1} . \tag{B.11}
\end{align*}
$$

After multiplying the DSE with those projectors and taking the trace, we end up with eqs. (3.17) through (3.19).

## B.2.2 Bethe-Salpeter equation

In analogy to the quark DSE, we can also separate the meson BSE into four equations, one for each of the four dressing functions $E$ through $H$. This is done by the exact same procedure as described above. The only difference is, that the
projectors are a bit more lengthy, since we are dealing with four tensor structures now. Also, we need to differentiate whether we are solving the BSE for a scalar or pseudoscalar meson. For a pseudoscalar, the projectors read explicitly

$$
\begin{align*}
p_{E} & =\frac{1}{4} \gamma^{5}  \tag{B.12}\\
p_{F} & =\frac{i}{4\left((p \cdot P)^{2}-p^{2} P^{2}\right)} \gamma^{5}\left(p^{2} \not P-(p \cdot P) \not p\right)  \tag{B.13}\\
p_{G} & =\frac{i}{4\left((p \cdot P)^{2}-p^{2} P^{2}\right)} \gamma^{5}\left(\frac{P^{2}}{p \cdot P} \not p-\not p\right)  \tag{B.14}\\
p_{H} & =\frac{1}{16\left((p \cdot P)^{2}-p^{2} P^{2}\right)} \gamma^{5}[\not p, \not p]_{-}, \tag{B.15}
\end{align*}
$$

while the projectors for a scalar meson are given by

$$
\begin{align*}
p_{E} & =\frac{1}{4} \mathbb{1}  \tag{B.16}\\
p_{F} & =\frac{i}{4\left((p \cdot P)^{2}-p^{2} P^{2}\right)}\left(\not p-\frac{p^{2}}{p \cdot P} \not p\right)  \tag{B.17}\\
p_{G} & =\frac{i}{4\left((p \cdot P)^{2}-p^{2} P^{2}\right)}\left((p \cdot P) \not P-P^{2} \not p\right)  \tag{B.18}\\
p_{H} & =\frac{1}{16\left((p \cdot P)^{2}-p^{2} P^{2}\right)}[\not p, P p]_{-} . \tag{B.19}
\end{align*}
$$

## B. 3 Chiral limit relations between dressing functions

In section 3.4 we have seen, that in the chiral limit the scalar dressing function of the quark propagator and the dressing function of the leading structure of the pseudoscalar Bethe-Salpeter amplitude are proportional to one another. We now want to investigate, why this is the case by proving this relation analytically. To begin with, recall the AVWTI (2.16)

$$
\begin{equation*}
P^{\mu} \Gamma_{5}^{\mu}(p, P)+2 m_{c} \Gamma_{5}(p, P)=S^{-1}\left(p_{+}\right) i \gamma^{5}+i \gamma^{5} S^{-1}\left(k_{-}\right) . \tag{B.20}
\end{equation*}
$$

We can again decompose the quark propagators appearing on the right-hand side into a scalar and a vector part. This leads to

$$
\begin{equation*}
P^{\mu} \Gamma_{5}^{\mu}(p, P)+2 m_{c} \Gamma_{5}(p, P)=i \gamma^{5}\left(B_{+}+B_{-}\right)-\left(\not p_{+} A_{+} \gamma^{5}+\gamma^{5} \not p_{-} A_{-}\right) . \tag{B.21}
\end{equation*}
$$

Here we introduced the abbreviations $A_{ \pm}=A\left(p_{ \pm}^{2}\right)$ and $B_{ \pm}=B\left(p_{ \pm}^{2}\right)$. Furthermore, we can again use the anticommutator relation between gamma matrices and expand the short-hand notation $p_{ \pm}=p \pm P / 2$ to get

$$
\begin{equation*}
P^{\mu} \Gamma_{5}^{\mu}(p, P)+2 m_{c} \Gamma_{5}(p, P)=i \gamma^{5}\left(B_{+}+B_{-}\right)+i \not p\left(A_{-}-A_{+}\right)-\frac{1}{2} \not P\left(A_{+}+A_{-}\right) . \tag{B.22}
\end{equation*}
$$

Since by equation (2.11) the Bethe-Salpeter amplitudes are only really meaningful object on their mass shells, we can evaluate this equation on-shell, where we can relate the axialvector amplitude to the pseudoscalar amplitude ${ }^{1}$ via [8]

$$
\begin{equation*}
\Gamma_{5}^{\mu}(p, P)=P^{\mu} \frac{2 i f_{\pi}}{P^{2}+m_{\pi}^{2}} \Gamma_{\pi}(p, P) \tag{B.23}
\end{equation*}
$$

We can contract this equation with the total momentum $P^{\mu}$ to get

$$
\begin{equation*}
P_{\mu} \Gamma_{5}^{\mu}(p, P)=P^{2} \frac{2 i f_{\pi}}{P^{2}+m_{\pi}^{2}} \Gamma_{\pi}(p, P) \tag{B.24}
\end{equation*}
$$

Since the pion is a Goldstone boson in the chiral limit, evaluating this equation in the chiral limit yields

$$
\begin{equation*}
P_{\mu} \Gamma_{5}^{\mu}(p, 0)=2 i f_{\pi} \Gamma_{\pi}(p, 0) . \tag{B.25}
\end{equation*}
$$

Since equation (B.22) still holds, we see that the pseudoscalar amplitude are related to the quark dressing functions via

$$
\begin{equation*}
2 i f_{\pi} \Gamma_{\pi}(p, 0)=2 i \gamma^{5} B\left(p^{2}\right) \tag{B.26}
\end{equation*}
$$

Here we have already used to fact, that in the chiral limit $m_{c}=0$ and $p_{+}=p_{-}=p$. Furthermore, the only term in $\Gamma_{\pi}(p, 0)$ that is not zero in the chiral limit is the leading tensor structure. Thus, the relation between between $E$ and $B\left(p^{2}\right)$ becomes

$$
\begin{equation*}
f_{\pi} \Gamma_{\pi}(p, 0)=f_{\pi} \gamma^{5} E(p, 0)=\gamma^{5} B\left(p^{2}\right) . \tag{B.27}
\end{equation*}
$$

[^13]
## Appendix C

## Numerical methods and software used

## C. 1 Numerical integration

All integrals appearing in the quark DSE and meson BSE have been evaluated numerically. There are many ways to approach the task of numerical integration. The simplest is the bar method, which approximates the integral of a function $f$ over a given interval $(a, b)$ by approximating the area under the function with rectangles of equal width, and their height being the functions value at the center of each rectangle's base. This can be written as the equation

$$
\begin{equation*}
\int_{a}^{b} f(x) \mathrm{d} x \approx \sum_{i=1}^{N} f\left(x_{i}\right) \cdot \Delta x, \tag{C.1}
\end{equation*}
$$

with $x_{i}=a+\frac{2 i-1}{2 N}(b-a)$ and $\Delta x=\frac{b-a}{N}$. This method however is not very efficient as it only converges slowly for increasing $N$. A faster method is the GauSSLegendre method, where the function is no longer evaluated at equidistant points, but instead is only evaluated at a specific set of points $x_{i}$ which all have their own weight factor $w_{i}$, so that we get

$$
\begin{equation*}
\int_{a}^{b} f(x) \mathrm{d} x \approx \sum_{i=1}^{N} f\left(x_{i}\right) \cdot w_{i} . \tag{C.2}
\end{equation*}
$$

With the correct choice of the abscissas and weights this method converges much faster, even for small values of $N$. In fact, the radial integrals in the DSE have
been evaluated using 600 abscissas, while the angular integrals used 128 abscissas. The integrals appearing in the BSE required even fewer abscissas, being 32 for the $\psi$ integration and 12 for the $\vartheta$ integration. In and close to the chiral limit, twice as many abscissas have been used for the angular integrals in the BSE, as this region requires some better precision in order to converge. A subroutine to calculate appropriate values for the abscissas and weights can be found in [22].

## C. 2 Iterating the quark DSE

The coupled integral equations for the dressing functions of the quark propagator have been solved iteratively. In detail this means, that we start by initializing a starting value for both $A\left(p^{2}\right)$ and $B\left(p^{2}\right)$ at each momentum site $p$. A fast convergence could be obtained by using the constant initial values

$$
\begin{align*}
& A_{0}=1.6  \tag{C.3}\\
& B_{0}=m_{c}+\Delta m \tag{C.4}
\end{align*}
$$

The exact initial values are not very important, they only play a role in how fast the iteration converge but do not change the result. The only important thing is to include a $\Delta m>0$ term as introduced above, when going to the chiral limit. Omitting it might lead to the iteration running into a fixed point where $B \equiv 0$ and the equations decouple. The exact value for $\Delta m$ is also not too important, but especially when going beyond the Rainbow-Ladder truncation it should not be too small, as this might lead to numerical instabilities in the chiral limit as well. A value of $\Delta m=1 \mathrm{MeV}$ has been used in this thesis and has not led to any major problems.
After the dressing functions have been initialized, they are plugged into the righthand side of the Dyson-Schwinger equation and the integrals are evaluated. The result are new functions $A_{1}\left(p^{2}\right)$ and $B_{1}\left(p^{2}\right)$, which now depend on the momentum $p$. These are then again plugged into the right-hand side of the DSE, yielding the functions $A_{2}\left(p^{2}\right)$ and $B_{2}\left(p^{2}\right)$ and so on. This process is then repeated, until the functions $A_{n}\left(p^{2}\right)$ and $A_{n+1}\left(p^{2}\right)$, or $B_{n}\left(p^{2}\right)$ and $B_{n+1}\left(p^{2}\right)$ respectively, only change by less than a given precision threshold with each new iteration step. In this thesis' calculations a precision of $10^{-9}$ in the ultraviolet has been required in order to terminate the iteration.

## C. 3 Interpolating the quark propagator

The integrals appearing in the $\kappa$-matrix (3.38) contain appearances of the quark's dressing functions evaluated in the complex plane. They occur, when the terms
$S\left(q_{ \pm}\right)$are evaluated. These in turn contain the expressions $q_{ \pm}^{2}$, which, written out explicitly, read

$$
\begin{equation*}
q_{ \pm}^{2}=q^{2}+\frac{1}{4} P^{2} \pm q \cdot P=q^{2}-\frac{1}{4} m^{2} \pm i m q \cos (\psi) \tag{C.5}
\end{equation*}
$$

This expression describes the area bordered by a parabola lying in the complex momentum plane with its vertex at the point $-m^{2} / 4$ on the real axis. The quark propagator can in principle be calculated in the complex plane by knowing it on the real axis. This can be done by taking the Dyson-Schwinger equation and evaluating the integral appearing on its right with a complex momentum argument in the gluon propagator. This is computationally not very efficient though. The integrals in the $\kappa$-matrix require us to do these evaluations a lot of times also. Therefore, we employ a much faster, but slightly less accurate way to evaluate the quark propagator in the complex plane. The idea behind this method is, to pre-calculate the quark propagator on a fixed logarithmic grid in the integration domain first and save the results in a lookup table. When the quark propagator for a specific complex momentum is then required, it can be estimated by doing two steps. First, the four momenta are determined, which span the smallest square, in which the wanted momentum lies. Then, the values for the propagator at these four momenta are used to interpolate to the momentum in question. The first step can be done in logarithmic runtime and thus only takes a minimal amount of time. There are multiple interpolation methods, that can be used to do the second step. In the calculations done in this thesis, a bilinear interpolation method was used. This method approximates the function within the square as a plane.
In order to see how this works explicitly, we want to look at the math behind it. ${ }^{1}$ Say we want to interpolate the function $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$ to a point $a=\left(a_{x}, a_{y}\right)$ and we know that

$$
\begin{align*}
& f_{1}=f\left(x_{1}, y_{1}\right)  \tag{C.6}\\
& f_{2}=f\left(x_{2}, y_{1}\right)  \tag{C.7}\\
& f_{3}=f\left(x_{2}, y_{2}\right)  \tag{C.8}\\
& f_{4}=f\left(x_{1}, y_{2}\right) \tag{C.9}
\end{align*}
$$

are the values of the function at the corners of the minimal square surrounding $a$, i.e. $x_{1} \leq a_{x} \leq x_{2}$ and $y_{1} \leq a_{y} \leq y_{2}$. Then, we can define the values

[^14]\[

$$
\begin{align*}
t & :=\frac{a_{x}-x_{1}}{x_{2}-x_{1}}  \tag{C.10}\\
u & :=\frac{a_{y}-y_{1}}{y_{2}-y_{1}}, \tag{C.11}
\end{align*}
$$
\]

with which we estimate the function at the point $a$ as

$$
\begin{equation*}
f\left(a_{x}, a_{y}\right) \approx(1-t)(1-u) f_{1}+t(1-u) f_{2}+t u f_{3}+(1-t) u f_{4} . \tag{C.12}
\end{equation*}
$$

## C. 4 Numerical root finding

To find the zeros of the function $f(m)=\lambda(m)-1$, the false position method, also called regula falsi, has been used. The method requires a continuous function $f$ given in an interval $\left[x_{1}, x_{2}\right]$ with the condition $f\left(x_{1}\right) \cdot f\left(x_{2}\right)<0$, assuring that a root can be found in the given interval. The idea behind this method is to approximate the function as linear in the interval. The linear approximation crosses the x -axis at a point $x_{3} \in\left(x_{1}, x_{2}\right)$, which then replace one of the original points $x_{1}$ or $x_{2}$, depending on the sign of $f\left(x_{3}\right)$. This procedure is iterated over and over until a certain accuracy $\Delta x$ is reached. The false position method converges faster than most simpler methods, such as the bisection method. There are still methods that are faster, such as the secant method, but some do not converge at all for certain functions, while the false position method always converges. An implementation of this method in $\mathrm{C} / \mathrm{C}++$ can be found in [22]. In the calculations done in this thesis, the implementation from the GNU scientific library (GSL) ${ }^{2}$ has been used though.

## C. 5 Extrapolating eigenvalue curves

As mentioned in section 4.4, the quark propagator has poles in the complex plane, which we can not numerically integrate over without running into problems. Therefore, the $\kappa$-matrix, eq. (3.38), can not be evaluated for too far out in the complex momentum plane. Thus, we are limited to only calculate the eigenvalue of the $\kappa$-matrix up to a meson mass of around 1 GeV . To go to higher meson masses, we used an extrapolation method, called the Schlessinger point method [34]. The idea behind the method is, to take a function, in our case the eigenvalue as a function of the meson mass, which is known at a set of $N$ points $x_{1} \ldots x_{N}$ and fit a rational

[^15]function to it and analytically continue it to points outside the scope, where the function is known. In particular, the function used is of the form
\[

$$
\begin{equation*}
C_{N}(x)=\frac{f_{1}}{1+\frac{a_{1}\left(x-x_{1}\right)}{1+\frac{a_{2}\left(x-x_{2}\right)}{\vdots}}}, \tag{C.13}
\end{equation*}
$$

\]

where $f_{i}=f\left(x_{i}\right)$. The coefficients $a_{i}$ can be calculated by initializing $a_{1}=\frac{f_{1} / f_{2}-1}{x_{2}-x_{1}}$ and then using the iterative formula

$$
\begin{equation*}
a_{i}=\frac{1}{x_{i}-x_{i+1}}\left(1+\frac{a_{i-1}\left(x_{i+1}-x_{i-1}\right)}{1+\frac{a_{i-2}\left(x_{i+1}-x_{i-2}\right)}{\vdots \frac{a_{1}\left(x_{i+1}-x_{1}\right)}{1-f_{1} / f_{i+1}}}}\right) . \tag{C.14}
\end{equation*}
$$

Having calculated the coefficients, we can just find the value for the meson mass, where the extrapolated eigenvalue curve intersects $\lambda=1$. This gives an estimate on the physical meson mass.

## C.5.1 Estimating the numerical error

Extrapolating the eigenvalue curve obviously introduces a numerical error to the results. However, this error can be estimated by doing the following procedure. First, the eigenvalue curve is calculated directly at numerous points below 1 GeV . From these points, a smaller subset of points is chosen. Using this subset of points, the eigenvalue curve is extrapolated and the physical mass is calculated. Then, a different subset of points is chosen and the physical mass is calculated again. Those steps are repeated many times, resulting in a distribution of masses. We can then evaluate the mean and standard deviation of this distribution. The mean is what we use as the result for the physical mass and the standard deviation gives an estimate on the uncertainty introduced by the extrapolation. This standard deviation is what is displayed as error bars in sections 4.4 and 5.1.

## C. 6 Software used

The software used to do the computations done in this thesis has been programmed in $\mathrm{C}++14$ using the JetBrains CLion $\mathrm{IDE}^{3}$ and the $\mathrm{C}++$ compiler from the GNU

[^16]Compiler Collection (GCC). ${ }^{4}$ The sourcecode of this software is available on my github. ${ }^{5}$ The two-dimensional plots of the results have been plotted using gnuplot version 5.2 patchlevel $8 .{ }^{6}$ The three-dimensional plots found in section 4.4 have been created using python3.8.5 ${ }^{7}$ and the matplotlib library. ${ }^{8}$

[^17]
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## Selbstständigkeitserklärung

Hiermit versichere ich, die vorgelegte Thesis selbstständig und ohne unerlaubte fremde Hilfe und nur mit den Hilfen angefertigt zu haben, die ich in der Thesis angegeben habe. Alle Textstellen, die wörtlich oder sinngemäß aus veröffentlichten Schriften entnommen sind, und alle Angaben die auf mündlichen Auskünften beruhen, sind als solche kenntlich gemacht. Bei den von mir durchgeführten und in der Thesis erwähnten Untersuchungen habe ich die Grundsätze guter wissenschaftlicher Praxis, wie sie in der ,Satzung der Justus-Liebig-Universität zur Sicherung guter wissenschaftlicher Praxis' niedergelegt sind, eingehalten. Gemäß $\S 25$ Abs. 6 der Allgemeinen Bestimmungen für modularisierte Studiengänge dulde ich eine Überprüfung der Thesis mittels Anti-Plagiatssoftware.

Gießen, den 22. Dezember 2020
(Unterschrift)


[^0]:    ${ }^{1}$ Using natural units, i.e. $\hbar=c=1$. For more information about the conventions used, chapter A in the appendix can be consulted.

[^1]:    ${ }^{2}$ The notation $\mathrm{d}^{n} q$ is an abbreviation for $\mathrm{d}^{n} q /(2 \pi)^{n}$.

[^2]:    ${ }^{3}$ We have to be careful though when choosing a regulator for divergent loop diagrams, as some regulators, such as a hard UV cutoff break Lorentz invariance.

[^3]:    ${ }^{1}$ Here we have added a subscript UV added to the cutoff so it is not to be confused with the interaction width $\Lambda$ in $\alpha\left(k^{2}\right)$

[^4]:    ${ }^{2}$ All functions in this equation also depend on the total momentum $P$. We have left them out for the sake of readability though.

[^5]:    ${ }^{3}$ Since we are setting the total momentum on-shell, we can relate it to the meson mass $m$ via $P^{2}=-m^{2}$.

[^6]:    ${ }^{4}$ Since this structure's dressing function is called $E(p)$, we are going to label it with an E synonymously.
    ${ }^{5}$ For a proof of this, section B. 3 can be consulted.

[^7]:    ${ }^{1}$ In fact, the set of equations arising from the Rainbow-Ladder truncation can be solved within a few seconds on a standard desktop CPU.

[^8]:    ${ }^{2}$ During this derivation, wherever lowercase subscripts with latin letters appear, they explicitly mark dirac indices.

[^9]:    ${ }^{3}$ Since the Rainbow-Ladder kernel is known by now, it is not written out once more.

[^10]:    ${ }^{4}$ Again, we are only focusing the tensor structures $L_{\mu}^{1}$ and $L_{\mu}^{3}$.

[^11]:    ${ }^{5}$ Here we are deviating from our convention of using $l^{\mu}$ as the sum of incoming and outgoing momenta to make it obvious, where the $p$ in the labels $\gamma p$ and $p p$ comes from.
    ${ }^{6}$ For more information about gamma matrices, section A. 2 in the appendix can be consulted.

[^12]:    ${ }^{7}$ The eigenvalues of the $\kappa$-matrix as a function of the meson mass.

[^13]:    ${ }^{1}$ Since the lightest pseudoscalar meson in nature is the pion, the subscript $\pi$ is used to label pseudoscalar quantities.

[^14]:    ${ }^{1}$ Even though the example shown here is for a function mapping from a two dimensional real space to the real numbers, it can canonically be generalized to a function $S: \mathbb{C} \rightarrow \mathbb{C}$.

[^15]:    ${ }^{2}$ https://www.gnu.org/software/gsl/

[^16]:    ${ }^{3}$ Student's license, https://www.jetbrains.com/

[^17]:    ${ }^{4}$ https://gcc.gnu.org/
    ${ }^{5}$ https://github.com/StHagel/MSc_code
    ${ }^{6} \mathrm{http}: / /$ www.gnuplot.info/
    ${ }^{7}$ https://www.python.org/
    ${ }^{8}$ https://matplotlib.org/

